PhDs in Logic IX
Program and Abstracts

May 2 - 4, 2017
Institute for Philosophy II
Ruhr-University Bochum
Organization and Sponsor

The conference is organized by:

⊙ Christopher Badura
⊙ AnneMarie Borg
⊙ Jesse Heyninck
⊙ Daniel Skurt

Further information on the conference can be found in the following link:
http://www.ruhr-uni-bochum.de/phdsinlogicix/

The conference is supported by by the Ruhr-University Research School PLUS (funded by Germany’s Excellence Initiative [DFG GSC 98/3]).
Conference Venue

The venue of the conference is **Veranstaltungszentrum**, located on the university campus. The conference will take place in Saal 3 (room 3).

- **Getting from Bochum Hauptbahnhof to the Ruhr-University of Bochum:** From Bochum Hauptbahnhof (central station) take the U35 towards Bochum Querenburg (Hustadt) and get out at stop “Ruhr Universität”. (Ticket needed: Preisstufe A, €2,70). On weekdays the subway U35 leaves every 5 minutes and reaches the university within 9 minutes.

- **Getting from the U-Bahn stop “Ruhr Universität” to the Veranstaltungszentrum:** From the train station of the U35 (“Ruhr Universität”) go up the pedestrian bridge, turn right from the exit and walk towards the university. Your route takes you directly to the building of the university library. Keep walking until you pass the University library and the building ”AudiMax” on your left, and continue until you reach the University “Mensa” building. The Veranstaltungszentrum is located directly under the Mensa. The conference takes place in Saal 3. (Around 10 min. walk in total.)
## Program

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Abstracts of Invited Talks

A Gentle Introduction to Abstract Algebraic Logic

Petr Cintula
Czech Academy of Sciences

Algebraic logic is the branch of mathematical logic that studies logical systems by giving them algebraic semantics. It mainly capitalizes on the standard Linbenbaum-Tarski proof of completeness of classical logic w.r.t. the two-element Boolean algebra, which can be analogously repeated in other logical systems yielding completeness w.r.t. other kinds of algebras. Abstract algebraic logic (AAL) determines what are the essential elements in these proofs and develops an abstract theory of the possible ways in which logical systems can be related to an algebraic counterpart. The usefulness of these methods is witnessed by the fact that the study of many logics, relevant for mathematics, computer science, linguistics or philosophical purposes, has greatly benefited from the algebraic approach, that allows to understand their properties in terms of equivalent algebraic properties of their semantics.

This course is a self-contained introduction to AAL. We start from the very basics of AAL, develop its general and systematic theory and illustrate the results with applications to particular examples of propositional logics.

Leon Henkin on Completeness

María Manzano
University of Salamanca

The Completeness of Formal Systems is the title of the thesis that Henkin presented at Princenton in 1947, and his director was Alonzo Church. His renowned results on completeness for both type theory and first order logic are part of his thesis. It is interesting to note that he obtained the proof of completeness of first order logic readapting the argument found for the theory of types.

In 1963 Henkin published a completeness proof for propositional type theory, A Theory of Propositional Types, where he devised yet another method not directly based on his completeness proof for the whole theory of types. It is surprising that the first-order proof of completeness that Henkin explained in class was not his own but was developed by using Herbrand’s theorem and the completeness of propositional logic.

In the book The Life and Work of Leon Henkin, recently published, there is a complete chapter devoted to this issue, Henkin on Completeness.
References


Identity, Equality, Nameability and Completeness

María Manzano
University of Salamanca

This contribution is divided into two parts. The first one is devoted to the concepts of identity and equality in a variety of logical systems that motivates the definition of our system of Equational Hybrid Propositional Type Theory (EHPTT). The language we have chosen contains a propositional type theory based on lambda and equality. It also contains hybrid resources as well as algebraic ones.

The second part concentrates on the relevant role played by names on the three completeness theorems Leon Henkin published last century. Our completeness proof for EHPTT owes much to the three and we will point out the more important debts we have undertaken.
References


Classic-like Analytic Tableaux for Non-Classical Logics

João Marcos
Federal University of Rio Grande do Norte

This tutorial will show how a single uniform classic-like proof-theoretical mechanism may be constructively applied to a very wide class of logics, exploiting the bivalence that show up in their meta-theory and at the same time making it easier to compare different non-classical systems. Tableaux with two labels are perfectly adequate for expressing such mechanism, and we will show how they may be obtained by a number of axiom-extraction procedures that operate over suitable semantic presentations for non-classical logics.
The Logic of Group Decision: An Introduction to Judgment Aggregation

Gabriella Pigozzi
University Paris-Dauphine

The aggregation of individual judgments on logically interconnected propositions into one collective judgment has drawn attention in economics, law, philosophy, logic and computer science. Classical social choice theoretic models focus on the aggregation of individual preferences into collective outcomes. Such models focus primarily on collective choices between alternative outcomes such as candidates, policies or actions. However, they do not capture decision problems in which a group has to form collectively endorsed beliefs or judgments on logically interconnected propositions. Such decision problems arise in expert panels, decision making bodies (and artificial agents!) seeking to aggregate diverse individual beliefs, judgments or viewpoints into a coherent collective opinion. Judgment aggregation fills this gap by extending earlier approaches of social choice theory. Despite the apparent simplicity of the problem, seemingly reasonable aggregation procedures cannot ensure a consistent collective judgment as result of the aggregation. This is the so-called discursive dilemma.

The bottom line is that it has been shown that no aggregation function can satisfy a number of desirable properties at the same time. Moreover, the aggregation problem has been generalized in a number of ways and several impossibility and possibility results have been proved.

Computer scientists also face the problem of combining different and potentially conflicting sources of information. Recently, methods that originated in computer science have been applied to judgment aggregation and, on the other hand, judgment aggregation has obtained attention from computer scientists as a fruitful paradigm for framing problems stemming from, in particular, distributed artificial intelligence.

Nonmonotonic Logic

Christian Straßer
Ruhr University Bochum

Nonmonotonic logics are attempts to understand defeasible reasoning in a formally precise way. A conclusion obtained by a strictly deductive inference is warranted to be true if it is derived from true premises. Not so for defeasible inferences: they are prone to exceptions. For instance, birds typically fly while penguins don’t, or: there is a good reason to infer that it rained when seeing a wet street, but it may have been cleaned just a moment ago, etc.

In this tutorial I will introduce students to nonmonotonic logics. Some central techniques and concepts will be explained. At least one family of nonmonotonic logics will be introduced in some depth.
Bi-Connexive Variants of Heyting-Brouwer Logic

Heinrich Wansing
Ruhr University Bochum

In this talk I will introduce connexive Heyting-Brouwer logic or bi-intuitionistic connexive logic, BCL. The system BCL is presented as a Gentzen-type sequent calculus, and some theorems are shown for embedding BCL into a Gentzen-type sequent calculus BL for bi-intuitionistic logic, BiInt. The completeness theorem with respect to a Kripke semantics for BCL is proved using these embedding theorems. The cut-elimination theorem and a certain duality principle are also shown for some subsystems of BCL. Moreover, a sound and complete triply-signed tableau calculus for BCL is presented.
Acyclicity, Simple Connectivity, and Unbranched Covers of Hypergraphs

Julian Bitterlich

1TU Darmstadt

The theory of finite, highly acyclic branched hypergraph covers has recently found a number of fruitful applications in finite model theory. Developed in [?], it was used, for example, to prove the finite model theoretic version of the theorem of Andréka-van Benthem-Németi [? , Theorem 4.2] that characterises the guarded fragment GF as the guarded bisimulation-invariant fragment of FO. A more sophisticated construction in [?] was then used for a strengthening of the celebrated result of Herwig-Lascar [? , Theorem 3.2], which says that certain classes of structures admit finite extensions of partial automorphisms.

On the surface level constructions of highly acyclic graph covers and highly acyclic hypergraph covers appear similar; the former can be constructed using acyclic groups, the latter using acyclic groupoids (however for groupoids one has to use an acyclicity notion stricter than just forbidding short generator cycles). On a deeper level we see more distinctions. Groups of high acyclicity can be obtained by a simple construction, e.g. [? ], but this and other constructions do not suffice for obtaining acyclic groupoids and more sophisticated constructions have to be used. Furthermore a highly acyclic hypergraph cover necessarily has to be branched in general.

Our research aims at characterising the class of hypergraphs that admit unbranched acyclic covers and finite, highly acyclic unbranched covers. To this end we apply techniques that are similar to the ones used for constructions of unbranched graph covers. Perhaps unsurprisingly, this question leads to a connection with basic algebraic topology.

In this talk we summarise the main points of our research:

1. We find a new, intuitive characterisation of acyclicity in hypergraphs.

2. We show that simple connectivity and acyclicity of hypergraphs do agree on the class of locally acyclic hypergraphs.

3. We find that the class of locally acyclic hypergraphs can also be characterised as those hypergraphs that admit unbranched acyclic covers. Furthermore we show that finite, locally acyclic hypergraphs admit finite, highly acyclic, unbranched covers.

4. At the heart of the proof of the aforementioned results lies the theory of what we call ‘granulated covers’, that is, graph covers that do not unravel certain cycles. We show that granulated covers provide universal elements and finite approximations of these. This result is interesting from a purely graph theoretic point of view.
1. Hypergraph Acyclicity

Several different notions of hypergraph acyclicity appear in the literature, all of which are generalisations of graph acyclicity, see e.g. [? ]. What we label hypergraph acyclicity is also more specifically called α-acyclicity, and has multiple equivalent characterisations. We choose a definition which is commonly used.

**Definition 1** A hypergraph is acyclic if its Gaifman graph is chordal and every clique in its Gaifman graph is contained in a hyperedge.

Despite its popularity in database theory and numerous equivalent characterisations, an intuitive characterisation of hypergraph acyclicity in terms of hypergraph cycles is yet to be given. We give a proposal of such hypergraph cycles which seems natural to us. We use these hypergraph cycles in the proof of Theorem ??.

2. Hypergraph Acyclicity and Simple Connectivity

There is a close correspondence between hypergraphs and (abstract) simplicial complexes. Technically speaking a simplicial complex is a hypergraph with a downward closed set of hyperedges. In the other direction one can easily complete a hypergraph to obtain a simplicial complex. For our purposes this completion process is harmless as it preserves all properties we are interested in.

It is a well-known fact that any simplicial complex $K$ has a geometric realisation $|K|$, that is, a topological space associated to $K$, unique up to homeomorphism e.g. [? ]. Also commonly known is a combinatorial description of the fundamental group $\pi(K, v_0)$ – the edge-path group $\pi[K, v_0]$; a group defined in purely combinatorial terms which is isomorphic to the fundamental group e.g. [? ].

Using the definition of the edge-path group it is easy to prove that every acyclic hypergraph is simply connected. The converse direction is false. An easy counterexample is the cartwheel, which consists of 4 vertices and 3 hyperedges of size 3 that all contain the center point.

Considering its localisation at the central vertex, that is, taking the subhypergraph consisting of those hyperedges that contain this vertex, we see that this is a cyclic hypergraph again. We show that this lack of local acyclicity is essentially what distinguishes simple connectivity from hypergraph acyclicity.

**Theorem 1** A connected, locally-acyclic hypergraph is acyclic iff it is simply-connected.

There is also a high-level argument as to why acyclicity and simple connectivity of hypergraphs are separate notions. In the case of finite hypergraphs, the former is decidable but the latter is not (this follows from the undecidability of the word problem for groups [? ]). As a corollary to Theorem ??, we obtain that a suitable combination of certain multiple instances of this undecidable problem is decidable.
**Corollary 1** A finite hypergraph is acyclic iff all connected components of $H$ and all connected components of iterated punctured localisations of $H$ are simply connected.

3. Unbranched Covers of Hypergraphs

In general, acyclic covers of hypergraphs have to be branched. We can see this in the example of the cartwheel. In any acyclic cover, a preimage of the central vertex necessarily has an infinite 1-neighbourhood. In fact, it is no coincidence that we see the cartwheel again.

**Theorem 2** A hypergraph $H$ admits an acyclic unbranched cover iff it is locally acyclic.

There is also a topological take on this. Actually, every unbranched cover $p : \hat{H} \to H$ induces a cover of the realisations $|p| : |\hat{H}| \to |H|$. Roughly speaking the most acyclic cover of a topological space is the simply connected cover. So the cartwheel, being simply connected, is its own most acyclic unbranched cover, and we cannot cover it acyclically in the hypergraph sense.

We also obtain a finite version of Theorem ??.

**Theorem 3** A finite, locally acyclic hypergraph $H$ admits highly acyclic, finite unbranched covers.

4. Granulated Covers of Graphs

In the proofs of Theorem ?? and especially Theorem ?? we want to use unbranched graph covers of the Gaifman graph of $H$. Clearly we can not just use acyclic covers, as this would ‘rip apart’ the hyperedges. To circumvent this we introduce the novel notion of a granular cover – an unbranched graph cover that does not unravel certain cycles.

**Definition 2** Let $G$ be a graph and $R$ a set of cycles in $G$. Then $p : \hat{G} \to G$ is an $R$-granular cover if every lift of an element of $R$ is a cycle in $\hat{G}$.

If $R$ is empty then an acyclic cover of $G$ is universal in the class of $R$-granular covers. We show that, for any choice of $R$, universal elements in the class of $R$-granular covers still exist, together with finite approximations of these covers, provided $R$ and $G$ are finite.

References


Through Many-valent Semantics

Carolina Blasio¹

¹ICFH/UNICAMP

According Suszko’s Thesis [? ? ], even though many-valued semantics could be used as a tool to understand logic systems and their properties, every logic is bivalent. Suszko’s Thesis is supported by the mathematical result known as Suszko Reduction that states every logic can be characterised by bivalent semantics. We can then distinguish two kinds of truth-values: the referential truth-values that make up many-valued semantics, and the inferential truth-values that are the proper truth-values, namely, the True and the False.

From the philosophical point of view, problems about non-determinism, probability, predictions and uncertainties have put logical bivalence into question. These issues motivated the creation of many-valued logics since the 1920s and, after Suszko’s Reduction result, these issues also motivated the creation of non-standard consequence relations as the quasi-entailment (q-entailment) [? ? ] and the plausible entailment (p-entailment) [? ], whose associated formalisms should be characterised by semantics with up to three inferential truth-values.

Let $S$ be a propositional language, a $q$-logic $L^q = \langle S, \models^q \rangle$ is associated with a kind of generalised valuation matrix that has two sets of distinguished truth-values, called $q$-matrix.

**Definition 3** A $q$-matrix for $L^q$ is a tuple $\Omega = \langle V, Y, N, O \rangle$ such that $V$ is a non-empty set — the set of truth-values — $Y$ and $N$ are disjoint subsets of $V$ — the sets of distinguished values, where $Y$ denotes the set of accepted truth-values and $N$ the set of rejected truth-values, and $O$ contains an $n$-ary truth-function $f_\circ: V^n \rightarrow V$ for each connective $\circ$ in $S$. A valuation based on $\Omega$ is a function $v: S \rightarrow V$ such that $v(\circ(\varphi_1, \ldots, \varphi_n)) = f_\circ(v(\varphi_1), \ldots, v(\varphi_n))$, for every $\circ$ in $S$. The semantics $SEM^q$ of $L^q$ is the set of all valuations based on $\Omega$.

Let $\Lambda := V - Y$ and $\mathcal{V} := V - N$. The notion of entailment for a $q$-logic is defined as follows:
Definition 4 \( \Gamma \vDash^q \Delta \) iff there is no \( v \in SEM^q \) such that \( v(\Gamma) \subseteq \mathcal{U} \) and \( v(\Delta) \subseteq \mathcal{L} \), for every \( \Gamma \cup \Delta \subseteq S \).

The notion of \( q \)-entailment is related to the reasoning by hypotheses, broadly adopted in empirical sciences. This kind of reasoning rejects some of the premisses if no statement of the conclusion is accepted.

Let a \( p \)-logic be the pair \( \mathcal{L}^p = (S, \vDash^p) \). \( \mathcal{L}^p \) is associated with a \( q \)-matriz and their semantics \( SEM^p \) is the set of all valuations based on \( \mathcal{Q} \). The notion of entailment for a \( p \)-logic is defined as follows:

Definition 5 \( \Gamma \vDash^p \Delta \) iff there is no \( v \in SEM^p \) such that \( v(\Gamma) \subseteq \mathcal{Y} \) and \( v(\Delta) \subseteq \mathcal{N} \), for every \( \Gamma \cup \Delta \subseteq S \).

The reasoning expressed by the \( p \)-entailment allows for the decrease of certainty from the premisses to the statements of the conclusion.

As one can prove from the definitions, the \( q \)-entailment is not reflexive — \( \alpha \vDash^q \alpha \) does not always hold — and the \( p \)-entailment is not transitive — given \( \Delta \vDash^p \varphi \) and, for every \( \psi \in \Delta \), \( \Gamma \vDash^p \psi \) it is not the case that \( \Gamma \vDash^p \varphi \) always holds. The lack of these fundamental properties found in the standard notion of entailment leads some authors to disregard the formalisms associated to non-standard notions of entailment as legitimate logics. However, if \( \mathcal{Y} \cup \mathcal{N} = \mathcal{V} \), the \( q \)-matrix turns out to be a standard matrix, and, hence the \( q \)-entailment (or \( p \)-entailment) restores the properties of the standard notion of entailment. This fact indicates these non-standard notions of entailment contains the standard reasoning.

A way to distinguish the kinds of reasoning using a single framework is given by a four-place consequence relation called \( B \)-entailment. A \( B \)-logic \( \mathcal{L}^B = (S, \bigr|\bigr|) \) is associated to a generalised \( q \)-matrix defined as below:

Definition 6 A generalised \( q \)-matrix for a \( B \)-logic \( \mathcal{L}^B \) is a \( q \)-matrix \( B = (V, Y, N, O) \), without the restriction \( Y \cap N = \emptyset \). The semantics \( SEM^B \) of \( \mathcal{L}^B \) is the set of all valuations based on \( B \).

Definition 7 Let \( SEM^B \) be the semantics of a given \( q \)-matrix \( B \). For every \( \Gamma, \Delta, \Phi, \Psi \subseteq S \), we set \( \Psi \bigr|\bigr| \Delta \bigr|\bigr| \Phi \) iff there is no \( v \in SEM^B \) such that \( v(\Gamma) \subseteq \mathcal{Y} \) and \( v(\Delta) \subseteq \mathcal{L} \) and \( v(\Phi) \subseteq \mathcal{N} \) and \( v(\Psi) \subseteq \mathcal{U} \).

The \( B \)-entailment has versions of all desirable properties of a consequence relation:

Proposition 1 The following holds in a \( B \)-logic:

\( B \)-reflexivities

\( (t) \bigr|\bigr| \alpha \) and \( (f) \bigr|\bigr| \alpha \)

\( B \)-monotonicity

If \( \Psi' \bigr|\bigr| \Delta', \Phi \), then \( \Psi'^{\prime}, \Psi'^{\prime} \bigr|\bigr| \Delta'^{\prime}, \Delta'^{\prime} \bigr|\bigr| \Phi' \)

\( B \)-transitivity
(t) If $\Psi \mid_{\Sigma, \Pi} \Delta, \Phi$ for every partition $\langle \Sigma, \Pi \rangle$ of a $\Theta \subseteq S$, then $\Psi \mid_{\Gamma} \Delta, \Phi$.

(f) If $\Sigma, \Psi \mid_{\Phi, \Pi} \Delta$ for every partition $\langle \Sigma, \Pi \rangle$ of a $\Theta \subseteq S$, then $\Psi \mid_{\Gamma} \Delta, \Phi$.

A $B$-logic can be characterised by tetravalent semantics. If $Y \cap N = \emptyset$, the generalised $q$-matrix turns out to be a trivalent $q$-matrix and it validates the reflexivity-like statement $\alpha \mid_{\alpha}$. If $Y \cap N = \emptyset$ and $Y \cup N = V$, than the reflexivity-like statement $\alpha \mid_{\alpha}$ is also valid and the $B$-entailment becomes the standard notion of entailment.

A $B$-logic can also present the different kinds of logical reasoning of standard and non-standard consequence relations as shown below.

$\Gamma \models_{t} \Delta$ iff there is no $s$ such that $v(\Gamma) \subseteq Y$ and $v(\Delta) \subseteq \Lambda$ iff $\Gamma \mid_{\Delta}$

$\Psi \models_{f} \Phi$ iff there is no $s$ such that $v(\Psi) \subseteq \mathcal{U}$ and $v(\Phi) \subseteq N$ iff $\Psi \mid_{\Phi}$

$\Psi \models_{q} \Delta$ iff there is no $s$ such that $v(\Psi) \subseteq \mathcal{U}$ and $v(\Delta) \subseteq \Lambda$ iff $\Psi \mid_{\Delta}$

$\Gamma \models_{p} \Phi$ iff there is no $s$ such that $v(\Gamma) \subseteq Y$ and $v(\Phi) \subseteq N$ iff $\Gamma \mid_{\Phi}$

Some other results comparing $B$, $q$, $p$-logics and also standard logics will be provide.

References


Bisimulation invariance may be regarded as the crucial semantic property of modal logics with their many and diverse applications that range from specification of process behaviours to reasoning about knowledge. As a notion of equivalence bisimulation captures the relevant properties of transition systems or Kripke structures that do not depend on some specific encoding of a structure. That makes bisimulation invariance the essential semantic property of any logic that is meant to deal with the relevant phenomena of transition systems. Various modal logics share this preservation property and can, moreover, often be characterised in relation to classical logics as precisely capturing the bisimulation invariant properties of transition systems. This turns bisimulation invariance into a criterion of expressive completeness.

Van Benthem’s Theorem [?] characterises basic modal logic \( \text{ML} \) as the bisimulation invariant fragment of first-order logic \( \text{FO} \) over the elementary class of all Kripke structures. For short we write \( \text{ML} \equiv \text{FO}/\sim \), where \( \text{FO}/\sim \) is the set of \( \text{FO} \)-formulae that are invariant under bisimulation equivalence \( \sim \).

Over the last years there have been many variations of this theorem. Rosen showed in [?] the finite model version, i.e. for every \( \text{FO} \)-formula \( \varphi \) that is bisimulation invariant over all finite Kripke structures there is an \( \text{ML} \)-formula \( \varphi' \) that is expressively equivalent to \( \varphi \) over all finite Kripke structures. This changes the meaning of the theorem completely and requires a whole new proof technique. A further constructive proof that works both in the classical as well as the finite case was given by Otto in [?]. The main idea is to show that a formula \( \varphi \in \text{FO} \) that is bisimulation invariant over a class of Kripke structures \( \mathcal{C} \) is in fact \( \sim^\ell \)-invariant over \( \mathcal{C} \) for some finite approximation of full bisimulation equivalence \( \sim \). Then the modal Ehrenfeucht-Fraisse theorem (cf. [?]) implies that \( \varphi \) is equivalent to some \( \text{ML} \)-formula of modal nesting depth \( \ell \) over \( \mathcal{C} \). The crucial part of the proof is an upgrading argument that links \( \sim^\ell \)-equivalence to finite levels \( \equiv_q \) of first-order equivalence, for some \( \ell \) that depends on \( q \). To be more precise: given a formula \( \varphi \in \text{FO} \) of quantifier depth \( q \) that is \( \sim \)-invariant over \( \mathcal{C} \) we show that there is some \( \ell = \ell(q) \) such that for all pointed Kripke structures \( \mathcal{M}, w \) and \( \mathcal{N}, v \) in \( \mathcal{C} \) with \( \mathcal{M}, w \sim^\ell \mathcal{N}, v \) we can construct bisimilar companions \( \mathcal{M}^*, w^* \sim \mathcal{N}, v \) over \( \mathcal{C} \) such that \( \mathcal{M}^*, w^* \equiv_q \mathcal{N}^*, v \). This immediately implies \( \sim^\ell \)-invariance of \( \varphi \) over \( \mathcal{C} \). The challenges in the argument lie in the construction of the suitable bisimilar companions and the proof of their \( \equiv_q \)-equivalence. This approach has been fruitful for many different variations of the van Benthem theorem, e.g. in [?], [? ] and [? ]. We use the same approach to prove van Benthem style modal characterisation theorems for epistemic modal logic with common knowledge modalities in the classical and finite case.

Epistemic modal logic with common knowledge modalities \( \text{ML}[\text{CK}] \) is usually introduced as an expansion of basic modal logic \( \text{ML} \) by common knowledge modalities, that can be described as a fixpoint construct, with semantics for S5 Kripke structures (cf. [?]). For our purposes we choose a different yet equivalent approach. We view common knowledge logic as basic modal logic with semantics over so-called Common Knowledge (CK) structures that are expansions of S5 Kripke structures. An S5 Kripke frame...
(W, (R_a)_{a \in \Gamma}) is a tuple where W is the set of possible worlds and the binary relations R_a are equivalence relations that are labelled by elements a from a finite set of agents \( \Gamma \); the \( a \)-equivalence class of a possible world \( w \in W \) is denoted by \([w]_a\). An S5 Kripke structure is an expansion of an S5 Kripke frame by a propositional assignment, for a given finite set of basic propositions \((P_i)_{i \in I}\). A CK frame (or structure) is obtained as the expansion of an S5 Kripke frame (or structure) by expanding the family \((R_a)_{a \in \Gamma}\) to the family \((R_a)_{a \subseteq \Gamma}\), where \( R_a \) is the transitive closure of the set \((\bigcup_{a \in \alpha} R_a)\). We note that the equivalence relations \( R_a \) are not FO-definable from the basic relations \( R_a \). Basic modal logic ML in our setting has atomic formulae \( \bot, \top \) and \( p_i \), for \( i \in I \), and is closed under the boolean connectives, \( \land, \lor, \neg \), as well as under the modal operators \( \Box \alpha \) and \( \Diamond \alpha \), for \( \alpha \subseteq \Gamma \). The semantics is the standard one, with an intuitive epistemic reading of \( \Box \{a\} \) as "agent a knows...", \( \Diamond \{a\} \) as "agent a regards it possible...", for \( a \in \Gamma \), and \( \Box \alpha \) as "it is common knowledge among the agents in \( \alpha \) that...", for \( \alpha \subseteq \Gamma \).

Our main contribution is the proof that we can upgrade \( \sim^\ell \)-invariance to \( \equiv_\ell \)-equality over the non-elementary classes of CK structures and finite CK structures. The first part of this proof involves the construction of suitable bisimilar companions. The companions \( \mathfrak{M}^* \) and \( \mathfrak{M}^* \) must avoid distinguishing features that are definable in FO\(\alpha\) but cannot be controlled by \( \sim^\ell \). In our case, as usual, these features are small multiplicities w.r.t. accessibility relations and basic propositions and short non-trivial cycles w.r.t. combinations of accessibility relations \( R_a \). A result from [?] shows that multiplicities can easily be boosted by taking the direct product with a finite clique. The situation is more intricate in the case of acyclicity. We need to construct bisimilar companions that are locally acyclic w.r.t. non-trivial overlaps between \( \alpha \)-classes \([w]_{\alpha}\), for various \( \alpha \). Simultaneously, every such class \([w]_{\alpha}\) of the structures must be locally acyclic in the same sense, w.r.t. to \( \beta \)-classes for \( \beta \subset \alpha \). The following notion from [?] is what we can use.

**Definition 1** Let \( \mathfrak{M} \) be a CK frame. A coloured cycle of length \( m \) in \( \mathfrak{M} \) is a cyclic tuple \((w_i, \alpha_i)_{i \in \mathbb{Z}_m}\), where, for all \( i \in \mathbb{Z}_m \), \((w_i, w_{i+1}) \in R_{\alpha_i}\), and

\[
[w]_{\alpha_{i-1} \cap \alpha_i} \cap [w_{i+1}]_{\alpha_i \cap \alpha_{i+1}} = \emptyset.
\]

A CK frame is acyclic if it does not contain a coloured cycle, and \( N \)-acyclic if it does not contain a coloured cycle of length up to \( N \). In [?] Otto constructed finite \( N \)-acyclic groups, in the sense of Definition [?], with generator set \( E \), for arbitrary finite \( E \) and arbitrary natural numbers \( N \). We use these groups to construct bisimilar companions that are based on rational encodings of their Cayley graphs.

**Lemma 2** For any (finite) CK structure, for any \( N, K \in \mathbb{N} \), there is a (finite) connected bisimilar companion \( \mathfrak{N}^* = (W, (R_a)_{a \subseteq \Gamma}, (P_i)_{i \in I}) \) such that every \([w]_{\alpha}\) has at least \( K \) distinct bisimilar copies of every world \( v \in [w]_{\alpha} \), and there are no coloured cycles of length up to \( N \).

The second part of the upgrade proof is the development of a novel structure theory for \( N \)-acyclic CK structures in order to prove \( \equiv_\ell \)-equivalence. In connected CK structures with accessibility relations \((R_a)_{a \subseteq \Gamma}\) every two worlds are connected via a \( \Gamma \)-edge because of the transitive nature of these combined relations. This rules out the straightforward use of any simple locality-based techniques.
CK frames possess a very intricate structure due to the combination of the basic relations \((R_a)_{a \in \Gamma}\), yet \(N\)-acyclicity imposes a high degree of regularity on the overlap patterns between different equivalence classes. This allows us to deal with the challenge of locality issues at different scales simultaneously. For example, in 2-acyclic structures there is always a unique minimal set of agents that connects two worlds, and two equivalence classes \([w]_\alpha\) and \([w]_\beta\) intersect exactly in the class \([w]_{\alpha \cap \beta}\). Furthermore, in \(N\)-acyclic structures two short paths between worlds \(w\) and \(w'\) are unique in the sense that they are just different decompositions of one and the same path. A detailed analysis of these and other properties allows for the proof of \(\equiv_q\)-equivalence for suitable CK structures that are \(\sim^f\)-equivalent. This implies our main result, a modal characterisation theorem for epistemic modal logic with common knowledge modalities.

**Theorem 3** \(\text{ML} \equiv \text{ML}[\text{CK}] \equiv \text{FO/}\sim\text{ over CK structures, both classically and in the sense of finite model theory.}\)

In other words, the logic ML[CK] is expressively equivalent to the bisimulation invariant fragment of a suitable extension of first-order logic over the classes of S5 structures and finite S5 structures.

**References**


Introduction

Reasoning with conflicting information is currently one of the main challenges in the field of Artificial Intelligence. While classical logic is good at modeling reasoning in the absence of conflicts, in the real world we are regularly confronted with imperfect information from different sources, which often lead to conflicting conclusions. Formal argumentation, one of the modeling techniques which have been developed to reason with conflicting information, aims at mimicking the way human beings argue and debate. However, attempting to study the connections between this formalism and real world arguments is tricky, as most real life debates often make use of implicit world knowledge, which is troublesome to model. In an attempt to facilitate this study, we propose to examine arguments about logical paradoxes, in which most people will not have much world knowledge to make use of.

The logical paradox we will focus on is the Liar paradox. A simple version of it would be: Let $L$ be the sentence “This sentence is false”. Then, if we suppose that $L$ is true, it must mean that it is true that $L$ is false. Hence, $L$ must be false, which is absurd. So $L$ cannot be true, therefore must be false, and hence it is true that $L$ is false. Thus, $L$ is true, giving us an absurdity under no assumption, from which we can derive absolutely everything, such as “The Earth is flat” or “The Moon is made out of cheese”.

The work of Dung [1] introduced the theory of abstract argumentation, in which one models arguments by abstracting away from their internal structure to focus on the relations of conflict between them. Sets of ‘winning’ arguments can then be computed based on different semantics, representing varying levels of skepticism. These frameworks have been extended in several ways, in particular with explanatory features. In [?], new elements called explananda are added, which represent phenomena requiring to be explained, and as well as an explanatory relation, which allows arguments to either explain these explananda, or deepen another argument’s explanation. The sets of acceptable arguments are then selected by also taking into account their explanatory power and depth.

The formalism we will present in this talk is based on the ASPIC+ framework for structured argumentation [?], where one starts with a knowledge base and a set of rules which allow one to make inferences from given knowledge. These rules are separated into two kinds, the first being strict rules, which represent safe universal inferences, such that if one accepts a strict rule’s antecedents, one is then forced to accept its conclusion as well. The other kind is the defeasible rules, representing weaker inference which are known to be generally true whilst also being subject to exceptions. Arguments are then built either by introducing an element of the knowledge base into the framework, or by making an inference based on a rule and the conclusions of previous arguments. Attacks are then constructed by either attacking the conclusion of a defeasible inference made within an argument, or by questioning the applicability of such a rule. An abstract argumentation
framework has then been built and acceptable arguments can then be selected using any abstract argumentation semantics.

**Overview of ASPIC-END**

Our formalism, ASPIC-END, features three main differences from ASPIC+. The first is an additional way of constructing arguments. In the philosophical literature on logical paradoxes, proofs by contradiction are often used. Hence, in ASPIC-END, we allow for arguments to follow a *proof by contradiction* construction, in which an assumption $\varphi$ is introduced in the hope of reaching an absurdity $\bot$, allowing one to retract the assumption $\varphi$ and deduce $\neg \varphi$. Note that in our case, we consider $\bot$ not simply as any contradiction, but as the conjunction of all possible formulas in the language. Hence, accepting $\bot$ would be equivalent to accepting that the Earth is flat and spherical at the same time.

Proofs by contradiction may be nested, and thus each argument must keep track of which assumptions it is working under. For intuitive reasons, we only allow arguments to attack within their scope of assumption.

The second difference between our formalism and the standard ASPIC+ is the weakening of strict rules. In ASPIC-END, we instead have *intuitively strict rules*, which represent rules of logic, such that if one accepts the antecedents and the applicability of the rule, one also has to accept its conclusion. Notice that in this case, we can question the applicability of such a rule, which is not allowed in ASPIC+. While the Liar Paradox is built using *prima facie* rules of logic which differ from the kind of fallible reasoning represented by defeasible rules, the different explanations to the paradox usually reject some of those prima facie rules of logic and motivate the acceptance of their own logic, such as paracomplete, paraconsistent or intuitionistic logic. The intuitively strict rules which are not rejected by an extension then behave similarly to strict rules in ASPIC+.

Finally, in the interest of studying the explanations of logical paradoxes, we allow for the construction of explananda and explanatory relations. In our application of the formalism, we want to be able to model not only the arguments about different theories regarding the paradoxes, but also the internal structure of the latter, which will allow us to provide more details on the different ways in which the rivaling theories explain the paradoxes. An explanandum can be constructed from an argument (the *source*) which derives $\bot$ from intuitively strict rules under no assumption, as such an argument constitutes a paradox in need of an explanation. An argument then explains one such explanandum if and only if it successfully attacks its source. By doing so, the argument points out the erroneous steps in the derivation of $\bot$ and thus provides an explanation as to how one can avoid running into trivial belief.

**Rationality postulates**

For a structured argumentation framework to be considered interesting, one must ensure that the results are intuitive for any input. As such, *rationality postulates* have been proposed by [?] to ensure no anomalies arise. All of the postulates below have been shown to hold for our formalism.
First, we have the closure under sub-arguments: for each extension of a framework, if an argument is included in the extension, so should all of its sub-arguments. This ensures that if one accepts an argument, one also accepts all of its parts, even though the sub-argument relation is not explicitly present at the abstract argumentation level.

Then, we have the postulate of closure under intuitively strict rules. However, since intuitively strict rules might be defeated during the analysis of the system, we have to consider closure under accepted intuitively strict rules only. This is the set of rules accepted and defended by a particular extension of arguments.

The last postulates concerns consistency. We have split it into two different formulations, depending on the application. The original consistency postulate states that if the input of the framework is consistent, then so is the output. This formulation makes sense when one expects the input to be consistent, and since we want our framework to be usable in these kinds of scenarios as well, we have shown such a formulation of the postulate. When considering logical paradoxes however, we expect some contradictions to arise from intuitively strict rules alone and hence the input is rarely consistent. Even though some logicians might allow inconsistent output, most will agree that it should be non-trivial, i.e. not contain $\bot$. Hence, we include a second formulation which is oriented towards our main application of the framework and show that no output will be trivial.

References


Nontransitive Approaches to Paradox and Compositional Principles of Truth

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Nontransitive approaches to paradox are typically motivated by arguing that they are able to both preserve all of classical logic and a fully transparent truth-predicate. This talk examines the interplay between nontransitive approaches and certain stronger principles about truth, the so-called compositional principles. We show that once these intuitive principles are added, the nontransitive approach renders them paradoxical - contra their intuitive status. I begin by laying out some formal preliminaries and the motivation for nontransitive approaches before showing that nontransitive approaches render compositional principles of truth paradoxical.

The goal of a solution to the truth-theoretic paradoxes is to non-trivially (and preferably consistently) add a transparent truth-predicate to an interesting arithmetical background theory. For proof-theoretic simplicity, we will restrict ourselves to a geometric formulation of Robinson Arithmetic ($\mathcal{Q}$), i.e. PA without the induction scheme, as our background theory which also serves as the syntax theory. We generally work with the operational and structural rules of LK. It is well known that this system of pure logic exhibit Cut-elimination, i.e. if the system proves a sequent, it also proves the sequent without an application of Cut.

$$\frac{\Gamma \vdash \Delta, \varphi}{\text{Cut}} \quad \frac{\Sigma, \varphi \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi}$$

To preserve Cut-elimination for our full arithmetical theory, we use the method of [?] for geometrical representations of first-order theories to translate the axioms of $\mathcal{Q}$ into sequent rules. The transparency of the truth-predicate is captured by the following rules (where the double-dashed lines indicate that the rule is applicable both ways):

$$\frac{\Gamma, \varphi \vdash \Delta}{\frac{\Gamma, T(\tau \varphi)}{\Delta}} \quad \frac{\Gamma \vdash \Delta, \varphi}{\frac{\Gamma \vdash \Delta, T(\tau \varphi)}{}}$$

By $\tau \varphi$ we mean the numeral of the code of $\varphi$ employing an effective and monotonic method of Gödel-coding. It is well known that $\mathcal{Q}$ includes a diagonalisation function $f_d(\tau \varphi) = \tau \varphi(\tau \varphi)$ which is recursive. We also know that for every recursive function $f$, there is a predicate in the language of $\mathcal{Q}$ such that $\mathcal{Q}$ proves that this predicate represents that function. A consequence of this fact is a (weak) diagonal lemma, i.e. our sequent calculus presentation of $\mathcal{Q}$ proves for any formula $\varphi$ the sequent $\vdash \varphi \leftrightarrow \psi(\tau \varphi)$. By appropriate choice of $\varphi$ and $\psi$, we receive a Liar sentence $\lambda \leftrightarrow \neg T(\tau \lambda)$.

Using an arithmetical background theory as ones syntax theory to be explicit about the diagonal lemma is a novelty in the literature on substructural approaches to paradox. Typically, names of sentences happens in the meta-language and paradoxical reasoning is partially represented in this meta-language as well. The current approach circumvents
these artificialities; we are able to represent the whole paradoxical argument in the object language.

By the invertibility of our sequent rules, we know that if \( \vdash \lambda \leftrightarrow \neg T(⌜\lambda⌝) \) is provable, so are \( \vdash \lambda, T(⌜\lambda⌝) \) and \( \lambda, T(⌜\lambda⌝) \vdash \). Thus we can give a derivation of the empty sequent as follows:

\[
\begin{align*}
\vdash \lambda, T(⌜\lambda⌝) &\quad \lambda, T(⌜\lambda⌝) \vdash \\
\vdash T(⌜\lambda⌝), T(⌜\lambda⌝) &\quad T(⌜\lambda⌝), T(⌜\lambda⌝) \vdash \\
\vdash T(⌜\lambda⌝) &\quad T(⌜\lambda⌝) \vdash
\end{align*}
\]

Any arbitrary \( \varphi \) may then be introduced by Weakening. Nontransitive approaches block the derivation of the empty sequent by excluding the Cut-rule from their system. Due to our Cut-free formulation of \( Q \), this only affects formulae containing \( T \). The resulting non-transitive system (NT) is provably non-trivial and if a sequent is classically provable, then it is also provable in NT [25]. The latter observation often motivates non-transitivists to claim that their solution preserves all of classical logic. Similarly, nontransitive theories can maintain full transparency, i.e. the above rules for \( T \) without restriction.\(^1\)

Transparency is itself a desirable property of the truth-predicate in one’s formal system. I agree that NT’s capability to non-trivially invoke such rules in the presence of all theorems of classical logic counts in its favour. Nevertheless, we may ask how NT fares with respect to more demanding principles governing the truth-predicate. Here I am concerned with the following compositional principles of truth:

\[(\text{Neg}) \forall x (\text{Sent}(x) \rightarrow (T(\neg x) \leftrightarrow \neg T(x)))\]
\[(\text{Con}) \forall x \forall y (\text{Sent}(x) \land \text{Sent}(y) \rightarrow (T(x \land y) \leftrightarrow T(x) \land T(y)))\]
\[(\text{Dis}) \forall x \forall y (\text{Sent}(x) \land \text{Sent}(y) \rightarrow (T(x \lor y) \leftrightarrow T(x) \lor T(y)))\]
\[(\text{Cond}) \forall x \forall y (\text{Sent}(x) \land \text{Sent}(y) \rightarrow (T(x \rightarrow y) \leftrightarrow T(x) \rightarrow T(y)))\]

Similar to transparency (or the \( T \)-biconditionals), these are commonly taken to be intuitive and desirable principles to govern the truth-predicate in one’s formal theory of truth. Note that these compositional principles, also in the presence of full transparency, are not provable in NT. Here, we assume that the (bi-)conditionals are added as initial sequents. They may be added as sequent rules as shown in [25] from which the (bi-)conditionals are derivable. Here we show that NT is inadequate with respect to these principles as it renders them paradoxical by its own standards.

By its solution to the paradoxes in terms of Cut-inadmissibility, NT is committed to a notion of paradoxicality s.t. \( \varphi \) in sequents \( \Gamma \vdash \varphi, \Delta \) and \( \Gamma', \varphi \vdash \Delta' \) is paradoxical relative to a rule system \( \mathcal{R} \) iff an application of Cut to the sequents \( \Gamma \vdash \varphi, \Delta \) and \( \Gamma', \varphi \vdash \Delta' \) is not eliminable with respect to \( \mathcal{R} \). Due to the provability of the sequents \( \vdash T(⌜\lambda⌝) \) and \( T(⌜\lambda⌝) \vdash \) in NT, however, we can show this behaviour for the compositional principles

\(^1\)Semantically, this amounts to say that \( I(T(⌜A⌝)) = I(A) \) where \( I \) is the interpretation function of one’s model.
above. Here we show this behaviour for the case of the left-to-right direction of (Neg)

\[
\begin{align*}
\lambda, T(\neg \lambda^\gamma) \vdash & \quad \vdash \lambda, T(\neg \lambda^\gamma) \\
\lambda, \lambda \vdash & \quad \vdash T(\neg \lambda^\gamma), T(\neg \lambda^\gamma) \\
\lambda \vdash & \quad \vdash \neg \lambda \\
\vdash T(\neg \neg \lambda^\gamma) & \quad \vdash T(\neg \lambda^\gamma) \\
\neg T(\neg \lambda^\gamma) & \quad \vdash
\end{align*}
\]

The derivation for the other direction of the biconditional of (Neg) as well as for both conditionals of the other compositional principles is straightforward. Due to issues of space, these details are omitted here.

By the nontransitivist’s notion of paradoxicality and the provability of the compositional principles on both sides of the sequent, she is committed to the claim that the compositional principles of truth are paradoxical. But surely, pretheoretically, they are not supposed to come out as paradoxical. This may be fleshed out by arguing that a notion of paradoxicality ought to be extensionally adequate. Call a notion of paradoxicality P extensionally adequate iff: \( \varphi \) is paradoxical iff \( \varphi \) exhibits P. Although the left-to-right direction seems to be satisfied (which may be shown by a consistency proof as in \[?\]), the right-to-left direction is not: Although the notion of paradoxicality applies to the compositional principles, they ought not to be regarded as such.

Thus we conclude that although NT is able to preserve all theorems of classical logic (and our arithmetical background theory) with a fully transparent truth-predicate it is inadequate with respect to stronger demands: the compositional principles.

References


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\(^2\)Where Sent\( (x) \) expresses that \( x \) is a sentence of our language which is well-known to be definable in \( Q \). In the following we make use of the obvious rule to introduce the predicate in derivations.
Translational Expressiveness Between Logics: Giving Adequacy Criteria

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A formal criterion for the notion of relative expressiveness for model-theoretic logics was given already in the 1970s (e.g. in [? ] and [? ]). This criterion is based on the capacity of characterizing structures and underlies each of the so-called Lindström-type theorems.

Despite being the basis for many important results, the referred criterion is very limited, as it requires the logics to be defined within the same class of structures. It is also desirable to be able to compare expressiveness of logics defined within different classes of structures. Then, a wider criterion for expressiveness would require, besides formula translations, also structure translations. Some generalizations have been proposed in [? ] and [? ], but they turned out to be unsatisfactory.

There have been also early claims outside abstract model-theory relating logics in a similar fashion, but no explicit definitions of the main concepts involved were given. Gödel used his result on the interpretation of classical into intuitionistic logic to infer that, contrary to the appearances, it is classical logic that is contained in intuitionistic logic [? , p. 295]. Since then, there followed many results of interpretations, embeddings, reconstructions, simulations, etc. among Tarskian and proof-theoretic logics. Such results have often been used to justify some statement of inclusion or relative expressiveness between the logics at issue.3 Naturally, this notion of expressiveness is no longer directly linked with the capacity of characterizing structures as in model-theoretic logics, rather it resides in the capacity of a logic to “encode” another. Let the framework of expressiveness based on such capacity be named “translational expressiveness”

As opposed to the case of model-theoretic logics, until recently there was no attempt to give a precise criterion for expressiveness in this framework. To the best of our knowledge, Mossakowski et al. [? ] were the first to give a formal definition of translational expressiveness. We will expose their definition and show that it is still not adequate. Some tentative adequacy criteria for expressiveness will be proposed and discussed. Among others, the criteria we would like to discuss are:

[Adequacy Criterion 1] \( L_2 \) is at least as expressive as \( L_1 \) only if everything that can be said in terms of the connectives of \( L_1 \) can also be said in terms of the connectives of \( L_2 \).

[Adequacy Criterion 2] It cannot hold that \( L_2 \) is more expressive than \( L_1 \) when

- \( L_2 \) is trivial and \( L_1 \) is non trivial;
- \( L_2 \) is decidable and \( L_1 \) is not decidable;
- \( L_1 \) satisfies the standard deduction theorem and the language fragment of \( L_2 \) purportedly as expressive as \( L_1 \) does not satisfy (not even) the general deduction theorem;4

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3E.g. [? , p. 154], [? , p. 67], [? , p. 163], [? , p. 441] and [? , p. 15].

4The required definitions will be given.
Adequacy Criterion 3 (taken from [? ]): The expressiveness relation should be a non-trivial pre-order, that is, it should be a transitive and reflexive relation, and there must be some pair of logics $L_1, L_2$ such that $L_2$ is not at least as expressive as $L_1$.

Then a formal criterion $E$ will be given and we will argue that $E$ satisfies the adequacy criteria above.

References


Our aim in this talk is to discuss several extensions of a conflict-tolerant deontic logic from [? ], with the goal of representing moral reasoning. We motivate and assess these extensions in terms of five desiderata. The first two desiderata are each underpinned by both a paradigm example and a philosophical justification. The last three desiderata are taken from Goble’s [? ].

The first paradigm example is based on [? ] and consists of two cases. In case 1(a) a mother (α) of a newborn infant has the obligation to feed her child. The only possible way for her to feed her child is by either formula feeding it or breastfeeding it (we assume that there is no other way). α also has the obligation to prevent her child from being exposed to situations that have a high chance of making the child seriously ill. α is HIV infected and it is thus not possible for her to breastfeed her child without the risk of infecting it. α also does not have access to clean drinking water. So she also cannot feed her child formula without risk. In case 1(b) a mother β is in the same situation as α, with the exception that β does have access to clean drinking water. α is faced with a moral dilemma: should she breastfeed her child, or should she feed it formula? In contrast, β does not face a moral dilemma. It is clear that she ought to feed her child formula.

The difference between the two cases in the first paradigm example shows that whether people face a moral conflict does not only depend on the obligations they have, but also on contingent, practical factors. The most natural way to explicate our deontic reasoning in such cases, is by adding alethic modal operators and having them interact with the deontic operators in such a way that both the situation of α and of β can adequately be described. So our first desideratum (d1) is that the language of our logics should contain alethic modal operators for “possible” and “necessary” and that the interaction between these operators and the deontic modal operators is such that our logics can explain our intuitions regarding moral reasoning involving these notions.

An additional, independent, reason to include alethic modal operators in our language is given by Beirlaen’s interpretation of Williams’ [? ]. In [? ] Beirlaen interprets Williams as saying that the real root of moral conflicts always lies in a contingent fact about the world and can thus best be characterised in the form $O A, O B, \Diamond (A \land B)$. In other words, any characterization of deontic conflicts by means of formulas of the form $O A, O \neg A$ fails to show the real roots of the conflict.
The second desideratum (d2) is supported by a second paradigm example. This second example is taken from [?]. A pair of conjoined twins (Mary and Jodie) is born to two religious parents. The doctors responsible for the case predict that, without surgery, the twins will both die quickly, probably within a year after birth. The doctors also predict that if a surgery to separate the twins is performed, then Jodie will be able to lead a normal life, with an average life expectancy. In contrast, Mary will certainly die during the surgery. Based on these predictions the doctors conclude that they have an obligation to perform the surgery, to save Jodie. The parents however disagree. They claim that choosing to perform the surgery amounts to killing Mary and they value the obligation not to kill as higher than the obligation to save a life.

We suggest that neither the reasoning of the doctors, nor the reasoning of the parents in this case is irrational. They merely assign different priorities to the different obligations they all share, and proceed to reach different conclusions based on that.\(^5\) In order to explain this paradigm example, our deontic logic has to be able to (correctly) handle obligations with different priorities. This is our second desideratum (d2).

To further support d2 we argue that our logics must be able to deal with both resolvable and irresolvable moral conflicts, as defined in [?]. Goble sees resolvable moral conflicts as conflicts between obligations with different priorities, while he sees irresolvable conflicts as conflicts between obligations with the same priority. Both sorts of conflict should not be trivialized by the logic and the logic should provide an all-things-considered ought in the case of resolvable conflicts, as would be expected by our intuition.\(^6\)

Goble’s three desiderata for conflict-tolerant deontic logics in [? ] are (d3) that the logic does not render normative conflicts trivial, (d4) that it is not too strong, in the sense that the logic does not validate any form of deontic explosion (of which several variants are presented in [?]) and (d5) that the logic is not too weak, in the sense that the logic can explain the five paradigm examples presented in [?]. We demand that our logics fulfill these desiderata as well.

The logics we present are based on the lexicographic adaptive logic MP\(_<\) presented in [?]. Like every adaptive logic, MP\(_<\) is characterised by a monotonic lower limit logic (LLL), a set of abnormalities and an adaptive strategy. In the case of MP\(_<\), the (monotonic) LLL is a combination of a non-aggregative deontic logic for the prima facie obligations and an aggregative deontic logic for the all-things-considered obligations. In [? ] this logic is called MP. The set of abnormalities and the adaptive strategy allow us to infer all-things-considered obligations from the prima facie obligations in a non-monotonic way.

Our main technical contribution consists in the enrichment of MP with normal modal operators that represent the alethic modalities “possible” (◊) and “necessary” (□), in order to fulfill d1. At the semantic level, this means that we need to enrich the models from [? ] with an accessibility relation for the alethic operators. As we will show, it is possible to construct several variants of this extension of MP. We will however show that not all of these variants fully satisfy d1, and argue that the principle RM (
\((O A \land \Box (A \to B)) \to OB\) should be valid in our new LLL in order to explain the paradigm examples.

Regarding the monotonic part we study how different interaction principles between the alethic and the deontic modal operators influence the new logic. Furthermore, as we will show, even for a fixed monotonic extension of MP, there are still various options for modeling the defeasible inference from prima facie to actual obligations. We will also compare those different options.

References


Three-and-a-half Semantics for Epsilon Terms

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Prerequisites

Epsilon terms were established in the context of an epsilon calculus developed by Hilbert and Bernays *Grundlagen der Mathematik* and first published in Ackermann’s Dissertation\(^7\) in 1924. The main intention was to create a basis for defining existential and universal quantifier for standard predicate logic. With the help of epsilon terms Hilbert and Ackermann sought out to find a finitary\(^8\) consistency proof of arithmetics.

\(^7\)cf. [? ]

\(^8\)meaning: quantifier-free
The meaning of an epsilon term $\varepsilon x P x$ is ,,some $x$ satisfying $P$, if there is one“. Epsilon terms are therefore the syntactical equivalent to generic quantification, e.g. ,,a(n)“ and ,,some“. I’ll present two paradigmatic semantics for epsilon terms, which have been developed by Leisenring\(^9\) and Claus-Peter Wirth\(^10\). Hermes\(^11\) uses essentially the same semantic as Leisenring, but adds an ,,,inconsistent“ axiom $\varepsilon x (x = x) = \varepsilon x \neg (x = x)$ and therefore adds a ,,half“ semantic to this topic. Finally I will present a game theoretic semantic (GTS) for epsilon terms.

**Leisenring**

Leisenring extends classical predicate logic with an $\varepsilon$-operator. This operator is meant to select an arbitrary, indeterminate member from a set of objects. His $\varepsilon$-operator is represented by a logical choice function and can entail - under certain circumstances - the axiom of choice (AC).

Leisenring uses the following example: The formula $\exists x (x = 1 \lor x = 2 \lor x = 3)$ is true, iff $x$ is 1 or 2 or 3. Checking the truth of this formula is considering, if this property applies to the selected individuals of a domain. On the other hand for the formula $\varepsilon x (x = 1 \lor x = 2 \lor x = 3)$ it is impossible to know which individual has been selected, denoting indefinite choice.

By adding an Axiom $P(t) \rightarrow P(\varepsilon x P)$, where $t$ is any term, to classical predicate logic, one can entail two formulae for reducing quantified sentences to terms: $\exists x P \leftrightarrow P(\varepsilon x P)$ and $\forall x P \leftrightarrow P(\varepsilon x \neg P)$. With the help of the $\varepsilon$-operator it is possible to obtain an easy and short completeness proof for this logic.

**Wirth**

Wirth argues, that Leisenrings semantics do not - and in general semantics, that interpret epsilon terms as choice functions - sufficiently capture Hilbert’s intentions. Seeing the epsilon-operator as a simple arbitrary choice gives rise to problems of repeated choices of the same element, e.g. pronouns which refer to indefinite descriptions in sentences like: ,,If Bill likes a donkey, then he beats it“. In the same way, if a variable in a proof gets substituted, it is pivotal to stick to this choice, whatever this arbitrary selected element might be.

Thus Wirth proposes a semantic, which views $\varepsilon$-operators as substitutions with indefinite, but comitted choices. Such choices can be represented with a liberalized $\delta^+$-substitution rule\(^12\). This treatment exposes some unexpected features of the epsilon-operator. The truth of sentences like ,,something is equal to something“, i.e. $\varepsilon x (true) = \varepsilon x (true)$, can not be analyzed the usual way trough compositionality of the the truth value of subterms, as the contradictory ,,something is unequal to something“ seems to have the same truth value. As it has been indicated by Hintikka: One problem of generic

\(^9\)cf. [? ]
\(^10\)cf. [? ]
\(^11\)cf. [? ]
\(^12\)cf. [? ]
quantification is the failure of substitutability\textsuperscript{13}. On the other hand, as Wirth admits, this liberalized $\delta^+$-substitution rule is unable to deal with Henkin-Quantification, i.e. the „slash“-operator from Hintikka and Sandu’s IF-Logic\textsuperscript{14}.

**Gametheoretical Semantics**

Instead GTS is capable of handling Henkin-Quantification. In GTS „slash“-operators are represented by games with imperfect information, meaning that information sets are restricted for each player. Also a game-theoretic treatment of generic quantifiers has been suggested by Hintikka \textsuperscript{15}, but no GTS semantics for an $\varepsilon$-operator has been developed. Hintikka suggested subgames as the game theoretical counterpart for generic quantifiers, which I’ll use for the $\varepsilon$-operator.

Again this raises the question about the nature of $\varepsilon$-terms, as the game theoretical counterpart to indefinite committed choices would be just a strategy of one of players. A short outlook on possible interpretations will close this presentation.

**References**


\textsuperscript{13}Tarski’s T-scheme (“snow is white” is true iff snow is white) does not apply to generic quantification, e.g. “Anyone can become a millionaire’ iff anyone can become a millionaire” is not a true sentence.

\textsuperscript{14}cf. [? ]

\textsuperscript{15}cf. [? ]
A Realistic View on Normative Conflicts

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While I was reading Antigone, I could not resist feeling compassion. Antigone was a good human being. She was no criminal, yet she was cruelly punished. Divine laws notwithstanding, she was justified in burying her brother Polyneices. On the other hand, the ruler Creon prohibited the burial, since Polyneices was a traitor who attacked the city. In spite of my sympathy to the rebellious Antigone, the Creon’s proclamation is understandable. From certain point of view, it is even justified.

What makes the whole situation tragic is the presence of a normative conflict. It may be important to distinguish moral dilemmas from normative conflicts (see for instance [?]), but I will speak only about normative conflicts. A normative conflict in its simplest form is a situation when some action is obligatory as well as forbidden (for a more extensive explanation, see [? ]). As regards the case of Antigone, if one tries to evaluate the overall normative situation, it seems as a clear example of normative conflict. Whichever decision the poor Antigone makes, she ends up doing something that is obligatory from one point of view, but forbidden from another point of view.

Recently, Kulicki and Trypuz proposed three systems of multivalued deontic logics aiming to resolve normative conflicts such as the above Antigone’s case (see [? ]). The first two systems employ three deontic values, \( o \) (obligatory), \( n \) (neutral: neither obligatory nor forbidden), and \( f \) (forbidden). The first system proposes a pessimistic view: an action obligatory from one point of view and forbidden from the other is finally regarded as forbidden. In other words, when combining (accumulating) deontic values \( o \) and \( f \), the value \( f \) overrides the value \( o \). The second system suggests an optimistic view. An action that is from one point of view obligatory and from the other forbidden is finally regarded as obligatory; in other words, the value \( o \) overrides the value \( f \). The last system aims to neutralize the conflicting elements and attempts to get rid of the tragic flavour. I will refer to the last system as the neutral view ([? ] uses the term ‘in dubio quodlibet’ view). The third system adds the fourth deontic value \( b \) (both obligatory and forbidden; the notation used in [? ] is slightly different). On this approach, accumulation of deontic values \( o \) and \( f \) results in the value \( b \). However, if some action \( \alpha \) takes the value \( b \), neither \( O(\alpha) \) nor \( F(\alpha) \) holds. To sum up, on the pessimistic view, Antigone did something forbidden; on the optimistic view, she did something obligatory; and finally, on the neutral view, she did something neutral.

Yet I think that the tragic flavour is something essential to cases like this: the burial of Polyneices (\( \alpha \)) was both forbidden and obligatory. Because of this, \( O(\alpha) \land F(\alpha) \) should hold. However, all three systems share the axiom \( \neg(O(\alpha) \land F(\alpha)) \). To capture the tragic aspect of normative conflicts, I will thus propose an alternative, a realistic view on normative conflicts. Yet note that I do not aim to criticize the above systems. Despite proposing and elaborating the fourth alternative, the logic will be built in the framework devised by Kulicki and Trypuz.

In short, the system I propose employs four deontic values, \( o, n, b, \) and \( f \), and it is closely related to the neutral view. The matrix semantics is used and the matrices of action complement and of accumulated actions are the same as in the neutral view. However, the main difference between the neutral view and the realistic view lies in the interpretation of
deontic atoms. In particular, the neutral view evaluates $O(\alpha)$ as true if $\alpha$ takes the value $o$ (and false otherwise); similarly, $F(\alpha)$ is true if $\alpha$ takes the value $f$ (and false otherwise); and $P(\alpha)$ is false if $\alpha$ takes the value $f$ (and true otherwise). The realistic view offers a different interpretation: $O(\alpha)$ is true if $\alpha$ takes the value $o$ or $b$ (and false otherwise); $F(\alpha)$ is true if $\alpha$ takes the value $f$ or $b$ (and false otherwise); and $P(\alpha)$ is false if $\alpha$ takes the value $f$ or $b$ (and true otherwise). I will explain how this semantic difference leads to some other formal differences.

The structure of my talk will be as follows. First, the multivalued deontic logics will be formally introduced. Second, my suggestion will be proposed and compared with the three systems from [? ]. Following the method from [? ], it can be easily shown that the proposed system is sound and complete. Third, it will be explained how the proposed logic captures the intuitions accompanying the case of Antigone.

References


Invariance of Metamathematical Theorems with regard to Gödel Numberings

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It is a classic maxim of metamathematics that arithmetical truth is not expressible in an adequate arithmetical language. The formal counterpart of this maxim is a version of Tarski’s Theorem (as formulated e.g in [? ], Theorem 21.5), commonly proven by employing a certain Gödel numbering. To justify the former informal claim by the latter theorem, it is essential that its validity does not depend on the choice of the employed Gödel numbering, i.e. the theorem is invariant with regard to numberings (see [? ]).

This situation is not peculiar to Tarski’s Theorem as the use of specific numberings is ubiquitous in metamathematical practice. However, interpretations of metamathematical theorems are often carried out without taking the arbitrariness of such numberings into account, or are accompanied by vague invariance claims regarding numberings.
In this talk, the invariance of several metamathematical theorems with regard to numberings will be examined in a formal and systematic framework. The talk will consist of three parts. Firstly, the notion of acceptability of a numbering will be discussed. It will be argued that the computability of a numbering is a necessary condition for its acceptability. The notion of computability will be made precise by work in [? ]. This allows the formerly vague invariance claims to be turned into (meta-)mathematical theorems, whose proofs will be outlined in the second part of the talk. Here emphasis will be given to Gödel’s Second Incompleteness Theorem. Finally, time permitting, the intensionality of the respective consistency statements will be discussed with reference to work by [? ] as well as [? ].

This work may be viewed as in line with other attempts to eliminate arbitrary choices in the process of arithmetisation, particularly in the context of Gödel’s Second Theorem. [? ] locates three sources of indeterminacy in the formalization of a consistency statement for a theory $U$:

I. The choice of a proof system.

II. The choice of a coding system.

III. The choice of a specific formula representing the axiom set of $U$.

According to [? ], “Feferman’s solution [? ] to deal with the indeterminacy is to employ a fixed choice for (I) and (II) and to make (III) part of the individuation of the theory” (p. 544). Visser’s (2011) own approach rests on fixed choices for (II) and (III) but is independent of (I). The primary result of the present work is to eliminate the dependency on (II).

### Acceptable Numberings

At first glance, any injective function from a set of expressions $\mathcal{E}$ to $\mathbb{N}$ qualifies as a numbering. However, it is not hard to construct a numbering of the set of arithmetical expressions which allows the expressibility of an arithmetical truth predicate, thus contradicting Tarski’s Theorem. In order to avoid trivialising the problem of invariance, certain adequacy conditions for acceptable numberings will be presented. It is sufficient for the purposes of this talk to require the following condition:

**(Comp)** Every acceptable numbering is computable.

Since a numbering assigns natural numbers to expressions, the explication of the notion *computable* for such functions by means of the Church-Turing Thesis is not straightforward. However, [? ] provide a suitable framework, basing the concept of computability on finite constructibility (this is further motivated in [? ]). In order to suit this framework, the expressions are taken here to be constructed from a finite “protoalphabet”, following [? ].

Since the aim of the present work is to eliminate arbitrary choices in the process of arithmetisation, it is undesirable to be restricted to a specific construction of the expressions from such a protoalphabet, i.e. whether expressions are finite sequences, finite trees, term algebras, etc. (cf. [? ]). In order to accommodate this need for generality, sets of expressions will be construed as free algebras, building on the work of [? ]. This allows
a unified account of Gödel numberings, independent of the specific structure of expressions. It may be further noted, that this approach does not require acceptable numberings to be monotone.

**Invariance Results**

Let $E$ be any set of expressions over a finite alphabet (as specified above). It can then be shown that for any two acceptable numberings $\alpha$ and $\beta$ of $E$, both $\alpha \circ \beta^{-1}$ and $\beta \circ \alpha^{-1}$ are recursive functions. In this case, $\alpha$ and $\beta$ are called (computably) equivalent.

Using basic recursion theoretic properties, this result yields the invariance of Tarski’s Theorem with regard to acceptable numberings (thereby generalising a similar result in [? ]):

**Theorem 4 (Invariance of Tarski’s Theorem)** For all acceptable numberings $\alpha$ of the set of arithmetical expressions, the set \{ $\alpha(\varphi) \mid N \models \varphi$ \} is not arithmetical.

The main (and original) result of this part is the invariance of Gödel’s Second Theorem with regard to acceptable numberings. In order to prove this theorem, certain properties of equivalence of numberings will be shown to be derivable in elementary arithmetic $EA$.\[16\]

Let $\varphi^-\alpha$ denote $\alpha(\varphi)$, i.e. the standard numeral of the $\alpha$-code of $\varphi$.

**Theorem 5 (Invariance of Gödel’s Second Theorem)** For all acceptable numberings $\alpha$, consistent, recursively enumerable theories $T \supseteq EA$ and arithmetical formulæ \(Pr^\alpha_T(x)\) satisfying the Hilbert-Bernays-Löb derivability conditions relative to $\alpha$ (for $T$), it holds that

$$T \not\vdash \neg Pr^\alpha_T(\varphi^-\alpha).$$

In the context of the above theorem, a formula $Pr(x)$ is said to satisfy the Hilbert-Bernays-Löb derivability conditions relative to $\alpha$ (for $T$), if for all sentences $\varphi$ and $\psi$:

1. $T \vdash \varphi$ implies $T \vdash Pr(\varphi^-\alpha)$;
2. $T \vdash Pr(\varphi^-\alpha) \land Pr(\varphi \rightarrow \psi^-\alpha) \rightarrow Pr(\varphi^-\alpha)$;
3. $T \vdash Pr(\varphi^-\alpha) \rightarrow Pr(Pr(\varphi^-\alpha)^\alpha)$.

It could be argued that a satisfying account of the invariance of Gödel’s Second Theorem also ensures that each acceptable numbering $\alpha$ allows the construction of a (non-trivial\[17\]) provability predicate satisfying the Hilbert-Bernays-Löb derivability conditions relative to $\alpha$. This will be shown by proving the following theorem:

**Theorem 6** For all acceptable numberings $\alpha$ and consistent, recursively enumerable theories $T \supseteq EA$, there exists a formula $Pr^\alpha_T(x)$ which satisfies the Hilbert-Bernays-Löb derivability conditions relative to $\alpha$ (for $T$) and numerates \{ $\alpha(\varphi) \mid T \vdash \varphi$ \} in $EA$.

\[16\] $EA = I\Delta_0 + \forall x, y \exists z e(x, y, z)$, where $e(x, y, z)$ is a binumeration of the exponentiation function and $I\Delta_0$ is PA with induction restricted to $\Delta_0$-formulae.

\[17\] Note that also the formula $x = x$ satisfies the above conditions.
Intensionality of Consistency Statements

Theorems 2 & 3 can be seen as a satisfying solution of eliminating the dependency on Gödel numberings in the formulation of Gödel’s Second Theorem. However, one can ask about the intensional correctness of the employed consistency statements, i.e. how well the formalized consistency statements express consistency. This question is central in Feferman’s seminal article of 1960 and was recently picked up, for instance, by [?] and [?]. Time permitting, the talk will be concluded with a proposal as to how the notion of a canonical provability predicate (and thus a canonical consistency statement) can be made precise as well as a discussion regarding the extensionality of such intensionally correct predicates with regard to numberings. This will refer to work in [?] and [?].

Acknowledgements

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References


The talk will present the results of my master thesis, which analyses four different possible ways of justifying the use of classical inferences in the meta-theory of non-classical logics: 1) To claim that the proposed logic does not aim to be the ‘true’ logic, and that the right logic for meta-linguistic discussions is classical; 2) Carnap’s tolerance principle; 3) Beall-Restall-Pluralism; 4) To claim that the metalinguistic talk is reduced to a determined class of proposition, which behave classically.

The common use of classical inferences in the proof of meta-theorems has been widely criticised. If those critiques are sufficient to discard a logical proposal as incoherent, the greater part of non-classical logics should be rejected. The only alternative would then be to prove their results in the proposed logic. However, if each logician uses their own non-classical logic in the meta-theory, the rational and scientific discussion between different logics would be compromised, since there would then be no common ground to set up the discussions. A weaker logic accepted by all parties in the discussion could be used, but in many cases this logic would not be strong enough to prove desired meta-theorems. This suggests that it would be very useful to develop an account to justify the coherence of the usage of classical inferences in the meta-theory of non-classical logics.

The first possibility analysed is the claim that the proposed logic in the object-level does not have the objective to be the ‘true’ logic, and therefore that there be no grounds to expect that it should be used in the meta-language. If the proposed logic has a technical purpose (as is common in computer science) or is just an object of pure mathematics, then it does not need to be used to model the inferences of the meta-language. Although this may justify the typical everyday praxis of logicians, it still leaves a question unanswered, as to why classical logic be considered the right one at the meta-level. Hence, the question concerning the usage of a classical meta-language in the proposal of a non-classical logic is only of interest provided the proposed logic aims to be the ‘true logic’ (that is, the right logic for the meta-level). In order to clarify the controversial concept of ‘true logic’, Priest’s definition of the ‘canonical application’ (the use of logic to analyse arguments) is used. Although the concept of ‘true logic’ is used, the thesis is not committed to the existence of a true logic, but only to the existence of logical proposals claiming to be the true logic.

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Secondly, Carnap’s tolerance principle is presented. This position is especially interesting because it combines the acceptance of a plurality of logics with the unity of science. His idea of an unrestricted freedom in the syntactic construction of logics is confronted with the question concerning the correctness of the meta-language (Syntax-Language). Although his position is very convincing, it is unable to avoid the problem concerning the ‘true logic’: it also needs to justify the choice of a determined Syntax-Language.

Thirdly, the Beall-Restall-Pluralism is analysed. This position is promising, since it enables one to consider different logics at the same time as true, viz. one in the meta-language and other in the object-language. Different from Carnap, Beall and Restall defend the acceptability of multiple consequence relations within the same language. Nevertheless, it is still not strong enough to justify the fact that the authors always appear to reason classically in the meta-theory: it is unclear how this kind of pluralism can defend itself from becoming a complete relativism.

The fourth position presented claims that one is allowed to reason classically in the meta-theory due to the classical behaviour of the restricted domain of the metalinguistic propositions (even considered by a non-classical logic). This position is able to provide a coherent theory for logicians who defend some non-classical logics (such as Ł3), but it suffers the setback of needing to be evaluated under each specific logic. The approach is not universally applicable. Indeed, in some cases, such as under fuzzy-logics, there are good reasons to consider that the domain of metalinguistic propositions does not behave classically (the phenomenon of higher-order-vagueness is presented as an example).

My thesis aims to close a gap in the literature: although there is much debate concerning this topic, there is no systematic and critical presentation of the main possible philosophical positions. My goal was not only to present those positions, but also to critically analyse them, pondering if they are able to give an answer to the question posed by me. Therefore, this research brings its own original contribution to the debate.

Regarding the short time for the presentation, it will not be possible to treat all topics in depth. The talk will be focused on position (2), Carnap’s tolerance principle.

References

A Variable as a Non-Rigidly Designating Modal Constant.
A Novel Framework in Reverse Correspondence Theory

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It is well known that modal languages in arbitrary similarity type correspond via the standard translation to one free variable fragment of first-order logic with bounded quantifiers. The aim of the so called reverse correspondence theory is to turn the tables and unveil the modal features of predicate logic with identity. The starting point of this endeavor are two natural observations. The axiom schemas of predicate logic have the same syntactic pattern as axiom schemas for the modal calculus K. Moreover, the rule of universal generalization mirrors the rule of necessitation. From the semantic angle the Tarskian satisfaction clause for quantified formulas invokes a binary relation of i-variance between assignments or sequences. Consequently quantifiers are interpretable as multi-modal operators. The minimal goal of reverse correspondence theory is to find a modal language which has the same expressive power as the first-order language at hand. In the existing literature there exist two frameworks which satisfy this desideratum. The first framework is the sorted modal logic of Kuhn. Whereas the second framework is the cylindric modal logic of Venema. We critically discuss these two frameworks. We emphasize their features, which can be interpreted as shortcomings. These pertain to the choice of the target language and the source language of respective translations and the metatheoretic properties of the resulting systems. Taking this discussion for granted we proceed towards ours original framework. We drop the assumption that first-order languages should be translated into propositional multi-modal languages. Moreover, we wish to account for the complexity of atomic formulas of ordinary first-order languages directly. In order to reach these objectives and satisfy the minimal goal of reverse correspondence theory we define the syntax of a specific first-order modal language. We denote this language by QFML. The characteristic property of QFML is the lack of variables and consequently quantifiers as syntactic constituents. The terms of QFML are just individual constants. Next we define the Kripke style semantics for QFML where constants are interpretable non-rigidly. Subsequently, we define the bisimulation relation appropriate for this language and compare it with the standard world-object bisimulation. With this at hand we show that QFML is indeed expressively weaker than its quantified extension. Next we proceed towards spelling out the details of our framework. Philosophically, it is motivated by arguing for similarity holding between semantics of a first-order variable and a non-rigidly designating individual constant. As expected our translation has two components. Accordingly, it consists of (a) the structure transformation function and (b) the syntactic translation. In order to define the former we borrow certain ideas from Rautenberg and introduce the notion of the space of models structures. We show that each space of model structures can be transformed into the unique constant domain Kripke structure with rigid interpretation of predicate symbols. Subsequently, we define
recursively our (gramatical) syntactic translation function. Finally we prove the semantic correctness of this translation.

In the last part of the talk we ask few metatheoretic questions. Firstly, we isolate two classes of structures for QFML. The first class is the class $X$ of all structures for QFML. The logic of $X$ is readily seen to be axiomatizable by Hilbert-style axiom system $K$. The proof of the strong completeness of this axiomatization with respect to $X$ is easily obtainable. The second class $Q$ is the class of structures whose members satisfy: (1) constancy of a domain (2) rigid interpretation of predicate symbols (3) cardinality constraints on the set of possible worlds determined by all possible interpretations of individual constants (4) coincidence of accessibility relations with $i$-variance relations. The elements of $Q$ are modal counterparts of the space of model structures for FOL. We ask whether the logic of $Q$ has a strongly complete axiomatization? An obvious attempt to answer this question is define a Hilbert-style axiom system which mirrors (via our translation) a Hilbert-style axiom system for FOL. Nevertheless, there is a technical problem with the execution of this idea. The standard Hilbert Style Axiom System for FOL critically depends on the notions of a free and a bound occurrence of a variable in a formula. When the attention is restricted to QFML we can at most inductively define the set of constants occurring in a formula. In order to deal with this issue we propose two solutions. The first solution is to employ the syntactic translation function in order to simulate the notions of a free and bound occurrence of a variable. The second solution is to choose a non-standard Hilbert-style axiom system for FOL defined by Tarski. This equivalent axiomatization of FOL employs only the replacement operation of a variable by a variable. In the present context the latter choice happens to be more natural. Finally, we aim to prove that the resulting axiomatization is strongly complete with respect to the class $Q$. We perform the standard canonical structure construction. Nonetheless, it turns out that the canonical structure does not belong $Q$. Namely, the domain of the canonical structure is not constant, the canonical accessibility relations are not $i$-variance relations and the interpretation of predicate symbols is not rigid. At first glance this situation looks disappointing. However, we find a reasonable explanation of these facts. The above mentioned properties are not expressible in QFML. Eventually, we conclude that ours Hilbert-style axiom system is complete with respect to $Q$. However the proof of this is indirect and employs properties of the previously defined translation.

References


Proof Mining for Nonexpansive Semigroups

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Proof mining is an ongoing research program in applied proof theory that involves logically analyzing proofs of mathematical statements and making their quantitative content explicit to derive new, constructive information. The logical form of a mathematical statement can, within a specific formal framework, a priori guarantee by certain logical metatheorems the extraction of additional information via a study of the underlying logical structure of its proof. This new information is usually of quantitative nature, in the form of effective (computable) bounds -even if the original proof is ineffective- and highly uniform i.e. it depends only on general bounds on the input data. The program was introduced by Ulrich Kohlenbach starting in [? ] though it finds its origins in the ideas of Georg Kreisel from the 1950's ([? ] [? ] ) who initiated it under the name unwinding of proofs. Proof mining has since been applied by Kohlenbach and his collaborators to works in analysis in general and more specifically to approximation theory, ergodic theory, fixed point theory, optimization theory and (recently for the first time) to the theory of differential equations, and has produced a vast number of results. A review of the results until 2008 can be found in the book ([? ] ) and after 2008 in the report [? ].

We will present here a recent application to nonlinear analysis and fixed point theory in which we extracted computable information on the approximate common fixed points of one-parameter nonexpansive semigroups on a subset of a Banach space (this was included in [? ] and [? ]).

We performed proof mining to the proof of the following theorem:

**Theorem 7** (Suzuki ([? ])) Let X be a Banach space and let \{T(t) : t \geq 0\} be a one-parameter nonexpansive semigroup on \( C \subseteq X \). Let \( \{\alpha_n\} \) be a sequence in \([0, \infty)\) converging to \( \alpha_\infty \in [0, \infty) \), and satisfying \( \alpha_n \neq \alpha_\infty \) for all \( n \in \mathbb{N} \). Suppose that \( z \in C \) satisfies \( T(\alpha_n)z = z \) for all \( n \in \mathbb{N} \). Then \( z \) is a common fixed point of \( \{T(t) : t \geq 0\} \).

A formalised version of the above statement is:

\[
\forall z \in C (\forall \delta > 0 \forall n \in \mathbb{N} \|T(\alpha_n)z - z\| \leq \delta \to \forall k \in \mathbb{N} \forall t \in [0, \infty)\|T(t)z - z\| < 2^{-k}).
\]

By prenexing the above we have

\[
\forall z \in C \forall k \in \mathbb{N} \forall t \in [0, \infty) \exists \delta > 0 \exists n \in \mathbb{N}(\|T(\alpha_n)z - z\| \leq \delta \to \|T(t)z - z\| < 2^{-k})
\]
i.e. (setting \( C_b := \{ z \in C : \| z \| \leq b \} \))

\[
\forall b \in \mathbb{N} \forall z \in C_b \forall k \in \mathbb{N} \forall M \in \mathbb{N} \forall t \in [0, M] \exists \delta > 0 \exists n \in \mathbb{N} \\
(\| T(\alpha_n)z - z \| \leq \delta \rightarrow \| T(t)z - z \| < 2^{-k}).
\]

The goal is to extract computable bounds on \( \delta \) and \( n \) from the proof of the statement. This bound extraction would then amount to obtaining a quantitative version of the statement.

We can show that we may write a version of a general logical metatheorem by Gerhardy and Kohlenbach (see Theorem 17.52 and Corollary 17.71 in [?]) adapted in particular explicitly for the mathematical setting and assumptions of Suzuki’s theorem above (such logical metatheorems have been shown using variations of Gödel’s functional Dialectica interpretation ([?])). Our adaptation reads:

**Metatheorem 1** Assume that we have a proof of a sentence in \( \mathcal{A}^\omega[X, \| \cdot \|, C]_{-b} \)

\[
\forall t \in \mathbb{R}^+ \forall z \in C \forall T \in C \times \mathbb{R}^+ \rightarrow C \forall \{ \alpha_n \} \subseteq \mathbb{R}^+ \forall \alpha_\infty \in \mathbb{R}^+
\]

\[
\forall \omega \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}
\]

\[
\forall \Phi, \Psi \in \mathbb{N} \rightarrow \mathbb{N} \forall m \in \mathbb{N} \exists k \in \mathbb{N} \exists n \in \mathbb{N}
\]

\[
(\forall t \in \mathbb{R}^+ \forall x, y \in C \| T(t)x - T(t)y \| \leq_\mathbb{R} \| x - y \|)
\]

\[
\wedge(\forall x \in C \forall t, s \in \mathbb{R}^+ T(s) \circ T(t)(x) =_X T(s + t)(x))
\]

\[
\wedge(\forall b \in \mathbb{N} \forall q \in C \forall m \in \mathbb{N} \forall K \in \mathbb{N} \forall t, t' \in [0, K]
\]

\[
(\| q \| <_R b \land | t - t' | <_R 2^{-\omega_{K,\lambda(m)}} \rightarrow \| T(t)q - T(t')q \| \leq_\mathbb{R} 2^{-m})
\]

\[
\wedge(\forall n \in \mathbb{N} | \alpha_n - \alpha_\infty | \geq_\mathbb{R} 2^{-\Psi(n)})
\]

\[
\wedge(\forall k \in \mathbb{N} \forall n \geq \Phi(k) | \alpha_n - \alpha_\infty | \leq_\mathbb{R} 2^{-k})
\]

\[
\wedge(\| T(\alpha_n)z - z \| \leq_\mathbb{R} 2^{-k}) \rightarrow \| T(t)z - z \| <_R 2^{-m}
\].

Then one can extract from the proof computable functionals \( W, \tilde{W} \) so that

\[
\forall M \in \mathbb{N} \forall t \in [0, M] \forall L \in \mathbb{N} \forall \{ \alpha_n \} \subseteq [0, L] \forall \alpha_\infty \in [0, L] \forall B \in \mathbb{N} \forall z \in C_B
\]

\[
\forall T \in C \times \mathbb{R}^+ \rightarrow C \forall \omega \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \forall \Phi, \Psi \in \mathbb{N} \rightarrow \mathbb{N}
\]

\[
\forall m \in \mathbb{N} \exists k \leq W(B, M, L, \Psi', \Phi', m, \omega') \exists n \leq \tilde{W}(B, M, L, \Psi', \Phi', m, \omega')
\]

\[
(\forall t \in \mathbb{R}^+ \forall x, y \in C \| T(t)x - T(t)y \| \leq_\mathbb{R} \| x - y \|)
\]

\[
\wedge(\forall x \in C \forall t, s \in \mathbb{R}^+ T(s) \circ T(t)(x) =_X T(s + t)(x))
\]

\[
\wedge(\forall b \in \mathbb{N} \forall q \in C \forall m \in \mathbb{N} \forall K \in \mathbb{N} \forall t, t' \in [0, K]
\]

\[
(\| q \| <_R b \land | t - t' | <_R 2^{-\omega_{K,\lambda(m)}} \rightarrow \| T(t)q - T(t')q \| \leq_\mathbb{R} 2^{-m})
\]

\[
\wedge(\forall n \in \mathbb{N} | \alpha_n - \alpha_\infty | \geq_\mathbb{R} 2^{-\Psi(n)})
\]

\[
\wedge(\forall k \in \mathbb{N} \forall n \geq \Phi(k) | \alpha_n - \alpha_\infty | \leq_\mathbb{R} 2^{-k})
\]

\[
\wedge(\| T(\alpha_n)z - z \| \leq_\mathbb{R} 2^{-k}) \rightarrow \| T(t)z - z \| <_R 2^{-m}
\]

holds for any nontrivial normed space \( X \) with a nonempty \( C \subseteq X \).
We stress that the purpose of a metatheorem is to serve only as a guideline to guarantee the extractability and uniformity of a computable bound from the proof of a mathematical statement that can be written in a certain logical form ($\forall \exists$). When applying proof mining on a given mathematical proof, the precise method of extracting the bound is not known a priori. Typically, this is done in three stages: (i) We write all the statements involved in a formal version using quantifiers. (ii) The mathematical objects involved must have the correct uniformity. To this end, we make explicit the quantitative content of their properties (i.e. modulus of continuity for uniform continuity, modulus of accretivity for uniform accretivity, modulus of convexity for uniform convexity, effective irrationality measure for irrationality etc). In that way we obtain quantitative versions of the statements/lemmas involved. (iii) Finally, we put the latter together in a deduction schema just like the one of the original proof, i.e. the structure of the original proof is typically preserved. However, the above process and the steps thereof are not automated and (even though not completely ad hoc) they are open to the manipulations of the mathematician(s) performing proof mining on a given proof.

As the above metatheorem predicts (and by following the above sketched method performing proof mining on Suzuki’s proof) we do succeed in extracting a computable and highly uniform bound. In particular, our result reads:

**Theorem 8** Let $X$ be a Banach space and let $\{T(t) : t \geq 0\}$ be a one-parameter uniformly equicontinuous semigroup of nonexpansive mappings on a subset $C$ of $X$, with a modulus of uniform equicontinuity $\omega$. Let $\{\alpha_n\}$ be a sequence of reals in $[0, \infty)$ converging to $\alpha_\infty \in [0, \infty)$ with a rate of convergence $\Phi : \mathbb{N} \to \mathbb{N}$, and so that $\forall n \in \mathbb{N}(|\alpha_n - \alpha_\infty| > 2^{-\Psi(n)})$ where $\Psi : \mathbb{N} \to \mathbb{N}$. Let $L \in \mathbb{N}$ be such that for all $n \in \mathbb{N} \{\alpha_n\}, \alpha_\infty \in [0, L]$. Then

$$\forall k \in \mathbb{N} \forall b \in \mathbb{N} \forall z \in C_b \forall M \in \mathbb{N} \forall L \in \mathbb{N} \exists n \leq \tilde{W} \quad \forall t \in [0, M] \quad \|T(t)z - z\| < 2^{-k}$$

with

$$\tilde{W} = \hat{W}(k, b, M, L, \Phi, \Psi, \omega) = \max\{\Phi(\omega_{b,M+1}(k+1)), \Phi(\omega_{b,L}(k+1 + \lceil \log_2(3M \sum_{i=1}^{\Phi(\omega_{b,M+1}(k+1))} 2^{\Psi(i)}) \rceil))\}$$

and

$$W = W(k, b, M, \Phi, \Psi, \omega) = \frac{2^{-(k+1)}}{3M \sum_{i=1}^{\Phi(\omega_{b,M+1}(k+1))} 2^{\Psi(i)}}.$$

**References**

Formalising Theories in Economics with Dynamic Epistemic Logic

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Of all of the economic bubbles that have been pricked [since 2008], few have burst more spectacularly than the reputation of economics itself. (The Economist 2009)[7, p.4].

As nicely stated in the above quote, the 2008 recession raised a lot of criticism towards economics: economists make unrealistic assumptions, their models were unable to predict the economic crisis and policymakers failed to prevent the collapse of Lehman Brothers and other financial institutions — small ones as well as giants. This presentation aims to reflect on the criticism of economics and argue that dynamic epistemic logic provides both the appropriate language and semantic models to reason about economic phenomena, especially when it comes to bubbles.
An often criticised cornerstone of theories in micro-economics is the infamous assumption of (common knowledge of) rationality. Indeed, such a highly idealised assumption may be misleading as it nowhere close characterises human nature. However, formal theories relying on such unrealistic assumptions can be useful, as long as their intended goals are not perfect prediction but rather structural insights and a formal understanding of the phenomena under study. Consider for instance No Trade theorems [? ? ] which are successful exactly because strong, unrealistic assumptions set the stage for a clear and concise understanding of the structural conditions for trade and liquid markets. The expectation to clarify concepts justifies the preservation of idealised abstractions. Clarifying basic concepts such as trade, liquidity and information updates, then combined with empirical observations, lab experiments and simulations, will contribute to an explanation of the complex economic world, the people in it and the way they deliberate, decide and act.

A case in point, consider asset price bubbles, which have been extensively studied both theoretically [? ? ?] and empirically [? ? ]. A financial bubble describes specific scenarios in which asset prices rise rapidly and way beyond fundamental value and in which eventually the market crashes, where the fundamental value is based on traders’ individual estimates of the asset’s future dividend. A proper understanding of bubbles, their environment and how investors reason about the market and each other during the build-up potentially plays a key role in preventing future crises (not only in finance, but for overheated assets in politics, media, science, art, etc., as described in [? ]).

In [? ], Brunnermeier divides literature on bubble phenomena into four groups, characterising different models on epistemic conditions for traders. This demonstrates the information-driven role in bubble formation, embodied by interacting multi-agents and higher-order epistemic structures. In particular, the epistemic concept of common knowledge plays a central role in economic theories about bubbles. Therefore, micro-economical theories may be adequately complemented by results from (dynamic) epistemic logic, because its language is suited to characterise epistemic conditions and its possible world semantics displays higher-order epistemic structures. A multi-agent Kripke model can map the circumstances under which common knowledge of a particular proposition holds, potentially revealing epistemic and doxastic interrelations between agents and atomic facts.

A concrete example where dynamic epistemic logic can help to understand underlying structures and relations of bubble phenomena is the greater fools theory. According to the theory, rational traders are willing to pay more for an asset than they deem it worth, because they anticipate they might be able to sell it to someone else for an even higher price, i.e., “the greater fool” [? ]. The greater fools theory sounds like a crisp and clear explanation for a mismatch between price and value, but it turns out to be a surprisingly difficult theory to model and analyze [? ]. At the heart of the greater fools theory are agents’ beliefs about other agents’ beliefs, thus epistemic logic seems like a suitable candidate framework for a formal analysis of the theory. Formally translated in [? ], a trader \( i \) may have two different kinds of motivation for buying an asset for price \( p \): either the agent believes the asset is worth more than the price, or the agent believes she can sell the
asset to another trader $j$ for a higher price $p' > p$ in the future:

$$\text{buy}_i(p) \leftrightarrow \left( B_i(v > p) \lor B_i \left( \bigvee_{j \in A, p' > p} \text{buy}_j(p') \right) \right)$$

Here $\text{buy}_i(p)$ says that agent $i$ is willing to buy the asset for price $p$, $B_i(v > p)$ reads that agent $i$ believes that the asset’s value is higher than $p$ and as $A$ is a given set of agents, the latter part expresses agent $i$’s belief in a “greater fool” $j$. Unfolding the higher-order content of the expression shows:

$$\text{buy}_i(p) \leftrightarrow \left( B_i(v > p) \lor B_i B_i(v > p + 1) \lor \ldots \lor B_i B_i \ldots B_k(v > p + k) \lor \ldots \right)$$

Another example is the work of Dégremont and Roy [10] and Demey [9], who use respectively plausibilistic epistemic logic and probabilistic dynamic epistemic logic to study Aumann’s agreeing theorem, with the explicit goal of clarifying conceptual issues. Succeeding in this, Demey argues that the role of common knowledge is less central to the agreement theorem than is often thought. In a broader sense, this ambition adds to the more common application of dynamic epistemic logic to social interaction and rationality and in specific those of bubble-fueling herding behavior, such as models of informational cascades [9,10].

Hopefully, studies in this fashion may help rebuilding the reputation of economics by supporting or adjusting the micro-economic theories of bubble formation with structural insights coming from this new angle of approach such that together we can avert future crashes and subsequent recessions — in theory as well as in practice.

References


As a consequence of his main theorem, \[?\], IV, §1.2] obtained a decision procedure for intuitionistic propositional logic. Gentzen’s procedure consisted basically in searching for proofs in the sequent calculus. In the early 1990s, issues related to computational efficiency led to the development of contraction-free sequent calculi \[?\]. These are sequent calculi especially tailored to ensure termination of backward (bottom-up) proof-search and thus avoid costly implementation measures like loop detection.

On the other hand, despite being more intuitive and more akin to informal reasoning, natural deduction systems are generally believed to be less suited as a framework for the development of automated decision procedures. The unidirectional character of natural deduction derivations together with its global approach to assumption discharge means that, in general, one has to appeal to both bottom-up and top-down methods of search.
which contrasts sharply with sequent calculi where a bottom-up method usually suffices. As a result, most decision procedures related to natural deduction are either based on hybrid systems [? ] or involve some translation between natural deduction and sequent calculi or some other formalism (e.g. Sieg’s intercalation calculus [? ] which provides a framework that combines bottom-up and top-down search).

Nevertheless, as a parallel to cut elimination in sequent calculi, many natural deduction systems enjoy normalization which also reduces the search space and thus facilitates automated proof-search. The main idea behind normalization (and, one could argue, also behind the corresponding cut elimination in sequent calculi) is what became known as the inversion principle. This idea is also known in the literature as proof-theoretic harmony, a notion which tries to elaborate some remarks of ? , II, § 5.13 and is a cornerstone of proof-theoretic semantics.

Although conceptually indebted to the same overall Gentzenian idea, decision procedures based on proof-search in sequent calculi are usually very sensitive to the underlying proof system. Thus, when comparing decision procedures, as is usual with proof systems in general, one is often faced with trade-offs between conceptual robustness, computational efficiency, simplicity of formulation and etc.

The relationship between harmony and normalization together with the constructive character of definitions of validity in proof-theoretic semantics make it reasonable to expect that these definitions would provide a method to decide the intuitionistic validity of arbitrary arguments from assumptions \( \Gamma \) to conclusion \( G \); that is, it is reasonable to expect that they would provide a decision procedure (at the very least for the propositional case). Such decision procedures would rest on a semantic basis and would therefore be independent of any particular proof system. Even though they may not be computationally very efficient, one may hope that they would be at least conceptually compelling (as is the case, for instance, with the truth table method in the classical propositional case).

? , Chapters 11–13 proposed two distinct proof-theoretic justification procedures for logical laws: one “verificationist” and the other “pragmatist”. The verificationist procedure is a proof-theoretic definition of validity on the basis of the introduction rules. The pragmatist procedure is a proof-theoretic definition of validity on the basis of the elimination rules. These procedures rely heavily on harmony, and their application to particular cases resembles the process of searching for analytic proofs (i.e. normal proofs in natural deduction or cut-free proofs in the sequent calculus).

Dummett’s pragmatist justification procedure, in particular, has a very intuitive and straightforward algorithmic interpretation. The main idea of the pragmatist procedure is that an argument is valid when any consequence that can be drawn in a canonical manner (i.e. by means of eliminations) from the conclusion can also be drawn in a canonical manner from the assumptions. This means that the procedure decides if an arbitrary argument from assumptions \( \Gamma \) to conclusion \( G \) is intuitionistically valid by appealing solely to the elimination rules for the logical constants. We can think of the algorithm as composed of two separate components: complementation (which determines what can be obtained from the conclusion) and search (which looks for a way to obtain the same thing from the assumptions). Here is very short (because of limitations of space) description of these components, in general lines:
Complementation To complement an argument $\Gamma : G$, we begin with $G$ as a major premiss and apply a series of elimination rules until we arrive at a conclusion $C$ which is either an atomic formula or $\bot$. However, the application of elimination rules figuring minor premisses may need auxiliary assumptions. Therefore, in order to uniquely fix their application, the rules of implication elimination ($\rightarrow E$) and disjunction elimination ($\lor E$) are always applied (in complementation as well as in search) according to their following instances:

\[
\frac{A \rightarrow B \quad A \rightarrow E}{A \lor B \quad \frac{A \rightarrow \chi}{\chi} \quad \frac{B \rightarrow \chi}{\chi} \lor E}
\]

where $A$ (in $\rightarrow E$) and $A \rightarrow \chi$ and $B \rightarrow \chi$ (in $\lor E$) are added to a set $\Delta$ of auxiliary assumptions and $\chi$ (in $\lor E$) is an atomic formula that does not occur in $\Gamma \cup \Delta$.

Search Here, for every conclusion $C$ of a complementation, we search for a canonical argument for $C$ from $\Gamma \cup \Delta$ (where $C$ and $\Gamma \cup \Delta$ are as determined by the complementation). For each assumption $A \in \Gamma \cup \Delta$, we apply a series of elimination rules until we arrive at an atomic formula or $\bot$. If there is no $A \in \Gamma \cup \Delta$ such that we can obtain $C$, then $\Gamma : G$ is invalid. In the case where a required auxiliary assumption is not in $\Gamma \cup \Delta$, we assume that it can be validly inferred from $\Gamma \cup \Delta$ and reiterate the procedure (recursive step).

As already remarked, a noteworthy feature of the procedure, which further emphasizes its semantic character, is the fact that it appeals solely to the eliminations rules for the constants of intuitionistic propositional logic. Perhaps it would be useful to consider a simple, but not trivial, example in order to illustrate how the procedure works and how it employs traditional notions from proof-theoretic semantics (like canonical argument). Consider the argument:

\[
\frac{A \rightarrow (B \land C)}{(A \rightarrow B) \land (A \rightarrow C)}
\]

The pragmatist procedure says that this argument would be valid if whatever consequences can be drawn from the conclusion in a canonical manner, can also be drawn from the assumptions in a canonical manner. In order to see what can be extracted canonically from the conclusion, we complement:

\[
\frac{A \rightarrow (B \land C)}{(A \rightarrow B) \land (A \rightarrow C)}\quad \frac{A \rightarrow (B \land C)}{(A \rightarrow B) \land (A \rightarrow C)}
\]

In order to show validity, we search canonical arguments from assumptions $A \rightarrow (B \land C)$ and $A$ to conclusion $B$, and from assumptions $A \rightarrow (B \land C)$ and $A$ to conclusion $C$:
A → (B ∧ C) A

\[\frac{B ∧ C}{B} \quad \frac{B ∧ C}{C}\]

Proof-theoretic definitions of validity based on the elimination rules seems to have received far less attention in the literature than their counterparts based on the introduction rules. Moreover, although the algorithmic interpretation of the pragmatist procedure is somewhat implicit in Dummett’s original presentation, its potential as an automated decision procedure seems to have passed unnoticed in the literature or, at least, to have been left unexplored.

In my talk, I shall present briefly the decision procedure for propositional intuitionistic logic based on Dummett’s pragmatist justifications (as sketched above) and work out some illustrative examples. I shall also discuss how to transform pragmatist justifications into natural deduction derivations and back (in this way, I aim to indicate how to obtain arguments for the adequacy of the procedure since I would not be able to discuss that in detail because of the time constraint). Furthermore, I shall compare the pragmatist procedure with decision procedures based on proof-search in the sequent calculus. Finally, I shall report preliminary results of running an implementation of the pragmatist procedure against the propositional part of the ILTP library of problems for testing and benchmarking automatic theorem provers [? ]. The implementation is called LONER (LONER Only Needs Elimination Rules) and is written in the C programming language.

References


Proof-Theoretic Analysis of the Quantified Argument Calculus

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This paper investigates the proof theory of the Quantified Argument Calculus (Quarc) as developed and systematically studied by Hanoch Ben-Yami \cite{Ben-Yami1}, \cite{Ben-Yami2}. Ben-Yami makes use of natural deduction (Suppes-Lemmon-style), we, however, have chosen a sequent calculus presentation, which allows for the proofs of a multitude of significant meta-theoretic results with minor modifications to the Gentzen’s original framework, i.e. LK. LK, although it has been developed in the 1930s, serves still (as a basis) for proof theoretic investigations \cite{Ben-Yami3}, \cite{Ben-Yami4}, \cite{Ben-Yami5}, \cite{Ben-Yami6}. The reason to use sequent calculus in this analysis is to provide a constructive proof of consistency, but first and foremost to prove an important, useful and interesting result of a cut elimination theorem and its corollaries. We are likewise able to straightforwardly extend the system with the identity, which does not appear in \cite{Ben-Yami1}.

Quarc is a system of quantified logic which does away with variables and unrestricted predicates, but nonetheless achieves results similar to the Predicate Calculus by employing quantifiers applied directly to predicates which appear as arguments of other predicates (hence the name Quantified Argument Calculus), along with anaphors and operators that attach directly to predicates. It is in this respect arguably closer to natural language. A goal of this paper is to show how and to what extent some of these results are achieved. Given that this is an interesting but not widely known system, it will be presented in some detail before proceeding with the proof-theoretic analysis of it.

Let us note that the quantifiers in Quarc do have particular import, a fact that is expressed semantically by the condition of instantiation – non-emptiness of (unary) predicates. This is in contrast to first-order predicate logic, where, as it is well known, (unary) predicates can be empty. On the level of theorems we make distinctions on the strength of particular import. On the basic level, i.e. LK-Quarc\textsubscript{B}, this is expressed by the following formulas (as stated previously, the notation of Quarc and its language will be explained in some detail): (1) \((\forall S)P \rightarrow ((\exists S)S \rightarrow (\exists S)P)\) – example: if all men are mortal, then if there are men, then some men are mortal; and (2) \((\forall S)P \rightarrow (aS \rightarrow aP)\), e.g. if all men are mortal, then if Socrates is a man, then Socrates is mortal. The strong version of particular import, that is, (3) \((\forall S)P \rightarrow (\exists S)P\) is a theorem of LK-Quarc\textsubscript{3}. Clearly, \((\exists S)S\), which can be read as “there are S”, is a theorem of Quarc\textsubscript{3} as well. However, this is not to be conflated with the existential construction “S exist”, as noted by Ben-Yami in \cite{Ben-Yami6} and discussed in more detail in \cite{Ben-Yami4}. Following that, the quantifier \(\exists\) is referred to as particular quantifier in this paper.
Concerning the other special symbols – Quarc introduces additional logical symbols and operations of Anaphora, Reorder and Negative Predication. Anaphora fulfills a role roughly similar, but broader, than that of the variables in Predicate calculus and is crucial in determining which parts of the formula a quantified argument governs. Reorder is an operation that replaces predicates with those which contain arguments in different order. Reordered predicates are interchangeable with identity-permutation ones in the basic case, but not in the quantified case, and are used to determine mutual governance in a multiply quantified formula. Negative Predication is an operation that switches between sentential negation (e.g. \( \neg(a)S \), it is not the case that a is S) and predication negation (e.g. \( (a)\neg S \), a isn’t S). Again, these two uses of negation are interchangeable in the basic, but not the quantum case (compare: \( \neg(\exists S)P \), it is not the case that some S are P and \( (\exists S)\neg P \), some S aren’t P), and are therefore used to determine the mutual scope of negations and quantifiers. Most of the proofs in this paper will focus on the quantifiers and the related additional special symbols, as those are the primary novelty of Quarc.

The way the research on Quarc is conducted here is as follows: we observe first that Ben-Yami’s Quarc is a rather rich system. In our analysis we split up Quarc into three distinct sub-systems, namely (1) LK-Quarc\(_B\), (2) LK-Quarc\(_2\), (3) LK-Quarc\(_3\), and finally, LK-Quarc - LK representation of full Quarc. LK-Quarc\(_B\) does not contain either the rules for identity or instantiation. LK-Quarc\(_2\) is an extension of LK-Quarc\(_B\) with identity, and LK-Quarc\(_3\) an extension of LK-Quarc\(_B\) with the rule for instantiation. Finally, LK-Quarc is obtained by combining LK-Quarc\(_2\) and LK-Quarc\(_3\). Each of the sub-systems is demonstrated to be deductively equivalent to its corresponding version of Quarc, and all of the sub-systems, so consequently the full LK-Quarc as well, will enjoy cut elimination and its corollaries (including subformula property and thus consistency which is not proven in \[?]\), although it follows almost immediately from the soundness proof present there).

References


Completeness via Correspondence for Extensions of First Degree Entailment Supplied with Classical Negation

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My presentation is zeroed in on a logic to be an extension of famous Belnap and Dunn’s logic $\text{FDE}$ by classical negation and labeled as $\text{BD}^+$ by De and Omori. I will present a uniform method for constructing natural deduction systems for all possible unary and binary truth-functional extensions of $\text{BD}^+$\textsuperscript{18}. In so doing I apply Kooi and Tamminga’s technique of correspondence analysis. First applied to Priest’s logic of paradox $\text{LP}$ (ref. [? ? ? ]) this approach was introduced in [? ], and later Tamminga in [? ] developed it for Kleene’s strong three-valued logic $\text{K}_3$ (ref. [? ]). My aim is to generalize correspondence analysis to the area of four-valued logics.

To begin with, consider some preliminary definitions\textsuperscript{19}. A language $\mathcal{L}$ of $\text{BD}^+$ is specified by the following grammar:

$$A := p \mid \neg A \mid \sim A \mid A \lor A \mid A \land A,$$

where $\neg$ is De Morgan negation and $\sim$ is classical (Boolean) negation\textsuperscript{20}. Let $\mathcal{L}_{(\star)_n(o)_m}$ be $\mathcal{L}$’s extension by unary $\star_1, \ldots, \star_n$ and binary $o_1, \ldots, o_m$ operators, respectively. Let $\text{BD}_+(\star)_n(o)_m$ be a logic built in $\mathcal{L}_{(\star)_n(o)_m}$. If $o$ is a truth-functional operator then $f_o$ is a truth table for $o$. Let $V_4$ be a set $\{1, b, n, 0\}$ of truth values “true”, “both true and false”, “neither true nor false”, and “false”. Let $x, y, z \in V_4$. Then $f_\star(x) = y (f_o(x, y) = z)$ stands for an entry of a such truth table $f_\star (f_o)$ that for each valuation $v$ if $v(A) = x$ then $v(\star A) = y$, for each $A \in \mathcal{L}_{(\star)_n(o)_m}$ (if both $v(A) = x$ and $v(B) = y$ then $v(A \circ B) = z$, for all $A, B \in \mathcal{L}_{(\star)_n(o)_m}$). For a four-valued case the following adaptation of Kooi and Tamminga’s definition 2.1 and Tamminga’s definition 1 holds:

\textsuperscript{18}Note Sano and Omori [? ] consider some first-order extensions of $\text{BD}^+$. I thank the second referee for attracting my attention to this paper.

\textsuperscript{19}I follow Kooi and Tamminga’s [? ] notation adopted for the case of $\text{BD}^+$.

\textsuperscript{20}$\text{FDE}$ is built in $\mathcal{L}$’s $\{\neg, \land, \lor\}$-fragment.
Definition 8 (Single Entry Correspondence) Let $\Gamma \subseteq \mathcal{L}_{(\star) n}(\circ)_m$ and let $A \in \mathcal{L}_{(\star) n}(\circ)_m$. Let $x, y, z \in V_4$. Let $\mathcal{E}$ be a truth-table entry of the type $f_1(x) = y$ or $f_0(x, y) = z$. Then the truth-table entry $\mathcal{E}$ is characterized by an inference scheme $\Gamma/A$, if

$\mathcal{E}$ if and only if $\Gamma \models A$.

I have found such inference schemes of the type $\Gamma/A$ that each possible truth-table entry $\mathcal{E}$ (mentioned in definition ??) is characterised by some inference scheme.

These inference schemes are in fact inference rules. By adding them to a natural deduction system for $\text{BD}^+$, one obtains natural deduction system $\mathcal{N} \mathcal{D}_{\text{BD}^+(\star) n}(\circ)_m$ for $\text{BD}^+(\star) n(\circ)_m$. Recently I have proved soundness and completeness theorems for $\mathcal{N} \mathcal{D}_{\text{BD}^+(\star) n}(\circ)_m$ [? ].

References


must and might in Questions

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Background

One focus of recent work in formal semantics lies on notions of meaning that comprise more than the informative content of a sentence. [?], and [?] present a framework integrating the inquisitive and informative content of sentences. [? ] further attempt to incorporate the attentive content expressed by might-sentences. A dynamic account of might and must is presented by [? ]. A key feature here is the introduction of ‘live possibilities’ as possibilities that are taken seriously by the agent. Extending this idea, [?] suggests ‘live necessities’ for a new treatment of must. [?] integrates live possibilities into the inquisitive framework to give, among others, a treatment of might in questions.

Based on these contributions, we propose a semantics that is inspired by Willer’s treatment of epistemic modalities using live possibilities and live necessities and implement this in an inquisitive fashion. The new result is a semantics that offers an arguably adequate treatment of must in questions, while still accounting for might in questions, free-choice effects, and epistemic contradictions.

Proposal

We propose a support conditional semantics that specifies the meaning of a sentence in terms of a support relation between information states and issues expressed by sentences. Intuitively, this relation holds if the information state contains enough information to resolve the issue. In the case of a declarative sentence $\varphi$, resolving the issue means having enough information to establish that $\varphi$. The issue expressed by a question can be resolved by a state containing enough information to establish one of the alternative ways of answering this question.

The information states of our semantics represent three aspects of an agent’s epistemic state. The first component encodes the agent’s knowledge as a set of worlds: the agent knows what is true in all those worlds. Besides her knowledge, a state also represents the agent’s live necessities, those propositions of which the agent thinks that they should, or must be the case, as well as her live possibilities.

Formally, an information state $s$ is a triple $(i, n, l)$ with $i, n \in \mathcal{P}(W)$, where $W$ is the set of possible worlds, and $l \in Up(\mathcal{P}(W))$ such that (1) $n \subseteq i$, (2) $n \in l$, and (3) if $n \neq \emptyset$ then $\forall j \in l : j \cap n \neq \emptyset$.

The first condition expresses our assumption that an agent is convinced of what she knows. The second condition forces that whatever she is convinced of is also relevant. The third condition expresses that whatever is not compatible with all live necessities is irrelevant for the agent.

The condition that $l$ is upwards closed expresses that an agent who takes the possibility that the actual world is contained in $j$ seriously also takes the possibility that it is contained in $k$ seriously for any $k \supseteq j$.
An extension $s$ of a state $t$ is a state with stronger or equal information, at least as many propositions as live necessities, and at least as many live possibilities.

The language consists of formulas built up from the atomic sentences using the binary connectives $\land$ and $\lor$ as well as the unary connectives $\neg$, $\Diamond$, $\Box$, and $!$. Further we use the abbreviation $?\varphi$ for $\varphi \lor \neg \varphi$.

The binary connective $\lor$ represents the inquisitive disjunction. This disjunction is interpreted similarly to an intuitionistic disjunction because supporting this disjunction requires to support one of its disjuncts. The formulas $\Diamond \varphi$ and $\Box \varphi$ represent the sentences ‘It might be that $\varphi$’ and ‘It must be the case that $\varphi$’ respectively. The $!$-operator removes the inquisitiveness of its argument, while $?\varphi$ expresses the question, whether $\varphi$.

Formulas express issues that are sets of states incomparable with respect to strength. These states represent the weakest ways to resolve the issue. An issue is inquisitive if it contains more than one information state.

Using these notions, we present a semantics with a natural notion of entailment, namely $\varphi \vDash \psi$ iff every state that resolves the issue expressed by $\varphi$ also resolves the issue expressed by $\psi$. In the case of finitely many atomic sentences we also have expressive completeness in the sense that any set of incomparable states is an issue expressed by a formula.

Predictions

**Prediction 1 (Negated Modalities)** The semantics captures the intuition that negation of modals works in the following way: ‘It is not the case that it must be $\varphi$’ is equivalent to ‘It might be the case that $\neg \varphi$’ and ‘It is not the case that it might be $\varphi$’ is equivalent to ‘It must be the case that $\neg \varphi$’. Accordingly, the following equivalences hold:

$$\neg \Diamond p \equiv \Box \neg p, \quad \text{and} \quad \neg \Box p \equiv \Diamond \neg p.$$  

**Prediction 2 (Failure of Contraposition)** While natural language suggests that $p$ entails $\Box p$, it is not the case that $\neg \Box p$ entails $\neg p$: If something might not be the case, then this does not imply that it isn’t. This failure of contraposition is predicted by our semantics, as we have that $p \vDash \Box p$ and $\neg \Box p \not\vDash \neg p$. Note that e.g. in [? ], this failure also arises - via a dynamic definition of entailment.

**Prediction 3 (Free Choice)** It has frequently been observed that might sentences allow for free choice (e.g. see [? ]). An example of this effect is that the following three sentences are equivalent.

1. It might rain or snow.
2. It might rain or it might snow.
3. It might rain and it might snow.

The semantics predicts narrow scope and wide scope free choice for might-sentences:

$$\Diamond (p \lor q) \equiv ! (\Diamond p \Diamond q) \equiv \Diamond p \land \Diamond q.$$  

The $!$-operator heading the formula $!(\Diamond p \Diamond q)$ is due to the fact that the formula is meant to be a translation of a declarative sentence with falling intonation. Consult [? ]
for more details. In the other formulas the \(!\)-operator is omitted since they are already non-inquisitive.

**Prediction 4 (Epistemic contradictions)** might-sentences and must-sentences can lead to contradictions. In particular the following four sentences seem contradictory:

1. It is not raining and it might be raining.
2. It is not raining and it must be raining.
3. It must not be raining and it might be raining.
4. It must not be raining and it must be raining.

[? ] argues that a semantic treatment of epistemic contradictions is favourable to a purely pragmatic one based on, among others, embedded appearances of epistemic contradictions. [? ] argues for the same point based on the interactions between free choice effects and epistemic contradictions.

Regarding epistemic contradictions we get the following prediction:

\[
p \land \Box \neg p \models \Box \varphi, \text{ for all formulas } \varphi, \quad \Box \neg p \models \Box \varphi, \text{ for all formulas } \varphi, \\
\models \Box \varphi, \text{ for all formulas } \varphi, \quad \models \Box \varphi, \text{ for all formulas } \varphi, \\
\not\models p, \quad \not\models p, \\
\not\models \neg p, \quad \not\models \neg p, \\
\not\models q, \text{ for any } q \neq p. \quad \not\models q, \text{ for any } q.
\]

Note the difference between flat out contradictions such as \( p \land \neg p \) and epistemic contradictions as above: while everything follows from \( p \land \neg p \), this is not the case with an epistemic contradiction. However, epistemic contradictions entail every modal formula (and its negation), which establishes their contradictory character.

Like [? ] this semantics correctly predicts the interaction between free choice and epistemic contradictions. The formula \( \neg p \land \Box (p \lor q) \), for example, turns out to be an epistemic contradiction.

**Prediction 5 (must and might in Questions)** The use of must and might in questions seems to allow answers along the following lines.

**Might Dr. Jekyll and Mr. Hyde be the same person?**

1. Yes, they might be the same person.
2. No, they must be different people.

**Must Dr. Jekyll and Mr. Hyde be different people?**

1. Yes, they must be different people.
2. No, they might be the same person.

These answers mirror the intuitions about negation of modals mentioned above. Accordingly, we get the following reasonable representation:

\[
\Box p \equiv \Box p \lor \Box \neg p \quad \text{and} \quad \Box p \equiv \Box p \lor \Box \neg p.
\]
In this contribution, we shall present an overview of some recent results about extensions of the four-valued Belnap–Dunn logic.

The four-valued Belnap–Dunn logic (denoted $B$ here and often called FDE for first-degree entailment) was introduced by Belnap [?] and Dunn [?] as a logic which a computer might use to deal with potentially inconsistent and incomplete information. Its semantics is very much like the semantics of classical logic $\mathcal{C}L$, except we allow for the possibility that a proposition may be both true and false or neither true nor false. Whether to interpret these truth values epistemically or ontologically is a question which will not concern us here.

Although there is now a sizeable body of research into the logic $B$ itself, until recently little was known about its non-classical extensions. There are two major exceptions: Priest’s Logic of Paradox $\mathcal{LP}$ [?] and Kleene’s strong three-valued logic $\mathcal{K}$. Both have been used by philosophers in gappy and glutty accounts of truth. More recently, Exactly True Logic $\mathcal{ETL}$ was introduced by Pietz & Rivieccio [?] as the logic defined by preserving truth-and-non-falsity.

References


Extensions of the Four-valued Belnap–Dunn logic

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In this contribution, we shall present an overview of some recent results about extensions of the four-valued Belnap–Dunn logic.

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It is Rivieccio who first suggested that extensions of Belnap logics, as he called them, were a subject deserving of systematic study (and proved that there are infinitely many such logics). Here we present some first results about this newly defined family of logics. In particular, the following theorem gives us a global picture of the landscape of super-Belnap logics.

**Theorem 9** Each non-trivial super-Belnap logic lies in one of the disjoint intervals \([B, LP], [ECQ, LP \lor ECQ], [ETL, CL]\), where \(ECQ\) extends \(B\) by the rule \(p, \neg p \vdash q\). Moreover, each of these intervals has the cardinality \(2^{\aleph_0}\).

The above result depends on viewing logics as single-conclusion consequence relations. If instead we adopt a multiple-conclusion perspective, the picture changes substantially, as the following theorem attests.

**Theorem 10** The only non-trivial multiple-conclusion super-Belnap logics are the multiple-conclusion versions of the logics \(B, LP \land K, LP, K,\) and \(CL\).

Even though there are many super-Belnap logics, relatively few of them are well-behaved from the point of view of abstract algebraic logic (AAL). In the statement of the following theorem, we use some of the standard terminology of abstract algebraic logic. (AAL is a field which deals with the systematic study of non-classical logics using the methods of universal algebra. The reader may consult e.g. the recent textbook of Font for an introduction into this field).

**Theorem 11** The only non-trivial protoalgebraic super-Belnap logic is \(CL\). The only non-trivial Fregean super-Belnap logic is \(CL\). The only non-trivial self-extensional super-Belnap logics are \(B, LP \cap K, CL\).

The fact that few super-Belnap logics satisfy these properties makes them interesting from the point of view of the theory of AAL, which is well-developed for protoalgebraic logics but has only recently started to systematically look outside of this family. Indeed, pathological examples of various sorts can be found among super-Belnap logics, for example non-finitary super-Belnap logics.

**Theorem 12** There is a non-finitary super-Belnap logic.

Another well-studied property in non-classical logics is interpolation. We shall say that a logic \(L\) enjoys \((L_1, L_2)\)-interpolation if \(\phi \vdash_L \chi\) implies that there is some formula \(\psi\) (an interpolant of \(\phi\) and \(\chi\)) such that \(\phi \vdash_{L_1} \psi\) and \(\psi \vdash_{L_2} \chi\) and each propositional atom of \(\psi\) occurs in both \(\phi\) and \(\chi\).

**Theorem 13** \(K\) enjoys \((K, B)\)-interpolation. \(LP\) enjoys \((B, LP)\)-interpolation. \(ETL\) enjoys \((ETL, B)\)-interpolation. \(CL\) enjoys \((K, LP)\)-interpolation.

The last claim of the previous theorem is in fact a recent refinement due to Milne of the interpolation theorem for classical logic. We can moreover show that this refinement is optimal, at least among super-Belnap logics.
Theorem 14  If $\mathcal{C}L$ enjoys $(\mathcal{L}_1, \mathcal{L}_2)$-interpolation for some super-Belnap logics $\mathcal{L}_1$ and $\mathcal{L}_2$, then $\mathcal{K} \subseteq \mathcal{L}_1$ and $\mathcal{L} \mathcal{P} \subseteq \mathcal{L}_2$.

Finally, we note that classical logic has a canonical decomposition as a join of two weaker logics in the lattice of super-Belnap logics. Classical logic $\mathcal{C}L$ may be axiomatized relative to $\mathcal{B}$ by the law of the excluded middle $p \lor -p$ and the rule of disjunctive syllogism $p, -p \lor q \vdash q$. These define the logics $\mathcal{L} \mathcal{P}$ and $\mathcal{E} \mathcal{T} \mathcal{L}$, respectively. In the lattice of extensions of $\mathcal{B}$, we therefore have $\mathcal{C}L = \mathcal{L} \mathcal{P} \lor \mathcal{E} \mathcal{T} \mathcal{L}$. It may be of interest even to the classical logician that this decomposition is in fact optimal.

Proposition 1  Suppose that $\mathcal{C}L = \mathcal{L}_1 \lor \mathcal{L}_2$ and $\mathcal{B} \subseteq \mathcal{L}_i \subsetneq \mathcal{C}L$ for $i \in \{1, 2\}$. Then $\mathcal{L} \mathcal{P} \subseteq \mathcal{L}_1$ and $\mathcal{E} \mathcal{T} \mathcal{L} \subseteq \mathcal{L}_2$ or vice versa.

References


Proof Mining in Convex Optimization

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Proof mining is a research program introduced by U. Kohlenbach in the 1990s ([? ] is a comprehensive reference, while [?] is a survey of recent results), which aims to obtain explicit quantitative information (witnesses and bounds) from proofs of an apparently ineffective nature. This offshoot of interpretative proof theory has successfully led so far to obtaining some previously unknown effective bounds, primarily in nonlinear analysis and ergodic theory. A large number of these are guaranteed to exist by a series of logical metatheorems which cover general classes of bounded or unbounded metric structures.

For the first time, this paradigm is applied to the field of convex optimization (for an introduction, see [ ]). We focus our efforts on one of its central results, the proximal point algorithm. This algorithm, or more properly said this class of algorithms, consists, roughly, of an iterative procedure that converges (weakly or strongly) to a fixed point of a mapping, a zero of a maximally monotone operator or a minimizer of a convex function. We select the middle case in order to give an illustration. Consider a Hilbert space $H$ and a multi-valued operator $A : H \rightarrow 2^H$. We call $A$ monotone if for all $x, y, u, v \in H$ such that $u \in A(x)$ and $v \in A(y)$ we have that

$$\langle x - y, u - v \rangle \geq 0.$$ 

In addition, we call it maximally monotone when it is maximal among the set of all monotone operators represented as binary relations $A \subseteq H \times H$ and ordered by set-theoretic inclusion. If we define the resolvent of $A$ as the relation

$$J_A := (id + A)^{-1}$$

where the operations are those involving binary relations, it is known from the general theory of monotone operators that when $A$ is maximally monotone, $J_A$ is a single-valued function with the whole of $H$ as its domain. Also, it is easy to check that the zeros of $A$ coincide with the fixed points of $J_A$. We shall call the resolvent of $A$ of order $\gamma > 0$ simply the resolvent of $\gamma A$. Such an operation preserves both the zeros, on the side of the monotone operator, and the fixed points, on the side of the resolvent. The classical result in this case says that if $A$ is a maximally monotone operator that has at least one zero and $(\gamma_n)_{n \in \mathbb{N}} \subseteq (0, \infty)$ is such that $\sum_{n=0}^{\infty} \gamma_n^2 = \infty$, then for any $x \in H$, setting $x_0 := x$ and for each $n \in \mathbb{N}$, $x_{n+1} := J_{\gamma_n A} x_n$, we have that the sequence $(x_n)_{n \in \mathbb{N}}$ converges weakly to a zero of $A$, i.e. an element $x \in H$ such that $0 \in A(x)$.

Similarly to other cases previously considered in nonlinear analysis, we may obtain rates of metastability or rates of asymptotic regularity. What is interesting here, however, is that for a relevant subclass of inputs to the algorithm – “uniform” ones, like uniformly convex functions or uniformly monotone operators – we may obtain an effective rate of convergence. The notion of convergence, being represented by a $\Pi_3$-sentence, has been usually excluded from the prospect of being quantitatively tractable, unless its proof exhibits a significant isolation of the use of reductio ad absurdum (see [ ? ? ]). Here, however, a peculiarity of the input, namely its uniformity, translates into a logical form that makes possible this sort of extraction.
We proceed to detail this phenomenon on the case outlined above. Let \( \varphi : [0, \infty) \to [0, \infty) \) be an increasing function which vanishes only at 0. We call \( A \) uniformly monotone with modulus \( \varphi \) on a ball \( C \) in \( H \) if for all \( x, y \in C \) and \( u, v \in H \) with \( u \in A(x) \) and \( v \in A(y) \) we have that:
\[
\langle x - y, u - v \rangle \geq \varphi(\|x - y\|).
\]
This class of mappings have the property that they possess a unique zero in \( C \), i.e. the resolvent, denoted generically by \( T \), has a unique fixed point, a property that can be formalized as:
\[
\forall x \forall y (T(x) = x \land T(y) = y \rightarrow x = y).
\]
Kohlenbach’s general logical metatheorem guarantee that this statement can be “uniformized” by a modulus of uniqueness \( \Phi \):
\[
\forall x \forall y \forall k \in \mathbb{N}(\|T(x) - x\| < 2^{-\Phi(k)} \land \|T(y) - y\| < 2^{-\Phi(k)} \rightarrow d(x, y) \leq 2^{-k}).
\]
Using this extracted modulus, one can is then led to obtain a rate of convergence for the iteration above. This technique was pioneered by Briseid [? ].

In our case, i.e. in Hilbert spaces, we use a vestigial form of the modulus, namely the following relation, valid for each \( \gamma > 0 \) and for all \( x \in C \) and any zero \( z \) in \( C \):
\[
\varphi(d(J_\gamma Ax, z)) \leq \frac{d(x, J_\gamma Ax)}{\gamma} \cdot d(J_\gamma Ax, z).
\]
Since, by the properties of the iteration, the fraction in the right hand side can be made arbitrarily small, using the properties of \( \varphi \) we obtain an appropriate bounding on \( d(x_n, z) \) – that is, an actual rate of convergence.

In addition to these quantitative results, we propose a natural definition that encompasses all known procedures which are commonly grouped under the name “proximal point algorithm”. Precursors to this idea may be found in previous studies of firmly non-expansive mappings [? ] and metric space linearization [? ]. We feel that this general notion helps to elucidate the reasons behind the form and the convergence of the various algorithms, and could lead to the development of similar algorithms, by identifying the respective pattern in new contexts.

These results are joint work with Laurențiu Leuștean and Adriana Nicolae.

References


