Constructive theories through a modal lens

MATTEO TESI*, *Scuola Normale Superiore di Pisa, Classe di Lettere e Filosofia, 56126 Pisa, Italy and Technische Universität Wien, Institute of Logic and Computation, A-1040 Vienna, Austria.*

Abstract

We present a uniform proof-theoretic proof of the Gödel–McKinsey–Tarski embedding for a class of first-order intuitionistic theories. This is achieved by adapting to the case of modal logic the methods of proof analysis in order to convert axioms into rules of inference of a suitable sequent calculus. The soundness and the faithfulness of the embedding are proved by induction on the height of the derivations in the augmented calculi. Finally, we define an extension of the modal system for which the result holds with respect to geometric intuitionistic.

Keywords: Modal embedding, proof theory, constructive, cut-elimination.

1 Introduction

In 1932, Gödel published a short note in which he presented an interpretation of intuitionistic logic into the modal propositional system **S4** [6]. He proved the soundness by induction on the height of derivation in the axiomatic calculus for intuitionistic logic and he conjectured that the converse statement held too.

It was only fifteen years later that McKinsey and Tarski proved the faithfulness of the embedding as well [9], by a semantic argument that exploited algebraic methods in combination with topological representation theorems by Stone [15]. The Gödel–McKinsey–Tarski translation and its equivalent formulations build a bridge between intuitionistic and modal logics, by interpreting the modal operator $\Box$ in terms of provability (in an informal sense).

The Gödel–McKinsey–Tarski translation can be used to view intuitionistic logic from a classical perspective (or better yet, as a fragment of an extension of classical logic). Further work on the Gödel–McKinsey–Tarski translation have extended the interpretation in more than one direction.

First, there are several modal logics in which intuitionistic logic can be soundly and faithfully embedded, e.g. **S4**, **Grz**, **GL** and also **S3** [1]. Second, it was shown how to extend the translation to the setting of first-order intuitionistic logic [14] and to various intermediate logics [1, 2].

In this paper, we take a different route and we study the soundness and the faithfulness of the embedding with respect to first-order intuitionistic theories. Previous works in this area focused on specific theories, specifically on the interpretation of Heyting arithmetic in Peano arithmetic extended with modal operators [5, 7, 10]. However, a general and uniform approach to the problem has not been developed yet. We identify a class of theories, determined by the shape of their axioms, for which the soundness and the faithfulness of the translation holds.

Proof analysis of first-order theories has obtained considerable results in the last twenty years. In particular, Negri and von Plato [11] showed how to convert mathematical axioms into sequent rules while preserving cut-elimination. The resulting system does not enjoy a full subformula property, but a weaker version thereof, which often allows a good structural analysis of the theory.

*E-mail: matteo.tesi@sns.it

Vol. 00, No. 0,
https://doi.org/10.1093/jigpal/jzad029

The methods of proof analysis allow us to present a uniform proof of the soundness and the faithfulness of the Gödel–McKinsey–Tarski embedding for first-order Horn theories. Furthermore, the proof that we offer is constructive, in the sense that it avoids appeal to Zorn's lemma or variants thereof, and it is also direct. In fact, the methods that we use are purely proof-theoretic and we explicitly define a proof transformation procedure that enables to obtain a modal proof from an intuitionistic one and vice versa.

This is interesting because it yields a modal interpretation of many constructive mathematical theories in terms of (informal) provability and furthermore it allows to exploit modal systems in order to obtain metalogical results. In particular, the relevance is both conceptual and technical. From a conceptual point of view, it allows to look at mathematical intuitionistic theories as expressing something in terms of provability and it is coherent with epistemic interpretations of intuitionistic logic.

From a technical point of view, we exploit the embedding result to obtain a syntactic proof of the disjunction property and of the witness property for first-order Horn theories, which would be harder to obtain working in a multisuccedent intuitionistic sequent calculus. Another interesting aspect is that this result connects geometric logic, i.e. a fragment of classical logic conservative over intuitionistic logic, with the modal embedding. Therefore, we can consider the results here presented as an attempt to unify various areas of logic.

Section 2 is devoted to the presentation of the sequent calculus for first order **S4** and to the extension of the methods of proof analysis to such system, establishing the usual desired structural properties, especially cut admissibility. The relation with the corresponding axiomatic presentation is investigated in Section 3. Section 4 discusses Horn theories, which are a subclass of universal theories and we describe some mathematical examples of theories that are axiomatized by Horn sentences. In Sections 5 and 6, we present the extension of Gödel–McKinsey–Tarski embedding to Horn theories. Such result is obtained by two separate (nontrivial) lemmas of soundness and faithfulness of the translation. We exploit the translation in order to give an alternative proof of the disjunction property and of the witness property for Horn theories. Finally, Section 7 deals with the extension of the embedding to geometric logic, by introducing an extension of the modal logic **S4**. We conclude the paper by sketching some possible future lines of research.

2 Theories based on S4

The language of first-order modal logic contains:

- Individual variables: $x_0, x_1, x_2, \cdots$
- Function symbols ($n_i \geq 0$): $f_0^{n_0}, f_1^{n_1}, f_2^{n_2}, \cdots$
- Predicate symbols ($n_i \geq 1$): $P_0^{n_0}, P_1^{n_1}, P_2^{n_2}, \cdots$
- The usual connectives $\wedge$, $\vee$ and $\rightarrow$, the universal and the existential quantifiers $\forall$ and $\exists$ and the unary modal operator $\Box$.

Formulas and terms are inductively defined as usual, we use P, Q, R to denote atomic formulas.

Sequents are syntactic objects of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas. $\Box\Gamma$ is the multiset that contains the formulas $\Box A$ for every A in Γ.

The degree of a formula is defined as the number of logical symbols occurring in it. The notations $\neg A$, $A \leftrightarrow B$ and $\Diamond A$ abridge $A \rightarrow \bot$, $(A \rightarrow B) \wedge (B \rightarrow A)$ and $\neg\Box\neg A$, respectively. We consider the sequent calculus **G3s4** for the quantified modal logic **S4** in Figure 1.

We show that the calculus **G3s4** can be extended with rules corresponding to certain axioms while preserving the structural properties of the original system.

Initial Sequents

$$\frac{}{\Gamma, P \Rightarrow P, \Delta}\ Ax \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ L\bot$$

Logical Rules

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A, B}\ R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ L\to \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ R\to$$

$$\frac{\Box A, A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\ L\Box \qquad \frac{\Box\Gamma, \Gamma \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Delta, \Box A}\ R\Box$$

$$\frac{\forall x A, A[x/t], \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ L\forall \qquad \frac{\Gamma \Rightarrow \Delta, A[x/y]}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall, y \text{ fresh}$$

$$\frac{A[x/y], \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ L\exists, y \text{ fresh} \qquad \frac{\Gamma \Rightarrow \Delta, \exists A, A[x/t]}{\Gamma \Rightarrow \Delta, \exists x A}\ R\exists$$

FIGURE 1 The **G3s4** sequent calculus.

DEFINITION 2.1
A *geometric* formula is a sentence of the form: $\forall\overline{x}(A \to B)$, where A and B do not contain $\to$, $\forall$ and $\Box$.

Any geometric formula can be equivalently reformulated as a sentence of the shape:

$$\forall\overline{x}(P_1 \wedge ... \wedge P_m \to \exists\overline{y}_1\mathbf{M}_1 \vee ... \vee \exists\overline{y}_n\mathbf{M}_n)$$

with $m, n \geq 0$ (when $m = 0$, then $P_1 \wedge ... \wedge P_m$ is $\top$, when $n = 0$, then $\exists\overline{y}_1\mathbf{M}_1 \vee ... \vee \exists\overline{y}_n\mathbf{M}_n$ is $\bot$) and where $\mathbf{M}_j$ is a finite conjunction of atomic formulas $Q_{j1} \wedge \ldots \wedge Q_{jk_j}$ and y_j are not free in P_i for every $i \in \{1, ..., m\}$ [12]. The expression $Q\overline{x}$, where Q is a quantifier, denotes the string of quantifiers $Q\overline{x}_1 \ldots Q\overline{x}_n$. A geometric theory is a theory whose axioms are all geometric sentences.

DEFINITION 2.2
For every geometric axiom:

$$\forall\overline{x}(P_1 \wedge ... \wedge P_m \to \exists\overline{y}_1\mathbf{M}_1 \vee ... \vee \exists\overline{y}_n\mathbf{M}_n)$$

its corresponding geometric rule scheme is:

$$\frac{\overline{Q}_1[\overline{z}_1/\overline{y}_1], \overline{P}, \Gamma \Rightarrow \Delta \quad \ldots \quad \overline{Q}_n[\overline{z}_n/\overline{y}_n], \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta}\,\text{Geom}$$

where $\overline{P} = P_1, ..., P_n$ and, for every j, $\overline{Q}_j = Q_{j1}, ..., Q_{jn_j}$, with $\mathbf{M}_j = Q_{j1} \wedge ... \wedge Q_{jn_j}$. $\overline{Q}_j[\overline{z}_j/\overline{y}_j]$ denotes the substitution of $\overline{z}_j$ with $\overline{y}_j$ in each Q_{jl_j} and $\overline{y}_k$ do not occur in the conclusion.

In order to ensure the admissibility of contraction, it may be necessary to add to the system the following closure condition.
Closure condition. Given a system of geometric rules, for every instance of the form:

$$\frac{\overline{Q}_1[\overline{z}_1/\overline{y}_1], P_1, ..., P_{m-2}, P, P, \Gamma \Rightarrow \Delta \quad \ldots \quad \overline{Q}_n[\overline{z}_n/\overline{y}_n], P_1, ..., P_{m-2}, P, P, \Gamma \Rightarrow \Delta}{P_1, ..., P_{m-2}, P, P, \Gamma \Rightarrow \Delta}\,\text{Geom}$$

We need to add its closure under contraction:

$$\frac{\overline{Q}_1[\overline{z}_1/\overline{y}_1], P_1, ..., P_{m-2}, P, \Gamma \Rightarrow \Delta \quad \ldots \quad \overline{Q}_n[\overline{z}_n/\overline{y}_n], P_1, ..., P_{m-2}, P, \Gamma \Rightarrow \Delta}{P_1, ..., P_{m-2}, P, \Gamma \Rightarrow \Delta}$$

To give a concrete example, consider the case of a theory $\mathcal{L} = \{R\}$, where R is euclidean, i.e. $\forall x \forall y \forall z(xRy \wedge xRz \rightarrow yRz)$, and consider the following instance:

$$\frac{xRy, xRy, yRy, \Gamma \Rightarrow \Delta}{xRy, xRy, \Gamma \Rightarrow \Delta}\,\text{Euc}$$

In this case the closure condition is:

$$\frac{xRy, yRy, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}\,\text{Euc c.c.}$$

For further discussion on geometric theories and examples thereof, the interested reader is referred to [12].

Given a set of geometric axioms **T**, we denote with **G3s4T** the sequent calculus obtained by adding to **G3s4** the corresponding geometric rules together with the rules obtained by the closure condition.

DEFINITION 2.3
The height of derivation is defined as usual as the number of sequents occurring in one of the maximally long branches.

REMARK
Before proceeding, we would like to spend a few words on the formulation of the rule R□. As it will be clear from the discussion in Section 3, the rule corresponds in a sense to the modal axiom $\Box A \rightarrow \Box\Box A$. The additional contexts Π and Δ in the conclusion are used to obtain admissibility of weakening with preservation of height.

We proceed with the structural analysis of the calculus.

LEMMA 2.4
For every variable x and every term t, the rule

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma[x/t] \Rightarrow \Delta[x/t]}\,\mathit{Sub}[x/t]$$

is height-preserving admissible in **G3s4T**.

PROOF. The proof follows the pattern of the corresponding proof in [12]. □

LEMMA 2.5
The rules

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ RW$$

are height-preserving admissible in **G3s4T**.

PROOF. By induction on the height of the derivations, exploiting Lemma 2.4 in order to avoid possible clashes of variables with respect to the rules $L\exists$, $R\forall$ and *Geom*. □

A rule is *invertible* if, whenever the conclusion is derivable so is (are) the premise(s).

LEMMA 2.6
Every rule except for $R\Box$ is height-preserving invertible in **G3s4T**.

PROOF. The rules $L\Box$ and *Geom* are invertible by Lemma 2.5. We limit ourselves to discuss the case of $R\forall$ as an example. If $n = 0$, then $\Gamma \Rightarrow \Delta, \forall x A$ is an initial sequent and so is $\Gamma \Rightarrow \Delta, A[x/t]$. If $n > 0$, we distinguish cases according to the last rule applied. If the last rule is any rule different from $R\Box$, apply the induction hypothesis to the premise(s) (together with height-preserving substitution to avoid clashes of variables) and then apply the rule again. If the last rule is $R\Box$, we have:

$$\frac{\Box\Gamma, \Gamma \Rightarrow B}{\Box\Gamma, \Gamma' \Rightarrow \Box B, \Delta, \forall x A}\ R\Box$$

In this case we apply again the rule $R\Box$ to obtain $\Box\Gamma, \Gamma' \Rightarrow \Box B, \Delta, A[x/t]$. □

LEMMA 2.7
The rules

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ LC \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ RC$$

are height-preserving admissible in **G3s4T**.

PROOF. By simultaneous induction on the height of the derivations.

We discuss the left rule of contraction. If $n = 0$, then $A, A, \Gamma \Rightarrow \Delta$ is an initial sequent and so is $A, \Gamma \Rightarrow \Delta$. If $n > 0$, then we distinguish cases according to the last rule applied. If A is not principal, or if it is principal in $L\forall$ or $L\Box$ or is an active formula in the antecedent of rule $R\Box$, apply the induction hypothesis to the premise(s) and then apply the rule again. If it is principal in a propositional rule or in $L\exists$ we apply invertibility of the corresponding rule by Lemma 2.6 and then we apply the induction hypothesis. If A is principal in a geometric rule we distinguish two subcases. If only one A is principal, we apply the induction hypothesis to the premise and then we apply the rule again. If both A's are principal, we exploit the closure condition.

The case of the right rule of contraction is similar, the most significant case to discuss is that in which the last rule applied is $R\Box$:

$$\frac{\Box\Gamma, \Gamma \Rightarrow A}{\Box\Gamma, \Gamma' \Rightarrow \Box A, \Box A, \Delta}\ R\Box$$

In this case the conclusion follows by applying again the rule to the premise. □

THEOREM 2.8 (Cut admissibility).
The rule

$$\dfrac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, \textit{Cut}$$

is admissible in **G3s4T**.

PROOF. The proof runs by double induction, with main induction hypothesis on the degree of the cut formula and secondary induction hypothesis on the sum of the height of the derivations. We distinguish cases.

1. **The left premise is the conclusion of an application of a geometric rule**. If it is a zeroary geometric rule, then the conclusion of the cut is an instance of the rule again. If it is the conclusion of an n-ary geometric rule, we have:

$$\dfrac{\dfrac{\Gamma, \overline{P}, \overline{Q}_1[z_1/y_1] \Rightarrow \Delta, A \qquad \ldots \qquad \Gamma, \overline{P}, \overline{Q}_n[z_n/y_n] \Rightarrow \Delta, A}{\Gamma, \overline{P} \Rightarrow \Delta, A}\,\text{Geom} \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma', \overline{P} \Rightarrow \Delta, \Delta'}\,\text{Cut}$$

In this case, the cuts are replaced by n cuts of lesser height and the conclusion is obtained by applying the rule again:

$$\dfrac{\dfrac{\Gamma, \overline{P}, \overline{Q}_1[z_1/y_1] \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma', \overline{P}, \overline{Q}_1[z_1/y_1] \Rightarrow \Delta, \Delta'}\,\text{Cut} \qquad \ldots \qquad \dfrac{\Gamma, \overline{P}, \overline{Q}_n[z_n/y_n] \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma', \overline{P}, \overline{Q}_n[z_n/y_n] \Rightarrow \Delta, \Delta'}\,\text{Cut}}{\Gamma, \Gamma', \overline{P} \Rightarrow \Delta, \Delta'}\,\text{Geom}$$

We can assume that no clashes of variables occur by height-preserving substitution. The n cuts are removed by the secondary induction hypothesis.

2. **The right premise of the cut is the conclusion of a geometric rule**. We distinguish two subcases:

2.1. If the cut formula is not principal, we consider two further subsubcases. If the geometric rule is a zeroary rule, then the conclusion is also an instance of it. If it is an n-ary rule, we have:

$$\dfrac{\Gamma \Rightarrow \Delta, A \qquad \dfrac{A, \overline{P}, \overline{Q}_1[\overline{z}_1/\overline{y}_1], \Gamma' \Rightarrow \Delta' \qquad \ldots \qquad A, \overline{P}, \overline{Q}_n[\overline{z}_n/\overline{y}_n], \Gamma' \Rightarrow \Delta'}{A, \overline{P}, \Gamma' \Rightarrow \Delta'}\,\text{Geom}}{\Gamma, \overline{P}, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Cut}$$

We construct the following derivation:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, A \qquad A, \overline{P}, \overline{Q}_1[\overline{z}_1/\overline{y}_1], \Gamma' \Rightarrow \Delta'}{\overline{P}, \overline{Q}_1[\overline{z}_1/\overline{y}_1], \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Cut} \qquad \ldots \qquad \dfrac{\Gamma \Rightarrow \Delta, A \qquad A, \overline{P}, \overline{Q}_n[\overline{z}_n/\overline{y}_n], \Gamma' \Rightarrow \Delta'}{\overline{P}, \overline{Q}_n[\overline{z}_n/\overline{y}_n], \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Cut}}{\overline{P}, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Geom}$$

we assume that no clashes of variables occur due to the admissibility of substitution with preservation of height.

2.2. If the cut formula is principal, we have:

$$\dfrac{\Gamma \Rightarrow \Delta, P_1 \qquad \dfrac{\Gamma', P_1, P_2, .., P_m, \overline{Q}_1[z_1/y_1] \Rightarrow \Delta' \qquad \ldots \qquad \Gamma', P_1, P_2, .., P_m, \overline{Q}_n[z_n/y_n] \Rightarrow \Delta'}{\Gamma', P_1, P_2, .., P_m \Rightarrow \Delta'}\,\text{Geom}}{\Gamma, \Gamma', P_2, .., P_m \Rightarrow \Delta, \Delta'}\,\text{Cut}$$

In this case, we reason by induction on the height of the left premise of the cut. If $\Gamma \Rightarrow \Delta, P_1$ is an initial sequent, then P_1 occurs in Γ. In this case, the proof is as follows:

$$\dfrac{\Gamma', P_1, P_2, .., P_n \Rightarrow \Delta'}{\Gamma, \Gamma', P_2, .., P_m \Rightarrow \Delta, \Delta'}\,\text{Weak}$$

If $\Gamma \Rightarrow \Delta, P_1$ is the conclusion of a rule, then P_1 is not principal. We distinguish cases according to the last rule applied.

2.2.1. The cases in which the last rule is a geometric one have been dealt with in 1.

2.2.2. The last rule applied is different from $R\Box$. We permute the cut upwards and we eliminate it by secondary induction hypothesis, applying height-preserving admissibility of substitution in order to avoid clashes of variables.

2.2.3. The last rule applied is $R\Box$ and we have:

$$\dfrac{\dfrac{\Box\Gamma'', \Gamma'' \Rightarrow B}{\Box\Gamma'', \Gamma''' \Rightarrow \Delta'', \Box B, P_1}\,R\Box \qquad \Gamma', P_1, P_2, .., P_n \Rightarrow \Delta'}{\Box\Gamma'', \Gamma''', \Gamma', P_2, .., P_n \Rightarrow \Delta'', \Box B, \Delta'}\,\text{Cut}$$

In this case, the desired result is obtained by applying rule $R\Box$ to $\Box\Gamma'', \Gamma'' \Rightarrow B$.

3. **The last rule applied is not a geometric rule in both premises**.

3.1. If the cut formula is not principal in the left premise of the cut in a rule different from $R\Box$, we permute the cut upwards, we eliminate it by secondary induction hypothesis and then we apply the rule again. If it is not principal in $R\Box$, the conclusion follows by applying again the rule $R\Box$ with weakening to the premise. The case in which the cut formula is not principal in the right premise of the cut is analogous.

3.2. If the cut formula is principal in both premises, we discuss only the modal cases (for the other cases the reader is referred to [19]). The possible combinations are $\langle R\Box, R\Box\rangle$ and $\langle R\Box, L\Box\rangle$. In the first case, we have:

$$\dfrac{\dfrac{\Box\Gamma, \Gamma \Rightarrow A}{\Box\Gamma, \Gamma'' \Rightarrow \Delta, \Box A}\,\text{R}\Box \qquad \dfrac{\Box A, A, \Box\Gamma', \Gamma' \Rightarrow B}{\Box A, \Box\Gamma', \Gamma''' \Rightarrow \Delta', \Box B}\,\text{R}\Box}{\Box\Gamma, \Box\Gamma', \Gamma'', \Gamma''' \Rightarrow \Delta, \Delta', \Box B}\,\text{Cut}$$

We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\Box\Gamma, \Gamma \Rightarrow A \qquad \dfrac{\dfrac{\Box\Gamma, \Gamma \Rightarrow A}{\Box\Gamma, \Gamma \Rightarrow \Box A}\,\text{R}\Box \qquad \Box A, A, \Box\Gamma', \Gamma' \Rightarrow B}{A, \Box\Gamma, \Box\Gamma', \Gamma, \Gamma' \Rightarrow B}\,\text{Cut}}{\Box\Gamma, \Box\Gamma, \Box\Gamma', \Gamma, \Gamma, \Gamma' \Rightarrow B}\,\text{Cut}}{\Box\Gamma, \Box\Gamma', \Gamma, \Gamma' \Rightarrow B}\,\text{Ctr}}{\Box\Gamma, \Box\Gamma', \Gamma'', \Gamma''' \Rightarrow \Delta, \Delta', \Box B}\,\text{R}\Box$$

The topmost cut is removed by secondary induction hypothesis on the sum of the height of the derivations, whereas the lowermost by main induction hypothesis on the complexity of the cut formula.

Finally, in the second case, we have:

$$\dfrac{\dfrac{\Box\Gamma,\Gamma \Rightarrow A}{\Box\Gamma,\Gamma'' \Rightarrow \Delta,\Box A}\,\text{R}\Box \qquad \dfrac{A,\Box A,\Gamma' \Rightarrow \Delta'}{\Box A,\Gamma' \Rightarrow \Delta'}\,\text{L}\Box}{\Box\Gamma,\Gamma',\Gamma'' \Rightarrow \Delta,\Delta'}\,\text{Cut}$$

The proof is transformed as follows:

$$\dfrac{\dfrac{\dfrac{\Box\Gamma,\Gamma \Rightarrow A \qquad \dfrac{\Box\Gamma,\Gamma'' \Rightarrow \Delta,\Box A \qquad A,\Box A,\Gamma' \Rightarrow \Delta'}{A,\Box\Gamma,\Gamma',\Gamma'' \Rightarrow \Delta,\Delta'}\,\text{Cut}}{\Box\Gamma,\Box\Gamma,\Gamma,\Gamma',\Gamma'' \Rightarrow \Delta,\Delta'}\,\text{Cut}}{\Box\Gamma,\Box\Gamma,\Box\Gamma,\Gamma',\Gamma'' \Rightarrow \Delta,\Delta'}\,\text{several L}\Box}{\Box\Gamma,\Gamma',\Gamma'' \Rightarrow \Delta,\Delta'}\,\text{Ctr}$$

The topmost cut is removed by secondary induction hypothesis on the sum of the height of the derivations and the lower cut is removed by main induction hypothesis on the degree of the cut formula. □

Let **G3s4** ⊕ **T** denote the sequent calculus obtained by adding every axiom of the theory **T** as an initial sequent.

COROLLARY 2.9
For every geometric theory **T**:

$$\mathbf{G3s4} \oplus \mathbf{T} \vdash \Rightarrow A \text{ if and only if } \mathbf{G3s4T} \vdash \Rightarrow A$$

PROOF. The direction from left to right easily follows by showing that every axiom of **T** is derivable in **G3s4T**. For the direction from right to left, we exploit the admissibility of cut and contraction. □

3 Equivalence with axiomatic system

In this section, we show that the analytic system here presented is equivalent to an axiomatic calculus. The calculus **QS4** is obtained by adding to an axiomatization of the modal logic **S4** the axiom $\forall x A \to A[t/x]$ and the rule scheme: $\vdash A \to B$ _ Gen $\vdash A \to \forall x B$, where x does not occur free in A. The notions of proof and derivation in **QS4** are defined as usual in the axiomatic calculi. The system **QS4** is sound and complete with respect to the class of first-order reflexive and transitive Kripke frames with increasing domains [8]. Indeed, the reader can easily observe that the converse Barcan formula $\Box\forall x A \to \forall x \Box A$ is provable both in the axiomatic system and in the sequent calculus. Within the sequent calculus, it is also easy to check that the Barcan formula is not derivable (it is sufficient to consider $\forall x \Box P \to \Box\forall x P$).

QS4

Axioms

1.1 $\vdash A \to (B \to A)$

1.2 $\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$

2.1 $\vdash A \wedge B \to A$

2.2 $\vdash A \wedge B \to B$

2.3 $\vdash (A \to B) \to ((A \to C) \to (A \to B \wedge C))$

3.1 $\vdash A \to A \vee B$

3.2 $\vdash B \to A \vee B$

3.3 $\vdash (A \to C) \to ((B \to C) \to (A \vee B \to C))$

4.1 $\vdash (A \to B) \to ((A \to \neg B) \to \neg A)$

4.2 $\vdash A \to (\neg A \to B)$

4.3 $\vdash A \vee \neg A$

5.1 $\forall x A \to A[t/x]$

5.2 $\exists x A \leftrightarrow \neg \forall x \neg A$

K $\vdash \Box(A \to B) \to (\Box A \to \Box B)$

T $\vdash \Box A \to A$

4 $\vdash \Box A \to \Box\Box A$

Inference Rules

$$\frac{\vdash A \qquad \vdash A \to B}{\vdash B}\ \text{MP} \qquad\qquad \frac{\vdash A}{\vdash \Box A}\ \text{RN}$$

$$\frac{\vdash A \to B}{\vdash A \to \forall x B}\ \text{Gen}$$

x not free in A

Given a finite set of geometric axioms **T**, **QS4T** is the system obtained by adding as axioms the formulas in **T**.

THEOREM 3.1
If **QS4T** $\vdash A$, then **G3s4T** $\vdash \Rightarrow A$.

PROOF. The proof is by induction on the height of the derivation. The modal axioms are easily seen to be provable and so are the classical tautologies. Modus ponens can be simulated via cut. We discuss the case of the rule *Gen*, to give an example:

$$\cfrac{\cfrac{\cfrac{\Rightarrow A \to B}{A \Rightarrow B}\ \text{Lemma 2.6}}{A \Rightarrow \forall x B}\ \text{R}\forall}{\Rightarrow A \to \forall x B}\ \text{R}\to$$

The axioms are seen to be provable using the geometric rules of **G3s4T**. □

The other direction is slightly more delicate and it requires to use the axiomatic calculus.

THEOREM 3.2
If **G3s4T** $\vdash \Gamma \Rightarrow \Delta$, then **QS4** $\vdash \bigwedge \Gamma \to \bigvee \Delta$, where $\bigwedge \Gamma$ ($\bigvee \Delta$) is the conjunction (disjunction) of the formulas in Γ (Δ).

PROOF. The proof is by induction on the height of the derivation in the system **G3s4T**. The cases of initial sequents and logical rules are routine. We limit ourselves to discussing the cases of the rule R□ and of the geometric rules.

(R$\Box$) We streamline the proof, leaving some details for the reader to fill out. We first construct the derivation $\mathcal{D}$:

$$\dfrac{\dfrac{\vdash \Box\bigwedge\Gamma \to \Box\Box\bigwedge\Gamma \qquad \vdash \Box\bigwedge\Gamma \to \Box\bigwedge\Gamma}{\vdash \Box\bigwedge\Gamma \to (\Box\Box\bigwedge\Gamma \wedge \Box\bigwedge\Gamma)}\,\text{adm} \qquad \vdash (\Box\bigwedge\Gamma \to \Box\Box\bigwedge\Gamma \wedge \Box\bigwedge\Gamma) \to (\Box\bigwedge\Gamma \to \Box(\Box\bigwedge\Gamma \wedge \bigwedge\Gamma))}{\vdash \Box\bigwedge\Gamma \to \Box(\Box\bigwedge\Gamma \wedge \bigwedge\Gamma)}\,\text{MP}$$

The leftmost formula is an instance of the axiom 4; $\vdash \Box\bigwedge\Gamma \to \Box\bigwedge\Gamma$ and $\vdash (\Box\bigwedge\Gamma \to \Box\Box\bigwedge\Gamma \wedge \Box\bigwedge\Gamma) \to (\Box\bigwedge\Gamma \to \Box(\Box\bigwedge\Gamma \wedge \bigwedge\Gamma)$ are easily seen to be derivable. The expression *adm* denotes an admissible rule; Cut is also a step that can easily be shown to be admissible. We then conclude the proof as follows:

$$\dfrac{\dfrac{\vdash \Box\bigwedge\Gamma \to \Box\Box\bigwedge\Gamma \qquad \dfrac{\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}}{\vdash \Box\bigwedge\Gamma \to \Box(\Box\bigwedge\Gamma \wedge \bigwedge\Gamma)} \qquad \vdash \Box(\Box\bigwedge\Gamma \wedge \bigwedge\Gamma) \to \Box\bigwedge\Gamma \wedge \bigwedge\Gamma}{\vdash \Box\bigwedge\Gamma \to \Box\bigwedge\Gamma \wedge \bigwedge\Gamma}\,\text{Cut} \qquad \overset{\vdots\,\text{IH}}{\vdash \Box\bigwedge\Gamma \wedge \bigwedge\Gamma \to A}}{\vdash \Box\bigwedge\Gamma \to A}\,\text{Cut}}{\vdash \Box\Box\bigwedge\Gamma \to \Box A}\,\text{adm}}{\vdash \Box\bigwedge\Gamma \to \Box A}\,\text{Cut}}{\vdash \bigwedge\Pi \wedge \Box\bigwedge\Gamma \to \Box A \vee \bigvee\Delta}\,\text{adm}$$

where the topmost formulas are instances of modal axioms or obtained through the induction hypothesis.

(Geom) The cases of geometric axioms are as follows:

$$\dfrac{\overline{Q}_1[\overline{z}_1/\overline{y}_1], \overline{P}, \Gamma \Rightarrow \Delta \qquad \ldots \qquad \overline{Q}_n[\overline{z}_n/\overline{y}_n], \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta}\,\text{Geom}$$

The induction hypothesis yields derivations of:

$$\vdash \bigwedge Q_i[\overline{z}_i/\overline{y}_i] \wedge \bigwedge \overline{P} \wedge \bigwedge \Gamma \to \bigvee \Delta$$

for $i \in \{1, \ldots, n\}$. We proceed as follows:

$$\dfrac{\dfrac{\vdash \bigwedge Q_i[\overline{z}_i/\overline{y}_i] \wedge \bigwedge \overline{P} \wedge \bigwedge \Gamma \to \bigvee \Delta}{\vdash \exists \overline{z}_i(\bigwedge Q_i \wedge \bigwedge \overline{P} \wedge \bigwedge \Gamma) \to \bigvee \Delta}\,\text{adm}}{\vdash \exists \overline{z}_i \bigwedge Q_i \wedge \bigwedge \overline{P} \wedge \bigwedge \Gamma \to \bigvee \Delta}\,\text{adm}$$

Hence, we can get a proof of:

$$\vdash (\bigvee_{1 \leq i \leq n} \exists \overline{z}_i \bigwedge Q_i) \wedge \bigwedge \overline{P} \wedge \bigwedge \Gamma \to \bigvee \Delta$$

with some propositional passages. Finally, we get the conclusion as follows:

$$\dfrac{\vdash \bigwedge \overline{P} \wedge \bigwedge \Gamma \to \bigvee \Delta \vee (\bigvee_{1 \leq i \leq n} \exists \overline{z}_i \bigwedge Q_i) \qquad \vdash (\bigvee_{1 \leq i \leq n} \exists \overline{z}_i \bigwedge Q_i) \wedge \bigwedge \overline{P} \wedge \bigwedge \Gamma \to \bigvee \Delta}{\vdash \bigwedge \overline{P} \wedge \bigwedge \Gamma \to \bigvee \Delta}\,\text{adm}$$

$\Box$

4 Horn theories and rules

In the second section, we have shown how to add rules corresponding to geometric axioms while preserving the structural properties of the underlying modal calculus. However, the class of geometric axioms is too large to establish the soundness of the Gödel–McKinsey–Tarski translation. Therefore, we focus our attention on a proper subclass of geometric theories.

DEFINITION 4.1
A Horn theory is a theory whose axioms are of the form

$$\forall \overline{x}(P_1 \wedge ... \wedge P_n \to Q)$$

where P_i are atomic for every i and Q is either an atomic formula or $\bot$.

Roughly speaking, Horn axioms are are universal closure of implications in which the succedent is an atomic formula and the antecedent is a conjunction of atomic formulas. There are numerous examples of mathematical Horn theories.

- **Groups** Consider the language $\mathcal{L} = \{\cdot, 1, ^{-1}, =\}$. The axioms are:
  1. $\forall xyz(x \cdot (y \cdot z) = (x \cdot y) \cdot z)$ associativity
  2. $\forall x(x \cdot 1 = x)$ right unit
  3. $\forall x(1 \cdot x = x)$ left unit
  4. $\forall x(x \cdot x^{-1} = 1)$ right inverse
  5. $\forall x(x^{-1} \cdot x = 1)$ left inverse

  In order to avoid the presence of existential quantifiers, we have considered an equivalent formulation of the theory obtained by expanding the language, adding the inverse and the unit as a unary and a zeroary operation symbol, respectively [18]. To obtain commutative groups, we add the axiom

$$\forall x \forall y(x = y \to y = x)$$

- **Rings** Consider the language $\mathcal{L} = \{\cdot, +, -, \cdot, =, 0, 1\}$. The axioms are:
  Addition
  1. $\forall xyz(x + (y + z) = (x + y) + z)$ associativity
  2. $\forall xy(x + y = y + x)$ commutativity
  3. $\forall x(x + 0 = x)$ unit
  4. $\forall x(x + (-x) = 0)$ inverse

  Multiplication
  1. $\forall xyz(x \cdot (y \cdot z) = (x \cdot y) \cdot z)$ associativity
  2. $\forall x(x \cdot 1 = x)$ right unit
  3. $\forall x(1 \cdot x = x)$ left unit

  Distributivity
  1. $\forall xyz(x \cdot (y + z) = (x \cdot y) + (x \cdot z))$ left distributivity
  2. $\forall xyz((y + z) \cdot x = (y \cdot x) + (y \cdot z))$ right distributivity

  A commutative ring is obtained by adding the axiom $\forall xy(x \cdot y = y \cdot x)$. Once again, we considered a suitable formulation of ring theory, by adding a specific function symbol for the inverse of the sum $+$.
- **Irreflexive graphs**. Consider the language $\{R\}$, where R is a binary relation symbol. The axioms are:
  1. $\forall x \neg R(x, x)$ irreflexivity
  2. $\forall x \forall y(R(x, y) \to R(y, x))$ symmetry

- **Partial orders**. Consider the language $\{\leqslant, =\}$. The axioms are:
  1. $\forall x(x \leqslant x)$ Reflexivity
  2. $\forall xyz(x \leqslant y \wedge y \leqslant z \to x \leqslant z)$ Transitivity
  3. $\forall xy(x \leqslant y \wedge y \leqslant x \to x = y)$ Antisymmetry

  Clearly, also strict orders, i.e. irreflexive and transitive orders can be treated.
- **Equivalence relations**. Consider the language $\{\sim\}$. The axioms are:
  1. $\forall x(x \sim x)$ reflexivity
  2. $\forall xyz(x \sim y \wedge y \sim z \to x \sim z)$ transitivity
  3. $\forall xy(x \sim y \to y \sim x)$ symmetry
- **Lattices**. Consider the language $\{\sqcap, \sqcup, =\}$. The axioms are dual for $\sqcap$ and $\sqcup$:
  1. $\forall xyz(x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z)$ associativity
  2. $\forall xy(x \sqcap y = y \sqcap x)$ commutativity
  3. $\forall xy(x \sqcap (x \sqcup y) = x)$ absorption

Since Horn axioms are a subclass of geometric ones, the rules obtained from Horn axioms are a particular case of geometric rules: in particular, they have a single premise (or no premises) and they do not contain variable restrictions.

DEFINITION 4.2
For every Horn axiom $\forall \overline{x}(P_1 \wedge ... \wedge P_n \to Q)$, the Horn rule scheme is as follows:

$$\frac{P_1, ..., P_n, Q, \Gamma \Rightarrow \Delta}{P_1, ..., P_n, \Gamma \Rightarrow \Delta}\text{ Horn}$$

if $Q = \bot$, then the rule is zeroary.

Since Horn rules are a subclass of geometric rules, the results of the previous section hold with respect to these rules as well.

COROLLARY 4.3
For every Horn theory **T**, the calculus **G3s4T** enjoys admissibility of weakening, contraction and cut. □

PROOF. Immediate from Theorem 2.8.

REMARK
Notice that intuitionistic Horn theories **T** with the equality schema, i.e. $\forall x \forall y(x = y \wedge A(x) \to A(y))$ for every formula A, correspond to sequent calculi extended with the rules corresponding to the axioms in **T** and with the Horn rules:

$$\frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\mathit{Ref} \qquad \frac{P(s), P(t), t = s, \Gamma \Rightarrow \Delta}{P(t), t = s, \Gamma \Rightarrow \Delta}\,\mathit{Repl}$$

see [11] for a proof of the result. This will be contrasted with the case of modal logic in Section 7 and, specifically, in Lemma 7.4.

Initial Sequents

$$\dfrac{}{P, \Gamma \Rightarrow \Delta, P}\ Ax \qquad \dfrac{}{\bot, \Gamma \Rightarrow \Delta}\ L\bot$$

Logical Rules

$$\dfrac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \dfrac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$$

$$\dfrac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \dfrac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee$$

$$\dfrac{A \to B, \Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ L\to \qquad \dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \to B}\ R\to$$

$$\dfrac{A[x/y], \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ L\exists, y \text{ fresh} \qquad \dfrac{\Gamma \Rightarrow \Delta, \exists x A, A[x/t]}{\Gamma \Rightarrow \Delta, \exists x A}\ R\exists$$

$$\dfrac{\forall x A, A[x/t], \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ L\forall \qquad \dfrac{\Gamma \Rightarrow A[x/y]}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall, y \text{ fresh}$$

FIGURE 2 The **G3i** sequent calculus.

5 Soundness of the translation

The language of first-order intuitionistic logic is defined as usual [18]. The sequent calculus for intuitionistic logic enjoys the formulation displayed in Figure 2, where sequents are built from multisets of formulas. A few comments to the formulation of **G3i** are in order. First, we opted for a multi-succedent version of the system as it is closer to the modal system **G3s4** and this is important in order to establish the faithfulness of the translation. Second, the principal formula of rule $L \to$ is repeated in the left premise and the rules $R \to$ and $R\forall$ have a context restriction on the premise (otherwise the rules would be unsound).

From now on, we denote by **G3iT** and **G3s4T** the extensions of **G3i** and of **G3s4** by rules corresponding to the Horn theory **T**, respectively. We summarize the results of proof analysis for the calculus **G3iT**.

THEOREM 5.1
The rules of substitution, weakening and contraction are height preserving admissible in **G3iT**. Every rule except for $R \to$ and $R\forall$ is height-preserving invertible. The cut rule is admissible.

PROOF. See [13]. □

We recall the formulation of the modal translation. The present formulation can be found in [1] and differs from the original from Gödel (see [6]).

DEFINITION 5.2
The Gödel–McKinsey–Tarski translation is a map from the language of intuitionistic logic to that of modal logic. It is inductively defined as follows:

- $(P)^* = \Box P$, for P atomic.
- $(\bot)^* = \bot$
- $(A\#B)^* = A^*\#B^*$, where $\# \in \{\wedge, \vee\}$
- $(A \to B)^* = \Box(A^* \to B^*)$
- $(\exists x A)^* = \exists x A^*$
- $(\forall x A)^* = \Box\forall x A^*$

In this section, we will show that every intuitionistic derivation can be transformed into a derivation in the modal calculus of the translation of the endsequent. We first prove an auxiliary lemma, see also [19].

LEMMA 5.3
The sequent $\Rightarrow A^* \leftrightarrow \Box A^*$ is provable in **G3s4**.

PROOF. One direction, namely $\Box A^* \Rightarrow A^*$ immediately follows by an application of rule $L\Box$. The other direction, i.e. $A^* \Rightarrow \Box A^*$, is proved by induction on the degree of A.

1. If A is atomic, then $\Box P \Rightarrow \Box\Box P$ is easily derivable by two applications of $R\Box$.
2. If A is $B \to C$, then $\Box(B^* \to C^*) \Rightarrow \Box\Box(B^* \to C^*)$ is easily seen to be provable; the same argument applies to the universal quantifier.
3. If A is of the form $B \wedge C$, then we proceed as follows:

$$\dfrac{\dfrac{\dfrac{\vdots_{\mathrm{IH}}}{B^* \Rightarrow \Box B^*} \qquad \dfrac{\dfrac{\vdots_{\mathrm{IH}}}{C^* \Rightarrow \Box C^*} \qquad \Box B^*, \Box C^* \Rightarrow \Box(B^* \wedge C^*)}{\Box B^*, C^* \Rightarrow \Box(B^* \wedge C^*)}\,\mathrm{Cut}}{B^*, C^* \Rightarrow \Box(B^* \wedge C^*)}\,\mathrm{Cut}}{B^* \wedge C^* \Rightarrow \Box(B^* \wedge C^*)}\,\mathrm{L}\wedge$$

   The topsequent on the right is easily derivable by applying rule $R\Box$ and $R\wedge$.
4. If A is of the form $B \vee C$, we have:

$$\dfrac{\dfrac{\dfrac{\vdots_{\mathrm{IH}}}{B^* \Rightarrow \Box B^*} \qquad \Box B^* \Rightarrow \Box(B^* \vee C^*)}{B^* \Rightarrow \Box(B^* \vee C^*)}\,\mathrm{Cut} \qquad \dfrac{\dfrac{\vdots_{\mathrm{IH}}}{C^* \Rightarrow \Box C^*} \qquad \Box C^* \Rightarrow \Box(B^* \vee C^*)}{C^* \Rightarrow \Box(B^* \vee C^*)}\,\mathrm{Cut}}{B^* \vee C^* \Rightarrow \Box(B^* \vee C^*)}\,L\vee$$

   The sequents $\Box B^* \Rightarrow \Box(B^* \vee C^*)$ and $\Box C^* \Rightarrow \Box(B^* \vee C^*)$ are derivable by applying rule $R\Box$ followed by $R\vee$.
5. If A is of the form $\exists x B$, then we proceed as follows.

$$\dfrac{\dfrac{\dfrac{\vdots_{\mathrm{IH}}}{B^*[x/y] \Rightarrow \Box B^*[x/y]} \qquad \Box B^*[x/y] \Rightarrow \Box\exists x B^*}{B^*[x/y] \Rightarrow \Box\exists x B^*}\,\mathrm{Cut}}{\exists x B^* \Rightarrow \Box\exists x B^*}$$

   Once again the topsequent on the right is easily derivable by applying rule $R\Box$ and then rule $R\exists$. □

We finally prove the soundness of the translation by a proof-theoretic argument based on induction on the height of the derivations. By Γ^*, we denote the multiset that contains the formulas A^* for every formula A in Γ.

THEOREM 5.4 (Soundness).
If **G3iT** $\vdash \Gamma \Rightarrow \Delta$, then **G3s4T** $\vdash \Gamma^* \Rightarrow \Delta^*$.

PROOF. The proof is by induction on the height of the derivations in **G3iT**. If $n = 0$, the proof is immediate. If $n > 0$, we distinguish cases according to the rule applied.

1. If the last rule applied is a rule whose principal formula is a finite conjunction, a disjunction or an existential quantifier, then apply the induction hypothesis to the premises and then apply the rule again.
2. If the last rule applied is $L\rightarrow$ or $L\forall$, then we apply the induction hypothesis to the premise, and then we apply the corresponding rule again (an extra weakening step is required only in the case of $L\rightarrow$ due to the repetition of the principal formula in the left premise of the rule). The desired conclusion follows by an application of rule $L\Box$. We give an example of this qualitative analysis:

$$\dfrac{A \to B, \Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ L{\to}$$

We transform the proof as follows:

$$\dfrac{\dfrac{\Box(A^* \to B^*), \Gamma^* \Rightarrow \Delta^*, A^* \qquad \dfrac{B^*, \Gamma^* \Rightarrow \Delta^*}{\Box(A^* \to B^*), B^*, \Gamma^* \Rightarrow \Delta^*}\ \text{Weak}}{\Box(A^* \to B^*), A^* \to B^*, \Gamma^* \Rightarrow \Delta^*}\ L\to}{\Box(A^* \to B^*), \Gamma^* \Rightarrow \Delta^*}\ L\Box$$

3. If the last rule applied is a Horn rule, we have:

$$\dfrac{\Gamma, P_1, P_2, .., P_n, Q \Rightarrow \Delta}{\Gamma, P_1, .., P_n \Rightarrow \Delta}\ \text{Horn}$$

By applying the induction hypothesis to the premise, we obtain a derivation of the sequent $\Gamma^*, \Box P_1, .., \Box P_n, \Box Q \Rightarrow \Delta^*$. We proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Box P_1, ..., \Box P_n, P_1, ..., P_n, Q \Rightarrow Q}\ \text{Ax}}{\Box P_1, ..., \Box P_n, P_1, ..., P_n \Rightarrow Q}\ \text{Horn}}{\Box P_1, ..., \Box P_n \Rightarrow \Box Q}\ \text{R}\Box \qquad \Gamma^*, \Box P_1, .., \Box P_n, \Box Q \Rightarrow \Delta^*}{\dfrac{\Gamma^*, \Box P_1, .., \Box P_n, \Box P_1, .., \Box P_n, \Rightarrow \Delta^*}{\Gamma^*, \Box P_1, .., \Box P_n \Rightarrow \Delta^*}\ \text{Ctr}}\ \text{Cut}$$

4. The other cases (R$\rightarrow$ and R$\forall$) are rather routine. In particular, consider the case of $R\forall$, we have:

$$\dfrac{\Gamma \Rightarrow A[x/y]}{\Gamma \Rightarrow \forall x A(x), \Delta}\ R\forall$$

By applying the induction hypothesis, we get a derivation of $\Gamma^* \Rightarrow A^*[x/y]$. We exploit the fact that, for every A, the sequent $A^* \Rightarrow \Box A^*$ is provable in **G3s4T** by Lemma 5.3 by invertibility of the rules $R\wedge$ and $R\rightarrow$. Then we complete the transformation as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma^* \Rightarrow A^*[x/y]}{\Gamma^* \Rightarrow \forall x A^*(x)}\, R\forall}{\Gamma^*, \Box\Gamma^* \Rightarrow \forall x A^*(x)}\, \text{Weak}}{\Box\Gamma^* \Rightarrow \Box\forall x A^*(x), \Delta^*}\, R\Box}{\Gamma^* \Rightarrow \Box\forall x A^*(x), \Delta^*}\, \text{admissible rule}$$

The rule is admissible via cuts with $A^* \Rightarrow \Box A^*$ for every formula A in the multiset Γ. □

Notice that soundness is a delicate passage, which requires the restriction of the class of geometric theories to the smaller class of Horn theories. In particular, it is necessary to exclude the presence of disjunctions and existential quantifiers in the succedent.

In fact, try to consider the case of the axiom of trichotomy in linear orders on the language $\{<, =\}$:

$$\forall x \forall y (x < y \vee y < x \vee x = y)$$

It is easy to observe that its $*$-translation is not provable in **G3s4T**, where **T** is the theory of linear orders.

LEMMA 5.5
The sequent

$$\Rightarrow \Box\forall x \Box\forall y (\Box(x < y) \vee \Box(y < x) \vee \Box(x = y))$$

is not derivable in **G3s4LO**, i.e. the sequent calculus obtained by adding the rule corresponding to the linearity axiom.

PROOF. If the sequent $\Rightarrow \Box\forall x \Box\forall y (\Box(x < y) \vee \Box(y < x) \vee \Box(x = y))$, via cuts we can easily infer the derivability of the sequent $\Rightarrow \Box(x < y), \Box(y < x), \Box(x = y)$. It is easy to observe that this sequent is derivable if and only if one among $\Rightarrow x < y$, $\Rightarrow y < x$ or $\Rightarrow x = y$, which is not the case. □

6 Faithfulness of the translation

A proof of the faithfulness of the embedding for pure logic was presented in [19]. Furthermore, embedding results of intuitionistic logic into modal logics have been obtained by exploiting the methodology of labelled sequent calculi [3, 4]. By adopting labelled system the faithfulness proof follows from a straightforward induction on the height of derivations in the modal calculus. Our proof extends these results to first-order Horn theories: we reason by induction on the height of the derivations and we use the standard cut-free sequent calculus **G3s4T**.

In order to prove the faithfulness lemma directly, i.e. by induction on the height of the derivations in the modal calculus, we need to devise a suitable strengthening of the induction hypothesis that takes into account the built-in contraction contained in the left rule for the universal quantifier.

LEMMA 6.1 (Faithfulness).
Let Π and Σ be multisets of atomic formulas, $\Gamma^{\forall}$ a multiset of formulas $\forall x A^*$, Λ and Δ multisets of formulas. Then:

$$\text{If } \mathbf{G3s4T} \vdash \Pi, \Gamma^{\forall}, \Lambda^* \Rightarrow \Delta^*, \Sigma, \text{ then } \mathbf{G3iT} \vdash \Pi, \Gamma^{\forall -}, \Lambda \Rightarrow \Delta, \Sigma$$

where $\Gamma^{\forall -}$ contains formulas $\forall x A$ for every $\forall x A^*$ in $\Gamma^{\forall}$.

PROOF. The proof is by induction on the height of the derivations in **G3s4T**. If $n = 0$, then the proof is immediate. If $n > 0$, we distinguish cases according to the last rule applied.

- If the last rule is different from $L\Box$ or $R\Box$, we can simply apply the induction hypothesis and then the rule again (if necessary, as in the case of $L\forall$, we add an extra step of contraction). In particular, if the last rule applied is a Horn rule, we apply the induction hypothesis and then the rule again, because the active formulas of the rule are all atomic.
- If the last rule is $L\Box$, we have:

$$\dfrac{\Pi, \Gamma^{\forall}, \Lambda^*, \Box A, A \Rightarrow \Delta^*, \Sigma}{\Pi, \Gamma^{\forall}, \Lambda^*, \Box A \Rightarrow \Delta^*, \Sigma}\,\text{Geom}$$

  where $\Box A = B^*$ for some formula B. By definition of the $*$-translation, B is of the shape $\forall x C, C \to D$ or P, with P atomic. If B is $\forall x C$ or P, we apply the induction hypothesis to the premise and then we apply height-preserving admissibility of contraction to obtain the desired conclusion. If $B = C \to D$, we have:

$$\dfrac{\Pi, \Gamma^{\forall}, \Lambda^*, \Box(C^* \to D^*), C^* \to D^* \Rightarrow \Delta^*, \Sigma}{\Pi, \Gamma^{\forall}, \Lambda^*, \Box(C^* \to D^*) \Rightarrow \Delta^*, \Sigma}\,L\Box$$

  In this case, we proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi, \Gamma^{\forall}, \Lambda^*, \Box(C^* \to D^*), C^* \to D^* \Rightarrow \Delta^*, \Sigma}{\Pi, \Gamma^{\forall}, \Lambda^*, \Box(C^* \to D^*) \Rightarrow \Delta^*, \Sigma, C^*}\,\text{Lemma 2.6}}{\Pi, \Gamma^{\forall-}, \Lambda, C \to D \Rightarrow \Delta, \Sigma, C}\,\text{IH}}{\Pi, \Gamma^{\forall-}, \Lambda, C \to D, C \to D \Rightarrow \Delta, \Sigma, C}\,\text{Weak} \qquad \dfrac{\dfrac{\Pi, \Gamma^{\forall}, \Lambda^*, \Box(C^* \to D^*), C^* \to D^* \Rightarrow \Delta^*, \Sigma}{\Pi, \Gamma^{\forall}, D^*, \Lambda^*, \Box(C^* \to D^*) \Rightarrow \Delta^*, \Sigma}\,\text{Lemma 2.6}}{\Pi, \Gamma^{\forall-}, D, \Lambda, C \to D \Rightarrow \Delta, \Sigma}\,\text{IH}}{\Pi, \Gamma^{\forall-}, \Lambda, C \to D \Rightarrow \Delta, \Sigma}\,\text{L}\to$$

  the application of the induction hypothesis (IH) is justified by the fact that invertibility preserves the height of the derivations.
- If the last rule is $R\Box$, then the principal formula is in Δ^* and we distinguish three subsubcases according to the shape of the principal formula in Δ^*, which can be $\Box P, \Box\forall x B^*$ or $\Box(B^* \to C^*)$ (the three cases are exhaustive due to the definition of the modal interpretation).
  - If it is of the form $\Box P$, we have:

$$\dfrac{\Box\Lambda''', \Lambda''' \Rightarrow P}{\Pi, \Gamma^{\forall}, (\Lambda'')^*, (\Lambda')^* \Rightarrow \Delta'^*, \Box P, \Sigma}\,R\Box$$

    with $\Lambda^* = (\Lambda'')^*, (\Lambda')^*$ and $(\Lambda'')^* = \Box\Lambda'''$. Now, formulas in Λ''' can be of three types: atomic formulas, implications and universal quantifiers. Namely,

$$\Lambda''' = Q_1, ..., Q_n, \forall x D_1^*(x), ..., \forall x D_l^*(x), B_1^* \to C_1^*, ..., B_j^* \to C_j^*$$

    with $l, j \geq 0$. We apply height-preserving invertibility of $L \to$ to reduce the complexity of the implication formulas. Then we apply the induction hypothesis to the 2^j derivations thus obtained. To simplify the explanation and the notation, we assume that there is a single occurrence for each of the three types of formulas, i.e. $n = l = j = 1$, the generalization is straightforward. Hence, we have $\Lambda''' = Q, \forall x D^*(x), B^* \to C^*$.

We proceed as follows:

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\Box\Lambda''', Q, \forall x D^*(x), B^* \to C^* \Rightarrow P}{\Box\Lambda''', Q, \forall x D^*(x) \Rightarrow P, B^*}\,\text{Lemma 2.6}}{\Lambda'', Q, \forall x D(x) \Rightarrow P, B}\,\text{IH}}{\Lambda'', Q, \forall x D(x), B \to C \Rightarrow P, B}\,\text{Weak}
\qquad
\dfrac{\dfrac{\Box\Lambda''', Q, \forall x D^*(x), B^* \to C^* \Rightarrow P}{\Box\Lambda''', Q, \forall x D^*(x), C^* \Rightarrow P}\,\text{Lemma 2.6}}{\Lambda'', Q, \forall x D(x), C \Rightarrow P}\,\text{IH}
}{
\dfrac{\dfrac{\Lambda'', Q, \forall x D(x), B \to C \Rightarrow P}{\Lambda'' \Rightarrow P}\,\text{Ctr}}{\Pi, \Gamma^{\forall -}, \Lambda \Rightarrow P, \Delta, \Sigma}\,\text{Weak}
}\,\text{L}\to
$$

The contraction step is justified because $\Lambda'' = \{Q, \forall x D(x), B \to C\}$. The applications of the induction hypothesis are justified because the applications of the invertibility lemma are height-preserving.

- The cases in which the formula in Δ^* is of the shape $\Box(A^* \to B^*)$ or $\Box\forall x A^*(x)$ are actually similar. In particular, we apply the invertibility of the right rule for $\to$ and $\forall$ in order to be able to apply the induction hypothesis and we repeat the procedure described for the atomic case. We sketch the case of the universal quantifier:

$$
\dfrac{\Box\Lambda''', \Lambda''' \Rightarrow \forall x B^*}{\Pi, \Gamma^{\forall}, (\Lambda'')^*, (\Lambda')^* \Rightarrow \Delta'^*, \Box\forall x B^*, \Sigma}\,R\Box
$$

We apply height-preserving invertibility of $R\forall$ and we obtain $\Box\Lambda''', \Lambda''' \Rightarrow B^*[x/y]$ where y is a fresh variable. Then we apply height-preserving invertibility of $L\to$ to the implicative formula in Λ''' (if present) and we can thus apply the induction hypothesis. Finally, we conclude the proof by an application of $R\forall$ and weakening admissibility to add the missing contexts.

This concludes the proof. □

Combining the faithfulness lemma with the results presented in the previous section, we obtain the embedding result.

THEOREM 6.2 (Embedding).
Let **T** be a Horn theory, then:

$$\mathbf{G3iT} \vdash \Rightarrow A \text{ if and only if } \mathbf{G3s4T} \vdash \Rightarrow A^*$$

PROOF. From left to right, we exploit the soundness theorem and from right to left we exploit the faithfulness lemma. □

We can exploit the soundness and faithfulness result in order to obtain an alternative proof of the disjunction property and of the witness property for Horn theories in a multisuccedent intuitionistic calculus.[1] Namely, instead of searching a proof in the multisuccedent intuitionistic system, we can solve the problem by working in the modal calculus.

THEOREM 6.3 (Disjunction and witness property).
For every Horn theory **T**, the following statements hold:

1. If **G3iT** $\vdash\Rightarrow A \vee B$, then **G3iT** $\vdash\Rightarrow A$ or **G3iT** $\vdash\Rightarrow B$.
2. If **G3iT** $\vdash\Rightarrow \exists x A(x)$, then **G3iT** $\vdash\Rightarrow A[t/x]$ for some term t.

[1] For similar result in the field of structural proof theory, the reader can consult [11, 20]

PROOF. The proofs are similar and we limit ourselves to discussing the first item. If **G3iT** $\vdash\Rightarrow A \vee B$, then by soundness we obtain **G3s4T** $\vdash\Rightarrow A^* \vee B^*$. By invertibility of rule R$\vee$ and cuts with $A^* \Rightarrow \Box A^*$ and $B^* \Rightarrow \Box B^*$, we get **G3s4T** $\vdash\Rightarrow \Box A^*, \Box B^*$. The derivation must have the following form:

$$\dfrac{\begin{array}{c}\vdots \\ \Rightarrow C\end{array}}{\Gamma \Rightarrow \Box A^*, \Box B^*}\ \text{R}\Box$$
$$\vdots\pi$$
$$\Rightarrow \Box A^*, \Box B^*$$

where π contains only applications of Horn rules, Γ is a multiset of atomic formulas and C is either A^* or B^*, depending on the principal formula of R$\Box$. This yields **G3s4T** $\vdash\Rightarrow A^*$ or **G3s4T** $\vdash\Rightarrow B^*$. By faithfulness of the translation, we get the desired conclusion. $\Box$

7 Geometric logic and the modal embedding

As we have already observed, the soundness of the modal translation breaks down in the presence of geometric axioms or, more in general, of axioms containing disjunctions or existential quantifiers in the succedent.

Indeed, the modal interpretation still holds for pure logic, in the sense that given an axiom A in first-order geometric logic, we have:

$$\textbf{G3i} \oplus A \vdash \Gamma \Rightarrow \Delta \text{ if and only if } \textbf{G3s4} \oplus A^* \vdash \Gamma^* \Rightarrow \Delta^*$$

However, this solution cannot be regarded as satisfactory. In general, the axiom A is not equivalent over **S4** to its $*$-translation. Therefore, we are actually considering a different theory and not the same theory over a modal base.

A very natural question consists in asking which kind of modal system is suitable to reach the following result:

$$\textbf{G3i} \oplus A \vdash \Gamma \Rightarrow \Delta \text{ if and only if } \textbf{G3?} \oplus A \vdash \Gamma^* \Rightarrow \Delta^*$$

To obtain such system, we need to properly extend **G3S4** with an infinite set of sequents (see also [16, 17] for a similar approach in the context of infinitary propositional logic and with labelled systems). In particular, we require:

$$P \Rightarrow \Box P \text{ for every atomic first-order formula } P$$

To obtain an analytic system for this logic, we need to slightly modify the rule governing the modal operator.

$$\dfrac{\Gamma^{at}, \Box\Pi, \Pi \Rightarrow A}{\Gamma^{at}, \Box\Pi, \Pi' \Rightarrow \Delta, \Box A}\ R\Box^+$$

In other words, we require that the atomic propositional formulas are not removed by the application of the rule for the modal operator. Let **G3s4T**$^+$ be the system obtained by replacing the rule R$\Box$ with the
rule R$\Box^+$.

LEMMA 7.1
The rules of substitution, weakening and contraction are height-preserving admissible in the calculus

G3s4T^{+}.
Every rule except for R$\Box^{+}$ is height-preserving admissible.

PROOF. The proofs run by induction and they are minor modifications of the ones for **G3s4T**, therefore we omit the details. $\Box$

THEOREM 7.2
The cut rule is admissible in **G3s4T**$^{+}$.

PROOF. The proof runs by double induction with main induction hypothesis on the degree of the cut formula and secondary induction hypothesis on the sum of the height of the derivations of the premises of the cut.

The new relevant case is the one in which the cut formula is atomic and principal in an application of the rule R$\Box^{+}$ in the right premise of the cut.

$$\dfrac{\Gamma \Rightarrow \Delta, P \qquad \dfrac{P, \Pi^{at}, \Box\Theta, \Theta \Rightarrow A}{P, \Pi^{at}, \Box\Theta, \Theta' \Rightarrow \Box A, \Lambda}\,\text{R}\Box^{+}}{\Gamma, \Pi^{at}, \Box\Theta, \Theta' \Rightarrow \Box A, \Lambda, \Delta}\,\text{Cut}$$

The cut cannot be simply permuted upwards as the rule R$\Box^{+}$ might not be applicable. Hence, we argue by induction on the left premise of the cut. The case in which it is an initial sequent is trivial. If it is the conclusion of a rule, then P is not principal. If the last rule applied is R$\Box^{+}$, we consider the premise and we apply the rule again to get the desired conclusion. If the last rule applied is any other rule, we permute the cut upwards and we apply the rule again. $\Box$

Although this section is devoted to a syntactic approach to the issue, we would like to point out that a very natural semantics for the system **G3s4T**$^{+}$ emerges by considering first-order Kripke models for modal logic with increasing domains and by imposing a monotonicity condition on atomic formulas.

LEMMA 7.3
Let **T** be a geometric theory. The following statements hold:

1. **G3s4T**$^{+}$ $\vdash \Rightarrow P \rightarrow \Box P$
2. There is not a collapse of the modality in **G3s4T**$^{+}$, i.e. there is a formula A such that **G3s4T**$^{+}$ does not prove $\Rightarrow A \leftrightarrow \Box A$.

PROOF. Item 1. follows from a routine root-first application of the rule R$\rightarrow$ and R$\Box^{+}$. Notice that the sequent is not provable in **G3S4T**.

Item 2. follows by noticing that $\Rightarrow (P \rightarrow Q) \rightarrow \Box(P \rightarrow Q)$ is not derivable. Suppose towards a contradiction that it is derivable, then by invertibility of the rule L$\rightarrow$ so is $\Rightarrow P, \Box(P \rightarrow Q)$. However, the only applicable rule is R$\Box^{+}$, which gives $\Rightarrow P \rightarrow Q$, an underivable sequent. $\Box$

We add a remark concerning equality. As it is well known, equality is characterized by the addition of the axiom of reflexivity $\forall x(x = x)$ and the axiom schema of replacement, i.e.

$$\forall x \forall y(x = y \wedge A(x) \rightarrow A(y))$$

for every formula A. As it is well-known, in first-order classical (and intuitionistic) logic the rules for equality:

$$\dfrac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\text{Ref} \qquad \dfrac{P(s), P(t), t = s, \Gamma \Rightarrow \Delta}{P(t), t = s, \Gamma \Rightarrow \Delta}\,\text{Geom}$$

can be added to a base sequent calculus while preserving the structural properties of the base system (see also Section 4). In first-order modal logic, the situation is more complex. Indeed, the addition of equality rules to the system **G3s4T** does not allow for the derivability of the sequent $t = s, A(t) \Rightarrow A(s)$ for every formula A, a quick counterexample is given by the sequent $t = s, \Box P(t) \Rightarrow \Box P(s)$. However, the stronger system $\mathbf{G3s4T}^{+}$ does the trick and we can prove the following proposition.

LEMMA 7.4
Given any geometric theory **T**, consider the system $\mathbf{G3s4T}^{+}$ extended with the rules for equality. The sequent:

$$t = s, A(t) \Rightarrow A(s)$$

is derivable.

PROOF. The proof is by induction on the degree of the formula A. We limit ourselves to discussing the case in which the formula A is of the shape $\Box B$. In this case, we have:

$$\dfrac{\dfrac{\vdots_{\text{IH}} \atop t = s, B(t) \Rightarrow B(s)}{t = s, B(t), \Box B(t) \Rightarrow B(s)}\,\text{Weak}}{t = s, \Box B(t) \Rightarrow \Box B(s)}\,\text{R}\Box^{+}$$

The topmost sequent is derivable by induction on the degree of the formula. □

REMARK
Let us observe that the underivability of the full equality schema in **G3s4T** extended with the usual rules for equality does not hinder the modal interpretation of Horn intuitionistic theories presented in Section 6 in Theorem 6.2. Indeed, **G3s4T** extended with the rules for equality is not equivalent to the an axiomatic modal calculus extended with the equality schema, but it is enough to obtain a sound and faithful interpretation of any Horn intuitionistic theory with the equality schema.

REMARK
We observe that the equivalence with an axiomatic system can be established for the system $\mathbf{G3s4T}^{+}$. In particular, it is enough to consider the system obtained by adding to the axiomatic system **QS4**, possibly extended with geometric axioms, the set of axioms $\{\vdash P \rightarrow \Box P \mid P \text{ atomic}\}$. This shows that the modification of the rule $\text{R}\Box^{+}$ is not arbitrary. We leave the details of the proof of the equivalence between the two systems to the reader.

The crucial result for $\mathbf{G3s4T}^{+}$ is that for every formula A, we have $A^{l} \leftrightarrow A^{*}$, where l is a light translation thus defined.

DEFINITION 7.5
The light Gödel–McKinsey–Tarski translation is a map from the language of intuitionistic logic to that of modal logic. It is inductively defined as follows:

- $(P)^{l} = P$, for P atomic.
- $(\bot)^{l} = \bot$
- $(A \# B)^{l} = A^{l} \# B^{l}$, where $\# \in \{\wedge, \vee\}$
- $(A \rightarrow B)^{l} = \Box(A^{l} \rightarrow B^{l})$

- $(\exists xA)^l = \exists xA^l$
- $(\forall xA)^l = \Box\forall xA^l$

LEMMA 7.6
G3s4T$^+ \vdash \Rightarrow A^l \leftrightarrow A^*$ for every formula A.

PROOF. Immediate by induction on the degree of the formula A. □

To complete our investigations, we show the following.

THEOREM 7.7
G3s4T$^+ \oplus A^l$ is equivalent to **G3s4T**$^+ \oplus A$.

PROOF. It is trivial to observe that $A^l \to A$, so one direction is easily established via suitable cuts. The converse does not hold in general, so we look at the structure of the derivations. Suppose we have a derivation employing an axiom $\Rightarrow A^l$ as initial sequent. It will be of the form:

$$\Rightarrow \Box\forall\overline{x}\Box(P_1 \wedge ... \wedge P_m \to \exists\overline{y}_1\overline{Q}_1 \vee ... \vee \exists\overline{y}_n\overline{Q}_n)$$

This can be simulated as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \forall\overline{x}(P_1 \wedge ... \wedge P_m \to \exists\overline{y}_1\overline{Q}_1 \vee ... \vee \exists\overline{y}_n\overline{Q}_n)}{\Rightarrow P_1 \wedge ... \wedge P_m \to \exists\overline{y}_1\overline{Q}_1 \vee ... \vee \exists\overline{y}_n\overline{Q}_n}\,\text{Inv R}\forall}{\Rightarrow \Box(P_1 \wedge ... \wedge P_m \to \exists\overline{y}_1\overline{Q}_1 \vee ... \vee \exists\overline{y}_n\overline{Q}_n)}\,\text{R}\Box^+}{\Rightarrow \forall\overline{x}\Box(P_1 \wedge ... \wedge P_m \to \exists\overline{y}_1\overline{Q}_1 \vee ... \vee \exists\overline{y}_n\overline{Q}_n)}\,\text{R}\forall}{\Rightarrow \Box\forall\overline{x}\Box(P_1 \wedge ... \wedge P_m \to \exists\overline{y}_1\overline{Q}_1 \vee ... \vee \exists\overline{y}_n\overline{Q}_n)}\,\text{R}\Box^+$$

□

The modal embedding is established for geometric axiomatic extensions by the following theorem.

THEOREM 7.8
For every geometric theory **T**, **G3iT** $\vdash \Gamma \Rightarrow \Delta$ if and only if **G3s4T**$^+ \vdash \Gamma^l \Rightarrow \Delta^l$.

PROOF. From left to right we argue by induction on the height of the derivation. The only new case to check is the one of the geometric rules. If the last rule applied is a geometric rule, we have:

$$\dfrac{\overline{Q}_1[\overline{z}_1/\overline{y}_1], \overline{P}, \Gamma^l \Rightarrow \Delta^l \quad ... \quad \overline{Q}_n[\overline{z}_n/\overline{y}_n], \overline{P}, \Gamma^l \Rightarrow \Delta^l}{\overline{P}, \Gamma^l \Rightarrow \Delta^l}\,\text{Geom}$$

Due to the definition of the translation, the atomic formulas are not modified. Hence, we proceed as follows:

$$\dfrac{\dfrac{\overline{Q}_1[\overline{z}_1/\overline{y}_1], \overline{P}, \Gamma^l \Rightarrow \Delta^l}{\overline{Q}_1[\overline{z}_1/\overline{y}_1], \overline{P}, \Gamma \Rightarrow \Delta}\,\text{IH} \quad ... \quad \dfrac{\overline{Q}_n[\overline{z}_n/\overline{y}_n], \overline{P}, \Gamma^l \Rightarrow \Delta^l}{\overline{Q}_n[\overline{z}_n/\overline{y}_n], \overline{P}, \Gamma \Rightarrow \Delta}\,\text{IH}}{\overline{P}, \Gamma \Rightarrow \Delta}\,\text{Geom}$$

From right to left, the strategy follows the pattern detailed in the case of **G3s4T**. □

THEOREM 7.9
For every geometric theory **T**, **G3iT** $\vdash \Gamma \Rightarrow \Delta$ if and only if **G3s4T**$^+ \vdash \Gamma^* \Rightarrow \Delta^*$.

PROOF. We consider the following chain of equivalences. Clearly, **G3iT** $\vdash \Gamma \Rightarrow \Delta$ if and only if **G3s4T**$^+ \vdash \Gamma^l \Rightarrow \Delta^l$. The latter is equivalent to **G3s4T**$^+ \vdash \Gamma^* \Rightarrow \Delta^*$, which yields the desired conclusion. □

8 Concluding remarks

We have applied the methods of proof analysis to systems of modal logics and we have proved an extension of the Gödel–McKinsey–Tarski embedding to first-order Horn theories.

There are various points that might be interesting future lines of research. First, it would be interesting to study a similar approach in terms of labelled sequent calculi.

Second, it is worth investigating the possibility of extending the approach to systems with first-order axioms containing modal formulas. For example, consider the formula: $\forall x(P(x) \rightarrow \Diamond\exists y Q(x,y))$, which inside a labelled sequent calculus might be converted into the rule [13]:

$$\frac{a \in D(w), w : P(a), wRo, b \in D(o), o : Q(a,b), \Gamma \Rightarrow \Delta}{a \in D(w), w : P(a), \Gamma \Rightarrow \Delta}\ \text{Geom}, o, b \text{ fresh}$$

with a double variable condition on worlds and elements of the domain. It is worth considering the scope of such an approach.

Third, in this paper, we have applied the methods of proof analysis to a domain that lies outside of classical and intuitionistic logic. This naturally poses the intriguing question whether the conversion of axioms into rules can be obtained also considering as a base calculus another non-classical system.

References

[1] A. Chagrov and M. Zakharyaschev. *Modal Logic*. Oxford University Press, 1997.

[2] M. A. E. Dummet and E. J. Lemmon. Modal logics between S4 and S5. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, **5**, 250–264, 1959.

[3] R. Dyckhoff and S. Negri. Proof analysis in intermediate logic. *Archive for Mathematical Logic*, **51**, 71–92, 2012.

[4] R. Dyckhoff and S. Negri. A cut-free sequent system for Grzegorczyk logic with an application to the Gödel–McKinsey–Tarski embedding. *Journal of Logic and Computation*, **26**, 169–187, 2016.

[5] R. C. Flagg and H. Friedman. Epistemic and intuitionistic formal systems. *Annals of Pure and Applied Logic*, **32**, 53–60, 1986.

[6] K. Gödel. Eine interpretation der intuitionistischen aussagenkalüls. *Ergebnisse eines Mathematischen Kolloquiums*, **4**, 39–40, 1933.

[7] N. D. Goodman. Epistemic arithmetic is a conservative extension of intuitionistic arithmetic. *Journal of Symbolic Logic*, **49**, 192–203, 1984.

[8] G. E. Hughes and M. J. Cresswell. *A New Introduction to Modal Logic*, p. 282. Routledge, 1996.

[9] J. C. McKinsey and A. Tarski. Some theorems about the sentential calculi of Lewis and Heyting. *Journal of Symbolic Logic*, **13**, 1–15, 1948.

[10] G. Mints. On Novikov hypothesis. *Selected Papers in Proof Theory*. Bibliopolis, 1992.

[11] S. Negri and J. von Plato. Cut elimination in the presence of axioms. *The Bulletin of Symbolic Logic*, **4**, 418–435, 1998.

[12] S. Negri. Contraction-free sequent calculi for geometric theories, with an application to Barr's theorem. *Archive for Mathematical Logic*, **42**, 389–401, 2003.

[13] S. Negri. Proof analysis in modal logic. *Journal of Philosophical Logic*, **34**, 507–544, 2005.
[14] H. Rasiowa and R. Sikorski. *The Mathematics of Metamathematics*. Polish Scientific Publishers, Warsaw, 1963.
[15] M. Stone. Topological representations of distributive lattices and Brouwerian logics. *Casopis pro Pestovan Matematiky a Fysiky*, **067**, 1–25, 1937.
[16] M. Tesi and S. Negri. Neighborhood semantics and labelled calculus for infinitary intuitionistic logic. *Journal of Logic and Computation*, **31**, 1608–1639, 2020.
[17] M. Tesi and S. Negri. The Gödel–McKinsey–Tarski embedding for infinitary intuitionistic logic and its extensions. *Annals of Pure and Applied Logic*, **174**, 103285, 2023.
[18] A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics*, vol. 2. North Holland, 1998.
[19] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, 2000.
[20] L. Viganò. *Labelled Non-Classical Logics*. Springer Science & Business Media, 2000.

Received 26 June 2023