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Towards an Argumentative Unification of Default Reasoning

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Abstract. We propose a novel knowledge representation method for the Default Logic paradigm by developing a proof calculus that yields arguments and counter-arguments in which defaults serve as explicit objects of reasoning. The proposed formalism allows for more transparent default reasoning and the use of explainability methods in formal argumentation. In particular, we provide a sound and complete argumentative characterization of Default Logic, by demonstrating that argumentation frameworks instantiated by the arguments derivable in our calculus yields the same inference relation as that of Default Logic. The modularity of our approach allows for various modifications of Default Logic. We demonstrate this by extending our calculus with a rule that enables disjunctive defeasible reasoning.

Keywords. default logic, logical argumentation, philosophy of reasons, sequent calculi, proof theory, disjunctive defeasible reasoning

1. Introduction

A central challenge of knowledge bases is to adequately reason with incomplete and inconsistent information. A multitude of defeasible reasoning formalisms has been developed to address these challenges. In these formalisms, prior inference may be retracted under extended context, yielding nonmonotonic (nm) inference relations. Formal argumentation, starting with the work of Dung [1], offers a unifying framework for defeasible reasoning and many nm formalisms can be embedded in it (see [2] for an overview.) A notably attractive feature of formal argumentation is its ability to explicitly model conflicts, which is particularly promising with respect to the challenge of explainability in knowledge representation and reasoning [3].

In the quest for more transparent and explainable formal systems, *reasons* and explicit reasoning with reasons play a pivotal role. The philosophical study of reasons is well-founded and, as pointed out by Horty [4], Reiter's [5] well-established Default Logic (and its many extensions) is particularly advantageous since it adopts rules as reasons. In recent work, van Berkel and Straßer [6] develop a proof-theoretic approach to logical argumentation in which reasons are modeled as explicit objects of reasoning.

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Their approach establishes strong ties between nm formalisms (such as classes of Input/Output logics [7]) and formal argumentation.

The work [6] by van Berkel and Straßer does not immediately extend to Default Logic, since the latter submits arguments (and implied attacks between them) to stricter conditions: all used defaults in the construction of a (counter)argument must be triggered. In this work, we develop a modular proof-calculus, referred to as Default Calculus (DC), that generates arguments of *normal Default Logic* and whose arguments, instantiated in an argumentation framework, yield an argumentative characterization of Default Logic.

Our contributions are conceptual as well as technical. From a conceptual point of view, we develop a novel knowledge representation method for the Default Logic paradigm that involves (i) a proof-theoretic formalism, (ii) the unifying formalism of formal argumentation, and (iii) the ever more important philosophy of reasons. With respect to the latter, DC constitutes a more transparent approach to default reasoning: it uses labels to indicate the role of formulas in the reasoning and yields derivations as a justification of which defaults are used for which conclusions. From a technical viewpoint, the main contribution of this paper is the characterization of Default Logic by means of DC-induced argumentation frameworks. The modularity of our approach is particularly promising for obtaining variants of Default Logic. We provide an extension of our calculus with a rule for "reasoning by cases," yielding a calculus whose induced argumentation frameworks are sound and complete for reasoning disjunctively in Default Logic.

Outline: Section 2 will see some preliminaries, recalling the essentials of Default Logic. Thereafter, in Section 3 we define our Default Calculus (DC). In Section 4, we define argumentation frameworks instantiated with DC-derivable default arguments which are shown sound and complete for Default Logic. We propose a sound and complete extension of DC with disjunctive reasoning in Section 5. In Section 6, we discuss related and future work. The proofs are provided in the technical appendix.

2. Preliminaries: Default Logic and a Running Example

In this paper, we focus on normal Default Logic [5] and work with the well-established extension building approach. We use a propositional language $\mathscr{L}$ containing the usual connectives $\neg, \wedge, \vee, \rightarrow$. We use $p, q, \ldots$ to denote atoms, $\varphi, \psi, \ldots$ to denote arbitrary formulas of $\mathscr{L}$, and $E, F, \ldots$ for arbitrary subsets of $\mathscr{L}$. Furthermore, let

$$\mathscr{L}^{\delta} = \{(\varphi, \psi) \mid \varphi, \psi \in \mathscr{L}\}$$

be the *language of defaults*. We give a default $\delta = (\varphi, \psi) \in \mathscr{L}^{\delta}$ an epistemic interpretation: "φ provides a (*prima facie*, defeasible) reason to believe ψ." We write $\varphi = \mathsf{body}(\delta)$ and $\psi = \mathsf{head}(\delta)$ for the default's body, respectively head. We use $\vdash_{\mathsf{CL}}$ for classical entailment and $\mathsf{Cn}(E) = \{\varphi \in \mathscr{L} \mid E \vdash_{\mathsf{CL}} \varphi\}$ for the deductive closure of $E \subseteq \mathscr{L}$.

Definition 1. A *default theory* is of the form $\mathbb{K} = \langle \mathscr{F}, \mathscr{D} \rangle$, where $\mathscr{F} \subseteq \mathscr{L}$ is a consistent factual context (i.e., $\mathscr{F} \nvdash_{\mathsf{CL}} \bot$) and $\mathscr{D} \subseteq \mathscr{L}^{\delta}$ a set of defaults.

Example 1. Consider the default theory $\mathbb{K}$ containing the facts $\mathscr{F} = \{w\}$ and defaults $\mathscr{D} = \{(w, \neg o), (w, h), (h, o)\}$. Although we use literals in $\mathbb{K}$, our approach generalizes to the full propositional language. We interpret o as "the window is open," w as "it is winter," and h as "the hearth is on." In an epistemic reading, $\mathbb{K}$ expresses a scenario in which it is winter, with the default $(w, \neg o)$ as "winter is a reason to believe that the window is not open" (this way warmth is preserved), (w, h) as "winter is a reason to believe the hearth is on," and (h, o) as "having the hearth on is a reason to believe the window is open" (this way fresh air gets in). Intuitively, $\mathbb{K}$ should give rise to the belief that the hearth is on, but leave us agnostic concerning the question whether the window is open. The defaults $(w, \neg o)$ and (h, o) (the latter triggered by (w, h)) are mutually exclusive. Our example is a variant of the order puzzle [4], without preferences.

Suppose that we find out that the window is open, that is, $\mathscr{F}' = \{w, o\}$. Then, in addition to $\mathscr{F}'$ we only conclude that the hearth is on. The default $(w, \neg o)$ must now not be applied since it conflicts with our knowledge that the window is open, i.e., $o \in \mathscr{F}$.

Definition 2. Let $\mathbb{K} = \langle \mathscr{F}, \mathscr{D} \rangle$ be a default theory. Let $\sigma = \langle \delta_i \rangle_1^n$ be a (possibly empty) sequence of defaults $\delta_i \in \mathscr{D}$. We define the following iterative *detachment* procedure $\mathsf{Det}(\sigma) = \bigcup_{i \geq 0} \mathsf{Det}^i_{\mathscr{F}}(\sigma)$, where $\mathsf{Det}^0_{\mathscr{F}}(\sigma) = \mathsf{Cn}(\mathscr{F})$ and $\mathsf{Det}^{i+1}_{\mathscr{F}}(\sigma) = \mathsf{Cn}(\mathscr{F} \cup \{\mathsf{head}(\delta) \mid \delta \in \sigma, \mathsf{body}(\delta) \in \mathsf{Det}^i_{\mathscr{F}}(\sigma)\})$.

A sequence $\sigma = \langle \delta_i \rangle_{i=1}^n$ is a *process* in case for each $1 \leq i < n$, $\mathsf{body}(\delta_{i+1}) \in \mathsf{Det}(\langle \delta_j \rangle_{j=1}^i)$ (in this case, $\mathsf{Det}(\sigma) = \mathsf{Cn}(\mathscr{F} \cup \{\mathsf{head}(\delta) \mid \delta \in \sigma\})$). σ is *consistent* iff for all $\delta \in \sigma$, $\neg\mathsf{head}(\delta) \notin \mathsf{Det}(\sigma)$, else it is *inconsistent*. σ is *complete* if for all $\delta \in \mathscr{D} \setminus \sigma$, if $\mathsf{body}(\delta) \in \mathsf{Det}(\sigma)$, $\neg\mathsf{head}(\delta) \in \mathsf{Det}(\sigma)$. σ is *proper* iff it is consistent and complete.

Definition 3. Let $\mathbb{K}$ be a default theory. A set of formulas $E = \mathsf{Det}(\sigma) \subseteq \mathscr{L}$ is a *default extension* of $\mathbb{K}$ iff σ is a proper process. We define skeptical (s) and credulous (c) *nonmonotonic inference* relations based on $\mathbb{K}$ as follows:

- $\mathbb{K} \mid\!\sim^s \varphi$ whenever for each default extension E of $\mathbb{K}$ we have $\varphi \in E$.
- $\mathbb{K} \mid\!\sim^c \varphi$ whenever there is a default extension E of $\mathbb{K}$ such that $\varphi \in E$.

Example 2. [Ex. 1 cont.] Given $\mathscr{F} = \{w\}$, we have two proper processes, $\sigma_1 = \langle (w,h), (h,o) \rangle$ and $\sigma_2 = \langle (w, \neg o), (w,h) \rangle$, yielding the extensions $E_1 = \mathsf{Cn}(\{w,h,o\})$ and $E_2 = \mathsf{Cn}(\{w,h,\neg o\})$. So, $\mathbb{K} \mid\!\sim^s h$, $\mathbb{K} \mid\!\sim^s w$, but $\mathbb{K} \not\mid\!\sim^s \neg o$. Similarly, $\mathbb{K} \mid\!\sim^c o, \neg o, h$, but $\mathbb{K} \not\mid\!\sim^c o \wedge \neg o$ since no extension concludes both o and $\neg o$. Given $\mathscr{F}' = \{w, o\}$, σ_2 is not a process since its detachable belief $\neg o$ contradicts the factual input $o \in \mathscr{F}'$. Since there is only one extension, skeptical and credulous inference coincide. We have $\mathbb{K} \mid\!\sim^s w \wedge h \wedge o$.

3. A Default Calculus

We now introduce a sequent-style proof calculus for Default Logic, called Default Calculus (DC). Sequent calculi are defined by sets of rules that stipulate under which conditions arguments (referred to as sequents) may be derived from other arguments [8]. An argument is of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite (possibly empty) sets of formulas and is read as "Γ provides support/reasons for concluding Δ" [6]. In order to enhance the modularity and transparency of our default reasoning, we use labels to differentiate the roles that formulas take in an argument: we use the *input* label 'i' for formulas given by the factual context and the *output* label 'o' for the formulas detached from input and defaults. Let $\mathscr{L}^i = \{\varphi^i \mid \varphi \in \mathscr{L}\}$ be the *input language* and $\mathscr{L}^o = \{\varphi^o \mid \varphi \in \mathscr{L}\}$ the *output language*. In our setting, the output formulas are beliefs, but in other contexts, such as deontic reasoning, they may be obligations [6]. We write $\Delta^x \subseteq \mathscr{L}^x$ to denote a set of x-labeled formulas (with $x \in \{i, o\}$). A *default argument* is an argument of the form

$$\mathbf{Ax}\ \frac{\vdash_{\mathsf{LK}} \Gamma \Rightarrow \Delta}{\Gamma^x \Rightarrow \Delta^x}\ \Gamma, \Delta \subseteq \mathscr{L} \qquad \mathbf{TP}\ \frac{}{\varphi^i \Rightarrow \varphi^o} \qquad \mathbf{SDet}\ \frac{\varphi^i, \Gamma \Rightarrow \psi^o}{\varphi^o, \Gamma \Rightarrow \psi^o}$$

$$\mathbf{FDet}\ \frac{}{\varphi^i, (\varphi, \psi) \Rightarrow \psi^o} \qquad \mathbf{Def}\ \frac{}{\varphi^i, \neg\psi^o \Rightarrow \neg(\varphi, \psi)} \qquad \mathbf{Cut}\ \frac{\Gamma_1 \Rightarrow \varphi^x \qquad \varphi^x, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Gamma_2 \Rightarrow \Delta}$$

Figure 1. Rules of DC (Def. 4). The rules **Ax**, **FDet**, and **TP** introduce initial sequents, and $x \in \{i, o\}$.

$$w^i, (w, \neg o) \Rightarrow \neg o^o$$

Recalling Ex. 1, the argument expresses that "we are licensed to believe that the windows are not open since it is winter (w^i) and winter is a reason to believe that the windows are not open $(w, \neg o)$." Furthermore, the use of defaults as objects of reasoning has the particular advantage that we can also construct arguments that conclude under which conditions certain defaults are *not applicable* (cf. [6]). We use a language of negated defaults $\overline{\mathscr{L}^\delta} = \{\neg(\varphi, \psi) \mid (\varphi, \psi) \in \mathscr{L}^\delta\}$ to express default inapplicability. The argument

$$w^i, o^o \Rightarrow \neg(w, \neg o)$$

states that "given that it is winter and the belief that the window is open, the default $(w, \neg o)$ is inapplicable" (since it implies contradictory beliefs). In DC, we also derive arguments like $w^i, (w, h), (h, o) \Rightarrow \neg(w, \neg o)$ stating that certain defaults jointly conclude the inapplicability of others. In defining DC, we use the language $\mathscr{L}^{\mathsf{DC}} = \mathscr{L}^i \cup \mathscr{L}^o \cup \mathscr{L}^\delta \cup \overline{\mathscr{L}^\delta}$ and assume the adequate sequent calculus LK for classical logic [8].

Definition 4 (Default Calculus (DC))**.** A DC-sequent is of the form $\Gamma \Rightarrow \Delta$ where $\Gamma \subseteq \mathscr{L}^{\mathsf{DC}}$ is a regular finite set and Δ is a set restricted to at most one element from $\mathscr{L}^{\mathsf{DC}}$. The calculus DC is defined by the rules **Ax**, **TP**, **FDet**, **SDet**, **Def**, **Cut** in Figure 1.

A DC-derivation of $\Gamma \Rightarrow \Delta$ is a tree-like structure where the leaves are initial sequents, whose root is $\Gamma \Rightarrow \Delta$, and whose rule-applications are instances of DC-rules. If $\Gamma \Rightarrow \Delta$ is DC-derivable, we write $\vdash_{\mathsf{DC}} \Gamma \Rightarrow \Delta$.

We briefly discuss the DC rules. All propositional formulas in DC-arguments are labeled i or o. The rule **Ax** takes labeled versions of LK-derivable sequent as initial sequent (hence, LK-rules are not required to be part of DC). **TP** ensures the property of identity, that is, all factual input is among the output. It is also referred to as "throughput." **FDet** expresses what is known as "factual detachment," stipulating how a fact φ^i and default (φ, ψ) enable the detachment of ψ^o. The rule **SDet** enables the "successive detachment" of formulas from defaults (see Ex. 3 for an illustration). The defeasible nature of default reasoning is captured by the rule **Def**, which expresses that in case a default δ is triggered (so, its body is derivable) but we have reason to believe that its head is false, then we should not apply δ. By concluding inapplicable defaults only on elementary default arguments, we assure that the defaults used as reasons in a defeating argument are properly triggered (this is illustrated in Ex. 3).[2] The rule **Cut** is defined as usual [8].

Example 3. Reconsider Ex. 1. Using DC, we may directly derive the following arguments from **FDet**: $a_1 : w^i, (w, \neg o) \Rightarrow \neg o^o$ and $a_2 : w^i, (w, h) \Rightarrow h^o$. Furthermore, we may derive the following complex argument using the **SDet** for successive detachment:

[2]This is the key difference to the calculus in [6] which allows, for instance, for arguments such as $\neg q^o, (p, q) \Rightarrow \neg(\top, p)$, where a non-triggered default provides a reason to not apply a triggered default.

$$\textbf{Cut}\ \dfrac{a_2 \qquad \textbf{SDet}\ \dfrac{h^i,(h,o) \Rightarrow o^o}{h^o,(h,o) \Rightarrow o^o}}{a_3 : w^i,(w,h),(h,o) \Rightarrow o^o}$$

We may also derive the inconsistent $a_4 : w^i,(w,h),(h,o),(w,\neg o) \Rightarrow \bot^o$. That these defaults cannot be jointly applied is expressed by the arguments $b_1 : w^i,(w,h),(h,o) \Rightarrow \neg(w,\neg o)$ and $b_2 : w^i,(w,\neg o),(w,h) \Rightarrow \neg(h,o)$. These arguments also show that the two processes of Ex. 2 are mutually exclusive in DC too. We show the derivation of b_2:

$$\textbf{Cut}\ \dfrac{\textbf{Cut}\ \dfrac{a_2 \qquad \textbf{Cut}\ \dfrac{a_1 \qquad \neg o^o, h^o, \Rightarrow (h\wedge\neg o)^o}{w^i,h^o,(w,\neg o) \Rightarrow (h\wedge\neg o)^o}}{w^i,(w,\neg o),(w,h) \Rightarrow (h\wedge\neg o)^o} \qquad \textbf{AND}^*\ \dfrac{\textbf{SDet}\ \dfrac{\textbf{Def}\ \dfrac{}{h^i,\neg o^o \Rightarrow \neg(h,o)}}{h^o,\neg o^o \Rightarrow \neg(h,o)}}{(h\wedge\neg o)^o \Rightarrow \neg(h,o)}}{b_2 : w^i,(w,\neg o),(w,h) \Rightarrow \neg(h,o)}$$

So far, all arguments are derivable for $\mathscr{F} = \{w\}$ and $\mathscr{F}' = \{w,o\}$ in Ex. 1. Using **TP**, in case of $\mathscr{F}'$ we may additionally derive the arguments $a_5 : w^i \Rightarrow w^o$, $a_6 : w^i,o^i \Rightarrow (w\wedge o)^o$, and the defeating argument $b_3 : w^i,o^i \Rightarrow \neg(w,\neg o)$ (which cannot be defeated).

The calculus in [6] allows for the derivation of $x : w^i,(w,\neg o),(h,o) \Rightarrow \neg(w,h)$, which is not allowed by default reasoning since it uses the untriggered (h,o) as a reason. That x is not DC-derivable follows from the soundness and completeness result in Prop. 1.

4. Formal Argumentation

DC generates two important types of argument: arguments that detach conclusions from (triggered) defaults and arguments that provide reasons for why, in light of logical consistency, some defaults are inapplicable in a given context. The second type of argument comprises the defeasible nature of default reasoning and imposes consistency restrictions on the process of detachment. We use both types to model conflicts in the knowledge base using formal argumentation. In brief, an argument of the form $\Delta \Rightarrow \neg(\varphi,\psi)$ attacks any argument $\Gamma,(\varphi,\psi) \Rightarrow \Sigma$ that uses the inapplicable (φ,ψ) as one of its reasons.

Formal argumentation, starting with the work of Dung [1], provides for an effective method for explicitly modeling conflicts in knowledge bases and has proven to be a unifying framework for the characterization of nm logics (e.g., see [2]). Its central concept is that of an argumentation framework $\mathscr{AF} = \langle \mathsf{Arg}, \mathsf{Att} \rangle$ which is a directed graph consisting of arguments $a,b,c,\ldots \in \mathsf{Arg}$ and an attack relation $\mathsf{Att} \subseteq \mathsf{Arg} \times \mathsf{Arg}$ holding between these arguments. Argumentation semantics deals with the identification of sets of arguments that are (under varying conditions) jointly defensible against attacks [9].

Below, we define the instantiation of $\mathscr{AF}$s with DC arguments, where attacks are based on default inapplicability. In constructing such an $\mathscr{AF}$ for a given knowledge base $\mathbb{K}$, we are only interested in those arguments that draw their support from $\mathbb{K}$.

Definition 5. Let DC be a calculus and $\mathbb{K} = \langle \mathscr{F}, \mathscr{D} \rangle$ a (labeled) knowledge base. A DC-instantiated argumentation framework $\mathscr{AF}(\mathbb{K}) = \langle \mathsf{Arg}, \mathsf{Att} \rangle$ is defined accordingly:

- $\Gamma \Rightarrow \Delta \in \mathsf{Arg}$ iff $\Gamma \Rightarrow \Delta$ is DC-derivable and $\Gamma \subseteq \mathscr{F} \cup \mathscr{D}$;
- $(a,b) \in \mathsf{Att}$ iff $a : \Gamma \Rightarrow \neg(\varphi,\psi) \in \mathsf{Arg}$ and $b : (\varphi,\psi),\Sigma \Rightarrow \Theta \in \mathsf{Arg}$.

Let $\mathsf{Arg}(\Sigma)$ denote the set of DC-derivable arguments $\Gamma \Rightarrow \Delta$ for which $\Gamma \subseteq \Sigma \subseteq \mathscr{L}^{\mathsf{DC}}$.

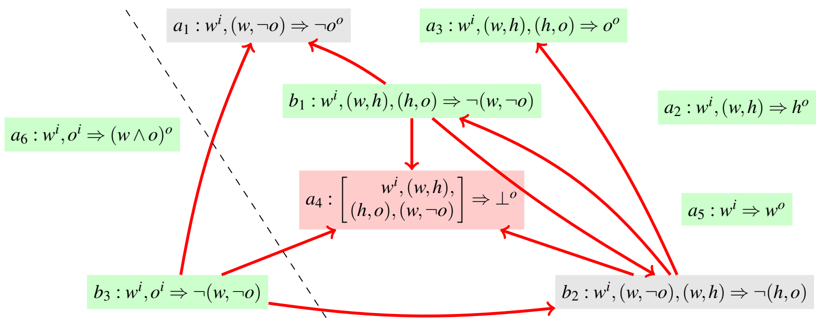

Figure 2. $\mathscr{AF}$ of DC-arguments from Ex. 3. The arrows denote attacks, e.g., the arrow from b_1 to a_1 denotes $(b_1, a_1) \in \mathsf{Att}$. The arguments for $\mathscr{F}$ and $\mathscr{F}'$ are $\mathsf{Arg} = \{a_1, a_2, a_3, a_4, a_5, b_1, b_2\}$, resp. $\mathsf{Arg}' = \mathsf{Arg} \cup \{a_6, b_3\}$.

Example 4. Fig. 2 contains the $\mathscr{AF}(\mathbb{K})$ induced by the DC-arguments from Ex. 3 (the framework is partial, which suffices for our purpose). The arguments for $\mathscr{F}$ and $\mathscr{F}'$ are $\mathsf{Arg} = \{a_1, a_2, a_3, a_4, a_5, b_1, b_2\}$, respectively $\mathsf{Arg}' = \mathsf{Arg} \cup \{a_6, b_3\}$ (differentiated by the dashed-line in Fig. 2). Arrows denote attacks, e.g., the arrow from b_1 to a_1 denotes $(b_1, a_1) \in \mathsf{Att}$. The arrow from b_3 to a_1, a_4 and b_2 only occur when $\mathscr{F}'$ is considered.

Our aim is to identify sets of arguments that yield an inference relation identical to entailment in Default Logic. The notion of *stable semantics* suffices for our purpose [2].

Definition 6. Let $\mathscr{AF} = \langle \mathsf{Arg}, \mathsf{Att} \rangle$ be an argumentation framework and let $\mathscr{A} \subseteq \mathsf{Arg}$:

- $\mathscr{A}$ is conflict-free if for each $(a,b) \in \mathsf{Att}$, if $a \in \mathscr{A}$, then $b \notin \mathscr{A}$.
- $\mathscr{A}$ is stable if $\mathscr{A}$ is conflict-free and $\forall b \in \mathsf{Arg} \backslash \mathscr{A}$, $\exists a \in \mathscr{A}$ such that $(a,b) \in \mathsf{Att}$.

We define credulous (c) and skeptic (s) nonmonotonic inference accordingly:

(1) $\mathscr{AF} \mid\!\sim^{c}_{\mathtt{stable}} \varphi$ iff there is a stable set $\mathscr{A}$ and $a = \Delta \Rightarrow \varphi \in \mathscr{A}$.
(2) $\mathscr{AF} \mid\!\sim^{s}_{\mathtt{stable}} \varphi$ iff for each stable set $\mathscr{A}$, there is an $a = \Delta \Rightarrow \varphi \in \mathscr{A}$.

Example 5 (Ex. 1 cont.). For $\mathscr{F}$, there are two stable extensions: $\mathscr{A}_1 = \{a_2, a_3, a_5, b_1\}$ (colored green in Fig. 2) and $\mathscr{A}_2 = \{a_1, a_2, a_5, b_2\}$. It is clear that the two extensions are mutually exclusive due to attacks between b_1 and b_2. Argument a_2 is part of each stable extension, and so $\mathscr{AF}(\mathbb{K}) \mid\!\sim^{s}_{\mathtt{stable}} h^o$ but $\mathscr{AF}(\mathbb{K}) \not\mid\!\sim^{s}_{\mathtt{stable}} o^o$ and $\mathscr{AF}(\mathbb{K}) \not\mid\!\sim^{s}_{\mathtt{stable}} \neg o^o$. We do have $\mathscr{AF}(\mathbb{K}) \mid\!\sim^{c}_{\mathtt{stable}} o^o$ and $\mathscr{AF}(\mathbb{K}) \mid\!\sim^{c}_{\mathtt{stable}} \neg o^o$. The inconsistent argument a_4 (colored red in Fig. 2) is part of no stable extension. In fact, $\mathscr{AF}(\mathbb{K}) \not\mid\!\sim^{c}_{\mathtt{stable}} (o \wedge \neg o)^o$.

For $\mathscr{F}'$ of Ex. 1, we include a_6 and b_3, where b_3 states that $(w, \neg o)$ is inapplicable w.r.t. $o \in \mathscr{F}'$. Since a_6 does not use any defaults, it cannot be attacked. So, there is only one stable extension $\mathscr{A}_3 = \{a_2, a_3, a_5, a_6, b_1, b_3\} \subseteq \mathsf{Arg}'$ and $\mathscr{AF}(\mathbb{K}) \mid\!\sim^{s}_{\mathtt{stable}} w^o, h^o, o^o$.

Ex. 5 shows that $\mathscr{AF}(\mathbb{K})$ gives the same outcome as the extension building method of Default Logic in Ex. 2. This is because the two approaches are sound and complete for each other. We prove this by a correspondence between default extensions and the propositional conclusions of DC-arguments in stable extensions (see technical appendix).

Proposition 1. *(1) For every default theory $\mathbb{K}$ and for every stable extension $\mathscr{A}$ of $\mathscr{AF}(\mathbb{K})$ there is a default extension E of $\mathbb{K}$ for which $E = \{\varphi \mid \Delta \Rightarrow \varphi^o \in \mathscr{A}\}$.*

(2) For every $\mathbb{K}$ and default extension E of $\mathbb{K}$ there is a stable extension $\mathscr{A}$ of $\mathscr{AF}(\mathbb{K})$ for which $E = \{\varphi \mid \Delta \Rightarrow \varphi^o \in \mathscr{A}\}$.

Corollary 1. $\mathbb{K} \mid\!\sim^s \varphi$ *iff* $\mathscr{AF}(\mathbb{K}) \mid\!\sim^s_{\texttt{stable}} \varphi^o$ *and* $\mathbb{K} \mid\!\sim^c \varphi$ *iff* $\mathscr{AF}(\mathbb{K}) \mid\!\sim^c_{\texttt{stable}} \varphi^o$.

5. Disjunctive Default Reasoning

Default Logic, as in [5], faces limitation when reasoning with disjunctions. For instance, if a default theory contains $\delta_1 = (\top, p \vee q)$, $\delta_2 = (p, u)$ and $\delta_3 = (q, u)$ we cannot conclude u on the basis of concluding $p \vee q$ from δ_1. In order to do so, we would have to *reason by cases*: By δ_1, we know that p is the case or q. In the first case, we can apply δ_2 to reason towards u, in the second case, we proceed analogously using δ_3. In this section, we set out to supplement default reasoning to reason more effectively with disjunctions. We do this in three steps: First, we define disjunctive Default Logic; we, then, generalize the default calculus; and, last, we show that the generalized calculus is sound and complete for the disjunctive Default Logic and, so, represents disjunctive default inference.

A natural way to generalize processes for disjunctive reasoning is to consider disjunctive triggering of sets of defaults. For instance, the set $\{\delta_2, \delta_3\}$ is triggered in the context of $(\top, p \vee q)$ since we can derive the disjunction of the bodies of δ_2 and δ_3 (namely, $p \vee q$). So, instead of letting processes be sequences of defaults, we consider sequences of (finite) sets of defaults $\langle \Xi_1, \ldots, \Xi_n \rangle$. Just like for processes (Def. 2), we demand that each member of the sequence is triggered in view of the previous members and, so, there is a simple way of checking the triggering condition: we only have to see whether $\mathsf{body}(\delta_{k+1}) \in \mathsf{Cn}(\mathscr{F} \cup \{\mathsf{head}(\delta_i) \mid 1 \leq i \leq k\})$. Leveraging this simple approach, we could simply check whether $\bigvee_{\delta \in \Xi_{k+1}} \mathsf{body}(\delta) \in \mathsf{Cn}(\mathscr{F} \cup \{\bigvee_{\delta \in \Xi_i} \mathsf{head}(\delta) \mid 1 \leq i \leq k\})$. However, caution is advised when considering examples like the following.

Example 6. Let $\mathbb{K} = \langle \mathscr{F}, \mathscr{D} \rangle$ with $\mathscr{D} = \{(p,u), (q,v), (p,v), (q,u), (u,s)\}$ and $\mathscr{F} = \{p \vee q\}$. Consider $\sigma = \langle \Xi_1 = \{(p,u), (q,v)\}, \Xi_2 = \{(p,v), (q,u)\} \rangle$. Should (u,s) be considered as triggered by σ? We have two options: (a) we don't consider it triggered since $u \notin \mathsf{Cn}(\{p \vee q, u \vee v\})$, or (b) we consider it triggered since $\bigcup \sigma$ already contains the defaults to obtain u, namely (p,u) and (q,u). So, implicitly, a reasoner committing to σ already has an argument for u and should commit to it.

To highlight the difference between (a) and (b) consider adding $\neg u$ to $\mathscr{F}$ resulting in $\mathbb{K}' = \langle \mathscr{F}', \mathscr{D} \rangle$. Should we consider σ consistent in $\mathbb{K}'$? According to (a) one may argue, yes, since the inconsistency is not apparent when considering $\mathsf{Cn}(\{p \vee q, u \vee v, v \vee u\})$. According to (b) one may argue, no, since the inconsistency is implicit in σ: any way of completing σ (e.g., by extending it with $\Xi_3 = \{(p,u), (q,u)\} \subseteq \bigcup \sigma$) makes the inconsistency apparent; a reasoner committing to σ is already on lost ground w.r.t. consistency.

In what follows, we approach default reasoning from the perspective of (b), according to which we consider what can be inferred from $\sigma = \langle \Xi_1, \ldots, \Xi_n \rangle$ by combining defaults present in $\bigcup \sigma$. It gives rise to the following generalized definitions of Section 2.

Definition 7. Let $\mathbb{K} = \langle \mathscr{F}, \mathscr{D} \rangle$ be a default theory and $\sigma = \langle \Xi_i \rangle_{i=1}^n$ be a sequence of finite $\Xi_i \subseteq \mathscr{D}$. We let $\mathsf{DetD}_{\mathscr{F}}(\sigma) = \bigcup_{i \geq 0} \mathsf{DetD}^i_{\mathscr{F}}(\sigma)$, where $\mathsf{DetD}^0_{\mathscr{F}}(\sigma) = \mathsf{Cn}(\mathscr{F})$ and

$$\mathsf{DetD}^{i+1}_{\mathscr{F}}(\sigma) = \mathsf{Cn}(\mathscr{F} \cup \{\bigvee_{\delta \in \Xi} \mathsf{head}(\delta) \mid \Xi \subseteq \bigcup \sigma, \bigvee_{\delta \in \Xi} \mathsf{body}(\delta) \in \mathsf{DetD}^i_{\mathscr{F}}(\sigma)\})$$

Let $\mathsf{trig}_{\mathscr{F}}(\sigma) = \{\Xi \subseteq \mathscr{D} \mid \bigvee_{\delta \in \Xi} \mathsf{body}(\delta) \in \mathsf{DetD}_{\mathscr{F}}(\sigma)\}$ (we omit reference to $\mathscr{F}$).

Example 7 (Ex. 6 cont.)**.** We have $\mathsf{DetD}^0_{\mathscr{F}} = \mathsf{Cn}(\{p \vee q\})$, $\mathsf{DetD}^1_{\mathscr{F}} = \mathsf{Cn}(\{p \vee q, u \vee v\})$ and $\mathsf{DetD}^2_{\mathscr{F}} = \mathsf{Cn}(\{p \vee q, u, v\})$. Note that $p \vee q \in \mathsf{DetD}^1_{\mathscr{F}}(\sigma)$ and $\{(p,u),(q,u)\}$, $\{(p,v),(q,v)\} \subseteq \bigcup \sigma$ So, $u, v \in \mathsf{DetD}(\sigma)$. For this reason, $(u,s) \in \mathsf{trig}(\sigma)$.

Definition 8. A sequence $\sigma = \langle \Xi_i \rangle_{i=1}^n$ is a $\vee$-*process* if for all $1 \leq i < n$, $\Xi_{i+1} \in \mathsf{trig}(\langle \Xi_j \rangle_{j=1}^i)$. σ is *proper* if

1. σ is *complete*: for all $\Theta \in \mathsf{trig}(\sigma)$, if $\Theta \setminus \bigcup \sigma \neq \emptyset$ then there is a $\delta \in \Theta \setminus \bigcup \sigma$ for which $\neg\mathsf{head}(\delta) \in \mathsf{DetD}(\sigma)$; and
2. σ is *consistent*: for all $\delta \in \bigcup \sigma$, $\neg\mathsf{head}(\delta) \notin \mathsf{DetD}(\sigma)$.

$E = \mathsf{DetD}(\sigma)$ is a *disjunctive default extension* of $\mathbb{K}$ iff σ is a proper $\vee$-process of $\mathbb{K}$.

Example 8 (Ex. 7 cont.)**.** σ is consistent in $\mathbb{K}$ but not proper since $\{(u,s)\} \in \mathsf{trig}(\sigma)$ and $(u,s) \notin \bigcup \sigma$, while $\neg\mathsf{head}((u,s)) \notin \mathsf{DetD}(\sigma)$. A proper $\vee$-process is $\sigma' = \sigma \circ \{(u,s)\}$.

Example 9. Consider a disjunctive generalization of Ex. 1 where p denotes "permission is asked" and r denotes "the radiator is on." We define $\vee\mathbb{K}$ as $\mathscr{F} = \{w, o\}$ and $\mathscr{D} = \{(w,\neg o),(w,h \vee r),(h,o),(r,\neg o),(h,p),(r,p)\}$. The disjunctive default $(w, h \vee r)$ states that the winter provides reasons to believe that the hearth or the radiator is on. When the radiator is on, the window is not likely to be open $(r,\neg o)$. The hearth or radiator being on, provides reason to believe that permission has been asked, i.e., (h,p) and (r,p). We have, e.g., the following proper $\vee$-process: $\sigma_1 = \langle \{(w,h \vee r)\}, \{(h,p),(r,p)\} \rangle$. Note that $\{(w,\neg o)\} \in \mathsf{trig}(\sigma_1)$, but $\neg o \in \mathsf{DetD}(\sigma_1)$ since $\neg o \in \mathscr{F}$. Similarly, while $\Xi = \{(h,o),(r,\neg o)\} \in \mathsf{trig}(\sigma_1)$ but $\Xi \not\subseteq \bigcup \sigma_1$. However, $\neg\neg o = \neg\mathsf{head}((r,\neg o)) \in \mathsf{DetD}(\sigma_1)$.

Now consider the knowledge $\vee\mathbb{K}'$ for which we remove o from the facts. Then, for instance, $\sigma_2 = \sigma_1 \circ \{(w,\neg o)\}$ is a proper $\vee$-process. There is, in fact, no proper $\vee$-process that does not contain $(w,\neg o)$ since there is no way to 'defeat' it by inferring $\neg\neg o$.

Remark 1. Input/Output (I/O) logic [7] offers another account of reasoning disjunctively with defaults (e.g., as norms). The idea is to close a default theory under meta-rules that generate new defaults. I/O disjunctive reasoning lets one combine two defaults, e.g., (φ_1, ψ) and (φ_2, ψ), to yield a new default in which the disjunction of the conclusions is obtained, e.g., $(\varphi_1 \vee \varphi_2, \psi)$. Other meta-rules are, e.g., right weakening (infer (φ, ψ) from (φ, ψ'), if $\psi' \vdash \psi$) and aggregation (infer $(\varphi, \psi_1 \wedge \varphi_2)$ from (φ, ψ_1) and (φ, ψ_2)).

For instance, given $\mathbb{K}$ with $\mathscr{F} = \{p \vee q\}$ and $\mathscr{D} = \{(p,s),(q,u),(p \vee q,\neg s)\}$, we obtain the default $(p \vee q, s \vee u)$. By aggregation with $(p \vee q, \neg s)$ and right weakening we then derive $(p \vee q, u)$. So, u can be detached. The outcome seems rather strong to us. Although $(p \vee q, \neg s)$ provides a good reason not to apply (p,s) when reasoning by cases on the basis of $p \vee q$, it does not provide an additional reason to still apply (q,u). For this one would have to reason contrapositively back from $\neg s$, via (p,s), to $\neg p$ which by disjunctive syllogism on $(p \vee q)$ would yield q, on the basis of which we could apply (q,u). This is why we opt for a more cautious form of disjunctive default reasoning.

A similar situation occurs in $\mathbb{K}'$ with $\mathscr{F} = \{p \vee q, \neg s\}$ and $\mathscr{D} = \{(p,s),(s,u),(q,t),(t,u)\}$. In I/O logic we obtain a default $(p \vee q, u)$ and can reason towards u. In our approach the reasoning path $(p,s),(s,u)$ is blocked due to the conflict with $\neg s$. In fact, $\sigma = \emptyset$ is the only proper $\vee$-process for $\mathbb{K}'$. Note that the only ($\subset$-minimal) set triggered by $\emptyset$ is $\Xi = \{(p,s),(q,t)\}$. However, $\neg s \in \mathsf{DetD}(\emptyset)$ and so it cannot be applied.

In order to accommodate disjunctive reasoning in an extension of the default calculus DC, we generalize two rules. First, we allow for disjunctive detachment via the rule:

$$\frac{}{(\bigvee_{i=1}^{n}\varphi_i)^i, \langle(\varphi_i,\psi_i)\rangle_{i=1}^{n} \Rightarrow (\bigvee_{i=1}^{n}\psi_i)^o}\ \textbf{DisDet}$$

For instance, in the introductory example of this section we can generate $(p \vee q)^i, (p,u), (q,u) \Rightarrow u^o$. We note that **FDet** is a special instance of **DisDet** for which $n = 1$.

We also adjust the rule **Def** managing argumentative defeat to be the following:

$$\frac{}{(\bigvee_{i=1}^{n}\varphi_i)^i, \langle(\varphi_i,\psi_i)\rangle_{i=2}^{n}, \neg\psi_1^o \Rightarrow \neg(\varphi_1,\psi_1)}\ \textbf{DisDef}$$

The rule expresses that a set of defaults Ξ should not be jointly applied if for one of its members $\delta \in \Xi$ we have $\neg\mathsf{head}(\delta)$.

Definition 9 (Disjunctive Default Calculus (DDC))**.** The calculus DDC is defined by the rules **Ax**, **TP**, **SDet**, **Cut**, **DisDet**, **DisDef**.

Example 10 (Ex. 9 cont.)**.** We model our previous example with DDC. Fig. 3 depicts some arguments and attack relations between them. Let us demonstrate a derivation:

$$\dfrac{\dfrac{}{w^i,(w,\neg o)\Rightarrow\neg o^o}\,\textbf{DisDet} \qquad \dfrac{\dfrac{}{w^i,(w,h\vee r)\Rightarrow(h\vee r)^o}\,\textbf{DisDet} \qquad \dfrac{\dfrac{}{(h\vee r)^i,(r,\neg o),\neg o^o\Rightarrow\neg(h,o)}\,\textbf{DisDef}}{(h\vee r)^o,(r,\neg o),\neg o^o\Rightarrow\neg(h,o)}\,\textbf{SDet}}{w^i,(w,h\vee r),(r,\neg o),\neg o^o\Rightarrow\neg(h,o)}\,\textbf{Cut}}{w^i,(w,h\vee r),(w,\neg o),(r,\neg o)\Rightarrow\neg(h,o)}\,\textbf{Cut}$$

There is one stable extension $\{a_1,a_2,a_3,b_1\}$ and, e.g., $\mathscr{AF}(\vee\mathbb{K}) \mid\!\sim^{s}_{\mathtt{stable}} p^o$.

DDC and stable semantics adequately represent disjunctive Default Logic (Prop. 2 is an immediate consequence of Prop. 3 and 4 in the Appendix.):

Proposition 2. *Let $\mathscr{AF}(\mathbb{K})$ be the argumentation framework induced by* DDC.

(1) For every stable extension $\mathscr{A}$ of $\mathscr{AF}$ there is a disjunctive default extension E of $\mathbb{K}$ for which $E = \{\varphi \mid$ there is $\Delta \Rightarrow \varphi^o \in \mathscr{A}\}$.

(2) For every disjunctive default extension E of $\mathbb{K}$ there is a stable extension $\mathscr{A}$ of $\mathscr{AF}$ for which $E = \{\varphi \mid$ there is $\Delta \Rightarrow \varphi^o \in \mathscr{A}\}$.

Let $\mid\!\sim^{c,\vee}$ [$\mid\!\sim^{s,\vee}$] be the credulous [skeptical] consequence relation induced by disjunctive Default Logic (analogous to Def. 3). Let $\mid\!\sim^{c,\vee}_{\mathtt{stable}}$ [$\mid\!\sim^{s,\vee}_{\mathtt{stable}}$] be the credulous [skeptical] consequence relation induced by stable semantics and DDC (analogous to Def. 6).

Corollary 2. $\mathbb{K} \mid\!\sim^{s,\vee} \varphi$ *iff* $\mathscr{AF}(\mathbb{K}) \mid\!\sim^{s,\vee}_{\mathtt{stable}} \varphi^o$ *and* $\mathbb{K} \mid\!\sim^{c,\vee} \varphi$ *iff* $\mathscr{AF}(\mathbb{K}) \mid\!\sim^{c,\vee}_{\mathtt{stable}} \varphi^o$.

6. Related and Future Work

There are several argumentative [1,10,11] and proof-theoretic accounts [12] of Default Logic. Our approach is different due to its focus on proof-theoretic modularity and unification. Our approach, based on a characterization of Input/Output logic in [6], adopts

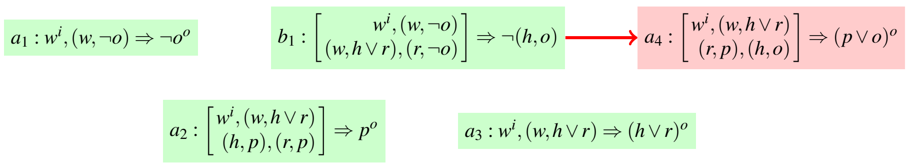

Figure 3. Partial Argumentation Framework induced by $\vee$DC-arguments for $\vee\mathbb{K}'$ of Ex. 10.

modifications to the defeat rule in [6] to represent Default Logic. To obtain a disjunctive variant of Default Logic, a generalization of the detachment and defeat rules sufficed. Our approach improves our understanding of the differences between paradigmatic methods in default reasoning, identifying exactly in which inference rules they differ.

While [12] handles inconsistency within the calculus, we outsource this to argumentation theory. The upshot is our focus on a transparent representation of the rationale underlying argumentative defeat: an argument $\Gamma \Rightarrow \neg(\alpha, \beta)$ provides reasons in view of which a default (α, β) should not be applied. Explicit reasons can be used for explanations (see [6]) and for integrating defeasible reasoning within the calculus (see [13]).

Reasoning disjunctively with defaults is still an open problem. We remind the reader of Remark 1 for a cautious note on disjunctive reasoning in Input/Output logic. Based on Reiter's approach, [14] provide an account which is highly syntax-dependent and is based on a specific way of building extensions. In contrast, our approach roots disjunctive reasoning in sequent-based inference rules. In this way we allow for the future exploration of variations of disjunctive reasoning. Additionally, we aim to obtain default calculi without the rule **TP** (i.e., identity) to capture deontic default reasoning. We additionally plan to include reasoning with priorities [11,15], specificity [16] and apply methods of causal reasoning [17], as well as the integration of different base logics [13].

In sum, this paper provides a unificatory framework for Default Logic in logical argumentation. The upshot of our approach is (i) the use of proof-calculi, (ii) defaults as explicit objects of reasoning, (iii) its uniformity and modularity, making the formalism promising for extensions of default reasoning, such as disjunctive and deontic reasoning.

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A. Appendix: Meta-Theory

In the context of the following lemmas we suppose an arbitrary knowledge base $\mathbb{K} = \langle \mathscr{F}, \mathscr{D} \rangle$, $\vdash$ to denote the derivability relation of DDC, and $\vdash_{\mathbf{CL}}$ to denote the derivability relation of classical logic (the subscript is omitted where the context disambiguates). We will use Θ, Ξ and Ω as placeholders for sets of defaults and δ as a placeholder for defaults in $\mathscr{D}$. For reasons of space some proofs will be only sketched or omitted.

Lemma 1.
1. *If* $\vdash \Gamma^i, \Delta^o, \Theta^n \Rightarrow \varphi^i$ *then* $\Delta \cup \Theta = \emptyset$.
2. $\vdash \Gamma'^i, \Delta'^o, \Theta' \Rightarrow \neg \mathsf{head}(\delta)^o$ *for some* $\Theta' \subseteq \Theta, \Gamma' \subseteq \Gamma, \Delta' \subseteq \Delta$, *if* $\vdash \Gamma^i, \Delta^o, \Theta \Rightarrow \neg\delta$.
3. *Let* $\vdash \Gamma^i, \Delta^o, \Xi \Rightarrow \neg\delta_2$ *and* $\delta_1 \in \Xi$. *Then there are* $x \in \{o, i\}$, $\Omega \subseteq \Xi \cup \{\delta_2\}$, $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$, *and* $\Xi' \subseteq \Xi$ *for which* $\delta_1 \in \Omega$ *and* $\vdash \Gamma'^i, \Delta'^o, \Xi' \setminus \Omega \Rightarrow (\bigvee_{\delta \in \Omega} \mathsf{body}(\delta))^x$.
4. *Let* $\vdash \Gamma^i, \Delta^o, \Xi \Rightarrow \psi^o$ *or* $\vdash \Gamma^i, \Delta^o, \Xi \Rightarrow$, *and* $\delta_1 \in \Xi$. *Then there are* $x \in \{o, i\}$, $\Omega \subseteq \Xi$, $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$, $\Xi' \subseteq \Xi$ *s.t.* $\delta_1 \in \Omega$ *and* $\vdash \Gamma'^i, \Delta'^o, \Xi' \setminus \Omega \Rightarrow (\bigvee_{\delta \in \Omega} \mathsf{body}(\delta))^x$.

Proof. Each proof is by induction on the length of the proof of the resp. sequents. □

Lemma 2.
1. *Let* σ *be a* $\vee$*-process for* $\mathbb{K}$ *such that* $\varphi \in \mathsf{DetD}(\sigma)$. *Then* $\vdash \Gamma^i, \Theta \Rightarrow \varphi^o$ *for some* $\Theta \subseteq \bigcup \sigma$ *and* $\Gamma \subseteq \mathscr{F}$. *If* $\mathscr{F} \vdash \varphi$, $\vdash \Gamma^i \Rightarrow \varphi^i$ *for some* $\Gamma \subseteq \mathscr{F}$.
2. *If* $\vdash \Gamma^i, \Delta^o, \Theta \Rightarrow \varphi^o$, *t.i. a* $\vee$*-process* σ *with* $\varphi \in \mathsf{DetD}_{\Gamma \cup \Delta}(\sigma)$ *and* $\bigcup \sigma = \Theta$.

Proof. The proof is by induction on the length of σ resp. the proof of $\Gamma^i, \Delta^o, \Theta \Rightarrow \varphi^o$. □

Lemma 3. *Let* $\mathscr{A}$ *be a stable ext. of* $\mathscr{AF}(\mathbb{K})$. *Then,* $\mathscr{A} = \mathsf{Arg}(\mathscr{F} \cup \bigcup_{a \in \mathscr{A}} \mathsf{defaults}(a))$.

Proof. Let $\mathscr{P} = \bigcup_{a \in \mathscr{A}} \mathsf{defaults}(a)$. Assume that $a \in \mathsf{Arg}(\mathscr{F} \cup \mathscr{P}) \setminus \mathscr{A}$. Thus, there is a $b \in \mathscr{A}$ that attacks a. So, $\mathsf{Con}(b) = \neg\delta$ for some $\delta \in \mathsf{defaults}(a)$. So, $\delta \in \mathscr{P}$ and there is a $c \in \mathscr{A}$ for which $\delta \in \mathsf{defaults}(c)$. But since b also attacks c this is a contradiction to the conflict-freeness of $\mathscr{A}$. So, $\mathsf{Arg}(\mathscr{F} \cup \mathscr{P}) \subseteq \mathscr{A}$. The other direction is trivial. □

Proposition 3. *Let* $\mathbb{K} = \langle \mathscr{F}, \mathscr{D} \rangle$, $\mathscr{A}$ *be a stable extension of* $\mathscr{AF}(\mathbb{K})$ *and* $\mathscr{A}^o = \{a \in \mathscr{A} \mid \mathsf{Con}(a) \in \mathscr{L}^o\}$. *There is a proper* $\vee$*-process for* $\mathbb{K}$ *for which* $\mathsf{Con}[\mathscr{A}^o] = \mathsf{DetD}(\sigma)$.

Proof. Let $\mathscr{P} = \mathsf{defaults}[\mathscr{A}^o]$ and let $\mathscr{A}^o_\top = \{a \in \mathscr{A}^o \mid \mathsf{Con}(a) = \top^o\}$. Since $\mathscr{D}$ is finite, $\mathscr{A}^o_\top$ is finite. Let $\mathscr{A}^o_\top = \{a_1, \ldots, a_m\}$. Also, $\mathsf{defaults}[\mathscr{A}^o] = \mathsf{defaults}[\mathscr{A}^o_\top]$ since, in view of Lem. 3, $\Gamma^i, \Theta \Rightarrow \varphi^o \in \mathscr{A}^o$ implies that $\Gamma^i, \Theta \Rightarrow \top^o \in \mathscr{A}^o_\top$. By Lem. 2.2, for each a_i there is a $\vee$-process $\sigma_i = \langle \Xi^i_j \rangle_{j=1}^{g_i}$ with $\bigcup \sigma_i = \mathsf{defaults}(a_i)$. Let $\sigma^i = \langle \Xi^j_i \rangle_{j=1}^m$ for each $1 \leq i \leq \max(\{g_i \mid 1 \leq i \leq m\})$, where we set $\Xi^i_k = \emptyset$ if $k > g_i$. We let $\sigma = \sigma^1 \circ \ldots \circ$

$\sigma^{\max(\{g_i \mid 1 \leq i \leq m\})}$. Since every σ_j is a $\vee$-process, it is easy to see that so is σ. Note that by the construction $\bigcup \sigma = \mathscr{P}$. We now show that σ is a proper $\vee$-process.

For *consistency* assume towards a contradiction that $\bot \in \mathsf{DetD}(\sigma)$. By Lem. 2.1, there is an $a \in \mathsf{Arg}(\mathbb{K})$ with $\mathsf{defaults}(a) \subseteq \mathscr{P}$ and $\mathsf{Con}(a) = \bot^o$. By Lem. 3, $a \in \mathscr{A}$. By Lem. 2.2, there is a process $\sigma' = \langle \Omega_i \rangle_{i=1}^m$ for which $\mathsf{defaults}(a) = \bigcup_{i=1}^m \Omega_i$ and $\bot \in \mathsf{DetD}(\sigma')$. So, Ω_1 is such that $\mathscr{F} \vdash \bigvee_{\delta \in \Omega_1} \mathsf{body}(\delta)$. So, there is an argument $b = \Gamma^i \Rightarrow (\bigvee_{\delta \in \Omega_1} \mathsf{body}(\delta))^i \in \mathscr{A}$. Let $\delta \in \Omega_1$ arbitrary. By **Ax** and Lem. 3, $\vdash \bot^o \Rightarrow \neg\mathsf{head}(\delta)^o$. We apply **Cut** to a, to derive $a' = \mathsf{facts}(a), \mathsf{defaults}(a) \Rightarrow \neg\mathsf{head}(\delta)^o$. By Lem. 3, $a' \in \mathscr{A}$. By **DisDef**, $\vdash (\bigvee_{\delta \in \Omega_1} \mathsf{body}(\delta))^i, \Omega_1 \setminus \delta, \neg\mathsf{head}(\delta)^o \Rightarrow \neg\delta$. By **Cut** with b and a' we obtain $c = \mathsf{facts}(a), \Gamma^i, \mathsf{defaults}(a), \Omega_1 \setminus \delta \Rightarrow \neg\delta$. By Lem. 3, $c \in \mathscr{A}$. Since c attacks a this is in contradiction to the conflict-freeness of $\mathscr{A}$.

We now show that σ is *proper*. Let for this $\Xi \in \mathsf{trig}(\sigma)$ such that $\Xi \setminus \bigcup \sigma \neq \emptyset$. By Lem. 2.1, $\vdash \Gamma^i, \Theta \Rightarrow (\bigvee_{\delta \in \Xi} \mathsf{body}(\delta))^x$ for some $\Theta \subseteq \mathscr{P}$, $\Gamma \subseteq \mathscr{F}$ and $x \in \{i, o\}$. By **DisDet** and **DDet**, $\vdash (\bigvee_{\delta \in \Xi} \mathsf{body}(\delta))^i, \Xi \Rightarrow (\bigvee_{\delta \in \Xi} \mathsf{head}(\delta))^y$ for $y \in \{o, i\}$. By **Cut**, we derive $a = \Gamma^i, \Theta, \Xi \Rightarrow (\bigvee_{\delta \in \Xi} \mathsf{head}(\delta))^o$. Note that $a \notin \mathscr{A}$. By the stability of $\mathscr{A}$, there is a $b \in \mathscr{A}$ that attacks a. So, $\mathsf{Con}(b) = \neg\delta$ for $\delta \in \mathsf{defaults}(a)$. By the conflict-freeness of $\mathscr{A}$, $\delta \in \Xi \setminus \bigcup \sigma$. By Lem. 1.2 and 3, there is a $c \in \mathscr{A}$ with $\mathsf{Con}(c) = \neg\mathsf{head}(\delta)^o$. In view of Lem. 2.2, $\neg\mathsf{head}(\delta) \in \mathsf{DetD}(\sigma)$.

Let now $\delta \in \mathscr{D}$ with $\neg\mathsf{head}(\delta) \in \mathsf{DetD}(\sigma)$. Assume $\delta \in \mathscr{P}$. So, there is an argument $b = \Gamma^i, \Xi \Rightarrow \top^o \in \mathscr{A}$ with $\delta \in \Xi$. Using the fact that $\neg\mathsf{head}(\delta) \in \mathsf{DetD}(\sigma)$ and Lem. 2 we construct an attacker of b in $\mathscr{A}$, in contradiction to the conflict-freeness of $\mathscr{A}$.

The fact that $\mathsf{Con}[\mathscr{A}^o] = \mathsf{DetD}(\sigma)$ follows directly in view of Lem. 2. □

Proposition 4. *Let σ be a proper $\vee$-process for $\mathbb{K} = \langle \mathscr{F}, \mathscr{D} \rangle$ and let $\Omega = \{\delta \in \mathscr{D} \mid \neg\mathsf{head}(\delta) \in \mathsf{DetD}(\sigma)\}$. Then $\mathscr{A} = \mathsf{Arg}(\mathscr{F} \cup (\mathscr{D} \setminus \Omega))$ is a stable set in $\mathscr{AF}(\mathbb{K})$ for which $\mathsf{Con}[\mathscr{A}^o] = \mathsf{DetD}(\sigma)$, where $\mathscr{A}^o = \{a \in \mathscr{A} \mid \mathsf{Con}(a) \in \mathscr{L}^o\}$.*

Proof. We omit the proof of conflict-freeness. For *stability* consider an $a \in \mathsf{Arg}(\mathbb{K}) \setminus \mathscr{A}$. WLOG, we assume that $\mathsf{defaults}(a)$ is $\subset$-minimal with the property of being in $\mathsf{Arg}(\mathbb{K}) \setminus \mathscr{A}$. So, there is a $\delta \in \mathsf{defaults}(a) \cap \Omega$. We show the case in which a has a conclusion of the form $\neg\delta_2$. By Lem. 1.3, there are $\Omega' \subseteq \mathsf{defaults}(a) \cup \{\delta_2\}$, $\Gamma^i \subseteq \mathsf{facts}(a)$, $\Xi \subseteq \mathsf{defaults}(a)$ for which $\delta \in \Omega'$ and $\vdash b$ with $b = \Gamma^i, \Xi \setminus \Omega' \Rightarrow (\bigvee_{\delta' \in \Omega'} \mathsf{body}(\delta'))^x$ for $x \in \{i, o\}$. By the minimality of a, $b \in \mathscr{A}$. Since $\delta \in \Omega$, $\neg\mathsf{head}(\delta) \in \mathsf{DetD}(\sigma)$. By Lem. 2.1 and since $\bigcup \sigma \subseteq (\mathscr{D} \setminus \Omega)$, there is a $c \in \mathscr{A}$ with $\mathsf{Con}(c) = \neg\mathsf{head}(\delta)^o$. By **DisDef** and **DDet**, $\vdash (\bigvee_{\delta' \in \Omega'} \mathsf{body}(\delta'))^y, \neg\mathsf{head}(\delta)^o, \Omega' \setminus \{\delta\} \Rightarrow \neg\delta$ for $y \in \{o, i\}$. By applying **Cut**s with b and c we obtain $d \in \mathscr{A}$ with $\mathsf{Con}(d) = \neg\delta$ which attacks a.

We now show that $\mathscr{A}^o = \{a \in \mathscr{A} \mid \mathsf{Con}(a) \in \mathscr{L}^o\}$. Let $\varphi \in \mathsf{DetD}(\sigma)$. We note that $\bigcup \sigma \subseteq \mathscr{D} \setminus \Omega$. By Lem. 2.1 there is a $a \in \mathscr{A}$ with $\mathsf{Con}(a) = \varphi^o$. The other direction is also shown with the help of Lem. 2. □

Prop. 2 follows from Prop. 3 and 4. Prop. 1 relies on proofs analogous to those of Prop. 3 and 4 for DC and non-disjunctive Default Logic. Cor. 1 and 2 are represented in Cor. 3, where items (iii) and (iv) follow from Prop. 3 and 4. Items (i) and (ii) again rely on analogous versions of Prop. 3 and 4 for DC and non-disjunctive Default Logic.

Corollary 3. *Let $\mathbb{K}$ be a knowledge base. Then,*

(i) $\mathbb{K} \mid\!\sim^s \varphi$ *iff* $\mathscr{AF}(\mathbb{K}) \mid\!\sim^s_{\mathtt{stable}} \varphi^o$; *(ii)* $\mathbb{K} \mid\!\sim^c \varphi$ *iff* $\mathscr{AF}(\mathbb{K}) \mid\!\sim^c_{\mathtt{stable}} \varphi^o$.

(iii) $\mathbb{K} \mid\!\sim^{s,\vee} \varphi$ *iff* $\mathscr{AF}(\mathbb{K}) \mid\!\sim^{s,\vee}_{\mathtt{stable}} \varphi^o$; *(iv)* $\mathbb{K} \mid\!\sim^{c,\vee} \varphi$ *iff* $\mathscr{AF}(\mathbb{K}) \mid\!\sim^{c,\vee}_{\mathtt{stable}} \varphi^o$.