Complexity of Lexicographic Strategies on Binary Cues

Michael Schmitt
Lehrstuhl Mathematik und Informatik, Fakultät für Mathematik
Ruhr-Universität Bochum, D–44780 Bochum, Germany
mschmitt@lmi.ruhr-uni-bochum.de

Laura Martignon
Center for Adaptive Behavior and Cognition, Max Planck Institute for Human Development, Lentzeallee 94, D–14194 Berlin, Germany
martignon@mpib-berlin.mpg.de

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We consider the problem to minimize the number of incorrect comparisons under the lexicographic strategy (see [2] for the motivation and background):

MIN-INCORRECT-LEX
INSTANCE: A set $B \subseteq \{0,1\}^n$, and a set $L \subseteq B \times B$ that represents a partial order (i.e. $b_i < b_j$ if and only if $[b_i, b_j] \in L$); a natural number k.

QUESTION: Is there a permutation of the components (or "cues") $1, \ldots, n$ of B such that the number of incorrect comparisons in L under the lexicographic strategy is at most k?

Theorem. *MIN-INCORRECT-LEX is NP-hard.*

Proof. We give a reduction from VERTEX COVER [1]. Given a graph $G = (V, E)$ where $V = \{v_1, \ldots, v_n\}$, define $B \subseteq \{0,1\}^{n+1}$ as follows:

1. Let $(1, \ldots, 1, 0) \in B$.
2. For $i = 1, \ldots, n$, let $(0_i, 1) \in B$ (where 0_i has a 0 in the i-th position and 1's elsewhere).
3. For every $\{v_i, v_j\} \in E$, let $(0_{i,j}, 1) \in B$ (where $0_{i,j}$ has a 0 at positions i and j, and 1's elsewhere).

Further, let L be defined by

$$\begin{aligned} L &= \{[(1,\ldots,1,0),(0_i,1)] : i = 1,\ldots,n\} \\ &\quad \cup \{[(0_{i,j},1),(1,\ldots,1,0)] : \{v_i,v_j\} \in E\}. \end{aligned}$$

Let us call the elements of the two sets from which L is constructed the vertex pairs and the edge pairs, respectively. Clearly, the reduction is computable in polynomial time. We establish its correctness as follows.

($\Rightarrow$) Let V' be a vertex cover for G of size k, $V' = \{v_{i_1},\ldots,v_{i_k}\}$. Define the permutation of the cues such that it begins with

$$i_1,\ldots,i_k,n+1$$

and continues with the remaining cues in arbitrary order. Then, the claim is that there are no more than k incorrect comparisons. As can easily be checked, the comparisons are correct for the edge pairs since V' is a vertex cover. They are also correct for those vertex pairs with $v_i \notin V'$. Finally, they are incorrect for $v_i \in V'$, of which we have at most k.

($\Leftarrow$) Let $i_1,\ldots,i_{n+1}$ be a permutation of the cues that produces at most k incorrect comparisons. Define the set of vertices as follows:

1. For every incorrect vertex pair, let $v_i \in V'$.

2. For every incorrect edge pair $\{v_i,v_j\}$, let one of $v_i,v_j \in V'$.

Clearly, V' has size at most k. Assume that V' is not a vertex cover, that is, there is an edge, say $\{v_i,v_j\}$, not covered. Then none of v_i,v_j is in V', implying that we have correct comparisons for the vertex pairs corresponding to v_i and v_j and for the edge pair corresponding to $\{v_i,v_j\}$. The fact that the edge pair yields a correct comparison implies that the permutation must rank the i-th or the j-th cue before the $(n+1)$-st. But then we have that one of the vertex pairs for v_i and v_j is compared incorrectly—a contradiction. $\square$

Remark. The same reduction works if the order is required to be total. In this case we complete L by all missing pairs in lexicographic order. The permutation is defined such that $i_1,\ldots,i_k$ are increasing and the remaining cues after the $(n+1)$-st are increasing as well.

References

[1] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W. H. Freeman, San Francisco, 1979.

[2] L. Martignon and M. Schmitt. Simplicity and robustness of fast and frugal heuristics. *Minds and Machines*, 9:565–593, 1999.