A SPLITTING OF $TMF_0(7)$ (PRELIMINARY VERSION)

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Abstract. We provide a splitting of $TMF_0(7)$ at the prime 3 as TMF-module into two shifted copies of TMF and two shifted copies of $TMF_1(2)$.

1. Introduction

The study of modules over the real K-theory spectrum KO has been central in Bousfield's work on the classification of K-local spectra [3]. If we localize further at a prime p, localization at K-theory becomes equivalent to localization at the first Johnson–Wilson theory $E(1)$. If we want to study $E(2)$-local spectra, topological modular forms are a natural substitute for KO.

Topological modular forms come in many variants. First, there is the periodic version TMF that is based on the moduli stack of elliptic curves $\mathcal{M}_{ell}$. It has the disadvantage that its homotopy groups are infinitely generated in most degrees, which is different in the refinement Tmf that is based on the compactified moduli stack $\overline{\mathcal{M}}_{ell}$. Its connective cover is called tmf. We refer to [10] as a basic reference for these spectra.

It has a long tradition in arithmetic geometry not only to consider the moduli stack of elliptic curves itself, but also to consider moduli of elliptic curves with level structures. A $\Gamma_0(n)$-level structure on an elliptic curve E/S is a sub-group scheme that is étale locally on S isomorphic to $(\mathbb{Z}/n)_S$. A $\Gamma_1(n)$-level structure on E is a sub-group scheme of E with a chosen isomorphism to $(\mathbb{Z}/n)_S$. This leads to moduli stacks $\mathcal{M}_0(n)$ and $\mathcal{M}_1(n)$ and to spectra $TMF_0(n)$ and $TMF_1(n)$. Hill and Lawson [15] were able to define spectra $Tmf_0(n)$ and $Tmf_1(n)$ based on the compactified moduli $\overline{\mathcal{M}}_0(n)$ and $\overline{\mathcal{M}}_1(n)$ as well. Note that n is here always inverted.

When studying Tmf-modules, $Tmf_0(n)$ and $Tmf_1(n)$ are among the first examples to consider. In [25], the first-named author has proven splittings for $Tmf_1(n)$ and $Tmf_0(n)$ in many cases if we localize at a prime p. If $p = 3$, the splittings are into shifted copies of $Tmf_1(2)_{(3)}$. As $\pi_*Tmf_1(2)_{(3)}$ is torsionfree, splittings into shifted copies of $Tmf_1(2)_{(3)}$ can only exist if $\pi_*Tmf_0(n)_{(3)}$ is also torsionfree, which is not expected if 3 divides $|(\mathbb{Z}/n)^\times|$. The first case where this occurs is $Tmf_0(7)$, where we prove nevertheless the following modified splitting result.

Theorem 1.1. *The $Tmf_{(3)}$-module $Tmf_0(7)_{(3)}$ decomposes as*

$$Tmf_{(3)} \oplus \Sigma^4 Tmf_1(2)_{(3)} \oplus \Sigma^8 Tmf_1(2)_{(3)} \oplus L,$$

where $L \in \mathrm{Pic}(Tmf_{(3)})$, i.e. L is an invertible $Tmf_{(3)}$-module.

The $TMF_{(3)}$-module $TMF_0(7)_{(3)}$ decomposes as

$$TMF_{(3)} \oplus \Sigma^4 TMF_1(2)_{(3)} \oplus TMF_1(2)_{(3)} \oplus \Sigma^{36} TMF_{(3)}.$$

Using unpublished work of M. Olbermann, one can deduce that L is actually an exotic Picard element, i.e. is not of the form $\Sigma^k Tmf_{(3)}$ for any k. The group $\mathrm{Pic}(Tmf_{(3)})$ was determined in [24] and one can explictly identify L. This shows that exotic Picard group elements of Tmf actually occur quite naturally.

Our main theorem is based on the following algebraic theorem.

Theorem 1.2. *Let $h\colon \overline{\mathcal{M}}_0(7)_{(3)} \to \overline{\mathcal{M}}_{ell,(3)}$ and $f\colon \overline{\mathcal{M}}_1(2)_{(3)} \to \overline{\mathcal{M}}_{ell,(3)}$ be the maps induced by forgetting the level structure on an elliptic curve and let $\mathcal{O}$ be the structure sheaf of $\overline{\mathcal{M}}_{ell,(3)}$. Then the quasi-coherent sheaf $h_*\mathcal{O}_{\overline{\mathcal{M}}_0(7)} \cong h_*h^*\mathcal{O}$ on $\overline{\mathcal{M}}_{ell,(3)}$ is a vector bundle of rank 8, which can be decomposed as a sum*

$$\mathcal{O} \oplus \underline{\omega}^{-6} \oplus f_*f^*\mathcal{O} \otimes \underline{\omega}^{-2} \oplus f_*f^*\mathcal{O} \otimes \underline{\omega}^{-4}.$$

Here, $\underline{\omega}$ is the generator of $\mathrm{Pic}(\overline{\mathcal{M}}_{ell})$ that can be constructed as the pushforward of the sheaf of differentials on the universal generalized elliptic curve.

Let us simultaneously describe the proof strategy and give an overview of the different sections. We will always work (implicitly) 3-locally.

Let $\mathrm{mf}_1(7)_* = \mathrm{mf}(\Gamma_1(7), \mathbb{Z}_{(3)})$ be the subring of the ring of holomorphic $\Gamma_1(7)$-modular forms $\mathrm{mf}(\Gamma_1(7), \mathbb{C})$ with coefficients in $\mathbb{Z}_{(3)}$. This can be identified with the sections of $\underline{\omega}^{\otimes *}$ on $\overline{\mathcal{M}}_1(7)$. It has an action by the automorphism group of $\mathbb{Z}/7$, namely $(\mathbb{Z}/7)^\times \cong \mathbb{Z}/2 \times \mathbb{Z}/3$. In Section 2, we exhibit an isomorphism $\mathrm{mf}_1(7)_* \cong \mathbb{Z}_{(3)}[z_1, z_2, z_3]/(z_1z_2 + z_2z_3 + z_3z_1)$ and identify the $(\mathbb{Z}/7)^\times \cong \mathbb{Z}/2 \times \mathbb{Z}/3$-action to be given by the sign action of $\mathbb{Z}/2$ and a cyclic permutation action of $\mathbb{Z}/3$ on the z_i. It is not hard to compute the group cohomology $H^*(\mathbb{Z}/6; \mathrm{mf}_1(7)_*)$, which already suggests a decomposition as in Theorem 1.2.

In the next step, we exhibit in Section 3 a Weierstraß equation for the elliptic curve over $\mathrm{mf}_1(7)_*[\Delta^{-1}]$ which corresponds to the composition

$$\operatorname{Spec} \mathrm{mf}_1(7)_*[\Delta^{-1}] \to \mathcal{M}_1(7) \to \mathcal{M}_0(7) \to \mathcal{M}_{ell}.$$

To do so, we use the Tate curve description as e.g. in [31], Theorem V.3.1. It turns out that the coefficients are indeed holomorphic modular forms, and can be identified with explicit elements of $\mathbb{Z}_{(3)}[z_1, z_2, z_3]/(z_1z_2 + z_2z_3 + z_3z_1)$.

Our strategy to show Theorem 1.2 is to show a statement about comodules. It is more convenient to do this not on $\overline{\mathcal{M}}_{ell}$, but on $\mathcal{M}_{cub}$ instead (as in [22]). The latter stack has a presentation by the Hopf algebroid (B, Γ) with $B = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$. To formulate a version of Theorem 1.2 on $\mathcal{M}_{cub}$ we define cubical versions of $\mathcal{M}_1(n)$ and $\mathcal{M}_0(n)$ by a normalization procedure in Section 4. We stress that $\mathcal{M}_0(n)_{cub}$ is not the stack quotient of $\mathcal{M}_1(7)_{cub}$ by the $(\mathbb{Z}/7)^\times$-action as this stack quotient is not representable over $\mathcal{M}_{cub}$. We also provide a flatness criterion for the map $\mathcal{M}_1(n)_{cub} \to \mathcal{M}_{cub}$.

The next step is to make the Hopf algebroids corresponding to $\mathcal{M}_1(7)_{cub}$ and $\mathcal{M}_0(7)_{cub}$ explicit. In Section 5, we produce explicit B-bases of B-algebras R_B and S_B, which are defined by

$$\operatorname{Spec} R_B \cong \mathcal{M}_1(7)_{cub} \times_{\mathcal{M}_{cub}} \operatorname{Spec} B \text{ and } \operatorname{Spec} S_B \cong \mathcal{M}_0(7)_{cub} \times_{\mathcal{M}_{cub}} \operatorname{Spec} B.$$

This allows us to prove in Section 6 a splitting of S_B as a comodule over (B, Γ), which implies Theorem 1.2. In Section 7 we apply standard techniques (the transfer and the descent spectral sequence) to deduce our topological main theorem.

We end with an appendix that gives an exposition of the theory of modular forms with level over general rings and their q-expansion. The reason for the length of this appendix is the subtle difference between so-called arithmetic and naive level structures, which only agree in the presence of an n-th root of unity. To achieve a q-expansion principle in the form we need, care is needed how to identify the sections of $\underline{\omega}^{\otimes *}$ on $\overline{\mathcal{M}}_1(n)_{\mathbb{C}}$ with holomorphic $\Gamma_1(n)$-modular forms in the classical sense.

After giving this overview, let us add two directions of further research. First, we can also ask for a splitting of the connective spectrum $tmf_0(7) = \tau_{\geq 0} Tmf_0(7)$. Indeed, our algebraic theorem suggests this as it works over not only over $\overline{\mathcal{M}}_{ell}$ but over $\mathcal{M}_{cub}$. It remains unclear though how $\mathcal{M}_0(7)_{cub}$ is exactly related to $tmf_0(7)$ as its complex bordism remains uncomputed. Secondly, our main topological theorem suggests the following optimistic conjecture:

Conjecture 1.3. *The spectrum $TMF_0(n)_{(3)}$ decomposes for every $n \geq 2$ into shifted copies of $TMF_{(3)}$ and of $TMF_1(2)_{(3)}$.*

This is related to a question asked in [27], namely whether all vector bundles on $\mathcal{M}_{ell,(3)}$ decompose (up to tensoring with powers of $\underline{\omega}$) into the structure sheaf $\mathcal{O}$, $f_*f^*\mathcal{O}$ and a certain vector bundle E_α of rank 2. Also note that $TMF_1(n)_{(3)}$ always decomposes into shifted copies of $TMF_1(2)_{(3)}$ after 3-completion as shown in [25].

1.1. **Conventions.** All quotients of schemes by group schemes (like $\mathbb{G}_m$) are understood to be stack quotients. Unless clearly otherwise, all rings and algebras are assumed to be commutative and unital. Tensor products of quasi-coherent sheaves are always over the structure sheaf.

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2. Modular forms of level 7

We refer for the basics about modular forms and in particular about the q-expansion principle to the appendix. We specialize mainly to the case of modular forms for $\Gamma_1(7)$. Our goal is to get an understanding of $mf(\Gamma_1(7); \mathbb{Z})$. We also want to determine the action of $(\mathbb{Z}/7)^\times \cong \mathbb{Z}/6$ on the ring of modular forms $\mathrm{mf}(\Gamma_1(7))$ with respect to the congruence group $\Gamma_1(7)$. Observe that the action of $\Gamma_0(7)/\Gamma_1(7) \cong (\mathbb{Z}/7)^\times$ by precomposition is the same as the action induced by the $(\mathbb{Z}/7)^\times \cong \mathbb{Z}/6$-action on the torsion points of precise order 7 in the modular interpretation.

Lemma 2.1. *The stack $\overline{\mathcal{M}}_1(n)$ is equivalent to $\mathbb{P}^1_{\mathbb{Z}[\frac{1}{n}]}$ for $5 \leq n \leq 10$ and $n = 12$. For $n = 7$, the line bundle $\underline{\omega}$ corresponds to $\mathcal{O}(2)$.*

Proof. The first statement is proven in Section 2 of [25]. The Picard group of $\mathbb{P}^1_{\mathbb{Z}[\frac{1}{n}]}$ is isomorphic to $\mathbb{Z}$. Indeed, by [13, Prop 6.5c], we have a short exact sequence

$$0 \to \mathbb{Z} \to \operatorname{Pic} \mathbb{P}^1_{\mathbb{Z}[\frac{1}{n}]} \to \operatorname{Pic} \mathbb{A}^1_{\mathbb{Z}[\frac{1}{n}]} \to 0,$$

where the first map is split by degree and $\operatorname{Pic} \mathbb{A}^1_{\mathbb{Z}[\frac{1}{n}]} \cong \operatorname{Pic} \mathbb{Z}[\frac{1}{n}] = 0$ by [13, Prop 6.6].

Thus, we have only to compute the degree of $\underline{\omega}$ on $\overline{\mathcal{M}}_1(7)$. In general, the degree of $\underline{\omega}$ on $\overline{\mathcal{M}}_1(n)$ is $\frac{1}{24}n^2\prod_{p|n}(1-\frac{1}{p^2})$ [25, Section 5]. Thus, for $n=7$, the degree is 2. □

First, we use Theorem 4.8.1 of [9] to identify $\mathrm{mf}(\Gamma_1(7);\mathbb{C})_1$ and the $\mathbb{Z}/6$-action on it. As explained in Section 3.9 of [9], the genus of $X_1(7) = \overline{\mathcal{M}}_1(7)_{\mathbb{C}}$ is 0, and using Theorem 3.6.1 of [9], we conclude that there are no cusp forms of weight 1 in $\mathrm{mf}(\Gamma_1(7);\mathbb{C})$; thus there is a basis of Eisenstein series of weight 1 for $\mathrm{mf}(\Gamma_1(7);\mathbb{C})_1$, given as follows. Fix a generator t for $\mathbb{Z}/6$ and the isomorphism $(\mathbb{Z}/7)^\times \cong \mathbb{Z}/6$ via

$$\begin{aligned} \alpha\colon \langle t \,|\, t^6 = 1\rangle &\to (\mathbb{Z}/7)^\times \\ t &\mapsto 3 \\ t^2 &\mapsto 2 \\ t^3 &\mapsto -1 \\ t^4 &\mapsto 4 \\ t^5 &\mapsto 5 \end{aligned}$$

(We list all the values for convenience.)

Then the three odd characters $\varphi_1,\varphi_2,\varphi_3\colon \mathbb{Z}/6\to\mathbb{C}^\times$ are described by

$$\begin{aligned} \varphi_1(t) &= \zeta_6, \\ \varphi_2(t) &= -1, \\ \varphi_3(t) &= -\zeta_6+1, \end{aligned}$$

where $\zeta_6 = \exp(\frac{2\pi i}{6})$ is a sixth primitive root of unity.

By Theorem 4.8.1 of [9], there are (modified) Eisenstein series $E(\varphi_1), E(\varphi_2), E(\varphi_3)$ which form the basis of $\mathrm{mf}(\Gamma_1(7);\mathbb{C})_1$ and on which the $\mathbb{Z}/6$-action is described exactly by the multiplication with the respective character. From Section 4.8 and Formula (4.33) of [9] (or [4]) we obtain

$$E(\varphi_j)(\tau) = -\frac{1}{14}\sum_{n=1}^{6} n\varphi_j(n) + \sum_{k=1}^{\infty}\left(\sum_{l|k,l>0}\varphi_j(l)\right)q^k, \text{ with } q=\exp(2\pi i\tau).$$

MAGMA-calculations suggest to consider the following modular forms in $\mathrm{mf}(\Gamma_1(7);\mathbb{C})_1$:

$$\begin{aligned} z_1 &= \frac{1}{3}(3\zeta_6-1)E(\varphi_1)+\frac{2}{3}E(\varphi_2)+\frac{1}{3}(-3\zeta_6+2)E(\varphi_3), \\ z_2 &= \frac{1}{3}(-\zeta_6-2)E(\varphi_1)+\frac{2}{3}E(\varphi_2)+\frac{1}{3}(\zeta_6-3)E(\varphi_3), \\ z_3 &= \frac{1}{3}(-2\zeta_6+3)E(\varphi_1)+\frac{2}{3}E(\varphi_2)+\frac{1}{3}(2\zeta_6+1)E(\varphi_3). \end{aligned}$$

Note that the base change matrix

$$\frac{1}{3}\begin{pmatrix} 3\zeta_6-1 & -\zeta_6-2 & -2\zeta_6+3 \\ 2 & 2 & 2 \\ -3\zeta_6+2 & \zeta_6-3 & 2\zeta_6+1 \end{pmatrix}$$

has determinant $\frac{2}{27}(84\zeta_6-42)$, which is invertible in $\mathbb{C}$, so that z_1,z_2,z_3 is a new $\mathbb{C}$-basis of $\mathrm{mf}(\Gamma_1(7);\mathbb{C})_1$.

Lemma 2.2. *The z_j have only $\mathbb{Z}$-coefficients in their q-expansion.*

Proof. Denote the coefficient of q^n in z_j by $c_n(z_j)$.

First, we compute $c_0(z_j)$. This calculation is somewhat different from the ones for higher coefficients:

$$\begin{aligned} c_0(z_1) =& \frac{1}{3}(3\zeta_6 - 1) \cdot \left(-\frac{1}{14}\sum_{n=1}^{6} n\varphi_1(n)\right) + \frac{2}{3} \cdot \left(-\frac{1}{14}\sum_{n=1}^{6} n\varphi_2(n)\right) \\ &+ \frac{1}{3}(-3\zeta_6 + 2) \cdot \left(-\frac{1}{14}\sum_{n=1}^{6} n\varphi_3(n)\right) \end{aligned}$$

Evaluating the sum for φ_1, we obtain

$$\sum_{n=1}^{6} n\varphi_1(n) = 1 + 2\zeta_6^2 + 3\zeta_6 + 4\zeta_6^4 + 5\zeta_6^5 + 6\zeta_6^3.$$

Using $\zeta_6^2 = \zeta_6 - 1$ and $\zeta_6^3 = -1$, we obtain

$$\begin{aligned} \sum_{n=1}^{6} n\varphi_1(n) =& \quad 1 + 2(\zeta_6 - 1) + 3\zeta_6 - 4\zeta_6 - 5(\zeta_6 - 1) - 6 \\ =& \quad -4\zeta_6 - 2. \end{aligned}$$

For φ_2, we obtain

$$\sum_{n=1}^{6} n\varphi_2(n) = 1 + 2 - 3 + 4 - 5 - 6 = -7.$$

For φ_3, recall that $1 - \zeta_6 = \zeta_6^5$, so we obtain

$$\sum_{n=1}^{6} n\varphi_3(n) = 1 + 2\zeta_6^4 + 3\zeta_6^5 + 4\zeta_6^2 + 5\zeta_6 + 6\zeta_6^3.$$

Using the properties of ζ_6 again, we obtain

$$\begin{aligned} \sum_{n=1}^{6} n\varphi_3(n) =& \quad 1 - 2\zeta_6 + 3(1 - \zeta_6) + 4(\zeta_6 - 1) + 5\zeta_6 - 6 \\ =& \quad 4\zeta_6 - 6. \end{aligned}$$

Inserting this values into the formula for $c_0(z_1)$, we obtain

$$\begin{aligned} c_0(z_1) =& \frac{1}{3}(3\zeta_6 - 1) \cdot \left(-\frac{1}{14}(-4\zeta_6 - 2)\right) + \frac{1}{3} \\ &+ \frac{1}{3}(-3\zeta_6 + 2) \cdot \left(-\frac{1}{14}(4\zeta_6 - 6)\right) = 0. \end{aligned}$$

Next, we use the values computed above to compute $c_0(z_2)$:

$$\begin{aligned} c_0(z_2) = \;& \frac{1}{3}(-\zeta_6 - 2) \cdot \left(-\frac{1}{14}\sum_{n=1}^{6} n\varphi_1(n)\right) + \frac{2}{3} \cdot \left(-\frac{1}{14}\sum_{n=1}^{6} n\varphi_2(n)\right) \\ & + \frac{1}{3}(\zeta_6 - 3) \cdot \left(-\frac{1}{14}\sum_{n=1}^{6} n\varphi_3(n)\right) \\ = \;& \frac{1}{3}(-\zeta_6 - 2) \cdot \left(-\frac{1}{14}(-4\zeta_6 - 2)\right) + \frac{1}{3} \\ & + \frac{1}{3}(\zeta_6 - 3) \cdot \left(-\frac{1}{14}(4\zeta_6 - 6)\right) = 0. \end{aligned}$$

Finally, we compute $c_0(z_3)$:

$$\begin{aligned} c_0(z_3) = \;& \frac{1}{3}(-2\zeta_6 + 3) \cdot \left(-\frac{1}{14}\sum_{n=1}^{6} n\varphi_1(n)\right) + \frac{2}{3}\left(-\frac{1}{14}\sum_{n=1}^{6} n\varphi_2(n)\right) \\ & + \frac{1}{3}(2\zeta_6 + 1) \cdot \left(-\frac{1}{14}\sum_{n=1}^{6} n\varphi_3(n)\right) \\ = \;& \frac{1}{3}(-2\zeta_6 + 3) \cdot \left(-\frac{1}{14}(-4\zeta_6 - 2)\right) + \frac{1}{3} \\ & + \frac{1}{3}(2\zeta_6 + 1) \cdot \left(-\frac{1}{14}(4\zeta_6 - 6)\right) = 1. \end{aligned}$$

Now we will show that $c_k(z_j)$ for $k > 0$ and $j \in \{1, 2, 3\}$ is always an integer. This is somewhat different from the previous argument. For z_1, we obtain

$$\begin{aligned} c_k(z_1) = \;& \frac{1}{3}(3\zeta_6 - 1) \cdot \left(\sum_{l|k, l>0} \varphi_1(l)\right) + \frac{2}{3} \cdot \left(\sum_{l|k, l>0} \varphi_2(l)\right) \\ & + \frac{1}{3}(-3\zeta_6 + 2) \cdot \left(\sum_{l|k, l>0} \varphi_3(l)\right) \\ = \;& \sum_{l|k, l>0} \frac{1}{3}\left((3\zeta_6 - 1)\varphi_1(l) + 2\varphi_2(l) + (-3\zeta_6 + 2)\varphi_3(l)\right). \end{aligned}$$

where l denotes also its congruence class in $\mathbb{Z}/7$.

We give the values of the summands depending on l: (Note we would only need to compute the values for one half because of the symmetry)

$l \mod 7$	0	1	2	3	4	5	6
Summand	0	1	-1	-2	2	1	-1

In particular, the sum we obtain has only integer summands, thus is itself an integer.

We now look at z_2:

$$\begin{aligned}
c_k(z_2) = &\ \frac{1}{3}(-\zeta_6 - 2)\cdot\left(\sum_{l|k,l>0}\varphi_1(l)\right) + \frac{2}{3}\cdot\left(\sum_{l|k,l>0}\varphi_2(l)\right)\\
&+\frac{1}{3}(\zeta_6-3)\cdot\left(\sum_{l|k,l>0}\varphi_3(l)\right)\\
= &\sum_{l|k,l>0}\frac{1}{3}\cdot\left((-\zeta_6-2)\varphi_1(l)+2\varphi_2(l)+(\zeta_6-3)\varphi_3(l)\right)
\end{aligned}$$

We give again the values of the summands depending on l:

$l \mod 7$	0	1	2	3	4	5	6
Summand	0	-1	2	-1	1	-2	1

Finally, for z_3 we obtain:

$$\begin{aligned}
c_k(z_3) = &\ \frac{1}{3}(-2\zeta_6 + 3)\cdot\left(\sum_{l|k,l>0}\varphi_1(l)\right) + \frac{2}{3}\left(\sum_{l|k,l>0}\varphi_2(l)\right)\\
&+\frac{1}{3}(2\zeta_6+1)\cdot\left(\sum_{l|k,l>0}\varphi_3(l)\right)\\
= &\sum_{l|k,l>0}\frac{1}{3}\cdot\left((-2\zeta_6+3)\varphi_1(l)+2\varphi_2(l)+(2\zeta_6+1)\varphi_3(l)\right).
\end{aligned}$$

Again, we put the values of the summands depending on l into a table:

$l \mod 7$	0	1	2	3	4	5	6
Summand	0	2	1	1	-1	-1	-2

Thus, we have seen that all coefficients of z_1, z_2, z_3 in the q-expansion are integers, so we have $z_1, z_2, z_3 \in \mathrm{mf}(\Gamma_1(7);\mathbb{Z})_1^q$. ☐

We want to show that $z_1, z_2, z_3 \in \mathrm{mf}(\Gamma_1(7);\mathbb{Z})_1$ is a basis. For this, we consider the q-expansions of z_1, z_2, z_3 modulo q^3:

$$\begin{aligned}
z_1 &\equiv & & q & &\mod q^3\\
z_2 &\equiv & & -q+q^2 & &\mod q^3\\
z_3 &\equiv & 1 & +2q+3q^2 & &\mod q^3
\end{aligned}$$

This is obviously a $\mathbb{Z}$-basis of $\mathbb{Z}[\![q]\!]/(q^3)$. Thus, $\mathrm{mf}(\Gamma_1(7);\mathbb{Z})_1 \to \mathbb{Z}[\![q]\!]/(q^3)$ is surjective. It is also injective as it is over $\mathbb{C}$. Alternatively, we can argue that by the following more general lemma the source is a free abelian group of rank ≤ 3 and this also implies the map to be injective.

Lemma 2.3. *The $\mathbb{Z}$-module $\mathrm{mf}(\Gamma_1(n);\mathbb{Z})_k$ is free of at most the same rank as the $\mathbb{C}$-dimension of $\mathrm{mf}(\Gamma_1(n);\mathbb{C})_k$ for every weight k.*

Proof. As $\mathrm{mf}(\Gamma_1(n);\mathbb{Z})$ is a subring of $\mathrm{mf}(\Gamma_1(n);\mathbb{C})$, it is $\mathbb{Z}$-torsionfree. As $\mathrm{mf}(\Gamma_1(n);\mathbb{C})_k$ is finite-dimensional, the composition

$$\mathrm{mf}(\Gamma_1(n);\mathbb{C})_k \to \mathbb{C}[\![q]\!] \to \mathbb{C}[\![q]\!]/q^N$$

is injective for N big enough. Thus, also the map

$$\mathrm{mf}(\Gamma_1(n);\mathbb{Z})_k \to \mathbb{Z}[\![q]\!]/q^N \to \mathbb{C}[\![q]\!]/q^N$$

is injective and hence the first map as well. Thus, $\mathrm{mf}(\Gamma_1(n);\mathbb{Z})$ is finitely generated free.

Tensoring the injection $\mathrm{mf}(\Gamma_1(n);\mathbb{Z})_k \to \mathbb{Z}[\![q]\!]/q^N$ with $\mathbb{C}$ gives an injection

$$\mathrm{mf}(\Gamma_1(n);\mathbb{Z})_k \otimes_{\mathbb{Z}} \mathbb{C} \to \mathbb{C}[\![q]\!]/q^N$$

and hence the map $\mathrm{mf}(\Gamma_1(n);\mathbb{Z})_k \otimes_{\mathbb{Z}} \mathbb{C} \to \mathrm{mf}(\Gamma_1(n);\mathbb{C})_k$ is also injective. □

This implies that $z_1, z_2, z_3 \in \mathrm{mf}(\Gamma_1(7);\mathbb{Z})_1$ is indeed a basis. The same is thus true in $\mathrm{mf}(\Gamma_1(7);\mathbb{Z}[\frac{1}{7}])_1$.

Our next goal is to understand all of $\mathrm{mf}_1(7)_*$ in terms of z_i's. We will prove the following proposition:

Proposition 2.4. *There is an isomorphism of rings*

$$\mathbb{Z}[\frac{1}{7}][z_1, z_2, z_3]/(z_1z_2 + z_2z_3 + z_3z_1) \to \mathrm{mf}(\Gamma_1(7);\mathbb{Z}[\frac{1}{7}]).$$

Proof. We will first show that the relation $z_1z_2 + z_2z_3 + z_3z_1 = 0$ is satisfied in $\mathrm{mf}(\Gamma_1(7);\mathbb{Z})_2$. For this, we will use an analoguous argument as for z_1, z_2, z_3 being a basis of weight 1 modular forms. More precisely, we will consider the q-expansion of the modular forms z_iz_j for $1 \le i \le j \le 3$ modulo q^5. This will be enough since using the formulae of Section 3.9 of [9], we conclude that $\mathrm{mf}(\Gamma_1(7);\mathbb{C})_2$ is 5-dimensional (and due to the form of the generators we will obtain) and because $\mathrm{mf}(\Gamma_1(7);\mathbb{Z})_2$ embeds into $\mathrm{mf}(\Gamma_1(7);\mathbb{C})_2$.

First, the z_i themselves are given via

$$\begin{aligned}
z_1 &\equiv & & q & & -q^3 + 2q^4 & & \mod q^5\\
z_2 &\equiv & & -q + q^2 & & -2q^3 + 2q^4 & & \mod q^5\\
z_3 &\equiv & 1 & +2q + 3q^2 & & +3q^3 + 2q^4 & & \mod q^5
\end{aligned}$$

One computes the following products of those:

$$\begin{aligned}
z_1^2 &\equiv & & q^2 & & -2q^4 & & \mod q^5\\
z_1z_2 &\equiv & & -q^2 + q^3 & & -q^4 & & \mod q^5\\
z_1z_3 &\equiv & q & +2q^2 + 2q^3 & & +3q^4 & & \mod q^5\\
z_2^2 &\equiv & & q^2 - 2q^3 & & +5q^4 & & \mod q^5\\
z_3^2 &\equiv & 1 + 4q & +10q^2 + 18q^3 & & +25q^4 & & \mod q^5\\
z_2z_3 &\equiv & -q & -q^2 - 3q^3 & & -2q^4 & & \mod q^5
\end{aligned}$$

First, observe that we immediately obtain that the q-expansion of $z_1z_2+z_2z_3+z_3z_1$ is $0 \mod q^5$. Next, we observe that the matrix mapping the basis $1, q, q^2, q^3, q^4$ to the first five truncated power series is

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 1\\
0 & 0 & 1 & 0 & 4\\
1 & -1 & 2 & 1 & 10\\
0 & 1 & 2 & -2 & 18\\
-2 & -1 & 3 & 5 & 25
\end{pmatrix}$$

The determinant of this matrix is 1; thus it is invertible over $\mathbb{Z}$ and the first 5 truncated power series above are a basis of $\mathbb{Z}[\![q]\!]/(q^5)$. This implies that a $\Gamma_1(7)$-modular form of weight 2 is zero iff its q-expansion is zero modulo q^5; this in turn implies the relation.

We have noted before that

$$\mathrm{mf}(\Gamma_1(7); \mathbb{Z}[\tfrac{1}{7}]) \cong H^0(\overline{\mathcal{M}}_1(7); \underline{\omega}^{\otimes *}) \cong H^0(\mathbb{P}^1_{\mathbb{Z}[\frac{1}{7}]}; \mathcal{O}(2*)).$$

Thus, this ring is abstractly isomorphic to polynomials of even degree in variables x_1, x_2 of degree 1 (and the degree of the modular form is half the degree of the polynomial). The ring of such polynomials is generated by the three monomials of degree 2 with one quadratic relation between those. Thus, the ring $\mathrm{mf}(\Gamma_1(7); \mathbb{Z}\left[\frac{1}{7}\right])$ is generated in degree 1 and so by the z_1, z_2, z_3. Thus, we get a surjective map

$$\mathbb{Z}[\tfrac{1}{7}][z_1, z_2, z_3]/(z_1z_2 + z_2z_3 + z_3z_1) \to \mathrm{mf}(\Gamma_1(7); \mathbb{Z}[\tfrac{1}{7}]),$$

which has to be an isomorphism by counting the ranks. □

Next, we want to identify the $(\mathbb{Z}/7)^\times$ action on the left-hand side.

Lemma 2.5. *Let the generator t of $\mathbb{Z}/6$ correspond to $[3] \in (\mathbb{Z}/7)^\times$ as above. Then $t.z_1 = -z_3$ and $t.z_2 = -z_1$ and $t.z_3 = -z_2$.*

Proof. Recall that we already know the action on the Eisenstein series by definition, so we can conclude as follows:

$$\begin{aligned}
t.z_1 &= \frac{1}{3}(3\zeta_6 - 1)\varphi_1(t)E(\varphi_1) + \frac{2}{3}\varphi_2(t)E(\varphi_2) + \frac{1}{3}(-3\zeta_6 + 2)\varphi_3(t)E(\varphi_3)\\
&= \frac{1}{3}(3\zeta_6 - 1)\zeta_6 E(\varphi_1) - \frac{2}{3}E(\varphi_2) + \frac{1}{3}(-3\zeta_6 + 2)(-\zeta_6 + 1)E(\varphi_3)\\
&= \frac{1}{3}(2\zeta_6 - 3)E(\varphi_1) - \frac{2}{3}E(\varphi_2) - \frac{1}{3}(2\zeta_6 + 1)E(\varphi_3)\\
&= -z_3,\\
t.z_2 &= \frac{1}{3}(-\zeta_6 - 2)\varphi_1(t)E(\varphi_1) + \frac{2}{3}\varphi_2(t)E(\varphi_2) + \frac{1}{3}(\zeta_6 - 3)\varphi_3(t)E(\varphi_3)\\
&= \frac{1}{3}(-\zeta_6 - 2)\zeta_6 E(\varphi_1) - \frac{2}{3}E(\varphi_2) + \frac{1}{3}(\zeta_6 - 3)(-\zeta_6 + 1)E(\varphi_3)\\
&= \frac{1}{3}(-3\zeta_6 + 1)E(\varphi_1) - \frac{2}{3}E(\varphi_2) + \frac{1}{3}(3\zeta_6 - 2)E(\varphi_3)\\
&= -z_1,\\
t.z_3 &= \frac{1}{3}(-2\zeta_6 + 3)\varphi_1(t)E(\varphi_1) + \frac{2}{3}\varphi_2(t)E(\varphi_2) + \frac{1}{3}(2\zeta_6 + 1)\varphi_3(t)E(\varphi_3)\\
&= \frac{1}{3}(-2\zeta_6 + 3)\zeta_6 E(\varphi_1) - \frac{2}{3}E(\varphi_2) + \frac{1}{3}(2\zeta_6 + 1)(-\zeta_6 + 1)E(\varphi_3)\\
&= \frac{1}{3}(\zeta_6 + 2)E(\varphi_1) - \frac{2}{3}E(\varphi_2) + \frac{1}{3}(-\zeta_6 + 3)E(\varphi_3)\\
&= -z_2.
\end{aligned}$$

□

Note that the resulting action on $\mathbb{Z}[z_1, z_2, z_3]$ makes it isomorphic as a $\mathbb{Z}/6 \cong \mathbb{Z}/2 \times \mathbb{Z}/3$-representation to $\mathbb{Z}^{\mathrm{sign}} \otimes \mathbb{Z}[z_1, z_2, z_3]$, where $\mathbb{Z}^{\mathrm{sign}}$ is the permutation representation of $\mathbb{Z}/2$ and $\mathbb{Z}/3$ acts on $\mathbb{Z}[z_1, z_2, z_3]$ now permuting the variables as indicated above. Thus in any degree k, the $\mathbb{Z}/3$-module $\mathbb{Z}[z_1, z_2, z_3]_k$ can be decomposed as $\mathbb{Z}/3$-module as a direct sum of permutation modules, generated by orbits of monomials of degree k. Note also that the orbit of a monomial $z_1^i z_2^j z_3^k$ has exactly three elements (and we have a free $\mathbb{Z}/3$-action) unless $i = j = k$. So the cohomology of the summands of the former type will be concentrated in cohomological degree 0, whereas the submodules $\mathbb{Z}\langle z_1^k z_2^k z_3^k\rangle$ have trivial $\mathbb{Z}/3$-action and hence

$$H^j(\mathbb{Z}/3, \mathbb{Z}\langle z_1^k z_2^k z_3^k\rangle) = \begin{cases} \mathbb{Z}, & \text{for } j = 0, \\ \mathbb{Z}/3, & \text{for } j > 0 \text{ even}, \\ 0, & \text{for } j \text{ odd}. \end{cases}$$

Now observe that $\sigma_2 := z_1z_2 + z_2z_3 + z_3z_1$ is an invariant element under this action (and also under the $\mathbb{Z}/2$-action), so we have a short exact sequence of $\mathbb{Z}[\mathbb{Z}/3]$-modules

$$0 \to \mathbb{Z}[z_1, z_2, z_3]\sigma_2 \to \mathbb{Z}[z_1, z_2, z_3] \to \mathbb{Z}[z_1, z_2, z_3]/(\sigma_2) \to 0.$$

Note that the module $\mathbb{Z}[z_1, z_2, z_3]\sigma_2$ admits a decomposition into permutation modules shifted from the one of $\mathbb{Z}[z_1, z_2, z_3]$. The long exact cohomology sequence yields the following, taking into account the internal grading:

Lemma 2.6. *For $j > 0$, we have:*

$$H^j(\mathbb{Z}/3, (\mathbb{Z}[z_1, z_2, z_3]/(\sigma_2))_k) \cong \begin{cases} \mathbb{Z}/3, & \textit{for } j \textit{ even and } k \equiv 0 \mod 3 \textit{ and } k \geq 0 \textbf{ or} \\ & j \textit{ odd and } k \equiv 2 \mod 3 \textit{ and } k \geq 0 \\ 0, & \textit{else.} \end{cases}$$

For the full group $\mathbb{Z}/6$, we obtain for $j > 0$ furthermore:

$$H^j(\mathbb{Z}/6, (\mathbb{Z}[z_1, z_2, z_3]/(\sigma_2))_k) \cong \begin{cases} \mathbb{Z}/3, & \textit{for } j \textit{ even and } k \equiv 0 \mod 6 \textit{ and } k \geq 0 \textbf{ or} \\ & j \textit{ odd and } k \equiv 2 \mod 6 \textit{ and } k \geq 0 \\ 0, & \textit{else.} \end{cases}$$

Last, after we invert Δ , we obtain for $j > 0$

$$H^j(\mathbb{Z}/6, (\mathbb{Z}[z_1, z_2, z_3]/(\sigma_2))_k) \cong \begin{cases} \mathbb{Z}/3, & \textit{for } j \textit{ even and } k \equiv 0 \mod 6 \textbf{ or} \\ & j \textit{ odd and } k \equiv 2 \mod 6 \\ 0, & \textit{else.} \end{cases}$$

Proof. The first statement was already explained above.

For the second statement, we use e.g. the Hochschild–Lyndon–Serre spectral sequence. Thus, the cohomology groups of $\mathbb{Z}/6$ are the $\mathbb{Z}/2$-fixed points of

$$H^j(\mathbb{Z}/3, (\mathbb{Z}[z_1, z_2, z_3]/(\sigma_2))_k),$$

i.e. just the even degrees k. □

Last, we compute the invariants of the $\mathbb{Z}/3$-action in $\mathbb{Z}[z_1, z_2, z_3]/(\sigma_2)$. For this, consider first the $\mathbb{Z}/3$-action on $\mathbb{Z}[z_1, z_2, z_3]$. This is done in the following well-known lemma:

Lemma 2.7. *Let $\sigma_1, \sigma_2, \sigma_3$ denote the elementary symmetric polynomials in z_1, z_2, z_3. Then the invariants of the $\mathbb{Z}/3 \cong \mathcal{A}_3$-action on $\mathbb{Z}[z_1, z_2, z_3]$ are a free module over $\mathbb{Z}[\sigma_1, \sigma_2, \sigma_3]$ with basis $1, p = z_1^2z_2 + z_2^2z_3 + z_3^2z_1$.*

Remark 2.8. *A similar/better description can be found in Lemma 3.4 of* [12].

Proof. We follow the proof of the Proposition 1.1.3 in [33]. Observe that $\mathbb{Z}[z_1, z_2, z_3]$ is a module over $\mathbb{Z}[\sigma_1, \sigma_2, \sigma_3]$, and that its submodule generated by $1, p$ is surely among $\mathbb{Z}/3$-invariants. We still have to show that all $\mathbb{Z}/3$-invariants lie in this submodule, and that $1, p$ are linearly independent over $\mathbb{Z}[\sigma_1, \sigma_2, \sigma_3]$.

Let $\tau \in S_3$ be any transposition. Then we observe that

$$\begin{aligned} p + \tau p = & \quad \sigma_1\sigma_2 - 3\sigma_3, \\ p - \tau p = & \quad (z_1 - z_2)(z_2 - z_3)(z_1 - z_3). \end{aligned}$$

In particular, it follows that $(z_1-z_2)(z_2-z_3)(z_1-z_3) = 2p-(\sigma_1\sigma_2-3\sigma_3)$. Let now f be any polynomial in $\mathbb{Z}[z_1, z_2, z_3]^{\mathbb{Z}/3}$. Then the polynomial $f+\tau f$ is symmetric, thus can be written as a polynomial $Q(\sigma_1, \sigma_2, \sigma_3)$ in elementary symmetric polynomials.

Next, $f' := f - \tau f$ has the property

$$f'(z_{\eta(1)}, z_{\eta(2)}, z_{\eta(3)}) = \operatorname{sgn}(\eta) f'(z_1, z_2, z_3)$$

for any $\eta \in S_3$ (i.e. f' is alternating) and thus is divisible by $(z_1 - z_2)(z_2 - z_3)(z_1 - z_3)$, and the quotient is then a symmetric polynomial, thus can be written as $Q'(\sigma_1, \sigma_2, \sigma_3)$.

So in $\mathbb{Q}[z_1, z_2, z_3]$, we obtain the following identity for f:

$$\begin{aligned} f = & \quad \frac{1}{2}((f + \tau f) + (f - \tau f)) \\ = & \quad \frac{1}{2}Q(\sigma_1, \sigma_2, \sigma_3) + \frac{1}{2}Q'(\sigma_1, \sigma_2, \sigma_3)(z_1 - z_2)(z_2 - z_3)(z_1 - z_3) \\ = & \quad \frac{1}{2}Q(\sigma_1, \sigma_2, \sigma_3) + \frac{1}{2}Q'(\sigma_1, \sigma_2, \sigma_3)(2p - (\sigma_1\sigma_2 - 3\sigma_3)) \\ = & \quad \frac{1}{2}(Q(\sigma_1, \sigma_2, \sigma_3) - Q'(\sigma_1, \sigma_2, \sigma_3)(\sigma_1\sigma_2 - 3\sigma_3)) + Q'(\sigma_1, \sigma_2, \sigma_3)p. \end{aligned}$$

Thus, the polynomial $f - Q'(\sigma_1, \sigma_2, \sigma_3)p$ is symmetric on the one hand, since it can be written as a polynomial in $\sigma_1, \sigma_2, \sigma_3$ over $\mathbb{Q}$. On the other hand, it is integral as a sum of two integral polynomials, thus in total, it lies in $\mathbb{Z}[\sigma_1, \sigma_2, \sigma_3]$, proving that $1, p$ are a generating system of all $\mathbb{Z}/3$-invariants as a a module over $\mathbb{Z}[\sigma_1, \sigma_2, \sigma_3]$.

Last, we have to check that $1, p$ are linearly independent over $\mathbb{Z}[\sigma_1, \sigma_2, \sigma_3]$. Assume on contrary that there are $P, Q \in \mathbb{Z}[\sigma_1, \sigma_2, \sigma_3]$ with $P + pQ = 0$. Thus we also have $P + \tau p \cdot Q = 0$, and this yields

$$(p - \tau p)Q = 0.$$

Since $p - \tau p \neq 0$ and $\mathbb{Z}[z_1, z_2, z_3]$ is an integral domain, we conclude that $Q = 0$ and thus $P = 0$. This completes the proof. □

Corollary 2.9. *The invariants $H^0(\mathbb{Z}/6, \mathbb{Z}[z_1, z_2, z_3]/\sigma_2)$ are the even degrees of the free $\mathbb{Z}[\sigma_1, \sigma_3]$-module on 1 and p.*

Proof. This follows, as $H^1(\mathbb{Z}/6, \mathbb{Z}[z_1, z_2, z_3]) = 0$ in every degree. □

Recall that we work over $\mathbb{Z}_{(3)}$ here and that we use the notation $\mathcal{O}$ for the structure sehaf of $\mathcal{M}_{ell}$.

Corollary 2.10. *The cohomology of $h_*h^*\mathcal{O} \otimes \underline{\omega}^{\otimes *}$ is given for $j > 0$ by*

$$H^j(\mathcal{M}_{ell}, h_*h^*\mathcal{O} \otimes \underline{\omega}^{\otimes k}) \cong \begin{cases} \mathbb{Z}/3, & \textit{for } j \textit{ even and } k \equiv 0 \mod 6 \textit{ \textbf{or}} \\ & j \textit{ odd and } k \equiv 2 \mod 6 \\ 0, & \textit{else.} \end{cases}$$

Moreover, $H^0(\mathcal{M}_{ell}, h_*h^*\mathcal{O}\otimes\underline{\omega}^{\otimes *})$ *is isomorphic to* $\mathbb{Z}_{(3)}[\sigma_1,\sigma_3,\Delta^{-1}]\oplus\mathbb{Z}_{(3)}[\sigma_1,\sigma_3,\Delta^{-1}]p$.

Proof. Since h is affine, h_* is an exact functor, and since h is separated and quasi-compact, we can also apply the projection formula. Altogether, we obtain

$$H^j(\mathcal{M}_{ell}, h_*h^*\mathcal{O}\otimes\underline{\omega}^{\otimes k}) \cong H^j(\mathcal{M}_0(7), \underline{\omega}^{\otimes k}_{\mathcal{M}_0(7)}).$$

We use Galois descent and the fact that $\mathcal{M}_1(7)$ is affine. This implies

$$H^j(\mathcal{M}_0(7), \underline{\omega}^{\otimes k}_{\mathcal{M}_0(7)}) \cong H^j((\mathbb{Z}/7)^\times, \Gamma(\underline{\omega}^{\otimes k}_{\mathcal{M}_1(7)})).$$

Last, we have $\Gamma(\underline{\omega}^{\otimes k}_{\mathcal{M}_1(7)}) \cong \mathrm{MF}_k(\Gamma_1(7),\mathbb{Z}_3) \cong \mathbb{Z}_3[z_1,z_2,z_3,\Delta^{-1}]/(\sigma_2)$. So the claim follows from Lemma 2.6 and Corollary 2.9. □

Remark 2.11. *Since the cohomology of the known indecomposable vector bundles* $\underline{\omega}^{\otimes *}, E_\alpha\otimes\underline{\omega}^{\otimes *}, f_*f^*\mathcal{O}\otimes\underline{\omega}^{\otimes *}$ *on* $\mathcal{M}_{ell}$ *over* $\mathbb{Z}_{(3)}$ *is known (see e.g.* [27], *Sections 4.1 and 4.2), we can conclude that if* $h_*h^*\mathcal{O}$ *can be decomposed as a sum of such, it has necessarily exactly the summands* $\mathcal{O}\oplus\underline{\omega}^{\otimes -6}$ *and two shifted copies of* $f_*f^*\mathcal{O}$.

3. q-expansion of α_i as $\Gamma_1(7)$-modular forms using Tate curve

The aim of this section is to obtain q-expansions for the coefficients of the Weierstraß equations of elliptic curve with a $\Gamma_1(7)$-level structure. It is known that for such curves, the coefficients of the Weierstraß equations yield at least meromorphic $\Gamma_1(7)$-modular forms. Our computations show that they are indeed holomorphic, and thus we can identify them unter the isomorphism of Proposition 2.4 with polynomials in z_1, z_2, z_3.

We start with the following classical theorem.

Theorem 3.1. *(*[31]*, Theorem V.1.1) For any* $q,u\in\mathbb{C}$ *with* $|q|<1$*, define the following quantities:*

$$\begin{aligned}
\sigma_k(n) &= \sum_{d|n} d^k,\\
s_k(q) &= \sum_{n\geq 1}\sigma_k(n)q^n = \sum_{n\geq 1}\frac{n^kq^n}{1-q^n},\\
a_4(q) &= -5s_3(q),\\
a_6(q) &= -\frac{5s_3(q)+7s_5(q)}{12},\\
X(u,q) &= \sum_{n\in\mathbb{Z}}\frac{q^nu}{(1-q^nu)^2} - 2s_1(q),\\
Y(u,q) &= \sum_{n\in\mathbb{Z}}\frac{(q^nu)^2}{(1-q^nu)^3} + s_1(q).
\end{aligned}$$

(1) Then the equation

$$y^2+xy = x^3+a_4(q)x+a_6(q)$$

defines an elliptic curve E_q over $\mathbb{C}$, and X, Y define a complex analytic isomorphism

$$\begin{aligned} \mathbb{C}^\times/q^{\mathbb{Z}} \to \quad & E_q \\ u \mapsto \quad & \begin{cases} (X(u,q), Y(u,q)), & \text{if } u \notin q^{\mathbb{Z}}, \\ O, & \text{if } u \in q^{\mathbb{Z}} \end{cases} \end{aligned}$$

(2) *As power series in q, both $a_4(q), a_6(q)$ have integer coefficients.*
(3) *Every elliptic curve over $\mathbb{C}$ is isomorphic to E_q for some q with $|q| < 1$.*

Remark 3.2. *The complex analytic isomorphism is automatically also an isomorphism of abelian groups (cf. e.g.* [30]*, Theorem VI.5.3).*

Our aim now is to compute the q-expansions of the coefficients α_i of the Tate normal form

$$y^2 + \alpha_1 xy + \alpha_3 y = x^3 + \alpha_2 x^2$$

for an elliptic curve with a chosen N-torsion point. Since these coefficients are unique after we have fixed an invariant differential, and transform accordingly, they indeed define (at least) meromorphic modular forms for $\Gamma_1(N)$ with $N \geq 4$.

We recall that we use the notion of $\Gamma_1(N)$ level structure on an elliptic curve called $\Gamma_{00}(N)^{naive}$ level structure in [17], Section 2. As explained there, the injective q-expansion map is given by evaluation of the modular form at the test object $(\mathrm{Tate}(q^N), \omega_{can}, q)$, where $q \in \mathbb{C}^\times/q^{N\mathbb{Z}}$ is the chosen point of exact order N and ω_{can} is the invariant differential coming from $\frac{dx}{x}$ on $\mathbb{C}^\times$. For more details, see appendix A.

First, observe from [30], formulae in Section III.1, that transforming any Weierstraß equation of the form

$$y^2 + xy = x^3 + a_4 x + a_6$$

into Tate normal form with a chosen torsionpoint (x_0, y_0) on this curve moving to $(0,0)$ has the transformation parameter (if they exist in the ring; setting $u = 1$)

$$\begin{aligned} r = \quad & x_0, \\ t = \quad & y_0, \\ s = \quad & \frac{a_4 - y_0 + 3x_0^2}{x_0 + 2y_0}, \end{aligned}$$

and the resulting coefficients of the Tate normal form are

$$\begin{aligned} \alpha_1 = \quad & 1 + 2s = \frac{x_0 + 6x_0^2 + 2a_4}{x_0 + 2y_0}, \\ \alpha_2 = \quad & -s - s^2 + 3r, \\ \alpha_3 = \quad & r + 2t = x_0 + 2y_0. \end{aligned}$$

Now we use methods from [31], Section V.3, to simplify the expressions for $X(uq^k, q^N)$ and $Y(uq^k, q^N)$ in our case, where $u \neq 0$ is a complex number and

$0 \leq k < N$.First, we reindex the sum over positive natural numbers:

$$\begin{aligned} X(uq^k, q^N) = & \sum_{n\in\mathbb{Z}} \frac{uq^{nN+k}}{(1-uq^{nN+k})^2} - 2s_1(q^N), \\ = & \frac{uq^k}{(1-uq^k)^2} + \sum_{n\geq 1}\left(\frac{uq^{nN+k}}{(1-uq^{nN+k})^2} + \frac{u^{-1}q^{nN-k}}{(1-u^{-1}q^{nN-k})^2} - 2\frac{q^{nN}}{(1-q^{nN})^2}\right). \end{aligned}$$

Recall the following formulae for $|x| < 1$, obtained e.g. by differentiating the geometric series:

$$\frac{x}{(1-x)^2} = \sum_{l\geq 1} lx^l \text{ and } \frac{x^2}{(1-x)^3} = \sum_{l\geq 1}\frac{l(l-1)}{2}x^l \text{ and } \frac{x}{(1-x)^3} = \sum_{l\geq 0}\frac{l(l+1)}{2}x^l.$$

Inserting this into the expression for $X(uq^k, q^N)$, we obtain

$$\begin{aligned} X(uq^k, q^N) = & \frac{uq^k}{(1-uq^k)^2} + \sum_{n\geq 1}\left(\frac{uq^{nN+k}}{(1-uq^{nN+k})^2} + \frac{u^{-1}q^{nN-k}}{(1-u^{-1}q^{nN-k})^2} - 2\frac{q^{nN}}{(1-q^{nN})^2}\right) \\ = & \sum_{l\geq 1} lu^l q^{kl} + \sum_{n\geq 1}\sum_{l\geq 1}\left(lu^l q^{(nN+k)l} + lu^{-l}q^{(nN-k)l} - 2lq^{nNl}\right) \end{aligned}$$

Similarly, for $Y(uq^k, q^N)$ we get

$$\begin{aligned} Y(uq^k, q^N) = & \sum_{n\in\mathbb{Z}} \frac{u^2q^{2(nN+k)}}{(1-uq^{nN+k})^3} + s_1(q^N), \\ = & \frac{u^2q^{2k}}{(1-uq^k)^3} + \sum_{n\geq 1}\left(\frac{u^2q^{2(nN+k)}}{(1-uq^{nN+k})^3} - \frac{u^{-1}q^{nN-k}}{(1-u^{-1}q^{nN-k})^3} + \frac{q^{nN}}{(1-q^{nN})^2}\right). \end{aligned}$$

Using again the formulae derived from geometric series, we obtain

$$\begin{aligned} Y(uq^k, q^N) = & \frac{u^2q^{2k}}{(1-uq^k)^3} + \sum_{n\geq 1}\left(\frac{u^2q^{2(nN+k)}}{(1-uq^{nN+k})^3} - \frac{u^{-1}q^{nN-k}}{(1-u^{-1}q^{nN-k})^3} + \frac{q^{nN}}{(1-q^{nN})^2}\right) \\ = & \sum_{l\geq 2}\frac{(l-1)l}{2}u^l q^{kl} + \sum_{n\geq 1}\sum_{l\geq 1}\left(\frac{(l-1)l}{2}u^l q^{(nN+k)l} - \frac{l(l+1)}{2}u^{-l}q^{(nN-k)l} + lq^{nNl}\right) \end{aligned}$$

Note that for every $u \neq 0$, both $X(uq^k, q^N)$ and $Y(uq^k, q^N)$ are not just Laurent series in q, but actually power series. In particular, so is $a_3 = X + 2Y$. For $0 \leq k < \frac{N}{2}$, the term uq^k is the lowest power of q occuring in both in $X(uq^k, q^N)$ and $Y(uq^k, q^N)$ is divisible by q^{k+1}; hence the lowest power of q occuring in a_3 is uq^k.

We would like to show that for every $u \neq 0$ and for every $0 < k < N$, also the resulting values of a_1 and a_2 turn out to be power series and not only general Laurent series in q. To this end, we determine the lowest power of q in a non-vanishing summand in a_3. In our Tate curve, we have

$$a_4(q^N) = -5s_3(q^N)\sum_{n\geq 1}\frac{n^3q^{nN}}{1-q^{nN}} = \sum_{n\geq 1}\left(n^3\sum_{l\geq 1}q^{nNl}\right),$$

so this power series has $N > k$ as lowest exponent of q.

Given the expressions for a_1, a_2, we only need to check that

$$s = \frac{a_4(q^N) - X(uq^k, q^N) + 3X(uq^k, q^N)^2}{X(uq^k, q^N) + 2Y(uq^k, q^N)}$$

is a power series. This follows from the analysis above.

Inserting these values into procedure above and comparing the results with our previously chosen basis of $\mathrm{mf}_*(\Gamma_1(7), \mathbb{Z}[\frac{1}{7}])$ using MAGMA, we obtain

$$\begin{aligned} \alpha_1 &= \quad z_1 - z_2 + z_3, \\ \alpha_2 &= \quad z_1z_2 + z_1z_3, \\ \alpha_3 &= \quad z_1z_3^2. \end{aligned}$$

4. The flatness of $\mathcal{M}_0(7)_{cub}$

We will work throughout this section over a $\mathbb{Z}[\frac{1}{n}]$-algebra A. Set

$$B = A[a_1, \ldots, a_4, a_6].$$

Let $\mathcal{M}_{cub} = \mathcal{M}_{cub,A}$ be the algebraic stack associated with the graded Weierstraß Hopf algebroid $(B, \Gamma = B[r,s,t])$ (see e.g. [1] for the precise structure maps).This contains $\mathcal{M}_{ell} = \mathcal{M}_{ell,A}$ as an open substack as it is the stack associated with the graded Weierstraß Hopf algebroid $(B[\Delta^{-1}], \Gamma[\Delta^{-1}])$. We want to extend the moduli stacks $\mathcal{M}_0(n) = \mathcal{M}_0(n)_A$ and $\mathcal{M}_1(n) = \mathcal{M}_1(n)_A$ to algebraic stacks that are finite over $\mathcal{M}_{cub}$ via a normalization construction.

Let us recall the notion of normalization. Let $\mathcal{X}$ be an Artin stack and $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_\mathcal{X}$-algebras. Let $\mathcal{A}' \subset \mathcal{A}$ be the presheaf that evaluated on any $\operatorname{Spec} C$ smooth over $\mathcal{X}$ consists of those elements in $\mathcal{A}(\operatorname{Spec} C)$ that are integral over A. This is an fpqc (and in particular étale) sheaf because being integral for an element can be tested fpqc-locally (as generating a finite module can be checked fpqc-locally). Thus, we obtain a sheaf on the lisse-étale site of $\mathcal{X}$ (see [21, Section 12] or [32, Tag 0786] for the definition). As relative normalization commutes with localization, $\mathcal{A}'$ is a quasi-coherent sheaf after pullback to every smooth $\operatorname{Spec} C \to \mathcal{X}$ and thus quasi-coherent by definition. We define the *normalization of $\mathcal{X}$ in $\mathcal{A}$* to be the relative Spec of $\mathcal{A}'$ over $\mathcal{X}$. For a quasi-compact and quasi-separated morphism $f\colon \mathcal{Y} \to \mathcal{X}$, we define the normalization of $\mathcal{X}$ in $\mathcal{Y}$ to be the normalization of $\mathcal{X}$ in $f_*\mathcal{O}_\mathcal{Y}$ (here, f_* denotes the pushforward of quasi-coherent sheaves as in [32, Tag 070A]).

Lemma 4.1. *Relative normalization commutes with smooth base change.*

Proof. The case of schemes is treated in [32, Tag 03GV], and the general case is similar. □

We define $\mathcal{M}_0(n)_{cub}$ and $\mathcal{M}_1(n)_{cub}$ as the normalizations of $\mathcal{M}_{cub}$ in $\mathcal{M}_0(n)$ and $\mathcal{M}_1(n)$. Note that the normalization maps are by definition affine. If we demand more of A, we get even finiteness. More precisely, we will need that A is quasi-excellent. The source [32, Tag 07QS] contains everything we will need about quasi-excellent rings; in particular, they show that $\mathbb{Z}[\frac{1}{n}]$ is quasi-excellent.

Lemma 4.2. *Assume that A is a quasi-excellent ring. Then $\mathcal{M}_1(n)_{cub} \to \mathcal{M}_{cub}$ and $\mathcal{M}_0(n)_{cub} \to \mathcal{M}_{cub}$ are finite.*

Proof. Let $\mathcal{X} \to \mathcal{M}_{ell}$ be an affine map of finite type from a reduced algebraic stack (note that reducedness is local in the smooth topology [32, Tag 034E]). We want to show that the normalization of $\mathcal{M}_{cub}$ in $\mathcal{X}$ is finite over $\mathcal{M}_{cub}$. The relevant cases for us are $\mathcal{X} = \mathcal{M}_1(n)$ and $\mathcal{X} = \mathcal{M}_0(n)$.

Let $B = A[a_1, \ldots, a_6] \to \mathcal{M}_{cub}$ be the usual smooth cover. Denote by T the global sections of the pullback $\mathcal{X} \times_{\mathcal{M}_{cub}} \operatorname{Spec} B$, which is an affine scheme. By the last lemma, we have to show that the normalization of B in T is finite over B (as finiteness can be checked after faithfully flat base change). By [32, Tag 03GR], we just have to check that B is a Nagata ring, $\operatorname{Spec} T \to \operatorname{Spec} B$ is of finite type and T is reduced. As B is a polynomial ring over a quasi-excellent ring, it is quasi-excellent again and hence Nagata. The second point is clear by base change. For the last one note that $\operatorname{Spec} T$ is equivalent to $\operatorname{Spec} B[\Delta^{-1}] \times_{\mathcal{M}_{ell}} \mathcal{X}$; this is reduced as $\operatorname{Spec} B[\Delta^{-1}] \to \mathcal{M}_{ell}$ is smooth (as smoothness can be checked after faithfully flat base change and $\operatorname{Spec} \Gamma[\Delta^{-1}] \simeq \operatorname{Spec} B[\Delta^{-1}] \times_{\mathcal{M}_{ell}} \operatorname{Spec} B[\Delta^{-1}]$ is smooth over $\operatorname{Spec} B[\Delta^{-1}]$) and being reduced is local in the smooth topology. Hence, we have finiteness. □

Lemma 4.3. *Let R be a graded normal domain and with a graded ring map $B \to R$. Consider the induced map*

$$\operatorname{Spec} R[\Delta^{-1}]/\mathbb{G}_m \to \operatorname{Spec} B[\Delta^{-1}]/\mathbb{G}_m \to \mathcal{M}_{ell}.$$

Then the normalization of $\mathcal{M}_{cub}$ in $\operatorname{Spec} R[\Delta^{-1}]/\mathbb{G}_m$ is equivalent to $\operatorname{Spec} R/\mathbb{G}_m$ if $R_B = R \otimes_B \Gamma$ is finite over B.

Proof. Note first that $\operatorname{Spec} B \times_{\mathcal{M}_{cub}} \operatorname{Spec} R/\mathbb{G}_m$ is equivalent to $\operatorname{Spec} R_B$. Thus, $\operatorname{Spec} R/\mathbb{G}_m$ is finite over $\mathcal{M}_{cub}$.

Let now $\operatorname{Spec} C \to \mathcal{M}_{cub}$ be any smooth map and denote by $\operatorname{Spec} R_C$ the fiber product $\operatorname{Spec} C \times_{\mathcal{M}_{cub}} \operatorname{Spec} R/\mathbb{G}_m$. As R_C is finite over C, every element of R_C is integral over C. As R is normal and R_C is smooth over R, also R_C is normal [32, Tag 033C]. Thus, every element that is integral over C (and hence R_C) in $R_C[\Delta^{-1}]$ is already in R_C. Thus, R_C is the normalization of C in $R_C[\Delta^{-1}]$. As $\operatorname{Spec} R_C[\Delta^{-1}]$ is the equivalent to the fiber product $\operatorname{Spec} C \times_{\mathcal{M}_{cub}} \operatorname{Spec} R[\Delta^{-1}]/\mathbb{G}_m$, this shows the result. □

Lemma 4.4. *Let R be a graded B-algebra that is Cohen–Macaulay. Assume furthermore that R is concentrated in nonnegative degrees and satisfies $R_0 = A$. Then R_B is flat over B if it is finite.*

Proof. As $R_B \cong R[r, s, t]$, we see that R_B is Cohen–Macaulay as well. By localizing A we can assume that A is local and thus B and R_B are graded local rings as well. As R_B is finite over B, we see that $\dim R = \dim R_B$. We obtain by a graded version of Hironaka's flatness criterion (see e.g. [11, Theorem 18.16]) that R_B is flat over B. □

Proposition 4.5. *The maps $\overline{\mathcal{M}}_1(n) \to \overline{\mathcal{M}}_{ell}$ and $\overline{\mathcal{M}}_0(n) \to \overline{\mathcal{M}}_{ell}$ are finite and flat.*

Proof. This is contained in Theorem 4.1.1 of [5]. □

Proposition 4.6. *Assume that $R = \operatorname{mf}(\Gamma_1(n), A)_* = mf_1(n)_*$ is normal and Cohen–Macaulay. Then $\mathcal{M}_1(n)_{cub} \to \mathcal{M}_{cub}$ is finite and flat and $\mathcal{M}_1(n)_{cub}$ is equivalent to $\operatorname{Spec} R/\mathbb{G}_m$.*

Proof. We can assume that $A = \mathbb{Z}[\frac{1}{n}]$ as this is the universal case. We know that $\mathcal{M}_1(n) = \operatorname{Spec} \operatorname{mf}(\Gamma_1(n))[\Delta^{-1}]/\mathbb{G}_m$ for all $n \geq 2$. Thus, we just have to show that R_B is finite over B.

Consider the two cartesian squares

$$\begin{tikzcd} \overline{\mathcal{M}}_1(n) \arrow[r, "j"] \arrow[d, "h"] & \operatorname{Spec} R/\mathbb{G}_m \arrow[d, "\tilde{h}"] \\ \overline{\mathcal{M}}_{ell} \arrow[r, "i"] & \mathcal{M}_{cub} \end{tikzcd}$$

and

$$\begin{tikzcd} U \arrow[r, "k"] \arrow[d, "q"] & \operatorname{Spec} B \arrow[d, "p"] \\ \overline{\mathcal{M}}_{ell} \arrow[r, "i"] & \mathcal{M}_{cub} \end{tikzcd}$$

As a quasi-coherent sheaf on $\operatorname{Spec} R/\mathbb{G}_m$ is determined by its graded global sections, we see that $j_*\mathcal{O}_{\overline{\mathcal{M}}_1(n)}$ is exactly $\mathcal{O}_{\operatorname{Spec} R/\mathbb{G}_m}$. We see that R_B are the global sections of

$$p^*\tilde{h}_*j_*\mathcal{O}_{\overline{\mathcal{M}}_1(n)} \cong p^*i_*h_*\mathcal{O}_{\overline{\mathcal{M}}_1(n)}.$$

As p is flat, we have an isomorphism $p^*i_*h_*\mathcal{O}_{\overline{\mathcal{M}}_1(n)} \cong k_*q^*h_*\mathcal{O}_{\overline{\mathcal{M}}_1(n)}$. As h is finite flat by the last proposition, $q^*h_*\mathcal{O}_{\overline{\mathcal{M}}_1(n)}$ is a vector bundle. As $\operatorname{Spec} B$ is normal and the complement $V(c_4, \Delta)$ of U is codimension 2, we see that $k_*\mathcal{F}$ is reflexive and hence coherent for any reflexive sheaf $\mathcal{F}$ on U. Indeed, we can extend $\mathcal{F}$ by a reflexive sheaf $\mathcal{E}$ on $\operatorname{Spec} B$ (by picking a coherent subsheaf $\mathcal{E}'$ of $k_*\mathcal{F}$ with $k^*\mathcal{E}' \cong \mathcal{F}$ and setting $\mathcal{E}$ to be the double-dual of $\mathcal{E}'$). By [14, Proposition 1.6], we see that $k_*\mathcal{F} \cong k_*k^*\mathcal{E} \cong \mathcal{E}$. If we apply this argument to $\mathcal{F} = q^*h_*\mathcal{O}_{\overline{\mathcal{M}}_1(n)}$, we see that $k_*q^*h_*\mathcal{O}_{\overline{\mathcal{M}}_1(n)}$ is coherent and hence its global sections R_B are finitely generated over B. □

Example 4.7. *Let $n = 7$. Then*

$$\operatorname{mf}_1(7)_* = \operatorname{mf}(\Gamma_1(7); A) \cong A[z_1, z_2, z_3]/\sigma_2.$$

This is a regular ring (and in particular normal and Cohen–Macaulay). Thus, the assumptions of Proposition 4.6 are true.

The flatness of $\mathcal{M}_0(n)_{cub} \to \mathcal{M}_{cub}$ seems to be more subtle in general and we will be content here to introduce some notation and make a simple observation.

We know that $\mathcal{M}_0(n)_{cub} \times_{\mathcal{M}_{cub}} \operatorname{Spec} B$ is affine and thus we can write it as $\operatorname{Spec} S_B$ for a B-algebra S_B. We claim that the map $S_B \to (R_B)^{(\mathbb{Z}/n)^\times}$ is an isomorphism. Indeed: By construction, $S_B \to S_B[\Delta^{-1}]$ is an injection (because S_B consists of those elements in $S_B[\Delta^{-1}]$ that are integral over B). We know that

$$R_B^{(\mathbb{Z}/n)^\times}[\Delta^{-1}] = (R_B[\Delta^{-1}])^{(\mathbb{Z}/n)^\times} \cong S_B[\Delta^{-1}].$$

Thus, the map

$$S_B \to R_B^{(\mathbb{Z}/n)^\times} \to R_B^{(\mathbb{Z}/n)^\times}[\Delta^{-1}] \cong S_B[\Delta^{-1}]$$

is an injection and thus also the first arrow. We know that $(R_B)^{(\mathbb{Z}/n)^\times}$ is finite over B (as B is noetherian) and thus every element is integral over B. Thus, $S_B = R_B^{(\mathbb{Z}/n)^\times}$.

5. Computation of invariants

We consider the elements $1, \sigma_1^2, \sigma_1^4, \sigma_3^2$ as well as

$$\begin{aligned}
n_4 &:= \sigma_1^2 r - z_1^3 z_3 - z_1 z_2^3 - z_1^2 z_3^2,\\
\sigma_1^2 n_4 &= \sigma_1^4 r - \sigma_1^2 \cdot (z_1^3 z_3 + z_1 z_2^3 + z_1^2 z_3^2),\\
n_6 &:= \sigma_1^2 r^2 - 2z_1^3 z_3 r - 2z_1 z_2^3 r - 2z_1^2 z_3^2 r + 2z_1^3 z_3^3 - z_1^2 z_3^4\\
&= 2n_4 r - \sigma_1^2 r^2 + 2z_1^3 z_3^3 - z_1^2 z_3^4,\\
\sigma_1^2 n_6 &= \sigma_1^2 \cdot (\sigma_1^2 r^2 - 2z_1^3 z_3 r - 2z_1 z_2^3 r - 2z_1^2 z_3^2 r + 2z_1^3 z_3^3 - z_1^2 z_3^4),
\end{aligned}$$

which are all elements in S_B. Our aim is to prove the following proposition.

Proposition 5.1. *The elements*

$$1, \sigma_1^2, \sigma_1^4, n_4, \sigma_1^2 n_4, n_6, \sigma_1^2 n_6, \sigma_3^2$$

form a B-basis of S_B; in particular, S_B is a free B-module of rank 8.

We can write the invariants with respect to our chosen B-basis of $R_B \cong B^{48}$ (found with MAGMA):

$$\begin{aligned}
&\{1, z_2, z_3, s, z_2 s, z_3^2, z_3 s, s^2, r, z_3^2 s, z_3 s^2, z_3 r, s^3, t, rs, z_2 r, z_3^2 s^2, z_3 rs, z_3 t, s^4, rs^2, r^2,\\
&z_3^2 r, z_2 rs, z_2 r^2, z_3^2 rs, z_3 rs^2, z_3 r^2, s^5, r^2 s, rt, rs^3, z_2 r^2 s, z_3^2 rs^2, z_3 r^2 s, z_3^2 r^2,\\
&z_3 rt, rs^4, r^2 s^2, z_3 r^2 s^2, z_3^2 r^2 s, rs^5, r^2 s^3, r^2 t, z_3^2 r^2 s^2, z_3 r^2 t, r^2 s^4, r^2 s^5\}.
\end{aligned}$$

We want to show that the 8 invariants listed above are B-linearly independent elements of R_B. To do so, we will show that they are linearly independent when viewed as elements of $R_B \otimes \mathbb{Q}/(a_1, a_2, a_3, a_4, a_6)$, which is enough since R_B is a free B-module (and B is an integral domain).

In $R_B \otimes \mathbb{Q}/(a_1, a_2, a_3, a_4, a_6)$, we have the following expressions in terms of basis elements (computed by MAGMA):

$$\begin{aligned}
1 =&1\\
\sigma_1^2 =&4z_3^2 - 8z_3 s + 12r\\
\sigma_1^4 =&48z_3^2 r + 16z_3^2 s^2 - 192z_3 rs + 384r^2 - 96rs^2 + 16s^4\\
n_4 =&4z_3^2 r + 4z_3^2 s^2 - 32z_3 rs + 4z_3 t + 54r^2 - 12rs^2 + 2s^4\\
n_6 =&60/7 z_2 r^2 s - 236/7 z_3^2 r^2 + 116/7 z_3^2 rs^2 - 38 z_3 r^2 s + 260/7 z_3 rt + 36/7 r^2 s^2 + 4rs^4\\
\sigma_1^2 n_4 =&1188/7 z_2 r^2 s - 5244/7 z_3^2 r^2 + 2644/7 z_3^2 rs^2\\
&- 864 z_3 r^2 s + 5400/7 z_3 rt + 1032/7 r^2 s^2 + 88 rs^4\\
\sigma_3^2 =&-15/7 z_2 r^2 s + 45/7 z_3^2 r^2 - 15/7 z_3^2 rs^2 + 3z_3 r^2 s - 72/7 z_3 rt + 12/7 r^2 s^2\\
\sigma_1^2 n_6 =&52768/355 z_3^2 r^2 s^2 - 406572/355 z_3 r^2 t + 61336/355 r^2 s^4
\end{aligned}$$

In the 48×8-matrix which has the following 8×8-submatrix, corresponding to basis elements $1, z_3^2, s^4, z_3 t, r^2 s^2, z_3 r^2 s, r s^4, r^2 s^4$:

$$\begin{pmatrix} 1 & * & * & * & * & * & * & * \\ 0 & 4 & * & * & * & * & * & * \\ 0 & 0 & 16 & * & * & * & * & * \\ 0 & 0 & 0 & 4 & * & * & * & * \\ 0 & 0 & 0 & 0 & \frac{36}{7} & \frac{1032}{7} & \frac{12}{7} & * \\ 0 & 0 & 0 & 0 & -38 & -864 & 3 & * \\ 0 & 0 & 0 & 0 & 4 & 88 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{61336}{355} \end{pmatrix}.$$

Here, by $*$ we denote (potentially different) ring elements which do not matter for the computation of the determinant. Again using MAGMA, this determinant can be computed to be

$$\frac{66325315584}{2485} \neq 0.$$

This implies that the 8 invariants listed above are B-linearly independent elements of R_B, so they generate a free sub-B-module of R_B of rank 8, which we denote by V.

Our next goal is to show that this module is already all of S_B, (identified with) invariants of R_B under $\mathbb{Z}/6$-action.

First, observe that the inclusion induces an isomorphism over $B \otimes \mathbb{Q}$. Indeed, recall first that $S_{B\otimes\mathbb{Q}} \cong S_B \otimes \mathbb{Q}$ using flatness of $\mathbb{Q}$ over $\mathbb{Z}_{(3)}$ and the fact that $H^1(\mathbb{Z}/6, \mathbb{Q}) = 0$. As the order of $\mathbb{Z}/6$ is invertible in $\mathbb{Q}$, we know that $S_{B\otimes\mathbb{Q}}$ is a direct summand of the free $B \otimes \mathbb{Q}$-module $R_{B\otimes\mathbb{Q}}$ and thus projective. By the Quillen-Suslin Theorem (see e.g. [20], Theorem XXI.3.7), it implies that $S_{B\otimes\mathbb{Q}}$ is also free, automatically of rank 8 as this is true after inverting Δ. We claim that $V \otimes \mathbb{Q} \to S_{B\otimes\mathbb{Q}}$ is surjective. It is enough to show this after quotiening by $a_1, \ldots, a_6$ by the graded Nakayama lemma. The 8×8-minor argument above shows that $V \otimes \mathbb{Q} \to R_B \otimes \mathbb{Q}$ is injective after quotiening by $a_1, \ldots, a_6$ and thus the same is true for

$$V \otimes_B B/(a_1, \ldots, a_6) \otimes \mathbb{Q} \to S_B \otimes_B B/(a_1, \ldots, a_6) \otimes \mathbb{Q}.$$

As an injective map between 8-dimensional $\mathbb{Q}$-vector spaces, it must be surjective as well, showing our claim.

Thus, it is enough to show that the map from the free B-module generated by the 8 invariants to S_B (or to R_B) is injective when we tensor it with $\mathbb{F}_3$. (This will imply surjectivity. Indeed, let $x \in S_B$ be some element. By the rational statement, we know that there is an element y in the B-span of the invariants above and $k \in \mathbb{N}$ s.t. $3^k x = y$. If $k = 0$, we are done; otherwise we can conclude that y is mapped to 0 in R_B after tensoring with $\mathbb{F}_3$, so by injectivity we assumed it can be divided by 3 in the module generated by our chosen invariants. Inductively, this implies the claim.)

Thus we do a similar computation over $\mathbb{F}_3$ as above rationally. We use the same basis for $R_B \otimes \mathbb{F}_3$ over $B \otimes \mathbb{F}_3$. As before, we use MAGMA to express the chosen invariants in terms of the basis.

$$\sigma_1^2 = a_1^2 + 2a_1 z_3 + a_2 + z_3^2 + z_3 s$$
$$\sigma_1^4 = a_1^4 + a_1^3 z_3 + 2a_1^2 a_2 + 2a_1^2 z_3^2 + 2a_1^2 z_3 s + a_1^2 r + a_1^2 s^2$$

$$
\begin{aligned}
&+ a_1a_2z_3 + a_1a_2s + a_1a_3 + a_1z_3^2s + 2a_1s^3 + 2a_2^2 + a_2z_3^2\\
&+ 2a_2z_3s + a_2r + a_2s^2 + 2a_4 + z_3^2s^2 + s^4\\
n_4 =&2a_1^3z_3 + a_1^2z_3^2 + a_1^2z_3s + 2a_1^2r + 2a_1^2s^2 + 2a_1a_2z_3\\
&+ 2a_1a_2s + a_1a_3 + a_1z_3^2s + a_1z_3r + a_1s^3 + 2a_2^2 + a_2z_3s + 2a_2r\\
&+ 2a_2s^2 + 2a_3z_3 + 2a_4 + z_3^2r + z_3^2s^2 + z_3rs + z_3t + 2s^4\\
n_6 =&2a_1^3z_2r + 2a_1^3z_3^2s + 2a_1^3z_3r + a_1^3rs + a_1^2a_2z_3^2\\
&+ 2a_1^2a_2r + 2a_1^2a_3z_2 + a_1^2a_3z_3 + a_1^2a_3s + a_1^2z_2rs\\
&+ 2a_1^2z_3^2s^2 + 2a_1^2z_3rs + 2a_1^2z_3t + 2a_1^2rs^2 + 2a_1a_2a_3\\
&+ 2a_1a_2z_2r + 2a_1a_2z_3^2s + 2a_1a_2z_3r + 2a_1a_2rs + a_1a_3z_2s\\
&+ a_1a_3z_3^2 + 2a_1a_3r + a_1a_3s^2 + a_1a_4z_2 + 2a_1a_4s + 2a_1z_3^2rs\\
&+ 2a_1rs^3 + a_2^2z_3^2 + a_2a_3z_3 + a_2a_4 + a_2z_2rs + a_2z_3^2r\\
&+ 2a_2z_3^2s^2 + 2a_2z_3rs + 2a_2z_3t + 2a_2rs^2 + a_3z_3r + 2a_4z_2s\\
&+ 2a_4z_3^2 + a_4r + 2a_4s^2 + z_3^2r^2 + 2z_3^2rs^2 + z_3r^2s + 2z_3rt + rs^4\\
\sigma_1^2n_4 =&2a_1^5z_3 + a_1^4z_3^2 + a_1^4z_3s + a_1^3a_2z_3 + 2a_1^3a_3 + a_1^3z_3^2s\\
&+ 2a_1^3z_3r + 2a_1^3rs + a_1^2a_2z_3^2 + 2a_1^2a_2z_3s + a_1^2a_2r\\
&+ a_1^2a_3z_3 + 2a_1^2a_3s + a_1^2a_4 + 2a_1^2z_3^2r + a_1^2z_3^2s^2\\
&+ a_1^2z_3rs + a_1^2r^2 + 2a_1a_2^2z_3 + a_1a_2z_3^2s + 2a_1a_2z_3r\\
&+ 2a_1a_3z_3s + a_1a_3r + 2a_1a_3s^2 + 2a_1a_4z_3 + a_1a_4s + a_1z_3^2rs\\
&+ 2a_1rs^3 + a_2^2z_3s + 2a_2^2r + a_2z_3^2r + a_2z_3^2s^2 + a_2z_3rs\\
&+ a_2r^2 + a_4z_3s + 2a_4r + a_4s^2 + z_3^2rs^2 + rs^4\\
\sigma_3^2 =&a_1^3z_2r + a_1^3z_3^2s + a_1^3z_3r + 2a_1^3rs + 2a_1^2a_2z_3^2 + a_1^2a_2r\\
&+ a_1^2a_3z_2 + a_1^2a_3z_3 + 2a_1^2a_3s + 2a_1^2z_2rs + a_1^2z_3^2s^2\\
&+ 2a_1^2z_3rs + 2a_1^2rs^2 + a_1a_2a_3 + a_1a_2z_2r + a_1a_2z_3^2s\\
&+ a_1a_2z_3r + 2a_1a_2rs + 2a_1a_3z_2s + 2a_1a_3z_3s + 2a_1a_3s^2\\
&+ 2a_1a_4z_2 + 2a_1a_4z_3 + a_1a_4s + 2a_2^2z_3^2 + a_2^2r + 2a_2a_4\\
&+ 2a_2z_2rs + a_2z_3^2s^2 + 2a_2z_3rs + 2a_2rs^2 + a_4z_2s + a_4z_3s + a_4s^2\\
\sigma_1^2n_6 =&2a_1^5rs + a_1^4a_2r + 2a_1^4a_3z_3 + 2a_1^4a_3s + 2a_1^4z_3rs + a_1^4z_3t\\
&+ a_1^4r^2 + 2a_1^4rs^2 + a_1^3a_2a_3 + a_1^3a_2rs + a_1^3a_3z_3^2+\\
&2a_1^3a_3r + 2a_1^3a_3s^2 + a_1^3a_4s + a_1^3z_3^2rs + a_1^3z_3r^2+\\
&a_1^3r^2s + 2a_1^2a_2^2r + a_1^2a_2a_3z_3 + 2a_1^2a_2a_3s + 2a_1^2a_2a_4+\\
&2a_1^2a_2z_3^2r + a_1^2a_2z_3rs + 2a_1^2a_2z_3t + a_1^2a_2r^2+\\
&a_1^2a_2rs^2 + 2a_1^2a_3^2 + 2a_1^2a_3z_3^2s + a_1^2a_3z_3r + a_1^2a_3rs\\
&+ 2a_1^2a_4z_3^2 + a_1^2a_4r + a_1^2a_4s^2 + 2a_1^2a_6 + 2a_1^2z_3^2r^2+\\
&a_1^2z_3^2rs^2 + 2a_1^2z_3rt + 2a_1^2r^2s^2 + a_1a_2^2a_3+\\
&2a_1a_2^2rs + 2a_1a_2a_3z_3^2 + a_1a_2a_3r + 2a_1a_2a_3s^2 + a_1a_2a_4s+
\end{aligned}
$$

$$
\begin{aligned}
&a_1a_2z_3^2rs + a_1a_2z_3r^2 + 2a_1a_2r^2s + a_1a_3^2z_3 +\\
&2a_1a_3z_3^2s^2 + a_1a_3z_3rs + 2a_1a_3z_3t + a_1a_3r^2 + a_1a_3rs^2 +\\
&a_1a_4z_3^2s + 2a_1a_4rs + a_1z_3^2r^2s + 2a_1r^2s^3 + a_2^3r +\\
&2a_2^2a_3z_3 + 2a_2^2a_4 + 2a_2^2z_3^2r + 2a_2^2z_3rs + a_2^2z_3t +\\
&a_2^2r^2 + 2a_2^2rs^2 + 2a_2a_3^2 + a_2a_3z_3r + a_2a_4z_3^2 + 2a_2a_4r +\\
&a_2a_4s^2 + 2a_2a_6 + a_2z_3^2r^2 + a_2z_3^2rs^2 + 2a_2z_3rt +\\
&2a_2r^2s^2 + 2a_3a_4z_3 + a_4z_3^2s^2 + 2a_4z_3rs + a_4z_3t + 2a_4r^2 +\\
&2a_4rs^2 + z_3^2r^2s^2 + r^2s^4
\end{aligned}
$$

There is a nonvanishing 8×8-minor in the corresponding 48×8-matrix and thus our claim follows.

6. Comodule Structures

Recall we denote by B the ring $B = A[a_1, a_2, a_3, a_4, a_6]$. Similarly to the case of elliptic curves, we obtain a Hopf algebroid (B, Γ), where Γ arises from the pullback diagram

$$
\begin{array}{ccc}
\operatorname{Spec}(\Gamma) & \xrightarrow{\eta_R} & \operatorname{Spec}(B) \\
\Big\downarrow{\scriptstyle \eta_L} & & \Big\downarrow{\scriptstyle 2nd} \\
\operatorname{Spec}(B) & \xrightarrow{1st} & \mathcal{M}_{cub}.
\end{array}
$$

By fpqc descent, evaluation at $\operatorname{Spec} B$ defines an equivalence between quasi-coherent sheaves on $\mathcal{M}_{cub,A}$ and (B, Γ)-comodules. Thus it suffices for our main algebraic theorem to provide an isomorphism of certain comodules, which will describe explictly.

6.1. The comodule corresponding to $f_*f^*\mathcal{O}$.

Recall that at the prime 3, we have $\mathrm{mf}_1(2)_* \cong \mathbb{Z}_{(3)}[b_2, b_4]$, and the B-module structure is given by

$$a_2 \mapsto b_2 \text{ and } a_4 \mapsto b_4 \text{ and } a_1, a_3, a_5 \mapsto 0.$$

Thecorresponding (B, Γ)-comodule is given by $\Gamma \otimes_B \mathrm{mf}_1(2)_*$ with extended comodule structure. In this tensor product, we use the right B-module structure of Γ.

Lemma 6.1. *There is a monic polynomial P of degree 3 over B so that there is a ring isomorphism $\Gamma \otimes_B \mathrm{mf}_1(2)_* \cong B[r]/P$, and the comodule structure is determined by $r \mapsto 1 \otimes r + r \otimes 1$.*

Forgetting the ring structure, we can identify this comodule with the free B-module $Bw_1 \oplus Bw_2 \oplus Bw_3$ with (B, Γ)-comodule structure given by

$$
\begin{aligned}
w_1 &\mapsto 1 \otimes w_1,\\
w_2 &\mapsto 1 \otimes w_2 + r \otimes w_1,\\
w_3 &\mapsto 1 \otimes w_3 + 2r \otimes w_2 + r^2 \otimes w_1.
\end{aligned}
$$

Proof. Identifying the B-module structures, we obtain a ring isomorphism

$$\Gamma \otimes_B \mathrm{mf}_1(2)_* \cong B[r, s, t, b_2, b_4]/(R),$$

where the relations R are generated by

$$\begin{aligned}
&a_1 + 2s = 0,\\
&a_2 - sa_1 + 3r - s^2 = b_2,\\
&a_3 + ra_1 + 2t = 0,\\
&a_4 - sa_3 + 2ra_2 - (t + rs)a_1 + 3r^2 - 2st = b_4,\\
&a_6 + ra_4 + r^2 a_2 + r^3 - ta_3 - t^2 - rta_1 = 0.
\end{aligned}$$

Here, we use the formulae for η_R from [1, Section 3]. Exploiting in particular the fact that 2 is invertible in $\mathbb{Z}_{(3)}$, we can express s, t, b_2, b_4 as polynomials in r over B. This yields the ring isomorphism

$$B[r, s, t, b_2, b_4]/(R) \cong B[r]/(R'),$$

where the relation R' is given by

$$a_6 + ra_4 + r^2 a_2 + r^3 + \frac{1}{2}(a_3 + ra_1)a_3 - \frac{1}{4}(a_3 + ra_1)^2 + \frac{1}{2}(a_3 + ra_1)ra_1 = 0,$$

which is a monic polynomial in r over B of degree 3, which we denoted by P in the statement.

The first statement about the comodule structure is immediate since $\Gamma \otimes_B \mathrm{mf}_1(2)_*$ carries the extended comodule structure.

For the second description of the comodule structure, observe that since P is a monic polynomial of degree 3, there is an isomorphism of B-modules $Bw_1 \oplus Bw_2 \oplus Bw_3 \to B[r]/(P)$ given by $w_i \mapsto r^{i-1}$. Thus, to identify the comodule structure, we only need to compute it on $1, r, r^2$ on the left-hand side, and transfer it via this isomorphism, using the compatibility of comodule structure with ring structure of $B[r]/(P)$. This yields the claim. □

6.2. **The comodule corresponding to $(f_7)_*(f_7)^*\mathcal{O}$.** Recall from Proposition 2.4 that 3-locally, $\mathrm{mf}_1(7)_* \cong \mathbb{Z}_{(3)}[z_1, z_2, z_3]/(\sigma_2)$. Weobserve that $R_B := \Gamma \otimes_B \mathrm{mf}_1(7)_*$ with the extended comodule structure.

Lemma 6.2. *There is a ring isomorphism $R_B \cong \mathbb{Z}_{(3)}[z_1, z_2, z_3, r, s, t]/(\sigma_2)$, and the (B, Γ)-comodule structure is completely determined by*

$$\begin{aligned}
z_i \mapsto&\quad 1 \otimes z_i, \text{ for } i \in \{1, 2, 3\},\\
s \mapsto&\quad 1 \otimes s + s \otimes 1,\\
r \mapsto&\quad 1 \otimes r + r \otimes 1,\\
t \mapsto&\quad 1 \otimes t + t \otimes 1 + s \otimes r.
\end{aligned}$$

6.3. **The comodule corresponding to $(h_7)_*(h_7)^*\mathcal{O}$.** Recall that we identified $S_B \cong (R_B)^{(\mathbb{Z}/7)^\times}$ as B-module with a free 8-dimensional B-module with basis

$$1, \sigma_1^2, \sigma_1^4, n_4, \sigma_1^2 n_4, n_6, \sigma_1^2 n_6, \sigma_3^2.$$

We will now describe the comodule structure on this B-module.

Lemma 6.3. *The (B,Γ)-comodule structure on S_B is given by*

$$\begin{aligned}
1 &\mapsto \quad 1 \otimes 1, \\
\sigma_1^2 &\mapsto \quad 1 \otimes \sigma_1^2, \\
\sigma_1^4 &\mapsto \quad 1 \otimes \sigma_1^4, \\
n_4 &\mapsto \quad 1 \otimes n_4 + r \otimes \sigma_1^2, \\
\sigma_1^2 n_4 &\mapsto \quad 1 \otimes \sigma_1^2 n_4 + r \otimes \sigma_1^4, \\
n_6 &\mapsto \quad 1 \otimes n_6 + 2r^2 \otimes \sigma_1^2 + r \otimes n_4, \\
\sigma_1^2 n_6 &\mapsto \quad 1 \otimes \sigma_1^2 n_6 + 2r^2 \otimes \sigma_1^4 + r \otimes \sigma_1^2 n_4, \\
\sigma_3^2 &\mapsto \quad 1 \otimes \sigma_3^2.
\end{aligned}$$

Proof. This follows from Lemma 6.2 and Proposition 5.1 by a straightforward computation. □

6.4. **The conclusion.** We continue to work 3-locally.

Proposition 6.4. *There is an isomorphism of comodules*

$$B \oplus (\Gamma \otimes_B \mathrm{mf}_1(2)_*)[2] \oplus (\Gamma \otimes_B \mathrm{mf}_1(2)_*)[4] \oplus B[6] \to S_B,$$

given by

$$\begin{aligned}
1_B &\mapsto \quad 1, \\
w_1[2] &\mapsto \quad \sigma_1^2, \\
w_2[2] &\mapsto \quad n_4, \\
w_3[2] &\mapsto \quad n_6, \\
w_1[4] &\mapsto \quad \sigma_1^4, \\
w_2[4] &\mapsto \quad \sigma_1^2 n_4, \\
w_3[4] &\mapsto \quad \sigma_1^2 n_6, \\
1_B[6] &\mapsto \quad \sigma_3^2.
\end{aligned}$$

Proof. This follows by inspection from Lemma 6.1 and Lemma 6.3. □

This implies our main algebraic theorem by the equivalence of (B,Γ)-comodules and quasi-coherent sheaves on $\mathcal{M}_{cub,A}$:

Theorem 6.5. *There is 3-locally an isomorphism*

$$(h')_*\mathcal{O}_{\mathcal{M}_0(7)_{cub}} \cong \mathcal{O}_{\mathcal{M}_{cub}} \oplus \underline{\omega}^{\otimes(-6)} \oplus (f')_*\mathcal{O}_{\mathcal{M}_1(2)_{cub}} \otimes (\underline{\omega}^{\otimes(-2)} \oplus \underline{\omega}^{\otimes(-4)}$$

of vector bundles on $\mathcal{M}_{cub}$.

By restricting to $\mathcal{M}_{ell,(3)}$, this implies Theorem 1.2.

7. Topological conclusions

In the following, we will work in the homotopy category of modules over the sheaf $\mathcal{O}^{top}$ of E_∞-ring spectra on $\overline{\mathcal{M}}_{ell} = \overline{\mathcal{M}}_{ell,R}$, where R is a localization of the integers (the case $R = \mathbb{Z}_{(3)}$ being the most important for us). We denote the derived smash product over $\mathcal{O}^{top}$ by $\otimes_{\mathcal{O}^{top}}$ and the internal Hom in this category by $\mathcal{H}om_{\mathcal{O}^{top}}$. We denote morphism sets by $[-,-]^{\mathcal{O}^{top}}$. We will call an $\mathcal{O}^{top}$-module $\mathcal{F}$ *locally free* if there is an étale covering $\{U_i \to \overline{\mathcal{M}}_{ell}\}$ such that $\mathcal{F}$ restricted to U_i is equivalent to $\bigoplus_J \mathcal{O}^{top}|_{U_i}$.

A related lemma already appears in [2, Lemma 2.2.2].

Lemma 7.1. *Let $\mathcal{A}$ be a sheaf of $\mathcal{O}^{top}$-algebras on $\overline{\mathcal{M}}_{ell}$ that is locally free of rank n as an $\mathcal{O}^{top}$-module. There is a trace map*

$$\mathrm{tr}_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{O}^{top}$$

such that the composite $\mathrm{tr}_{\mathcal{A}}\, u$ with the unit map

$$u \colon \mathcal{O}^{top} \to \mathcal{A}$$

equals multiplication by n.

Proof. Consider the composite

$$\mathrm{tr}_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{H}om_{\mathcal{O}^{top}}(\mathcal{A}, \mathcal{A}) \xleftarrow{\simeq} \mathcal{A} \otimes_{\mathcal{O}^{top}} \mathcal{H}om_{\mathcal{O}^{top}}(\mathcal{A}, \mathcal{O}^{top}) \xrightarrow{\mathrm{ev}} \mathcal{O}^{top}.$$

Here, the middle map is an equivalence because $\mathcal{A}$ is locally free. We claim that the composite

$$\mathrm{tr}_{\mathcal{A}}\, u \colon \mathcal{O}^{top} \to \mathcal{O}^{top}$$

equals multiplication by n.

Note first that the map

$$\pi_0 \colon [\mathcal{O}^{top}, \mathcal{O}^{top}]^{\mathcal{O}^{top}} \to \mathrm{Hom}_{\pi_0\mathcal{O}^{top}}(\pi_0\mathcal{O}^{top}, \pi_0\mathcal{O}^{top})$$

is a bijection. Indeed, the source agrees with $\pi_0\Gamma(\mathcal{O}^{top}) = \pi_0 TMF$ and the morphism is the edge homomorphism of the descent spectral sequence. It can be deduced from [19] that this edge homomorphism is an isomorphism.

As $\mathcal{A}$ is locally free, $\pi_0\,\mathrm{tr}_{\mathcal{A}} \colon \pi_0\mathcal{A} \to \pi_0\mathcal{O}^{top}$ agrees with the trace map of $\pi_0\mathcal{A}$ over $\pi_0\mathcal{O}^{top}$. Its precomposition with $\pi_0 u$ equals n as it does locally (as we get exactly the trace of the identity map of a free module of rank n). This shows the claim. $\square$

We will need the following variant of Lemma 5.2.2 from [26]

Lemma 7.2. *Let $\mathcal{F}$ be a locally free $\mathcal{O}^{top}$-module on $\overline{\mathcal{M}}_{ell}$ of finite rank. Let $g_{alg} \colon f_*f^*\underline{\omega}^{\otimes(-i)} \to \pi_0\mathcal{F}$ be a split inclusion. Then g_{alg} can be uniquely realized by a split map*

$$g \colon \Sigma^{2i} f_*f^*\mathcal{O}^{top} \to \mathcal{F}$$

with $\pi_0 g = g_{alg}$.

Proof. If $\mathcal{F}$ and $\mathcal{G}$ are locally free $\mathcal{O}^{top}$-modules, then

$$\pi_k \mathcal{H}om_{\mathcal{O}^{top}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\pi_*\mathcal{O}^{top}}(\pi_*\mathcal{F}, \pi_{*+k}\mathcal{G})$$

is an isomorphism as this is true locally; moreover, the target is isomorphic to $\mathcal{H}om_{\pi_0\mathcal{O}^{top}}(\pi_0\mathcal{F}, \pi_k\mathcal{G})$ as $\mathcal{F}$ and $\mathcal{G}$ are even periodic.

For simplicity we assume now that $i = 0$. The dual of the vector bundle $f_*f^*\mathcal{O}_{\overline{\mathcal{M}}_{ell}} \cong f_*\mathcal{O}_{\overline{\mathcal{M}}_1(2)}$ is isomorphic to $\underline{\omega}^{\otimes 4} \otimes_{\mathcal{O}_{\overline{\mathcal{M}}_{ell}}} f_*\mathcal{O}_{\overline{\mathcal{M}}_1(2)} \cong f_*f^*\underline{\omega}^{\otimes 4}$ as can be deduced from Lemma 6.1.

This implies that

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_{\overline{\mathcal{M}}_{ell}}}(f_*f^*\mathcal{O}_{\overline{\mathcal{M}}_{ell}}, \pi_k\mathcal{F}) &\cong f_*f^*\underline{\omega}^{\otimes 4} \otimes_{\overline{\mathcal{M}}_{ell}} \pi_k\mathcal{F} \\ &\cong f_*f^*(\underline{\omega}^{\otimes 4} \otimes_{\overline{\mathcal{M}}_{ell}} \pi_k\mathcal{F}).\end{aligned}$$

As f is affine and every quasi-coherent sheaf on $\overline{\mathcal{M}}_1(2) \simeq \mathcal{P}^1_R(2,4)$ (the weighted projective line) has cohomology at most in degrees 0 and 1, the descent spectral sequence

$$H^q(\overline{\mathcal{M}}_{ell}; \pi_p \mathcal{H}om_{\mathcal{O}^{top}}(f_*f^*\mathcal{O}^{top}, \mathcal{F}) \Rightarrow \pi_{p-q} \operatorname{Hom}_{\mathcal{O}^{top}}(f_*f^*\mathcal{O}^{top}, \mathcal{F})$$

is concentrated in the lines 0 and 1. Moreover, the E_2-term is zero for p odd and thus the edge homomorphism

$$[f_*f^*\mathcal{O}^{top}, \mathcal{F}]^{\mathcal{O}^{top}} = \pi_0 \operatorname{Hom}_{\mathcal{O}^{top}}(f_*f^*\mathcal{O}^{top}, \mathcal{F}) \to \operatorname{Hom}_{\mathcal{O}_{\overline{\mathcal{M}}_{ell}}}(f_*f^*\mathcal{O}_{\overline{\mathcal{M}}_{ell}}, \pi_0\mathcal{F})$$

is an isomorphism.

Similarly, one shows that

$$[\mathcal{F}, f_*f^*\mathcal{O}^{top}]^{\mathcal{O}^{top}}) = \pi_0 \operatorname{Hom}_{\mathcal{O}^{top}}(\mathcal{F}, f_*f^*\mathcal{O}^{top}) \to \operatorname{Hom}_{\mathcal{O}_{\overline{\mathcal{M}}_{ell}}}(\pi_0\mathcal{F}, f_*f^*\mathcal{O}_{\overline{\mathcal{M}}_{ell}})$$

is an isomorphism. The lemma follows. $\square$

Theorem 7.3. *We can decompose $Tmf_0(7)_{(3)}$ as*

$$Tmf_{(3)} \oplus \Sigma^4 Tmf_1(2)_{(3)} \oplus \Sigma^8 Tmf_1(2)_{(3)} \oplus L,$$

where $L \in \operatorname{Pic}(Tmf_{(3)})$, i.e. L is an invertible $Tmf_{(3)}$-module.

Proof. Throughout this proof, we will implicitly localize at 3. Denote as before the map $\overline{\mathcal{M}}_0(7) \to \overline{\mathcal{M}}_{ell}$ by h. By Lemma 7.1, the unit map $\mathcal{O}^{top} \to h_*h^*\mathcal{O}^{top}$ splits off as an $\mathcal{O}^{top}$-module; denote the cofiber by $\mathcal{F}$. Note that $\pi_k\mathcal{F} = 0$ for k odd. By Theorem 6.5,

$$\pi_0\mathcal{F} \cong \underline{\omega}^{\otimes(-6)} \oplus f_*f^*\underline{\omega}^{\otimes(-2)} \oplus f_*f^*\underline{\omega}^{\otimes(-4)}.$$

By Lemma 7.2, we obtain a decomposition

$$\mathcal{F} \cong \mathcal{L} \oplus \Sigma^4 f_*f^*\mathcal{O}^{top} \oplus \Sigma^8 f_*f^*\mathcal{O}^{top}$$

with $\pi_0\mathcal{L} \cong \underline{\omega}^{\otimes(-6)}$. Thus, $\mathcal{L}$ is an invertible $\mathcal{O}^{top}$-module. We obtain our result by taking global sections because the global sections of $h_*h^*\mathcal{O}^{top}$ are $Tmf_0(7)$. To see that $L = \Gamma(\mathcal{L})$ is an invertible Tmf-module, we use that the global sections functor

$$\Gamma\colon \operatorname{QCoh}(\overline{\mathcal{M}}_{ell}, \mathcal{O}^{top}) \to Tmf\text{−mod}$$

is a symmetric monoidal equivalence of ∞-categories by one of the main results of [23]. $\square$

Remark 7.4. *In* [24]*, the Picard group* $\operatorname{Pic}(Tmf_{(3)})$ *is identified with* $\mathbb{Z} \oplus \mathbb{Z}/3$*, where*

$$\mathbb{Z} \to \operatorname{Pic}(Tmf_{(3)})$$

is the map $k \mapsto \Sigma^k Tmf_{(3)}$. The image of the generator of $\mathbb{Z}/3$ is called $\Gamma(\mathcal{J})$ [24, Construction 8.4.2]*. As the homotopy groups of all $\Sigma^k \mathcal{J}^{\otimes l}$ can be easily calculated from those of Tmf, one can deduce the identity of L in the previous theorem by calculating $\pi_* Tmf_0(7)$. This was done in unpublished work by Martin Olbermann and he shows that $L \simeq \Sigma^{36}\Gamma(\mathcal{J}^{\otimes 2})$.*

Note that $\Gamma(\mathcal{J}^{\otimes l})$ is in the kernel of $\operatorname{Pic}(Tmf_{(3)}) \to \operatorname{Pic}(TMF_{(3)})$ *so that L becomes $\Sigma^{36} TMF_{(3)}$ after base changing to $TMF_{(3)}$.*

Appendix A. Modular forms and q-expansions

The aim of this appendix is to give the reader a quick introduction to the different flavors of modular forms and to introduce and justify the q-expansion principle in the specific form we will need. We refer to [9] for an introduction to modular forms and to [8], [16] and [17, Section 2] for treatments closer to our needs. We also refer to [6] for a thorough treatment of the analytic side.

A.1. **Modular forms.** In this section, we will give three definitions of modular forms and compare them.

We start with the classical definition and denote by $\mathrm{MF}_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C})$ the set of holomorphic functions $f\colon \mathbb{H} \to \mathbb{C}$ satisfying for every $z \in \mathbb{H}$ and every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ the compatibility condition

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \tag{A.1}$$

and meromorphic at ∞. (To make this last condition precise, recall that the $\mathrm{SL}_2(\mathbb{Z})$-compatibility implies in particular that f is 1-periodic, and so there is a well-defined holomorphic function $g\colon \mathbb{D}^2 \setminus \{0\} \to \mathbb{C}$ satisfying $f(z) = g(e^{2\pi i z})$, and we require this g to be meromorphically extended to 0. We will say that the Laurent expansion of g at 0 is the *classical q-expansion of f* at ∞.) Elements of $\mathrm{MF}_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C})$ are called *meromorphic modular forms*. Denote by $\mathrm{MF}_k(\mathrm{SL}_2(\mathbb{Z}), R_0)$ for a subring R_0 of $\mathbb{C}$ the subset of $\mathrm{MF}_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C})$ of modular forms with coefficients of classical q-expansion of f lying in R_0.

For the algebro-geometric definitions of modular forms, we denote for an elliptic curve $p\colon E \to T$ the quasi-coherent sheaf $p_*\Omega^1_{E/T}$ by ω_E.

Proposition A.1 (Proposition II.1.6 of [7]). *Let $p\colon E \to T$ be an elliptic curve, and denote its chosen section by $e\colon T \to E$. Then the sheaf $\omega_E = p_*\Omega^1_{E/T}$ is a line bundle on T. Moreover, the adjunction counit*

$$p^*p_*\Omega^1_{E/T} \to \Omega^1_{E/T}$$

is an isomorphism, implying also $p_*\Omega^1_{E/T} \cong e^*\Omega^1_{E/T}$.

An *invariant differential* for E is a nowhere vanishing section of $\Omega^1_{E/T}$ or equivalently a trivialization of ω_E.

Our second definition of modular forms will define them as a certain kind of natural transformations. Fix a (commutative) ring R_0. For any R_0-algebra R, denote by $\mathrm{Ell}^1(R)$ the set of isomorphism classes of pairs (E, ω) consisting of an elliptic curve E over R together with an invariant differential. This defines (together with pullback of elliptic curves and of invariant differentials) a functor

$$\mathrm{Ell}^1(-)\colon (\mathrm{AffSh}\,/\,\mathrm{Spec}(R_0))^{op} \to \mathrm{Sets}\,.$$

As in [16], Section 1.1, we can consider a notion of a modular form of level 1 and weight k over R_0 as the subset of the set of natural transformations $f \in \mathrm{Nat}(\mathrm{Ell}^1(-), \Gamma(-))$ with the following scaling property: For any R_0-algebra R, elliptic curve with chosen invariant differential (E, ω) and any $\lambda \in R^\times$, we have

$$f(E, \lambda\omega) = \lambda^{-k} f(E, \omega). \tag{A.2}$$

Denote the set of such natural transformations by $\mathrm{Nat}_k(\mathrm{Ell}^1(-), \Gamma(-))$.

For the third definition, let $\mathcal{M}_{ell,R_0}$ be the moduli stack of elliptic curves over $\mathrm{Spec}(R_0)$ (see e.g. [7] or [28]). On its big étale site, one defines a line bundle $\underline{\omega}$ as follows. For a morphism $t\colon T \to \mathcal{M}_{ell,R_0}$ from a scheme T, let $p\colon E \to T$ be the corresponding elliptic curve with unit section e. We associate with (T,t) the line bundle ω_E on T. To check that this actually defines a line bundle consider a cartesian square

$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ T' \arrow[r, "f"] & T \end{tikzcd}$$

with unit section $e'\colon T' \to E'$. We obtain a chain of natural isomorphisms

$$f^*\omega_E \cong f^*e^*\Omega^1_{E/T} \cong (e')^*\tilde{f}^*\Omega^1_{E'/T'} \cong (e')^*\Omega^1_{E'/T'} \cong \omega_{E'}$$

as required.

The third definition of the meromorphic modular forms over R_0 of weight k is $H^0(\mathcal{M}_{ell,R_0}; \underline{\omega}_{R_0}^{\otimes k})$.

A.1.1. *Comparision of definitions of modular forms.* There is an easy map

$$\alpha\colon H^0(\mathcal{M}_{ell,R_0}, \underline{\omega}^{\otimes k}_{\mathcal{M}_{ell,R_0}}) \to \mathrm{Nat}_k(\mathrm{Ell}^1(-), \Gamma(-)),$$

constructed as follows. Given an element $f \in H^0(\mathcal{M}_{ell,R_0}, \omega^{\otimes k}_{\mathcal{M}_{ell,R_0}})$, an R_0-algebra R and an elliptic curve E/R together with an invariant differential ω. If E is classified by $\varphi\colon \mathrm{Spec}(R) \to \mathcal{M}_{ell,R_0}$, we have $\varphi^*(\underline{\omega}^{\otimes k}_{\mathcal{M}_{ell,R_0}}) \cong \omega_E^{\otimes k}$. By definition, f defines an element in $\Gamma(\varphi^*(\underline{\omega}^{\otimes k}_{\mathcal{M}_{ell,R_0}}))$, which via the previous isomorphism and via the isomorphism $\omega^{\otimes k}$ from $\mathcal{O}_R^{\otimes k}$ to $\omega_{E/R}^{\otimes k}$ is identified with

$$\Gamma(\varphi^*(\underline{\omega}^{\otimes k}_{\mathcal{M}_{ell,R_0}})) \cong \Gamma(\omega_E^{\otimes k}) \cong \Gamma(\mathcal{O}_R^{\otimes k}) \cong \Gamma(\mathcal{O}_R) = R.$$

Define $\alpha(f)(E,\omega)$ to be the image in R of the element defined by f in the left-hand side. The naturality of $\alpha(f)$ is clear. Replacing ω by $\lambda\omega$ for $\lambda \in R^\times$ multiplies the chosen isomorphism above by λ^k, so we obtain

$$\alpha(f)(E, \lambda\omega) = \lambda^{-k}\alpha(f)(E,\omega).$$

Let us sketch why α is an iso. By definition, the section f corresponds to a compatible choice of sections in $H^0(T; \omega_E^{\otimes k})$ for all $T \to \mathcal{M}_{ell,R_0}$ classifying an elliptic curve E. As ω_E is locally trivial, f is uniquely determined by its values on those T where ω_E is already trivial and $T = \mathrm{Spec}\, R$ is affine. In this case, a section of $\omega_E^{\otimes k}$ corresponds exactly to associating with each trivialization ω of ω_E an element $f(E,\omega)$ such that $f(E,\lambda\omega) = \lambda^{-k}f(E,\omega)$. This describes $\mathrm{Nat}_k(\mathrm{Ell}^1(-), \Gamma(-))$.

Moreover, for any subring R_0 of $\mathbb{C}$ we also have an easy map

$$\beta\colon \mathrm{Nat}_k(\mathrm{Ell}^1(-), \Gamma(-)) \to \mathrm{MF}_k(\mathrm{SL}_2(\mathbb{Z}), R_0),$$

defined as follows. For any $f \in \mathrm{Nat}_k(\mathrm{Ell}^1(-), \Gamma(-))$ and any $\tau \in \mathbb{H}$, set $\beta(f)(\tau) = f(\mathbb{C}/\mathbb{Z}\cdot 1 \oplus \mathbb{Z}\tau, dz) \in \mathbb{C}$.

We will check $\mathrm{SL}_2(\mathbb{Z})$-compatibility of $\beta(f)$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ be given. Observe that we have a biholomorphism

$$\begin{aligned} \psi\colon \mathbb{C}/\left(\mathbb{Z}\cdot 1 \oplus \mathbb{Z}\tau\right) \to \quad & \mathbb{C}/\left(\mathbb{Z}\cdot 1 \oplus \mathbb{Z}\frac{a\tau+b}{c\tau+d}\right), \\ [z] \mapsto \quad & \left[\frac{z}{c\tau+d}\right]. \end{aligned}$$

This shows that, since f is well-defined on isomorphism classes and the scaling property,

$$\begin{aligned} \beta(f)\left(\frac{a\tau+b}{c\tau+d}\right) &= f\left(\mathbb{C}/\left(\mathbb{Z}\cdot 1 \oplus \mathbb{Z}\frac{a\tau+b}{c\tau+d}\right), dz\right) \\ &= (c\tau+d)^k f(\mathbb{C}/\mathbb{Z}\cdot 1 \oplus \mathbb{Z}\tau, dz) = (c\tau+d)^k \beta(f)(\tau). \end{aligned}$$

We will later sketch why this is holomorphic in the interior and meromorphic at the cusp and why β is an isomorphism.

A.1.2. *Holomorphic modular forms.* In each of the three definitions above, we can also restrict to modular forms that are "holomorphic at the cusps". In the classical definition we just require function g to be holomorphic at 0. By requiring that the classical q-expansion is in $R_0[\![q]\!]$, we obtain the R_0-module $mf_k(SL_2(\mathbb{Z}), \mathbb{C})$ of *holomorphic modular forms*.

For the algebro-geometric version, we have to work with generalized elliptic curves instead [7, Definition II.1.12]. We can define ω_E in the same way as for usual elliptic curves and the analogue of Proposition A.1 is still valid. This defines a line bundle $\underline{\omega}$ on the compactified moduli stack $\overline{\mathcal{M}}_{ell}$ (which is our notation for $\mathfrak{M}_1$ from [7, Remarque III.2.6]). Our algebro-geometric definition of holomorphic modular forms of weight k is $H^0(\overline{\mathcal{M}}_{ell,R_0}, \underline{\omega}^{\otimes k})$.

We will later sketch the comparison between these two definitions.

A.2. **Level structures.** Throughout this section, let R_0 be a $\mathbb{Z}[\frac{1}{n}]$-algebra.

We begin with the classical definition of modular forms with level structure. Let $\Gamma_1(n) \subset SL_2(\mathbb{Z})$ be the subgroup of matrices that reduce to a matrix of the form $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ modulo n.

A *meromorphic/holomorphic modular form of level n and weight k* is a holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ satisfying the transformation formula (A.1) and is meromorphic/holomorphic at all cusps. We will say more about cusps later, but for the moment see [9, Section 1.2] for details. Note that modular forms of level n are still 1-periodic and thus the classical q-expansion still makes sense. Assume that $R_0 \subset \mathbb{C}$. We will denote by $\mathrm{MF}_k(\Gamma_1(n); R_0)$ the meromorphic modular forms of level n and weight k that have classical q-expansion with coefficients in R_0 and by $mf_k(\Gamma_1(n); R_0)$ the analogue for holomorphic modular forms.

For the algebro-geometric definitions of modular forms with level structure, we have to distinguish between two different ways to phrase them, the *naive* and the *arithmetic* level structures.

A.2.1. *Naive level structures.*

Definition A.2 ([7], Construction 4.8)**.** *For an R_0-algebra R, let $\mathrm{Ell}^1_{\Gamma_1(n)}(R)$ denote the set of isomorphism classes of triples (E, ω, j), where E is an elliptic curve over R, further ω is a chosen trivialization of the line bundle ω_E, and $j\colon \mathbb{Z}/n\mathbb{Z}_R \to E$ is a morphism of group schemes over* $\mathrm{Spec}(R)$ *and a closed immersion. This morphism j is called a* $\Gamma_1(n)$-level structure*.*

Recall that $\mathbb{Z}/n\mathbb{Z}_R = \coprod_{\mathbb{Z}/n\mathbb{Z}} \mathrm{Spec}(R)$ as a scheme, with the obvious map to $\mathrm{Spec}\, R$ and group structure coming from the group structure on $\mathbb{Z}/n\mathbb{Z}$. The group structure on the elliptic curve is explained in [18], Section 2.1. We can identify j with the image $P = j(1) \in E(R)$ since it determines j completely.

Remark A.3. *We should remark that this variant of level structures is often called "naive" in the literature. Note also that the analogous definition in* [8]*, Section 8.2, looks slightly different, but is equivalent by using that being closed immersion can be checked for proper schemes on geometric points.*

Using again the scaling condition (A.2) we can define $\mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_1(n)}(-), \Gamma(-))$ analogously to our definition without level in Section A.1.

We can also define a moduli stack $\mathcal{M}_1(n)$ classifying elliptic curves over $\mathbb{Z}[\frac{1}{n}]$-schemes with $\Gamma_1(n)$-level structure. We obtain a morphism $f_n\colon \mathcal{M}_1(n) \to \mathcal{M}_{ell}$ by forgetting the level structure. As in Section A.1.1 we obtain a comparison isomorphism

$$\alpha\colon H^0(\mathcal{M}_1(n); (f_n)^*\underline{\omega}^{\otimes k}) \to \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_1(n)}(-), \Gamma(-)).$$

There are different ways to compare modular forms with and without level structure. The particular form of compatibility is expressed in the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_1(n)}(-), \Gamma(-)) & \xrightarrow{(E,P)\mapsto E/\langle P\rangle} & \mathrm{Nat}_k(\mathrm{Ell}^1(-), \Gamma(-)) \\ \downarrow{\scriptstyle (\mathbb{C}/\mathbb{Z}+n\tau\mathbb{Z}, dz, \tau)} & & \downarrow{\scriptstyle (\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}, dz)} \\ \mathrm{MF}(\Gamma_1(n), R_0) & \longrightarrow & \mathrm{MF}_k(SL_2(\mathbb{Z}), R_0) \end{array}$$

We will denote the left vertical morphism by β_1. The reason for our particular choice of $\beta_1(n)$ might become clearer in the next subsection and even clearer when we discuss q-expansions. Note that we have not shown yet that the vertical morphism actually land in the indicated target, but we will later.

A.2.2. *Arithmetic level structures.* Now we would like to discuss a different variant of level structures, called "arithmetic" in the literature.

Definition A.4. *For an R_0-algebra R, let $\mathrm{Ell}^1_{\Gamma_\mu(n)}(R)$ denote the set of isomorphism classes of triples (E, ω, ι), where E is an elliptic curve over R, again ω is a chosen trivialization of the line bundle ω_E, and $\iota\colon \mu_{n,R} \to E$ is a morphism of group schemes over* $\mathrm{Spec}(R)$ *and a closed immersion. Here, $\mu_{n,R}$ is a group scheme given by the spectrum of the bialgebra $R[t]/(t^n - 1)$ with comultiplication determined by $t \mapsto t \otimes t$. The morphism ι is called an* arithmetic (or $\Gamma_\mu(n)$-) level structure *on E.*

One can check that for a $\mathbb{Z}\left[\frac{1}{n}, \zeta_n\right]$-algebra R, both group schemes $\mu_{n,R}$ and $\mathbb{Z}/n\mathbb{Z}(R)$ are isomorphic, but this is not true in general. Now we can define the set of weight k modular forms with arithmetic level structure to be $\mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-), \Gamma(-))$

with the same scaling condition as before. Likewise, we can define a moduli stack $\mathcal{M}_\mu(n)$ of elliptic curves with $\Gamma_\mu(n)$-level structure (over bases with n invertible). Denoting the forgetful map $\mathcal{M}_\mu(n) \to \mathcal{M}_{ell}$ by f'_n we obtain as before comparison isomorphisms

$$\alpha\colon H^0(\mathcal{M}_\mu(n); (f'_n)^*\underline{\omega}^{\otimes k}) \to \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-), \Gamma(-)).$$

We need to discuss a relation between $\Gamma_1(n)$- and $\Gamma_\mu(n)$-level structures. Observe we have a morphism $\varphi\colon \mathcal{M}_1(n) \to \mathcal{M}_\mu(n)$ sending $(E \to S, P)$ to $(E/\langle P\rangle \to S, \alpha)$, where we define $\alpha\colon \mu_{n,S} \to E/\langle P\rangle$ as follows: As explained in [18, Section 2.8] there is an alternating pairing

$$\langle -,-\rangle_\pi\colon\ \ker(\pi) \times \ker(\pi^t) \to \mathbb{G}_{m,S}$$

for $\pi\colon E \to E/\langle P\rangle$ the projection and π^t the dual isogeny. This induces an isomorphism

$$\ker(\pi^t) \to \mathrm{Hom}_{S-\mathrm{gp}} \ker(\pi), \mathbb{G}_{m,S}) \xrightarrow{\mathrm{ev}_P} \mu_{n,S}$$

and α is the composition of the inverse of this isomorphism with the natural inclusion $\ker(\pi^t) \to E$ *composed with* $[-1]$. The reasons for composing with $[-1]$ will be apparent in the example below.

We remark that an analogous construction provides an inverse of φ (using the isomorphism $E/E[n] \cong E$ induced by $[n]$, the multiplication-by-n morphism) and thus $\varphi\colon \mathcal{M}_1(n) \to \mathcal{M}_\mu(n)$ is an equivalence of stacks. See also [17, Section 2.3].

One can compute ϕ in terms of the Weil pairing as follows: As $\pi\pi^t = [n]$, [18, 2.8.4.1] implies that $\langle Q, \pi(R)\rangle_\pi$ for $Q \in \ker(\pi)(T)$ and $R \in E[n](T)$ with $T \to S$ can be computed as $e_n(Q,R)$, where e_n denotes the Weil pairing.

Example A.5. *Let $E = \mathbb{C}/(\mathbb{Z}+n\tau\mathbb{Z})$ with chosen n-torsion point τ. We clam that $\phi(E,\tau) = (\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}, \zeta_n \mapsto \frac{1}{n})$ with $\zeta_n = e^{\frac{2\pi i}{/} n}$.*

Indeed, we have $e_n(\tau, \frac{1}{n}) = \zeta_n^{-1}$ by [18, 2.8.5.3] *and thus $\langle \tau, \frac{1}{n}\rangle_\pi = \zeta_n^{-1}$. The claim follows.*

Under the isomorphism

$$\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z} \to \mathbb{C}^\times/q^{\mathbb{Z}}, \qquad z \mapsto e^{2\pi i z}$$

with $q = e^{2\pi i\tau}$ the morphism α corresponds thus just to the obvious inclusion of μ_n.

The example implies directly the following lemma.

Lemma A.6. *The following diagram commutes:*

$$\begin{tikzcd}
H^0(\mathcal{M}_\mu(n)_{R_0}, \omega^{\otimes k}_{\mathcal{M}_\mu(n)_{R_0}}) \arrow[r, "\varphi^*"] \arrow[d, "\alpha"] & H^0(\mathcal{M}_1(n)_{R_0}, \omega^{\otimes k}_{\mathcal{M}_1(n)_{R_0}}) \arrow[d, "\alpha"] \\
\mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-), \Gamma(-)) \arrow[r, "\varphi^*"] \arrow[dr, "{(\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}, dz, \zeta_n \mapsto \frac{1}{n})}"'] & \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_1(n)}(-), \Gamma(-)) \arrow[d, "{(\mathbb{C}/\mathbb{Z}+n\tau\mathbb{Z}, dz, \tau)}"] \\
 & \mathrm{MF}(\Gamma_1(n), R_0)
\end{tikzcd}$$

We will denote the diagonal arrow by β_{mu}. As a last point, we mention the following lemma.

Lemma A.7. *Let E/S be an elliptic curve and n be invertible on S. Let $\iota\colon \mu_{n,S} \to E$ be a $\Gamma_\mu(n)$-structure. Then we have a short exact sequence*

$$1 \to \mu_{n,S} \xrightarrow{\iota} E[n] \xrightarrow{\kappa} (\mathbb{Z}/n)_S \to 1$$

of étale sheaves of abelian groups.

Proof. The Weil pairing discussed in [18, Section 2.8] induces an isomorphism

$$E[n] \xrightarrow{\cong} \mathrm{Hom}_{S-\mathrm{gp}}(E[n], \mathbb{G}_{m,S}).$$

Postcomposing with ι^* induces a surjection

$$E[n] \to \mathrm{Hom}_{S-\mathrm{gp}}(\mu_{n,S}, \mathbb{G}_{m,S}) \cong (\mathbb{Z}/n)_S,$$

which we call κ. The composition $\kappa\iota$ is zero by [18, 2.8.7]. $\square$

A.2.3. *Compactifications and comparison of algebraic and analytic theory.* In this section we discuss how to compactify $\mathcal{M}_1(n)$ and also the comparison of the algebraic and the analytic theory. The basic sources are [7] and [6] and we will just give a short summary.

The moduli stack $\mathcal{M}_{ell}$ has a compactification $\overline{\mathcal{M}}_{ell}$ classifying suitable generalized elliptic curves (this is $\mathfrak{M}_1$ in the sense of [7, Section III]). The moduli stack $\mathcal{M}_1(n)$ has as well a compactification $\overline{\mathcal{M}}_1(n)$. It can be defined as the normalization of $\overline{\mathcal{M}}_{ell}$ in $\mathcal{M}_1(n)$ (see ... for the normalization construction). It is shown in [7, Section IV] that $\overline{\mathcal{M}}_1(n) \to \operatorname{Spec} \mathbb{Z}[\frac{1}{n}]$ is proper and smooth of relative dimension 1.

For $n \geq 5$, the stack $\overline{\mathcal{M}}_1(n)$ is representabe by a projective scheme (see ... or ...). It is shown in [6, Thm 2.2.2?] that the Riemann surface associated with $\overline{\mathcal{M}}_1(n)_{\mathbb{C}}$ is isomorphic to a more classical construction, namely the compactification $X_1(n)$ of the quotient $Y_1(n)$ of the upper half plane $\mathbb{H}$ by $\Gamma_1(n)$. Indeed, Conrad shows that both $\overline{\mathcal{M}}_1(n)_{\mathbb{C}}$ and $X_1(n)$ classify generalized elliptic curves over complex analytics spaces with $\Gamma_1(n)$-level structure. The family of elliptic curves $(\mathbb{C}/\mathbb{Z}+n\tau\mathbb{Z}, \tau)$ with $\Gamma_1(n)$-level structure over $\mathbb{H}$ descends to $Y_1(n)$ and extends to $X_1(n)$. This specifies a possible isomorphism $\overline{\mathcal{M}}_1(n)_{\mathbb{C}} \to X_1(n)$.

The compactification $X_1(n)$, for example, studied in [6] and in [9, Chapter 2]. In particular, [6, Lemma 1.5.7.2?] shows that $mf_k(\Gamma_1(n); \mathbb{C}) \cong H^0(X_1(n); \underline{\omega}^{\otimes k})$, where $\underline{\omega}$ on $X_1(n)$ is the analytification of a line bundle $\underline{\omega}$ on $\overline{\mathcal{M}}_1(n)$ that extends $\underline{\omega}$ on $\mathcal{M}_1(n)$ as we defined it before. By GAGA, holomorphic and algebraic sections of $\underline{\omega}^{\otimes k}$ agree.

The complement of $Y_1(n)$ in $X_1(n)$ is a finite set consisting of the *cusps*. One can see that $MF_k(\Gamma_1(n); \mathbb{C})$ corresponds to those meromorphic sections of $\underline{\omega}^{\otimes k}$ that are holomorphic on $Y_1(n)$ (but have possibly poles at the cusps). Likewise, sections of $\underline{\omega}^{\otimes k}$ on $\mathcal{M}_1(n)$ correspond to meromorphic sections of $\underline{\omega}^{\otimes k}$ on $\overline{\mathcal{M}}_1(n)$ that are algebraic on $\mathcal{M}_1(n)$. This implies that our comparison map

$$H^0(\mathcal{M}_1(n)_{\mathbb{C}}; \underline{\omega}^{\otimes k} \xrightarrow{\cong} \mathrm{Nat}_k(\mathrm{Ell}_{\Gamma_1(n)}(-), \Gamma(-)) \to MF(\Gamma_1(n); \mathbb{C})$$

is an isomorphism, where we restricted the domain of $\mathrm{Ell}_{\Gamma_1(n)}$ and Γ to $\mathbb{C}$-algebras.

For $n < 5$, $\overline{\mathcal{M}}_1(n)$ is no longer a scheme. In these case, one can analogously use a GAGA theorem for stacks as, for example, proven in [29]. In our situation the proof should be considerably simplified though as $\overline{\mathcal{M}}_1(n)_{\mathbb{C}}$ has a finite faithfully flat cover by a scheme (e.g. by $\overline{\mathcal{M}}_1(5n)_{\mathbb{C}}$) and one should be able to deduce a sufficiently strong GAGA theorem just by descent from the scheme case.

A.3. **The Tate curve.** In this section, we will discuss the Tate curve, which will give us an algebraic way to define q-expansions of modular forms. We first discuss the situation over the complex numbers.

Theorem A.8. *([31], Theorem V.1.1) For any $q, u \in \mathbb{C}$ with $|q| < 1$, define the following quantities:*

$$\begin{aligned}
\sigma_k(n) &= \sum_{d|n} d^k, \\
s_k(q) &= \sum_{n\geq 1} \sigma_k(n) q^n = \sum_{n\geq 1} \frac{n^k q^n}{1-q^n}, \\
a_4(q) &= -5s_3(q), \\
a_6(q) &= -\frac{5s_3(q) + 7s_5(q)}{12}, \\
X(u,q) &= \sum_{n\in\mathbb{Z}} \frac{q^n u}{(1-q^n u)^2} - 2s_1(q), \\
Y(u,q) &= \sum_{n\in\mathbb{Z}} \frac{(q^n u)^2}{(1-q^n u)^3} + s_1(q).
\end{aligned}$$

(1) *Then the equation*

$$y^2 + xy = x^3 + a_4(q)x + a_6(q) \tag{A.3}$$

defines an elliptic curve E_q over $\mathbb{C}$, and X, Y define a complex analytic isomorphism

$$\begin{aligned}
\mathbb{C}^\times/q^{\mathbb{Z}} &\to E_q \\
u &\mapsto \begin{cases} (X(u,q), Y(u,q)), & \text{if } u \notin q^{\mathbb{Z}}, \\ O, & \text{if } u \in q^{\mathbb{Z}} \end{cases}
\end{aligned}$$

(2) *As power series in q, both $a_4(q), a_6(q)$ have integer coefficients.*
(3) *The power series $a_4(q)$ and $a_6(q)$ define holomorphic functions on the open unit disk $\mathbb{D}$.*
(4) *The discriminant of E_q is given by*

$$\Delta(q) = q\prod_{n\geq 1}(1-q^n)^{24} \in \mathbb{Z}[\![q]\!].$$

(5) *Every elliptic curve over $\mathbb{C}$ is isomorphic to E_q for some q with $|q| < 1$.*

Let $\mathrm{Conv} \subset \mathbb{Z}[\![q]\!]$ be the subset of "convergent" power series, i.e. those that define holomorphic functions on $\mathbb{D}$; in particular, $a_4, a_6 \in \mathrm{Conv}$. By the explicit description of the discriminant, we can use the Weierstraß equation (A.3) to define an elliptic curve $\mathrm{Tate}(q)$ over Conv.

Let $q_0 \in \mathbb{D}$ be a nonzero point and consider the morphism $\mathrm{ev}_{q_0}\colon \mathrm{Conv} \to \mathbb{C}$. By the theorem above, we see that the analytic space associated with $\mathrm{ev}_{q_0}^* \mathrm{Tate}(q)$ is isomorphic to $\mathbb{C}^\times/q_0^{\mathbb{Z}}$. The invariant differential η^{can} associated to the Weierstraß equation corresponds under this isomorphism to $\frac{dq}{q}$.

Next, we want to describe a group homomorphism $\iota\colon \mu_{n,\mathrm{Conv}} \to \mathrm{Tate}(q)[n]$ for $n \geq 2$. Here, we denote for an abelian group scheme G over a scheme S by $G[n]$ the n-torsion, i.e. the pullback $G \times_G S$, where we use the multiplication-by-n-map

$[n]\colon G \to G$ and the unit map $S \to G$. For simplicity, we will only describe it over $\mathrm{Conv}[\frac{1}{n}]$. We first describe ι after base change to $\mathrm{Conv}[\frac{1}{n}, \zeta_n] = \mathrm{Conv} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{n}\zeta_n]$. As μ_n is isomorphic to $\mathbb{Z}/n$ over this ring, it suffices to give an n-torsion point in $\mathrm{Tate}(q)[n](\mathrm{Conv}[\frac{1}{n}, \zeta_n])$; we take $[X(\zeta_n, q), Y(\zeta_n, q), 1]$. This is compatible with the Galois action and thus, we obtain a morphism $\mu_{n,\mathrm{Conv}[\frac{1}{n}]} \to \mathrm{Tate}(q)[n]_{\mathrm{Conv}[\frac{1}{n}]}$. Note that we can check that this is indeed a group homomorphism into the n-torsion by evaluating at infinitely many points in $\mathbb{D}$. For a nonzero $q_0 \in \mathbb{D}$, this ι corresponds under the isomorphism of $\mathrm{ev}_{q_0}^* \mathrm{Tate}(q)$ with $\mathbb{C}^\times/q_0^{\mathbb{Z}}$ exactly to the composite $\mu_n(\mathbb{C}) \to \mathbb{C}^\times \to \mathbb{C}^\times/q_0^{\mathbb{Z}}$. Note that ι defines a $\Gamma_\mu(n)$-structure on $\mathrm{Tate}(q)$.

We remark that there are other possible choices to define ι, corresponding to different constructions of the Tate curve. To avoid possible ambiguity, we show the following uniqueness statement.

Proposition A.9. *The morphism*

$$\mathbb{Z}/n \to \mathrm{Hom}(\mu_{n,\mathrm{Conv}}, \mathrm{Tate}(q)), \qquad k \mapsto k\iota$$

is a bijection.

Proof. The group $\mathrm{Tate}(q)[n]_{\mathrm{Conv}_C[q^{1/n}]}$ is isomorphic to $(\mathbb{Z}/n)^2$ with

$$(X(\zeta^a q^{b/n}, q), Y(\zeta^a q^{b/n}, q), 1)$$

as the non-trivial torsion points, where $\zeta = e^{2pi/n}$; indeed these are all n-torsion points as we can check on infinitely many points in $\mathbb{D}$ (away from some chosen ray so that $q^{1/n}$ makes sense as a holomorphic function) and there cannot be more n-torsion points.

We see that only n of these torsion points have coordinates in $\mathrm{Conv}_{\mathbb{C}}$ and thus $\mathrm{Tate}(q)[n]_{\mathrm{Conv}_{\mathbb{C}}} \cong \mathbb{Z}/n$. We obtain that $\mathrm{Hom}(\mu_{n,\mathrm{Conv}_{\mathbb{C}}}, \mathrm{Tate}(q)_{\mathrm{Conv}_{\mathbb{C}}}) \cong \mathbb{Z}/n$ and the existence of ι shows that the injective map

$$\mathrm{Hom}(\mu_{n,\mathrm{Conv}}, \mathrm{Tate}(q)) \to Hom(\mu_{n,\mathrm{Conv}_{\mathbb{C}}}, \mathrm{Tate}(q)_{\mathrm{Conv}_{\mathbb{C}}})$$

is also a surjection. □

By Lemma A.7, we obtain for each $n \geq 1$ a short exact sequence

$$0 \to \mu_{n,\mathrm{Conv}[\frac{1}{n}]} \xrightarrow{\iota'} \mathrm{Tate}(q)[n] \xrightarrow{\kappa} (\mathbb{Z}/n\mathbb{Z})_{\mathrm{Conv}[\frac{1}{n}]} \to 0$$

of étale sheaves of abelian groups. We can normalize κ in the following way: For any $\mathrm{Conv}[\frac{1}{n}]$-algebra R, any $\zeta \in \mu_n(R)$ and $X \in \mathrm{Tate}(q)[n](R)$, the Weil pairing $e_n(\iota(\zeta), X)$ equals $\zeta^{\kappa(X)}$.

We remark that by comparing the explicit equations , one sees that our definition of the Tate curve agrees with the one discussed in [18, Section 8.8] (and e.g. in [7] before).

A.4. q**-expansions.** Our goal in this section is to define the q-expansion both in the holomorphic and in the algebraic context, to compare them and to obtain a q-expansion principle.

Given a modular form f in $MF(\Gamma_1(n), \mathbb{C})$ (possibly $n = 1$ so $\Gamma_1(n) = SL_2(\mathbb{Z})$), we note that $f(\tau) = f(\tau + 1)$ and thus f factors through a meromorphic function $\tilde{f}\colon \mathbb{D} \to \mathbb{C}$, where $\mathbb{D}$ denotes the open disk with radius 1; more precisely, we have $\tilde{f}(q) = f(\tau)$, where $q = q(t) = e^{2\pi i \tau}$. Taylor expansion of $\tilde{f}$ at 0 yields a map

$$\Phi^{hol}\colon MF_k(\Gamma_1(n), \mathbb{C}) \to \mathbb{C}((q)).$$

On the algebraic side, we obtain a map

$$\Phi^{\mu,R_0}\ \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-),\Gamma(-)) \to R_0((q))$$

(for Ell living over a fixed $\mathbb{Z}[\frac{1}{n}]$-algebra R_0 again) by evaluating the natural transformation at the Tate curve $(\mathrm{Tate}(q),\eta_{can},\iota)$ from the last section. More precisely, we evaluating on the pullback of the Tate curve to $R_0((q))$.

We want to show that Φ^{hol} and $\Phi^{\mu,\mathbb{C}}$ correspond to each other under β_μ. Note first that both have actually image in the subring $\widetilde{\mathrm{Conv}} \subset \mathbb{C}((q))$ of Laurent series that converge on $\mathbb{D}\setminus\{0\}$. We can check the agreement of $\Phi^{hol}\beta_\mu(n)$ with $\Phi^{\mu,\mathbb{C}}$ after postcomposing these two maps with $\mathrm{ev}_{q_0}\colon \mathrm{Conv} \to \mathbb{C}$ for infinitely many $q_0 \in \mathbb{D}\setminus\{0\}$.

Choose $\tau_0 \in \mathbb{H}$ with $e^{2\pi i\tau_0} = q_0$. By definition, $\mathrm{ev}_{q_0}\Phi^{hol}(f) = \tilde{f}(q_0) = f(\tau_0)$. Using that $\mathbb{C}/(\mathbb{Z}+\tau_0\mathbb{Z}) \cong \mathbb{C}^\times/q_0^{\mathbb{Z}}$, we observe that $\mathrm{ev}_{q_0}\Phi^{hol}\beta_\mu(g)$ (with $g \in \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-),\Gamma(-))$) equals $g(\mathbb{C}^\times/q_0^{\mathbb{Z}},\frac{dq}{q}),\iota^{can})$, where ι^{can} denotes the composition $\mu_n(\mathbb{C}) \to \mathbb{C}^\times \to \mathbb{C}^\times/q_0^{\mathbb{Z}}$.

On the other hand, $\mathrm{ev}_{q_0}\Phi^{\mu,\mathbb{C}}(g)$ equals $(\mathrm{ev}^*_{q_0}\mathrm{Tate}(q),\mathrm{ev}^*_{q_0}\iota,\mathrm{ev}^*_{q_0}\eta^{can})$. We have seen in the last section that this triple is isomorphic to $(\mathbb{C}^\times/q_0^{\mathbb{Z}},\iota^{can},\frac{dq}{q})$, what was to be shown. Thus, the following triangle commutes:

$$\begin{tikzcd}
\mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-),\Gamma(-)) \arrow[r, "\Phi^{\mu,\mathbb{C}}"] \arrow[d, "\beta_\mu"] & \mathbb{C}((q)) \\
MF_k(\Gamma_1(n),\mathbb{C}) \arrow[ur, "\Phi^{hol}"] &
\end{tikzcd}$$

We obtain the q-expansion morphism

$$\Phi^{1,R_0}\colon\ \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-),\Gamma(-)) \to \mathrm{Conv}[q^{-1}]\otimes R_0$$

as the composition $\Phi^{\mu,R_0}(\varphi^*)^{-1}$, where φ is as in Subsection A.2.2.

Lemma A.10. *Assume that $R_0 \subset \mathbb{C}$ and let $q_0 \neq 0$ be a point in the open unit disk. Evaluating at q_0 yields a morphism $\mathrm{ev}_{q_0}\colon \mathrm{Conv}[q^{-1}]\otimes R_0 \to \mathbb{C}$. Then*

$$\mathrm{ev}_{q_0}\Phi^{1,R_0}(g) = g(\mathbb{C}^\times/q^{n\mathbb{Z}},\frac{dq}{q},q)$$

for every $g \in \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-)$.

Proof. It suffices to show that

$$\varphi(\mathbb{C}^\times/q^{n\mathbb{Z}},\frac{dq}{q},q) = (\mathbb{C}^\times/q^{\mathbb{Z}},\frac{dq}{q},\iota^{can}).$$

This follows from Example A.5. □

Note that these discussions actually show that β_1 and β_μ actually have target $MF(\Gamma_1(n);R_0)$, i.e. that the classical q-expansion of β_1 of a modular form over R_0 actually has coefficients in R_0 and similarly for β_μ.

Theorem A.11 (q-expansion principle)**.** *Let R_0 be a subring of $\mathbb{C}$. The morphisms*

$$\beta_\mu\colon\ \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_\mu(n)}(-),\Gamma(-)) \to MF(\Gamma_1(n);R_0)$$

and

$$\beta_q\colon\ \mathrm{Nat}_k(\mathrm{Ell}^1_{\Gamma_q(n)}(-),\Gamma(-)) \to MF(\Gamma_1(n);R_0)$$

are isomorphisms. In other words: If the coefficients of the q-expansion of a complex modular form are in R_0, it is actually already defined over R_0.

Proof. By the considerations above, it suffices to show the first statement. For $R_0 = \mathbb{C}$, this was discussed in Subsection A.2.3. The general case follows by the q-expansion principle as stated in [8, Theoem 12.3.4]. $\square$

A.5. **Summary.** Let R be any ring. We can define holomorphic modular forms for $\Gamma_1(n)$ of weight k over R as $H^0(\overline{\mathcal{M}}_1(n); \underline{\omega}^{\otimes k})$ and meromorphic modular forms as $H^0(\mathcal{M}_1(n)_R; \underline{\omega}^{\otimes k})$. We have a morphism $\operatorname{Spec}\mathbb{C} \to \mathcal{M}_1(n)$ classifying the elliptic curve $C/\mathbb{Z} + n\tau\mathbb{Z}$ with chosen point τ of order n. Pulling $f \in H^0(\mathcal{M}_1(n); \underline{\omega}^{\otimes k})$ back to $\operatorname{Spec}\mathbb{C}$ and using the trivialization $\underline{\omega}^{\otimes k}$ induced by the choice of differential dz, defines a holomorphic function of $\tau \in \mathbb{H}$ that is a meromorphic modular form for $\Gamma_1(n)$ in the classical sense. This defines an isomorphism

$$\beta_1\colon H^0(\overline{\mathcal{M}}_1(n)_{\mathbb{C}}; \underline{\omega}^{\otimes k}) \to MF_k(\Gamma_1(n); \mathbb{C}).$$

The q-expansion of $\beta_1(f)$ lies in $R \subset \mathbb{C}$ if and only if f is in the image of the injection

$$H^0(\overline{\mathcal{M}}_1(n)_R; \underline{\omega}^{\otimes k}) \to H^0(\overline{\mathcal{M}}_1(n)_{\mathbb{C}}; \underline{\omega}^{\otimes k}).$$

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