In: Proceedings of the Sixth Italian Workshop on Neural Networks WIRN VIETRI-93, E. Caianiello (Ed.), World Scientific, Singapore, 1994, pages 241–246.

On the Size of Weights for McCulloch-Pitts Neurons

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ABSTRACT

The maximum absolute value of integral weights sufficient to represent any linearly separable Boolean function is investigated. It is shown that upper bounds exhibited by Muroga (1971) for rational weights satisfying the so-called normalized system of inequalities also hold for integral weights. This improves previously known upper bounds.

1 Introduction

The logical behavior of a McCulloch-Pitts neuron with n binary inputs is described by a linearly separable Boolean function $f : \{0,1\}^n \to \{0,1\}$ (see e.g. [1]). It is represented by a real vector $(w_1, \ldots, w_n, t)$ of weights such that for all $x \in \{0,1\}^n$

$$w_1 x_1 + \cdots + w_n x_n \geq t \qquad \text{iff} \qquad f(x_1 \ldots x_n) = 1. \tag{1}$$

Function f is also called threshold function and t is known as the threshold. It is a well-known fact that the possibly infinite information contained in the real components of the weight vector can be made finite without restricting the class of representable functions by requiring all weights to be integers (see e.g. [6, 8, 12] for proofs). In the past, there has been considerable interest to bound the maximum absolute integral value sufficient for the weights from above for various reasons. Predominant was the search for a polynomial upper bound on the length of a weight in binary representation leading to $O(n \log n)$ [5, 11, 12, 13]. The tightest result has been given by Muroga [9]. Investigating weight vectors satisfying the so-called normalized system of inequalities

$$\begin{array}{lcll} w_1 x_1 + \cdots + w_n x_n & \geq & t & \text{if} \quad f(x) = 1 \\ w_1 x_1 + \cdots + w_n x_n & \leq & t-1 & \text{if} \quad f(x) = 0 \end{array} \tag{2}$$

he obtained the following results [9, Section 9.3.2]:

- Every linearly separable Boolean function $f : \{0,1\}^n \to \{0,1\}$ can be represented by integral weights satisfying

$$|w_i| \leq 2^{-n}(n+1)^{(n+1)/2}, \qquad |t| \leq 2^{-n} n (n+1)^{(n+1)/2}. \tag{3}$$

- If rational numbers are permitted then there exist weights satisfying (2) and

$$|w_i| \leq 2^{-n}(n+1)^{(n+1)/2}, \qquad |t| \leq 2^{-n-1}(n+1)^{(n+3)/2} + 1/2. \tag{4}$$

- Furthermore, if we denote the truth value 0 by -1 then for a linearly separable function $g : \{-1,1\}^n \to \{-1,1\}$ rational weights can be found such that (2) and

$$|w_i| \leq 2^{-n}(n+1)^{(n+1)/2}, \qquad |t| \leq 2^{-n}(n+1)^{(n+1)/2} \tag{5}$$

hold simultaneously.

In this paper we show that inequalities (4) and (5) can be met even if we require all weights (including the threshold) to be integers. Thus we obtain an improvement of (3) by approximately a factor of $1/2$.

In the following section we demonstrate some old and new properties of integral separating weight vectors. These will be used in the proof of the main theorem in Section 3. In the final section we make some remarks on the methods and refer to results concerning lower bounds.

2 Properties of Integral Separating Weight Vectors

The existence of an integral vector $(w_1, \ldots, w_n, t)$ satisfying the system of linear inequalities (2) is sufficient and necessary for a Boolean function $f : \{0,1\}^n \to \{0,1\}$ to be linearly separable. The same holds if we replace 0 by -1 and consider Boolean functions $g : \{-1,1\}^n \to \{-1,1\}$. Given an integral separating vector for g we obtain a – not necessarily integral – vector for the corresponding f by the following method.

Lemma 1 *Let $g : \{-1,1\}^n \to \{-1,1\}$ be linearly separable, represented by the vector $(w_1, \ldots, w_n, t)$, and let $f : \{0,1\}^n \to \{0,1\}$ be defined by*

$$2f(x_1, \ldots, x_n) - 1 \;=\; g(2x_1 - 1, \ldots, 2x_n - 1) \qquad \textit{for all } x \in \{0,1\}^n.$$

Then f is represented by $(w_1, \ldots, w_n, (t + \sum w_i)/2)$.

Proof. The proof is due to Muroga [9, Theorem 1.3.1]. In the inequality system (2) corresponding to g

$$\begin{array}{lcll} w_1 y_1 + \cdots + w_n y_n & \geq & t & \text{if } g(y) = 1 \\ w_1 y_1 + \cdots + w_n y_n & \leq & t-1 & \text{if } g(y) = -1 \end{array} \qquad (y \in \{-1,1\}^n)$$

we add $\sum w_i$ on both sides of each inequation and divide by 2. Then we obtain the equivalent system

$$\begin{array}{lcll} w_1 \cdot (y_1+1)/2 + \cdots + w_n \cdot (y_n+1)/2 & \geq & (t + \sum w_i)/2 & \text{if } g(y) = 1 \\ w_1 \cdot (y_1+1)/2 + \cdots + w_n \cdot (y_n+1)/2 & \leq & (t + \sum w_i - 1)/2 & \text{if } g(y) = -1. \end{array}$$

From that it can be derived that

$$\begin{array}{lcll} w_1 x_1 + \cdots + w_n x_n & \geq & (t + \sum w_i)/2 & \text{if } f(x) = 1 \\ w_1 x_1 + \cdots + w_n x_n & < & (t + \sum w_i)/2 & \text{if } f(x) = 0 \end{array} \qquad (x \in \{0,1\}^n),$$

in other words, (1) holds for $(w_1, \ldots, w_n, (t + \sum w_i)/2)$. □

As can be seen, we could easily obtain an integral vector for f from an integral vector for g by multiplying by 2 the representation for f constructed in the proof. However, as we will show in the next lemma, either adding 1 to or subtracting from the threshold of g is always possible, thereby making $t + \sum w_i$ even.

Lemma 2 *Let $g : \{-1,1\}^n \to \{-1,1\}$ be linearly separable by $(w_1, \ldots, w_n, t) \in \mathbb{Z}^{n+1}$.*

1. *There exists* $t' \in \mathbb{Z}$ *such that each of the following holds:*
   (a) $|t' - t| \leq 1$
   (b) $t' + \sum w_i$ *is even.*
   (c) $(w_1, \ldots, w_n, t')$ *represents* g.
2. *There exists* $t'' \in \mathbb{Z}$ *such that for all* $y \in \{-1, 1\}^n$

$$\begin{array}{lcll} w_1 y_1 + \cdots + w_n y_n & \geq & t'' + 1 & \text{if} \quad g(y) = 1 \\ w_1 y_1 + \cdots + w_n y_n & \leq & t'' - 1 & \text{if} \quad g(y) = -1. \end{array} \tag{6}$$

Proof. Both assertions are proved if we show that $w \cdot y$ has the same parity for all y. This can be seen from

$$w_1 y_1 + \cdots + w_i(-y_i) + \cdots + w_n y_n \;=\; w \cdot y - 2w_i.$$

Therefore, either an increase or a decrease of t of at most 1 still yields a representation for g. □

By $\|(w_1, \ldots, w_n, t)\|_\infty$ we denote the maximum of $|w_1|, \ldots, |w_n|, |t|$. With the previous lemma we immediately have the following.

Lemma 3 *Let* $g : \{-1, 1\}^n \to \{-1, 1\}$ *be linearly separable by* $(w_1, \ldots, w_n, t) \in \mathbb{Z}^{n+1}$ *and let* $\gamma = \|(w_1, \ldots, w_n, t)\|_\infty$. *If we replace* -1 *by* 0 *then the corresponding* $f : \{0, 1\}^n \to \{0, 1\}$ *is representable by a vector* $(w'_1, \ldots, w'_n, t') \in \mathbb{Z}^{n+1}$ *such that*

$$|w'_i| \;\leq\; \gamma, \qquad |t'| \;\leq\; ((n+1) \cdot \gamma + 1)/2.$$

Proof. By Lemma 1, f can be represented by $(w_1, \ldots, w_n, (t + \sum w_i)/2)$. If $t + \sum w_i$ is not divisible by 2 we can increase or decrease t by 1 by virtue of Lemma 2. Therefore the absolute value of the new threshold t' is bounded by

$$|t'| \;\leq\; (\gamma + 1 + n \cdot \gamma)/2 \;=\; ((n+1) \cdot \gamma + 1)/2.$$

□

3 Upper Bounds on the Size of Weights

Before we state the theorem, we report a property of finite systems of linear inequalities in real linear spaces demonstrated by K. Fan [3] and employed in our proof, the so-called *principle of bounding solutions.*

Theorem 1 [3, Theorem 2] *Let* X *be a real linear space of arbitrary dimension, let* $F_1, \ldots, F_p$ *be linear functionals on* X, *and* $\alpha_1, \ldots, \alpha_p$ *real numbers, and let* r *be the maximum number of linearly independent linear functionals among* $F_1, \ldots, F_p$. *If the system*

$$F_i(x) \leq \alpha_i \qquad (1 \leq i \leq p) \tag{7}$$

has a solution then there exist r *linearly independent functionals* $F_{\nu_1}, \ldots, F_{\nu_r}$ *among* $F_1, \ldots, F_p$ *such that every solution of the system*

$$F_{\nu_k}(x) = \alpha_{\nu_k} \qquad (1 \leq k \leq r) \tag{8}$$

is also a solution of (7).

For the proof we refer the reader to reference [3].

Theorem 2 *Every linearly separable function* $g : \{-1,1\}^n \to \{-1,1\}$ *is representable by a vector* $(w_1, \ldots, w_n, t)$ *of integers satisfying*

$$\|(w_1, \ldots, w_n, t)\|_\infty \;\leq\; 2^{-n}(n+1)^{(n+1)/2}.$$

Proof. Let $g : \{-1,1\}^n \to \{-1,1\}$ be linearly separable. According to assertion 2 of Lemma 2 the system

$$\begin{array}{rcll} -w_1y_1 - \cdots - w_ny_n + t & \leq & -1 \quad \text{if} \quad g(y) = 1 \\ w_1y_1 + \cdots + w_ny_n - t & \leq & -1 \quad \text{if} \quad g(y) = -1 \end{array} \qquad (y \in \{-1,1\}^n) \tag{9}$$

has a solution in $(w_1, \ldots, w_n, t)$. For the sake of simplicity, we denote the threshold by w_{n+1} and let $y_{n+1} = -1$ for the rest of the proof. Then (9) appears as

$$\begin{array}{rcll} -w_1y_1 - \cdots - w_{n+1}y_{n+1} & \leq & -1 \quad \text{if} \quad g(y) = 1 \\ w_1y_1 + \cdots + w_{n+1}y_{n+1} & \leq & -1 \quad \text{if} \quad g(y) = -1 \end{array} \qquad (y \in \{-1,1\}^n) \tag{10}$$

having a solution in $(w_1, \ldots, w_{n+1})$. Let r denote the rank of the coefficient matrix on the left hand side of (10). By virtue of Theorem 1 there exist r elements $y^{(1)}, \ldots, y^{(r)} \in \{-1,1\}^n$ such that every solution of the system of equalities

$$\begin{array}{rcll} -w_1y_1^{(k)} - \cdots - w_{n+1}y_{n+1}^{(k)} & = & -1 \quad \text{if} \quad g(y^{(k)}) = 1 \\ w_1y_1^{(k)} + \cdots + w_{n+1}y_{n+1}^{(k)} & = & -1 \quad \text{if} \quad g(y^{(k)}) = -1 \end{array} \qquad (1 \leq k \leq r) \tag{11}$$

is also a solution of (10). Further, the rank of the coefficient matrix on the left hand side of (11) is equal to r, i.e. it contains an $(r \times r)$-submatrix consisting of columns, say $\lambda_1, \ldots, \lambda_r$, such that the system of equalities

$$\begin{array}{rcll} -w_{\lambda_1}y_{\lambda_1}^{(k)} - \cdots - w_{\lambda_r}y_{\lambda_r}^{(k)} & = & -1 \quad \text{if} \quad g(y^{(k)}) = 1 \\ w_{\lambda_1}y_{\lambda_1}^{(k)} + \cdots + w_{\lambda_r}y_{\lambda_r}^{(k)} & = & -1 \quad \text{if} \quad g(y^{(k)}) = -1 \end{array} \qquad (1 \leq k \leq r) \tag{12}$$

has a unique solution $(\widehat{w}_{\lambda_1}, \ldots, \widehat{w}_{\lambda_r})$. We can extend this solution of (12) to a solution of (11) (and thereby of (10)) by letting the components $i \notin \{\lambda_1, \ldots, \lambda_r\}$ be equal to 0. Because of the matrix on the left hand side of (12) being regular, we obtain the solution by Cramer's Rule

$$\widehat{w}_i = \Delta_i/\Delta, \qquad i \in \{\lambda_1, \ldots, \lambda_r\}$$

where Δ is the determinant of the coefficient matrix on the left hand side of (12) and Δ_i is the determinant obtained by replacing the i-th column by the right hand side of (12). We then have

$$(\Delta_1, \ldots, \Delta_{n+1})/\Delta \qquad \text{with } \Delta_i = 0 \text{ if } i \notin \{\lambda_1, \ldots, \lambda_r\}$$

as solution of (11) and thereby of (10) and, therefore,

$$(\Delta_1, \ldots, \Delta_{n+1})$$

as integral representation for g. Each $\Delta_i \neq 0$ is the determinant of an $(r \times r)$-matrix consisting solely of -1 and 1. Now we show how to get the factor 2^{r-1} from each of them. We multiply each row having -1 in the last component by -1 and obtain matrices with the last column consisting solely of 1. After adding this column to each of the remaining $r-1$ columns, columns $1, \ldots, r-1$ contain only 0 and 2 and we can factor 2^{r-1} outside of each determinant. If we reverse the previous multiplications we have

$$\Delta_i \;=\; 2^{r-1} \cdot w_i \qquad \text{for } i \in \{\lambda_1, \ldots, \lambda_r\}$$

where w_i is the determinant of an $(r \times r)$-matrix with entries $-1, 0, 1$. With the vector

$$(w_1, \ldots, w_{n+1}) \qquad \text{with } w_i = 0 \text{ if } i \notin \{\lambda_1, \ldots, \lambda_r\}$$

we obtain again an integral representation for g. Hadamard's Determinant Theorem implies $|\Delta_i| \leq r^{r/2}$, therefore

$$|w_i| \;\leq\; 2^{-r+1} r^{r/2} \qquad (1 \leq i \leq n+1).$$

Taking into account that r, the rank of the coefficient matrix on the left hand side of (10), is not greater than $n+1$ we have the statement of the theorem. □

Having shown that (5) holds for integral weight vectors we can now easily derive that (4) is also satisfied by integers. The following statement is implied by Theorem 2 and Lemma 3.

Corollary 1 *Every linearly separable function $f : \{0,1\}^n \to \{0,1\}$ is representable by a vector $(w_1, \ldots, w_n, t)$ of integers satisfying*

$$|w_i| \;\leq\; 2^{-n}(n+1)^{(n+1)/2}, \qquad |t| \;\leq\; 2^{-n-1}(n+1)^{(n+3)/2} + 1/2.$$

4 Remarks

In the three coarse steps

1. reducing a system of inequalities to a system of equalities by Fan's Theorem
2. solving this system by Cramer's Rule
3. bounding the solution by Hadamard's Theorem

the proof of Theorem 2 closely follows the original proof given by Muroga *et al.* [10, Theorem 16] for linearly separable Boolean functions $f : \{0,1\}^n \to \{0,1\}$. A later version [9, Theorem 9.3.2.1] makes no longer use of Fan's Theorem but employs the existence of extreme points in convex sets.

Polyhedral Theory is also suitable to make the essential inference in the proof. Minimal faces and vertices of polyhedra are obtained by equating inequalities (see e.g. [14]). In Linear Programming they are closely related to the so-called *basic feasible solutions* [7]. We chose to employ Fan's Theorem because of its generality and because it is purely algebraic and does not presuppose any further background.

The improvement we achieved by our proof is essentially based on observations on weight vectors for functions over $\{-1,1\}^n$ that were expressed in Lemma 2. Furthermore, we fell back upon a property of determinants exhibited by Williamson [15]: Every determinant of an $(n \times n)$-matrix consisting solely of -1 and 1 is divisible by 2^{n-1}. A proof of this fact can also be found in [2, p. 332].

It can be shown by a simple counting argument that $\Omega(n)$ is a lower bound for the binary length of a weight [12]. Recently, this was improved by Håstad [4] exhibiting a function $F_n : \{-1,1\}^n \to \{-1,1\}$ that when n is a power of 2 requires $|w_i| \geq \Omega(\frac{1}{n} e^{-4n^\beta} 2^{(n \log n)/2-n})$ for all i, where $\beta = \log_2 \frac{3}{2}$. Moreover, Håstad also obtained a proof of (5), independently from our work.

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