To the left of the sphere spectrum[1]

§1. Bivariant theories & correspondences

1.1 Let $E \to F$ be a morphism of nice (eg symmetric) commutative ring spectra.

Definition The (enriched) category Corr_E of E-correspondences has finite CW-spectra X, Y as objects, with morphisms

$$[X, Y]_E := X^D \wedge Y \wedge E \sim [X, Y \wedge E] \; ;$$

where $X^D = [X, S]$ is the Spanier-Whitehead dual. If $f \in [X, Y]_E$ and $g \in [Y, Z]_E$ then

$$g \circ f : X \to Y \wedge E \to Z \wedge E \wedge E \to Z \wedge E \; .$$

is the composition $(1 \wedge m_E) \cdot (g \wedge 1_E) \cdot f$. This is a symmetric monoidal category, with a concretification enriched over graded abelian groups defined by taking homotopy groups of its morphism spectra. The functor which is the identity on objects, and the obvious map

$$[X, Y \wedge E] \to [X, Y \wedge F]$$

on morphisms, preserves the monoidal structure.

[[On the trip up the Rhine K. Hess explained to me that these are Kleisi categories (as described in MacLane's category textbook; she credits HRM with emphasizing their interest.]]

1.2 This is close to, but not quite the same as, the classical definition, which starts with something like compact closed oriented manifolds as objects. The graph of a map $f : X \to Y$ defines a class

$$(\text{graph of } f)_!(1) \in H^{\dim Y}(X \times Y, k) \cong H_*(X) \otimes H^*(Y) \cong [X, Y \wedge H(k)]$$

by Poincaré duality (taking k to be a field).

This construction encodes extra data (eg orientations) in a homotopy-theoretic framework; it also fits nicely with duality. There are **many** variations:

- Restrict to a subclass of morphisms (eg algebraic maps)
- Invert objects, add kernels to projections, ...

More generally, any suitable **bivariant** theory suggests an associated category of correspondences; cf early work of Goresky-McPherson based on algebraic

[1] Based on recent (last 2-3 years) conversations with HRM, but with roots going back to conversations of us both with John Moore, forty years ago. Maybe some old wine in a spiffy new plastic bottle ...

K-theory; more recently Connes, Consani, and Marcolli use Kasparov's KK-theory to study noncommutative motives. I've been inspired by work of Bruce Williams (on A-theory) and Dundas-Østvær (on TC).

§2 Change of rings

2.1 Tannakian theory studies the automorphism group of the monoidal functor

$$X \mapsto X_F : \mathrm{Corr}_E \to \mathrm{Corr}_F \ ,$$

(or rather of its concretification), by trying to represent the functor defined by varying F through ring spectra flat above it. There is a natural homotopy-theoretic candidate for such a representing object, given by the comonoid

$$(\overline{F}, \overline{F} \wedge_E \overline{F})$$

in ring spectra; where

$$E \to \overline{F} \to F$$

(thanks to JHS) is a factorization through a cofibration (ie, some kind of Adams/bar/cobar construction) followed by a weak equivalence. This is an analog of the Hopf algebroids in Adams' blue book; comultiplication, for example, is the composition

$$\overline{F} \wedge_E \overline{F} \to \overline{F} \wedge_E E \wedge_E \overline{F} \to \overline{F} \wedge_E \overline{F} \wedge_E \overline{F}$$

$$= (\overline{F} \wedge_E \overline{F}) \wedge_{\overline{F}} (\overline{F} \wedge_E \overline{F}) \ .$$

When $E = S$ and $F = MU$ this is classical, but I want to focus on cases at the opposite extreme: Waldhausen A-theory (equivalently: the K-theory of the sphere spectrum)

$$A = S \vee \mathrm{Wh} \to S$$

(Wh is Waldhausen's Whitehead spectrum) and the topological cyclic homology

$$TC = S \vee \Sigma \mathbb{C}P^{\infty}_{-1} \to S$$

of S (up to a profinite completion).

2.2 The Tannakian principle that a functor takes values in a category of representations of its own automorphism group implies very generally that there is a lift

$$\begin{tikzcd} & (\overline{F} \wedge_E \overline{F} - \mathrm{Comod\ in\ Corr}_F) \arrow[d] \\ \mathrm{Corr}_E \arrow[ur, dotted] \arrow[r] & \mathrm{Corr}_F \end{tikzcd} .$$

This uses the ring homomorphism

$$E \wedge_S E \to \overline{F} \wedge_E \overline{F}$$

to define the composition

$$[X, Y \wedge E] \to [X, Y \wedge E \wedge E] = [X, Y \wedge E] \wedge_E (E \wedge E) \to$$

$$\to [X, Y \wedge \overline{F}] \wedge_{\overline{F}} (\overline{F} \wedge_E \overline{F}) \to [X, Y \wedge F] \wedge_{\overline{F}} (\overline{F} \wedge_E \overline{F}) \ .$$

In interesting cases this leads to a 'descent' spectral sequence

$$\mathrm{RHom}_{\overline{F} \wedge_E \wedge \overline{F} - \mathrm{Comod}}(X_{\overline{F}}, Y_{\overline{F}}) \Rightarrow [X, Y]_E \ ,$$

with Hess-style coinvariants [cf also Rognes] of a suitable cofibrant replacement for $X^D \wedge Y \wedge F$ on the left.

[[Thanks to HRM and B. Richter for pointing out that Rognes' Hopf-Galois objects are ring spectra with E_∞ coproduct. This issue needs much more attention in my fantasies.]]

2.3 Here's a classical example:

Homology with $\mathbb{F}_p$ coefficients is a monoidal homological functor from the tensor triangulated category $D(\mathbb{Z}_p - \mathrm{Mod})$ to graded vector spaces. The Bockstein operation

$$\beta : H_*(-, \mathbb{F}_p) \to H_{*+1}(-, \mathbb{F}_p)$$

defines a coaction of the elementary Hopf algebra $E(\beta)$, so we can describe the mod p homology as a representation of a super-groupscheme $\mathrm{Spec}(E(\beta)) \rtimes \tilde{\mathbb{G}}_m$, and there is an associated 'descent' spectral sequence

$$\mathrm{RHom}_{E(\beta)-\mathrm{Comod}}(H_*(X, \mathbb{F}_p), H_*(Y, \mathbb{F}_p) \Rightarrow \mathrm{RHom}_{D(\mathbb{Z}_p)}(X, Y) \ .$$

There is a similar story for more general local rings $A \to k$, going back to Tate in the 50's, with Hopf algebra

$$\mathrm{Tor}^*_A(k, k) = k \otimes^L_A k$$

generalizing $E(\beta) = \mathrm{Tor}^*_{\mathbb{Z}_p}(\mathbb{F}_p, \mathbb{F}_p)$. In the equi-characteristic case, when $A = k \oplus I$, this Tor can be calculated as the homology of a bar construction; for a square-zero extension it's the cotensor algebra on I, suitably graded. A similar but dual calculation identifies $\mathrm{Ext}^*_A(k, k)$ as a tensor algebra on the dual of I.

§3 What good is this?

To explain what all this might be used for requires some digressions, which unfortunately run in opposite directions:

3.1 The first comes the **arithmetic** theory of motives. **Geometric** motives start with projective (complete) varieties over a nice field k, with morphisms defined by correspondences coming from algebraic cycles; the Hom objects are suitable quotients of

$$\mathrm{gr}^* K^{\mathrm{alg}}(X \times Y) \otimes \mathbb{Q} \ .$$

One constructs from this a **semisimple** $\mathbb{Q}$-linear abelian tensor category of **pure** motives, equivalent to the category of representations of some motivic group-scheme.

There is a generalization to a category of **mixed** motives, built from **quasi**-projective varieties (roughly, $X - Y$ for X, Y complete). This is a nice category, but no longer semisimple: it has nontrivial extensions. For example, in this category projective space

$$\mathbb{P}^n = 1 \oplus L \oplus \ldots \oplus L^{\otimes n}$$

decomposes into a sum of CW-cell analogs.

In the early 80's Deligne proposed the study of an abelian category of arithmetic or **mixed Tate** motives, associated to a very restricted class of geometric objects over $\mathbb{Z}$: iterated extensions of the $L^{\otimes n} \sim \mathbb{Z}(n)$ as above. The existence of a conjectured spectral sequence

$$\mathrm{Ext}^*_{\mathbf{mtm}}(\mathbb{Z}(0), \mathbb{Z}(n)) \Rightarrow K^{\mathrm{alg}}_{2n-*}(\mathbb{Z}) \otimes \mathbb{Q}$$

was proved recently by Deligne and Goncharov.

[[These notes very sloppily confuse triangulated categories of motives with their (sometimes hypothetical) abelian hearts. Deligne's work sees the cohomology theories of arithmetic geometry (l-adic, p-adic, Archimedean ...) as analogs of the Euler factors of zeta-functions; Voevodsky, on the other hand, enlarges the field of play over a field by model and derived category techniques.]]

This seems strikingly like the Adams spectral sequence

$$\mathrm{Ext}^*_{\Psi'\mathrm{s}}(K(S^0), K(S^n)) \Rightarrow \mathrm{Im}\ J_*$$

for K-theory: the target groups of the Deligne-Goncharov sseq are generated (via Borel regulators) by the zeta values $\zeta(1 + 2k)$, while J_{2k-1} is roughly the cyclic group

$$\langle \zeta(1 - 2k) \rangle \subset \mathbb{Q}/\mathbb{Z} \ .$$

It is tempting [JM, Newton §4.7] to interpret the Deligne-Goncharov theorem as a change-of-rings spectral sequence for

$$K^{\mathrm{alg}}(\mathbb{Z}) \otimes \mathbb{Q} \to K^{\mathrm{top}}(\mathbb{C}) \otimes \mathbb{Q} \ .$$

3.2 The other direction of interest comes from differential topology (cf. Igusa's book, or the 06 Talbot workshop): the homotopy groups

$$\pi_{2i-1}(\mathrm{Diff}(D^{2n+1} \text{ rel } \partial)) \otimes \mathbb{Q} \cong K^{\mathrm{alg}}_{2i+1}(\mathbb{Z}) \otimes \mathbb{Q}$$

of the group of diffeomorphisms of a high odd-dimensiona disk (fixing the boundary) are isomorphic (via higher Reidemeister torsion invariants **also** related to odd zeta-values) to the algebraic K-theory of the integers; more precisely,

$$B\mathrm{Diff}_c(\mathbb{R}^{\mathrm{odd}}) \sim \Omega\mathrm{Wh} \ .$$

One wonders what the algebraic K-theory of $\mathbb{Z}$ could have to do with differential topology; Waldhausen's answer

$$K(\mathbb{Z}) \otimes \mathbb{Q} \ \sim \ K(S) \otimes \mathbb{Q}$$

is that rationally it's the same as the K-theory of the sphere spectrum.

Turning this around, one can ask if the category of mixed Tate motives might be detecting not the K-theory of the integers, but the K-theory of S. This suggests regarding (some version of) the category of A-correspondences as a category of **base** motives, characterized by a ring-change spectral sequence

$$\mathrm{RHom}^*_{\overline{S} \wedge_A \overline{S}-\mathrm{Comod}}(S^0, S^n) \Rightarrow A_*(S^n)$$

associated to the Waldhausen-Bökstedt trace morphism $A \to S$. What's at issue seems not to be the existence of such a spectral sequence, but its relation, if any, to its purported arithmetic analog: that is, the existence of a functor assigning something like an underlying space to a mixed Tate motive.

[[This hypothetical correspondence would seem to associate to a stable disk bundle over the $2i$-sphere, something like the Thom complex of a vector bundle over the S^{2i+3}-sphere. Perhaps someone with more geometric smarts than me will be able to explain the anomalous factor of three . . .]]

3.3 Arguably the strongest algebraic evidence for such a connection comes from these categories' motivic groups. This becomes clearer if we work not with A but with the closely related topological cyclic homology of S. [There is a cofibration

$$j \vee \Sigma^{-2} kO \to \mathrm{Wh} \to \Sigma \mathbb{C}P^\infty_{-1}$$

at regular odd primes [Rognes].]

I can't say I know what to do about the negative-dimensional cell in the latter spectrum, but it seems plausible that **something like**

$$\overline{S} \wedge_{TC} \overline{S} \ \sim \ \text{cotensor algebra on } \Sigma \mathbb{C}P^\infty_{-1}$$

(and, correspondingly)

$$\mathrm{Hom}_{TC}(\overline{S}, \overline{S}) \ \sim \ \text{tensor algebra on } \Sigma \mathbb{C}P^\infty_{-1}$$

might be the case. The objects on the right are closely related to work of Baker and Richter, who show that the homology of the ring-spectrum $S[\Omega \Sigma \mathbb{C}P^\infty_+]$ is the universal enveloping algebra of a graded free Lie algebra, dual to the algebra of quasi-symmetric functions.

Over $\mathbb{Q}$, this is about twice the size of the Hopf algebra of the pro-unipotent group Deligne associates to the category of mixed Tate motives; it's closer to the algebra appearing in the Connes-Kreimer-Marcolli theory of renormalization. The A-theoretic version has a Lie algebra closer to Deligne's, whose generators are expected to correspond somehow to the odd zeta-values.