Recent Investigations into Inconsistency: from Measurement to Reasoning

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Part I (joint work with B. Raddaoui and S. Jabbour)

Inconsistency measure

$$\mathcal{I} : K \mapsto [0, \infty]$$

that satisfies some requirements, e.g.,

- if K is consistent, then $\mathcal{I}(K) = 0$

Inconsistency measures - two rationales

Inconsistency measures may reflect ...

1. *minimizing cost and minimal intervention to fix* a knowledge base
2. *structural transparency and diagnosis* of a knowledge base

Example

- $K_1 = \{\neg p, p\}$
- $K_2 = \{\neg p, p, q, \neg(p \wedge q)\}$

1. minimal costs: Minimal repair: remove p
2. structural diagnosis: structurally K_2 seems more conflicting and we expect that it has more inconsistency weight

This is informal talk, but we'll soon capture these intuitions in formal properties.

Some terminology … in funny notation

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- *problematic formulas* ("gangsters"): 🤠$(K) = \{p, \neg p\}$

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- *minimal conflicts* ("gangs"): 👥$(K) = \{\{p, \neg p\}\}$
- *free formulas* ("innocent bystanders"): 👼$(K) = \{q\}$

Some simple measures

- $\mathcal{I}_d : K \mapsto \begin{cases} 0 & K \not\vdash \bot \quad \text{(no crime)} \\ 1 & K \vdash \bot \quad \text{(crime)} \end{cases}$

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- $👼(K) = \{q\}$
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- $\mathcal{I}_{👥}(K) = \frac{1}{2}$

Some core desiderata for inconsistency measures

Consistency

🤠$(K) = \emptyset \implies \mathcal{I}(K) = 0$ “no 🤠, no 💀”

Guiltlessness

🤠$(K) \neq \emptyset \implies \mathcal{I}(K) > 0$ “if 🤠, then 💀”

Together

🤠$(K) = \emptyset \iff \mathcal{I}(K) = 0$ “🤠, iff, 💀”

Monotonicity

$\mathcal{I}(K) \subseteq \mathcal{I}(K \cup K')$ “more citizens, at least as much 💀”

Free Formula Independence

$K' \subset$ 👼$(K \cup K') \implies \mathcal{I}(K \cup K') = \mathcal{I}(K)$ “👼 cause no 💀”

Back to our example

- $K_1 = \{\neg p, p\}$
- $K_2 = K_1 \cup \{q, \neg(p \wedge q)\}$

1. rationale minimal repair costs: $\mathcal{I}_{\text{hit}}(K_1) = \mathcal{I}_{\text{hit}}(K_2) = 1$
   - where $\mathcal{I}_{\text{hit}}(K)$ is the minimal amount of arrests it takes to get 💀-free (ie., consistent)
   - equivalently: $\mathcal{I}_{\text{hit}}(K) = \min\{|M| : M$ is a hitting set over 👥$(K)\}$
   - 👥$(K_1) = \{\{\neg p, p\}\} \subset$ 👥$(K_2) = \{\{\neg p, p\}, \{p, q, \neg(p \wedge q)\}\}$

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2. rationale structural diagnosis:
   - strict monotonicity: strictly more 🤠 $\implies$ strictly more 💀. [1]
   - formally: 🤠$(K) \subset$ 🤠$(K \cup K') \implies \mathcal{I}(K) < \mathcal{I}(K \cup K')$

[1] A similar property has been proposed by Matthias (2009) under the name *Penalty*.

Are there other properties tracking structural diagnosis of inconsistency?

Gradual scaling (integer-based measures)

Gradual Scaling

For each weight degree, there is a *witnessing set.*

$$\forall (k \leq \mathcal{I}(K))\ \exists (S \subseteq K) : \mathcal{I}(K \setminus S) = k$$

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- This comes natural for correction-based measures. For instance, $\mathcal{I}_{\text{hit}}$ fulfills it.
- But $\mathcal{I}_{\#} : K \mapsto |$👥$(K)|$ violates it.
- Take: $K = \{p, \neg p, \neg\neg p, \neg\neg\neg p\}$.
- So, 👥$(K) = \{\{p, \neg p\}, \{p, \neg\neg\neg p\}, \{\neg p, \neg\neg p\}, \{\neg\neg p, \neg\neg\neg p\}\}$.
- So, $\mathcal{I}_{\#}(K) = 4$, but there is no $S \subset K$ s.t. $\mathcal{I}_{\#}(K \setminus S) = 3$.

A weaker variant: Fairness

Fairness

If $\mathcal{I}(K) = \mathcal{I}(K')$, then for all $S \subset K$ there is a $S' \subset K'$: $\mathcal{I}(K \setminus S) = \mathcal{I}(K' \setminus S')$.

- "Same 💀 rate, same clustering".

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- "Same 💀 rate, same clustering".
- Gradual scaling implies Fairness.

Separability

Strict Separability (Jabbour et al., 2016):

"If K and K′ don't share 🤠 and 👥, then 💀 separates."[2]

$$🤠(K) \cap 🤠(K') = \emptyset \quad \text{and} \quad 👥(K \cup K') = 👥(K) \cup 👥(K')$$
$$\implies \mathcal{I}(K \cup K') = \mathcal{I}(K) + \mathcal{I}(K')$$

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Counter-example:

- $\mathcal{I}_{\text{ms}} : K \mapsto (|\text{ms}(K)| + |\bot(K)|) - 1$, where $\text{ms}(K)$ are maximal consistent subsets of K and $\bot(K)$ are contradictions in K.
- $\mathcal{I}_{\text{ms}}(\{p, \neg p\}) = 1 = \mathcal{I}_{\text{ms}}(\{q, \neg q\})$ but $\mathcal{I}_{\text{ms}}(\{p, \neg p, q, \neg q\}) = 3$.

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Separability (2/2)

ϕ separates K if

- $\phi \notin K$ and
- for all $S \in$ 👥$(K \oplus \phi)$, either $S \ominus \phi \subseteq$ 🤠(K) or $S \ominus \phi \subseteq$ 👼(K).

Every time ϕ forms a 👥, it either only uses 🤠 or only 👼 from K .
In other words, ϕ does not involve 🤠 with 👼 in a 👥.

- $K = \{p, \neg p, \neg(p \wedge q)\}$ and $\phi = q$
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- q does not separate K. Why?
- 👼$(K) = \{\neg(p \wedge q)\}$ and 🤠$(K) = \{p, \neg p\}$
- $\{p, q, \neg(p \wedge q)\} \in$ 👥$(K \oplus q)$ where $p \in$ 🤠(K) and $\neg(p \wedge q) \in$ 👼(K).
- q mixes up 🤠 and 👼 in a newly formed 👥.

Example

- Let $K = \{\neg u, \neg p, \neg\neg p, \neg\neg\neg p\}$ and $\phi = p \wedge u$.
- $p \wedge u$ separates K into
- $K_1 = \{p \wedge u, \neg u\}$ (new 🤠) and
- $K_2 = \{p \wedge u, \neg p, \neg\neg p, \neg\neg\neg p\}$ (old 🤠)

Formula Separability

Let $K_1 = \text{🤠}(K \oplus \phi) \setminus \text{🤠}(K)$ (“newly converted 🤠”) and $K_2 = \bigcup_{\psi \in \text{🤠}(K)} \text{👥}(\psi, K \oplus \phi)$ (“veteran 🤠”).

If ϕ separates K, then $\mathcal{I}(K \oplus \phi) = \mathcal{I}(K_1) + \mathcal{I}(K_2)$.

Idea: *We measure the two separated sets of problematic formulas.*

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Counter-example

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- Recall: $K_1 = \{p \wedge u, \neg u\}$ (new 🤠) and $K_2 = \{p \wedge u, \neg p, \neg\neg p, \neg\neg\neg p\}$ (old 🤠)
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- Recall: $K_1 = \{p \wedge u, \neg u\}$ (new 🤠) and $K_2 = \{p \wedge u, \neg p, \neg\neg p, \neg\neg\neg p\}$ (old 🤠)
- $\mathcal{I}_{\text{hit}}(K_1) = 1$ and $\mathcal{I}_{\text{hit}}(K_2) = 2$,
- but $\mathcal{I}_{\text{hit}}(K \oplus (p \wedge u)) = 2$ (for the repair remove $\{p \wedge u, \neg\neg p\}$).

Let Structural Additivity be the joint property of:

- strict monotonicity
- gradual scaling
- strict separability
- formula separability

Theorem

Consistency, Guiltlessness, Free Formula Independence, Monotonicity and Structural Additivity are compatible.

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- How? We'll propose a concrete correction-based measure.
- … but let's first see how measures from the literature perform.

How do Measures from the Literature perform?

$$\mathcal{I}_d(K) = \begin{cases} 0 & \text{if } |\text{🤠}(K)| = 0 \\ 1 & \text{else} \end{cases} \qquad \mathcal{I}_\#(K) = |\text{👥}(K)|$$

$$\mathcal{I}_{\text{👥}}(K) = \sum_{M \in \text{👥}(K)} \frac{1}{|M|} \qquad \mathcal{I}_{\text{🤠}}(K) = |\text{🤠}(K)|$$

$$\mathcal{I}_{\mathsf{hit}}(K) = \min_{M \in \mathsf{mc}(K)} |M| \qquad \mathcal{I}_{\mathsf{mcsc}}(K) = |K| - |\bigcap \mathcal{C}|$$

$$\mathcal{I}_{\mathsf{ms}}(K) = (|\mathsf{ms}(K)| + |\bot(K)|) - 1$$

Concerning: $\mathcal{I}_{\mathsf{mcsc}}$ (Ammoura et al., 2017): where $\mathcal{C}$ is a $\subset$-minimal family of minimal corrections sets such that $\bigcup \mathcal{C} = \text{🤠}(K)$, and among these it maximizes $|\bigcap \mathcal{C}|$.

Overview

Property	$\mathcal{I}_d$	$\mathcal{I}_\#$	$\mathcal{I}_{👥}$	$\mathcal{I}_{🤠}$	$\mathcal{I}_{\text{hit}}$	$\mathcal{I}_{\text{ms}}$	$\mathcal{I}_{\text{mcsc}}$	$\mathcal{I}_{\text{scs}}$
Consistency	✓	✓	✓	✓	✓	✓	✓	✓
Guiltlessness	✓	✓	✓	✓	✓	✓	✓	✓
Free Formula Independence	✓	✓	✓	✓	✓	✓	✓	✓
Monotonicity	✓	✓	✓	✓	✓	✓	✓	✓
Strict Monotonicity	×	✓	✓	✓	×	×	×	✓
Strict Separability	×	✓	✓	✓	✓	×	✓	✓
Fairness	✓	×	×	×	✓	×	×	✓
Gradual Scaling	✓	×	×	×	✓	×	×	✓
Formula Separability	×	✓	✓	×	×	×	×	✓

So, what's the mysterious $\mathcal{I}_{\mathsf{SCS}}$?

Correction Set (CS) A set $S \subseteq$ 🤠(K) is a *correction set* in K iff $K \setminus S \nvdash \bot$.

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Sequential Correction Sets (SCS). An ordered correction set $S = \{\phi_1, \ldots, \phi_n\}$ is a *sequential correction set* of K, iff,

$$\emptyset = \text{👥}(K \setminus S) \subset \ldots \subset \text{👥}(K \setminus \{\phi_1, \ldots, \phi_i\}) \subset \text{👥}(K \setminus \{\phi_1, \ldots, \phi_{i-1}\}) \subset \ldots \subset \text{👥}(K)$$

Example

Let

$$K = \{\, p,\ \neg p \wedge q,\ \neg q \wedge \neg r,\ r \,\}$$

- Consider $S_1 = \{p, \neg p \wedge q, \neg q \wedge \neg r\}$. Then,

$$\underbrace{\text{👥}(K \setminus S_1)}_{\emptyset} \subset \underbrace{\text{👥}(K \setminus \{p, \neg p \wedge q\})}_{\{\{\neg q \wedge \neg r, r\}\}} \subset \underbrace{\text{👥}(K \setminus \{p\})}_{\left\{\begin{matrix}\{\neg p \wedge q, \neg q \wedge \neg r\},\\ \{\neg q \wedge \neg r, r\}\end{matrix}\right\}} \subset \underbrace{\text{👥}(K)}_{\left\{\begin{matrix}\{p, \neg p \wedge q\},\\ \{\neg p \wedge q, \neg q \wedge \neg r\},\\ \{\neg q \wedge \neg r, r\}\end{matrix}\right\}}$$

- So, S_1 is an SCS.

Example

Let

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- Consider $S_1 = \{p,\ \neg p \wedge q,\ \neg q \wedge \neg r\}$. Then,

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- Consider $S_2 = S_1 \setminus \{p\}$. It is a minimal correction set.

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- Consider $S_1 = \{p, \neg p \wedge q, \neg q \wedge \neg r\}$. Then,

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- So, S_1 is an SCS.
- Consider $S_2 = S_1 \setminus \{p\}$. It is a minimal correction set.
- Also, $\emptyset = \text{👥}(K \setminus S_2) \subset \text{👥}(K \setminus \{\neg p \wedge q\}) \subset \text{👥}(K)$. So, S_2 is an SCS.

Some properties

- Every minimal correction set is an SCS (for any ordering of formulas), but not vice versa.

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- Every minimal correction set is an SCS (for any ordering of formulas), but not vice versa.
- $\mathrm{SCS}(K) = \emptyset$ iff 🤠$(K) = \emptyset$.

Two Competing Rationales for an inconsistency measure

1. *minimizing cost and minimal intervention to fix*
   - minimal amounts of repairs to fix K
   - other measures in this spirit: minimum hitting sets ($\mathcal{I}_{\text{hit}}$),
2. *structural transparency and diagnosis*
   - strict monotonicity: $\mathcal{I}_{\#}$, $\mathcal{I}_{🤠}$, $\mathcal{I}_{👥}$
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While aiming for (2) it also bridges the repair idea with the aim for structural transparency.

Theorem

$\mathcal{I}_{\text{SCS}}$ satisfies Consistency, Guiltlessness, Free Formula Independence, Monotonicity, and Structural Additivity.

A link to graph theory

- Recall: *hypergraphs* have the structure $\mathcal{H} = (V, E)$ with $E \subseteq 2^V \setminus \{\emptyset\}$.

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- A set $\mathcal{E} \subseteq E$ is α-cyclic iff for all $x \in \mathcal{E}$, $x \subseteq \bigcup \mathcal{E} \setminus \{x\}$.
- A *minimum feedback hyperedge set* on $\mathcal{H}$ is a minimum subset E' of E such that $E \setminus E'$ is α-acyclic. (NP-hard)
- Complementary problem: find a maximal α-acyclic subset of E

The link to hypergraph theory

- Our hypergraph: $(K,$ 👥$(K))$.
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The link to hypergraph theory

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Theorem

$\mathcal{I}_{\text{SCS}}(K) = m$ where $m = |Q|$ and $Q \subseteq \text{👥}(K)$ is maximally acyclic in $(K, \text{👥}(K))$.

Examples (1/2)

We look back at $K = \{\, p,\ \neg p \wedge q,\ \neg q \wedge \neg r,\ r \,\}$. We have 👥$(K) = \{M_1, M_2, M_3\}$ with

- $M_1 = \{p, \neg p \wedge q\}$
- $M_2 = \{\neg p \wedge q, \neg q \wedge \neg r\}$
- $M_3 = \{\neg q \wedge \neg r, r\}$

Note, 👥(K) is acyclic. (For instance, $M_1 \not\subseteq M_2 \cup M_3$.)

- So, $\mathcal{I}_{\text{scs}}(K) = 3 = |$👥$(K)| = \mathcal{I}_{\#}(K)$.

Examples (2/2)

Consider $K = \{\phi_1 : p \wedge q, \phi_2 : \neg p \wedge q, \phi_3 : p \wedge \neg q, \phi_4 : \neg p \wedge \neg q\}$.

- We have 👥$(K) = \{M_{ij} \mid j \neq j\}$ where $M_{ij} = \{\phi_i, \phi_j\}$.
- 👥(K) is cyclic.
- each 👥$(K) \setminus \{M_{ij}\}$ is acyclic
- so, $\mathcal{I}_{\text{scs}}(K) = 3$, while $\mathcal{I}_{\#}(K) = 4$ and $\mathcal{I}_{\text{hit}}(K) = 2$.

Results

Corollary.

1. $\mathcal{I}_{\text{scs}}(K) = \mathcal{I}_{\#}(K)$ if $(K,$ 👥$(K))$ is acyclic.
2. $\mathcal{I}_{\text{scs}}(K) \leq \mathcal{I}_{\#}(K)$.

In sum:

- $\mathcal{I}_{\text{hit}}(K) \leq \mathcal{I}_{\text{scs}}(K) \leq \mathcal{I}_{\#}(K)$.

Part II (joint work with O. Arieli and B. Raddaoui)

Free formulas, how innocent are you?

Let $K = \{m, \neg m, e, m \vee p\}$

- Fred: The museum is open. (m)
- Wilma: The museum is closed. ($\neg m$)
- There is an event today. (e)
- Bernie: The museum or the park is open. ($m \vee p$)

We have

- 🤠$(K) = \{m, \neg m\}$
- 👼$(K) = \{e, m \vee p\}$
- How innocent is $m \vee p$?

Example 2

Let $K = \{m, \neg m \wedge s, m \wedge \neg s, \neg m \vee \neg s, a\}$

- The patient has migraine. (m)
- No migraine, but stroke. ($\neg m \wedge s$)
- No stroke, but migraine. ($m \wedge \neg s$)
- Not migraine and stroke. ($\neg m \vee \neg s$)
- The patient is 15y old. (a)

We have:

- 🤠$(K) = \{m, \neg m \wedge s, m \wedge \neg s\}$
- 👼$(K) = \{a, \neg m \vee \neg s\}$
- How innocent is $\neg m \vee \neg s$?

Reasoning with inconsistent information (1/2)

1. Rescher and Manor, for instance:
    - $K \mid\!\sim^{\text{👼}} \phi$ iff 👼$(K) \vdash \phi$
    - $K \mid\!\sim^{\text{strong}} \phi$ iff $K' \vdash \phi$ for all $K' \in \mathrm{ms}(K)$

Example:

- $\{m, \neg m, e, m \vee p\} \mid\!\sim^{\text{👼}} (e \wedge m) \vee (e \wedge p)$
- $\{m, \neg m \wedge s, m \wedge \neg s, \neg m \vee \neg s, a\} \mid\!\sim^{\text{👼}} (a \wedge \neg m) \vee (a \wedge \neg s)$

Reasoning with inconsistent information (2/2)

2. paraconsistent logic
   - LP (3-valued, $0 < i < 1$), $\vee \approx \max$, $\wedge \approx \min$ and

ϕ	$\neg\phi$
0	1
i	i
1	0

   - minimal models:
     - $M \prec M'$ iff $i(M) \subset i(M')$, where $i(M) = \{p \in \text{Atoms} \mid v_M(p) = i\}$.
     - $K \mid\!\sim^{LP} \phi$ iff $v_M(\phi) \in \{1, i\}$ for all $M \in \min_{\prec}(\text{MOD}(K))$.

Example

For $K = \{m, \neg m, e, m \vee p\}$:

	m	p	e	
M_1	i	1	1	minimal
M_2	i	0	1	minimal
M_3	i	i	1	
M_4	i	1	i	
M_5	i	0	i	
M_6	i	i	i	

So, $K \mid\!\sim_{LP} m \wedge \neg m \wedge e$.

Towards even more cautious reasoning via filtering

- upper limit: free formulas

We filter free sets as follows:[3]

- 👼$^{\forall}_{CL\downarrow LP}(K) = \{\phi \in$ 👼$_{CL}(K) \mid \forall M \in \min^{LP}_{\prec}(K), v_M(\phi) = 1\}$

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And then define:

- $K \mid\!\sim^{\forall}_{CL\downarrow LP} \phi$ iff $\text{👼}^{\forall}_{CL\downarrow LP} \vdash_{CL} \phi$
- $K \mid\!\sim^{\exists}_{CL\downarrow LP} \phi$ iff $\text{👼}^{\exists}_{CL\downarrow LP} \vdash_{CL} \phi$

[3] This can be generalized to other logics instead of CL and LP.

Example

For $K = \{m, \neg m, e, m \vee p\}$:

	m	p	e	$m \vee p$
M_1	i	1	1	1
M_2	i	0	1	i

So,

- $K \not\mid\!\sim^{\forall}_{CL\downarrow LP} m \vee p$
- $K \mid\!\sim^{\exists}_{CL\downarrow LP} m \vee p$
- $K \mid\!\sim^{\forall}_{CL\downarrow LP} e$
- $K \mid\!\sim^{\exists}_{CL\downarrow LP} e$

More refined example

$$K = \{p \wedge \neg q,\ \neg p \wedge q,\ p \vee q,\ p \vee s,\ r\}$$

- 🤠$(K) = \{p \wedge \neg q,\ \neg p \wedge q\}$
- 👼$(K) = \{p \vee q,\ p \vee s,\ r\}$

Minimal models:

	p	q	s	r	$p \vee s$	$p \vee q$
M_1	i	i	1	1	1	i
M_2	i	i	0	1	i	i

We have:

- $K \mid\!\sim^{\forall}_{CL\downarrow LP} r$ and $K \mid\!\sim^{\exists}_{CL\downarrow LP} r$
- $K \not\mid\!\sim^{\forall}_{CL\downarrow LP} p \vee s$ but $K \mid\!\sim^{\exists}_{CL\downarrow LP} p \vee s$
- $K \not\mid\!\sim^{\exists}_{CL\downarrow LP} p \vee q$ and $K \not\mid\!\sim^{\forall}_{CL\downarrow LP} p \vee q$

A filtering variant

- $M \sqsubset M'$ iff $i^\star(M) \subset i^\star(M')$, where $i^\star(M) = \{\phi \in K \mid v_M(\phi) = i\}$
- 👼$^{\star\forall}_{CL\downarrow LP}(K) = \{\phi \in$ 👼$_{CL}(K) \mid \forall M \in \min^{LP}_{\sqsubset}(K), v_M(\phi) = 1\}$
- 👼$^{\star\exists}_{CL\downarrow LP}(K) = \{\phi \in$ 👼$_{CL}(K) \mid \exists M \in \min^{LP}_{\sqsubset}(K), v_M(\phi) = 1\}$

Example

Let $K = \{p \wedge \neg q, \neg p \wedge q, p \vee q, p \vee s, r\}$.

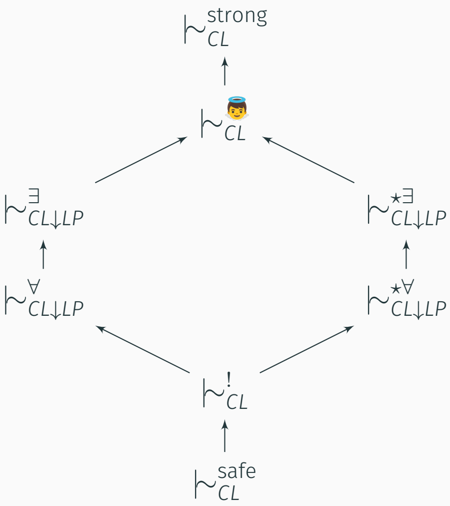
	p	q	s	r	$p \wedge \neg q$	$\neg p \wedge q$	$p \vee q$	$p \vee s$	
M_1	i	i	1	1	i	i	i	1	minimal
M_2	i	i	0	1	i	i	i	i	
M_3	i	i	i	1	i	i	i	i	
M_4	i	i	1	i	i	i	i	1	
M_5	i	i	0	i	i	i	i	i	
M_6	i	i	i	i	i	i	i	i	

So, $K \mid\!\sim^{\star\forall}_{CL\downarrow LP} p \vee s$.

Two more extreme variants

- Safe inference:
  - ϕ is *safe* in K iff $\mathrm{Atoms}(\phi) \cap \mathrm{Atoms}(K \setminus \{\phi\}) = \emptyset$ and $\phi \notin$ 🤠(K)
  - $K \vdash^{\mathrm{safe}} \phi$ iff $\mathrm{Safe}(K) \vdash \phi$
- 👼$^{!}(K) = \{\phi \in$ 👼$(K) \mid \mathrm{Lit}(\phi) \subseteq$ 👼$(\mathrm{Lit}(K))\}$
  - $r \in$ 👼$^{!}(\{\neg p \vee \neg q, r\})$
  - $p \notin$ 👼$^{!}(\{\neg p \vee \neg q, p\})$
  - Definition: $K \vdash^{!} \phi$ iff 👼$^{!}(K) \vdash \phi$.

A hierarchy


$\mid\!\sim^{strong}_{CL}$
$\mid\!\sim^{👼}_{CL}$
$\mid\!\sim^{\exists}_{CL\downarrow LP}$
$\mid\!\sim^{\star\exists}_{CL\downarrow LP}$
$\mid\!\sim^{\forall}_{CL\downarrow LP}$
$\mid\!\sim^{\star\forall}_{CL\downarrow LP}$
$\mid\!\sim^{!}_{CL}$
$\mid\!\sim^{safe}_{CL}$

Some properties …

- $\wedge E$: $\phi_1 \wedge \phi_2 \in \mathsf{Fr}(K \cup \{\phi_1 \wedge \phi_2\}) \implies \phi_1 \in \mathsf{Fr}(K \cup \{\phi_1\})$ and $\phi_2 \in \mathsf{Fr}(K \cup \{\phi_2\})$

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- $\wedge I$: $\phi_1 \in \mathrm{Fr}(K \cup \{\phi_1\})$ and $\phi_2 \in \mathrm{Fr}(K \cup \{\phi_2\}) \implies \phi_1 \wedge \phi_2 \in \mathrm{Fr}(K \cup \{\phi_1 \wedge \phi_2\})$
- $\vee E$: $\phi_1 \vee \phi_2 \in \mathrm{Fr}(K \cup \{\phi_1 \vee \phi_2\}) \implies \phi_1 \in \mathrm{Fr}(K \cup \{\phi_1\})$ or $\phi_2 \in \mathrm{Fr}(K \cup \{\phi_2\})$
- $\vee I$: $\phi_1 \in \mathrm{Fr}(K \cup \{\phi_1\})$ or $\phi_2 \in \mathrm{Fr}(K \cup \{\phi_2\}) \implies \phi_1 \vee \phi_2 \in \mathrm{Fr}(K \cup \{\phi_1 \vee \phi_2\})$
- s-$\vee E$: $\phi_1 \vee \phi_2 \in \mathrm{Fr}(K \cup \{\phi_1 \vee \phi_2\}) \implies \phi_1 \in \mathrm{Fr}(K \cup \{\phi_1\})$ and $\phi_2 \in \mathrm{Fr}(K \cup \{\phi_2\})$
- w-$\vee I$: $\phi_1 \in \mathrm{Fr}(K \cup \{\phi_1\})$ and $\phi_2 \in \mathrm{Fr}(K \cup \{\phi_2\}) \implies \phi_1 \vee \phi_2 \in \mathrm{Fr}(K \cup \{\phi_1 \vee \phi_2\})$

- $\wedge E$: $\phi_1 \wedge \phi_2 \in \mathrm{Fr}(K \cup \{\phi_1 \wedge \phi_2\}) \implies \phi_1 \in \mathrm{Fr}(K \cup \{\phi_1\})$ and $\phi_2 \in \mathrm{Fr}(K \cup \{\phi_2\})$
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	$\wedge$E	$\wedge$I	$\vee$E	$\vee$I	s-$\vee$E	w-$\vee$I
👼	✓	×	×	✓	×	✓
👼$^{\exists}$	×	×	×	×	×	×
👼$^{\star\exists}$	✓	×	×	✓	×	✓
👼$^{\forall}$	×	×	×	×	×	✓
👼$^{\star\forall}$	✓	×	×	✓	×	✓
👼$^{!}$	✓	✓	✓	✓	✓	✓
Safe	✓	✓	✓	✓	✓	✓

- Our study motivates ‘ultra-sensitive measures’ that violate the free formula independence ($\mathcal{I}(K) = \mathcal{I}(K \setminus$ 👼$(K))$) for global measures, or $\mathcal{I}(\phi, K) = 0$ where $\phi \in$ 👼(K) for local measures.

Outlook: Inconsistency measures

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$K = \{p \wedge \neg q,\ \neg p \wedge q,\ p \vee q,\ p \vee s,\ r\}$

- $\mathcal{I}(r) = 0$
- $\mathcal{I}(p \vee s) = \frac{1}{2}$
- $\mathcal{I}(p \vee q) = 1$

	p	q	s	r	$p \vee q$	$p \vee s$
M_1	i	i	1	1	i	1
M_2	i	i	0	1	i	i
M_3	i	i	i	1		
M_4	i	i	1	i		
M_5	i	i	0	i		
M_6	i	i	i	i		

Thank you! 🤠

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POLIZEI
Nordrhein-Westfalen
Bochum

Not really ... 👼