New Designs for the Descartes Rule of Signs*

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The question of how to construct polynomials having as many roots as allowed by the Descartes rule of signs has been the focus of interest recently [1],[2]. For a given real polynomial

$$p(x) \quad = \quad a_0 + a_1 x + \cdots + a_n x^n,$$

Descartes's rule of signs says that the number of positive roots of $p(x)$ is equal to the number of sign changes in the sequence $a_0, a_1, \ldots, a_n$, or is less than this number by a positive even integer. Investigating which of the possible numbers of roots permitted by Descartes actually occur, Anderson, Jackson, and Sitharam [1] show that the rule cannot be improved. For any sign sequence not containing zero, they construct polynomials with this sign sequence in the coefficients and any prescribed number of positive roots that is in accord with Descartes's rule. Analogously, since the negative roots are the roots of $p(-x)$, all allowable numbers of negative roots are realized. Grabiner [2] establishes further examples. In particular, he provides polynomials for sign sequences that may contain zero and shows how to achieve certain numbers of positive and negative roots simultaneously.

In this note we add to the previous work by presenting more general designs for given numbers of roots that yield a richer class of functions. Their construction might be more intuitive than earlier methods and, once one has the technique at hand, easier to do. Further, they give rise to a counterexample to a conjecture by Grabiner [2] concerning the numbers of positive and negative roots. The basic idea follows Grabiner, who uses the fact that a continuous function must have a root between points of opposite sign. Instead of constructing polynomials directly, however, we first look at a different class of functions. We

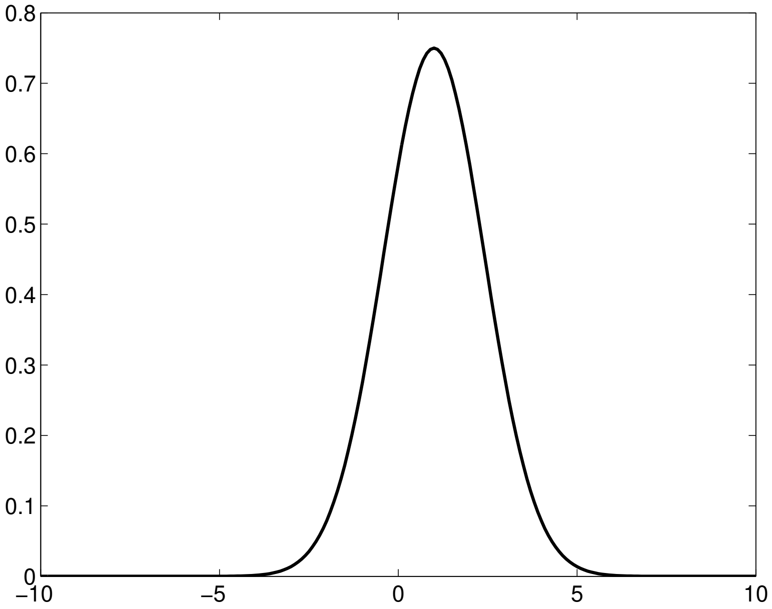

Figure 1. The radial basis function $r(x) = (3/4)\exp(-(x-1)^2/4)$.

consider the functions

$$f(y) \quad = \quad b_0 \exp\left(-\frac{(y-c_0)^2}{s^2}\right) + \cdots + b_n \exp\left(-\frac{(y-c_n)^2}{s^2}\right) \tag{1}$$

that incorporate three types of parameters: coefficients $b_0, \ldots, b_n$, centers $c_0, \ldots, c_n$, and a width $s > 0$. This function scheme is well known as a radial basis function (neural) network [3, chap. 5], an RBF network for short. In the multivariate case, such networks provide powerful methods in approximation theory [5]. Furthermore, they are widely employed as artificial neural networks, where each term of the sum models the computations performed by a biological nerve cell, and mechanisms for learning are available such that the network adapts to a desired behavior [3],[4]. For the single variable case, upon which we focus here, a variant of the Descartes rule of signs has been established recently [6, Theorem 1] (we say that a function is *computed by an RBF network* if it can be written as in (1)):

Theorem 1. *Suppose that f is computed by an RBF network with centers satisfying $c_0 \leq \cdots \leq c_n$. Let S be the number of changes of sign of the sequence $b_0, \ldots, b_n$ and Z the number of roots of f. Then $S - Z$ is even and nonnegative.*

Figure 1 shows a single summand of an RBF network, the radial basis function $r(x) = b\exp(-(x-c)^2/s^2)$ with $b = 3/4$, $c = 1$, and $s = 2$. A few useful facts about radial basis functions are easy to verify: that $r(c \pm ds) = be^{-d^2}$, that in particular $r(c) = b$, and that $r(x) \to 0$ as $x \to \pm\infty$.

RBF networks have a property that is very useful for creating roots. Assume without loss of generality that all centers are different. For a given sequence of signs not containing zero, any sequence of coefficients with these signs gives rise to a function that has a root between neighboring centers, provided the corresponding coefficients have opposite signs. We just have to choose the width sufficiently small.

Theorem 2. *Suppose that f is computed by an RBF network with nonzero coefficients $b_0, \ldots, b_n$, centers $c_0 < \cdots < c_n$, and sufficiently small width $s > 0$. Then the number of roots of f is equal to the number of sign changes in the sequence $b_0, \ldots, b_n$.*

Proof. By Theorem 1, it suffices to show that the number of roots is not smaller than the number of sign changes. Let

$$b = \min_{i=0,\ldots,n} |b_i|, \quad B = \sum_{i=0}^{n} |b_i|, \quad c = \min_{i=0,\ldots,n-1} (c_{i+1} - c_i)^2,$$

and choose s so that

$$s^2 \ < \ \frac{c}{\ln(B/b)}.$$

(Since we may assume that $n \geq 1$ without loss of generality, we have $B > b$.) Then

$$b \ > \ B \exp\left(-\frac{c}{s^2}\right),$$

which, by the definitions of b, B, and c, implies that

$$|b_i| \ > \ \sum_{\{j : b_i b_j < 0\}} |b_j| \exp\left(-\frac{(c_j - c_i)^2}{s^2}\right)$$

for $i = 0, \ldots, n$. Since at the point $y = c_i$ the term of f with center c_i has value b_i, this term dominates the sum of all terms with sign opposite to that of b_i. Hence, c_i is not a root and $f(c_i)$ has the same sign as b_i. Consequently, for any change of sign from some b_i to b_{i+1} there must be a root in (c_i, c_{i+1}). $\square$

Thus a center may represent a hill or a valley depending on whether its coefficient is positive or negative, and functions with as many roots as there are sign changes in a given sequence can be constructed as easily as drawing a landscape: the length of the sequence determines the number of centers, and the signs specify for each of them whether it is to become the top of a hill or the bottom of a valley.

As far as polynomials are concerned, it is not difficult to establish a link between RBF networks, polynomials, and the roots of both.

Lemma 3. *Let f be computed by an RBF network with arbitrary coefficients $b_0, \ldots, b_n$ (possibly containing zero), centers $0, 1, \ldots, n$, and width $s > 0$. Consider the polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ defined by taking*

$$a_i \quad = \quad b_i \exp\left(-\frac{i^2}{s^2}\right).$$

Then y is a root of f if and only if $x = \exp(2y/s^2)$ is a root of p.

Proof. Being a root of

$$b_0 \exp\left(-\frac{y^2}{s^2}\right) + b_1 \exp\left(-\frac{(y-1)^2}{s^2}\right) + \cdots + b_n \exp\left(-\frac{(y-n)^2}{s^2}\right)$$

is, by multiplication with $\exp(y^2/s^2)$, equivalent to being a root of

$$b_0 + b_1 \exp\left(\frac{2y-1}{s^2}\right) + \cdots + b_n \exp\left(\frac{2ny-n^2}{s^2}\right)$$

and, hence, of

$$b_0 + b_1 \exp\left(-\frac{1}{s^2}\right) \exp\left(\frac{2y}{s^2}\right) + \cdots + b_n \exp\left(-\frac{n^2}{s^2}\right) \exp\left(\frac{2ny}{s^2}\right).$$

With $x = \exp(2y/s^2)$ we see that y is a root of the former if and only if x is a root of

$$b_0 + b_1 \exp\left(-\frac{1}{s^2}\right) x + \cdots + b_n \exp\left(-\frac{n^2}{s^2}\right) x^n.$$

□

Before we apply this result for the design of polynomials, we employ it in the reverse direction. Grabiner's Theorem 2 [2] shows that for every sequence of signs (including sequences containing zero), there are polynomials with these signs in the coefficients and a number of positive roots less than the number of sign changes by any prescribed positive multiple of two. Using Lemma 3 we may infer for RBF networks that all numbers of roots given by Theorem 1 are indeed attainable.

Corollary 4. *Let $\sigma_0, \sigma_1, \ldots, \sigma_n$ be any sequence with terms -1, 0, or 1. For every number of roots allowed by Theorem 1 there is a function with that number of roots that is computed by an RBF network with centers $0, 1, \ldots, n$ and coefficients $b_0, \ldots, b_n$ that have the signs of $\sigma_0, \ldots, \sigma_n$.*

Concerning the maximum number of positive roots admitted by the Descartes rule of signs, Anderson et al. [1] provide polynomials that achieve this maximum. As an improvement, polynomials where the numbers of positive and negative roots attain the maxima simultaneously are constructed by Grabiner [2]. The following consequence of Theorem 2 and Lemma 3 yields further instances of the latter type. Moreover, it shows not only that the signs of the coefficients can be specified, but also that their values can be related to any given sequence of numbers.

Corollary 5. *Let $b_0, \ldots, b_n$ be any sequence of numbers, and let $s > 0$ be sufficiently small. Then the polynomial with coefficients $a_0, \ldots, a_n$ satisfying*

$$a_i \quad = \quad b_i \exp\left(-\frac{i^2}{s^2}\right)$$

has the same number of positive roots as there are sign changes in the sequence $b_0, \ldots, b_n$ and the same number of negative roots as there are sign changes in the sequence $b_0, -b_1, \ldots, (-1)^n b_n$.

Grabiner also studies which numbers of positive and negative roots, in addition to the maxima, are possible for a given sign sequence. He shows that for any sequence $\tau_0, \ldots, \tau_n$ that arises from a sign sequence $\sigma_0, \ldots, \sigma_n$ by replacing some σ_i $(1 \leq i \leq n-1)$ with zero, there is a polynomial with coefficients having the signs of $\sigma_0, \ldots, \sigma_n$, as many positive roots as there are sign changes in $\tau_0, \ldots, \tau_n$, and as many negative roots as there are sign changes in $\tau_0, -\tau_1, \ldots, (-1)^n \tau_n$. He conjectures that this might be the only way to obtain combined numbers of positive and negative roots. In the following, we provide a counterexample to this hypothesis. We consider the case where no σ_i is changed to zero, that is, $\tau_0, \ldots, \tau_n$ is equal to $\sigma_0, \ldots, \sigma_n$. It is straightforward to extend the construction to the general case.

Let f be the function computed by the RBF network with coefficients $b_0 = 1, b_1 = 1/2$, and $b_2 = 1$, centers $c_0 = 0, c_1 = 1$, and $c_2 = 2$, and width $s = 1$. Let g have the same parameters, except for b_1, which is defined as $-1/2$. Then the coefficients of f and g have the sign sequences $+, +, +$ and $+, -, +$, respectively. Neither f nor g, shown in Figure 2, has a root. By Lemma 3, f gives rise to the polynomial

$$q(x) \quad = \quad 1 + \frac{1}{2e}x + e^{-4}x^2,$$

which has no real root, and g gives rise to $q(-x)$. In particular, the number of negative roots of $q(x)$ is different from the number of sign changes in the coefficients of $q(-x)$.

Grabiner [2] further shows that not all combinations of numbers of positive and negative roots are possible, thereby giving a negative answer to a question of Anderson et al. [1]. The function q designed here implies that more combinations can be achieved than assumed by Grabiner. Thus, the true number of combinations lies somewhere strictly between these two conjectures.

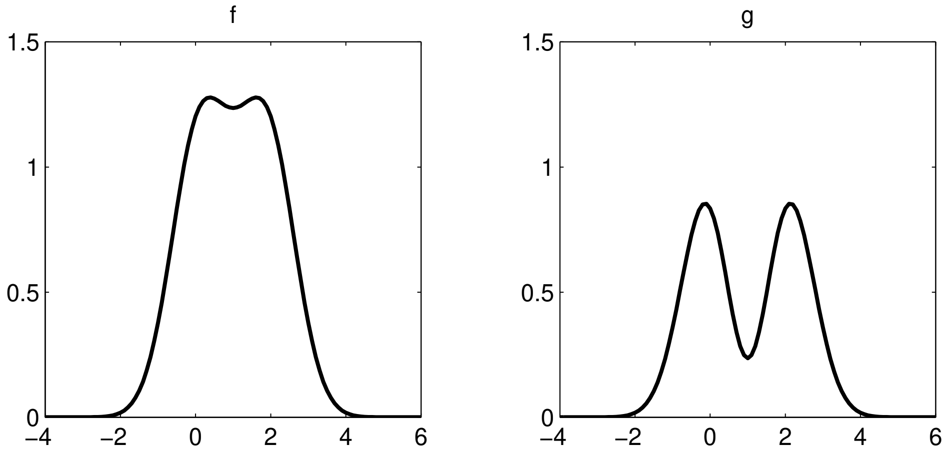

Figure 2. A counterexample: going from f to g, the sign of one coefficient is changed from $+$ to $-$ without creating a root.

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