NERVES VS PUSHOUTS

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1. Pushouts along Dwyer morphisms

The aim of this section is to prove the following result.

Proposition 1.1. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "i"] \arrow[d, "F"'] & \mathcal{B} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{D} \end{tikzcd}$$

be a pushout of categories, and assume i to be a Dwyer morphism. Then the induced map $N\mathcal{B} \coprod_{N\mathcal{A}} N\mathcal{C} \to N(\mathcal{B} \coprod_{\mathcal{A}} \mathcal{C})$ is a Joyal *weak equivalence.*

We will use the description of a pushout along a Dwyermorphism from [BMO⁺15, Proof of Lemma 2.5]; cf. also [Sch19, Construction 1.2] and [AM14, §7.1]. Let $\mathcal{W}$ be the cosieve generated by $\mathcal{A}$ and $r\colon \mathcal{W} \to \mathcal{A}$ be its right adjoint with $r|_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}}$ and the counit $\varepsilon\colon r \Rightarrow \mathrm{id}_{\mathcal{W}}$ being identity on $\mathcal{A}$. In particular, the triangle identity asserts $r(\varepsilon_{-}) = \mathrm{id}_{r(-)}$. We will write $\xi\colon \mathcal{B} \to [1]$ for the functor with $\xi^{-1}(0) = \mathcal{A}$. We also write $\mathcal{V} := \xi^{-1}(1)$.

Remark 1.2. In $\mathcal{B}$, the counit $\varepsilon_z\colon rz \to z$ is a monomorphism for any z in $\mathcal{W}$. Indeed, let morphisms $f, g\colon x \to rz$ be given so that $\varepsilon_z \circ f = \varepsilon_z \circ g$. First, we observe that since $rz \in \mathcal{A}$ and $\mathcal{A}$ is a sieve, then x is necessarily an object of $\mathcal{A}$, and both morphisms of f, g are thus also in $\mathcal{A}$. We apply r to the equation given, remembering $r|_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}}$ and $r(\varepsilon_z) = \mathrm{id}_{r(z)}$. This implies exactly $f = g$, as desired.

Note that the objects of $\mathcal{D}$ can be identified with $\mathrm{Ob}\,\mathcal{C} \sqcup \mathrm{Ob}\,\mathcal{V}$ (we are using that Ob has a right adjoint, as well as the fact that $\mathcal{A} \to \mathcal{B}$ is injective on objects). Under this identification, we can describe hom-sets in a pushout $\mathcal{D}$ as follows:

$$\mathcal{D}(x,y) = \begin{cases} \mathcal{C}(x,y) \text{ if } x, y \in \mathrm{Ob}\,\mathcal{C}, \\ \mathcal{B}(x,y) = \mathcal{V}(x,y) \text{ if } x, y \in \mathrm{Ob}\,\mathcal{V}, \\ \varnothing \text{ if } x \in \mathrm{Ob}\,\mathcal{V}, y \in \mathrm{Ob}\,\mathcal{C}, \\ \varnothing \text{ if } x \in \mathrm{Ob}\,\mathcal{C}, y \in \mathrm{Ob}\,\mathcal{V} \setminus \mathrm{Ob}\,\mathcal{W}, \\ \mathcal{C}(x, Fr(y)) \text{ if } x \in \mathrm{Ob}\,\mathcal{C}, y \in \mathrm{Ob}\,\mathcal{W}, \end{cases}$$

and the last bijection is given by postcomposition with $\varepsilon_y \colon Fr(y) \sim r(y) \to y$.

Note that all simplices of $N(\mathcal{B}\coprod_{\mathcal{A}}\mathcal{C})$ can be classified as follows. Either all vertices are in $\operatorname{Ob}\mathcal{C}$, or all vertices are in $\operatorname{Ob}\mathcal{V}$, and in these cases the simplices are in $N\mathcal{B}\coprod_{N\mathcal{A}}N\mathcal{C}$, or the vertices are mixed. In this latter case, no object of $\mathcal{C}$ can appear after an object of $\mathcal{V}$, so we have some vertices of $\mathcal{C}$ followed by some vertices of $\mathcal{V}$, all of which are even necessarily in $\mathcal{W}$.

Definition 1.3. We define the *suspect index* of a d-simplex $\sigma \colon [d] \to \mathcal{B}\coprod_{\mathcal{A}}\mathcal{C}$ of $N\mathcal{D}$ as the maximal integer $0 \leq r \leq d$ so that $\sigma(i)$ is in $\operatorname{Ob}\mathcal{C}$ for $0 \leq i \leq r$. If no such integer exists, we set suspect index to be -1. A non-degenerate simplex of suspect index $r < d$ is called *suspect* if $\sigma(r(r+1)) = \varepsilon_{\sigma(r+1)}$.

Remark 1.4. Any d-simplex of suspect index -1 has all its vertices in $\mathcal{V}$ and is in particular contained in $N\mathcal{B}$. Any d-simplex of suspect index d has all its vertices in $\mathcal{D}$ and is in particular contained in $N\mathcal{C}$. In particular, any $(d+1)$-suspect simplex of $N(\mathcal{B}\coprod_{\mathcal{A}}\mathcal{C})$ not contained in $N\mathcal{B}\coprod_{N\mathcal{A}}N\mathcal{C}$ has a suspect index $0 < r+1 < d+1$.

Lemma 1.5. *Let $\widetilde{\sigma} \colon [d+1] \to \mathcal{B}\coprod_{\mathcal{A}}\mathcal{C}$ be a non-degenerate suspect simplex of index $0 < r+1 < d+1$. For $0 \leq a \leq d+1$, we have*

$$d_a\widetilde{\sigma} = \begin{cases} \textit{suspect} & \textit{if } 0 \leq a \leq r, \\ \textit{non-suspect simplex of suspect index } r & \textit{if } a = r+1, \\ \textit{simplex of suspect index } r+1 & \textit{if } a = r+2, \\ \textit{suspect} & \textit{if } r+3 \leq a \leq d+1. \end{cases}$$

Proof of Proposition 1.1. We define a filtration

$$N\mathcal{B}\coprod_{N\mathcal{A}}N\mathcal{C} =: X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \ldots \hookrightarrow X_d \hookrightarrow \ldots \hookrightarrow N(\mathcal{B}\coprod_{\mathcal{A}}\mathcal{C})$$

as follows: For $d \geq 1$, let X_d be the smallest simplicial subset of $N(\mathcal{B}\coprod_{\mathcal{A}}\mathcal{C})$ containing X_{d-1} and all simplices of dimension d as well as all suspect simplices of dimension $d+1$. Note that X_0 in particular contains all 0-simplices and all suspect simplices of dimension 1 in $N(\mathcal{B}\coprod_{\mathcal{A}}\mathcal{C})$.

We define a filtration

$$X_{d-1} =: Y_d \hookrightarrow Y_{d-1} \hookrightarrow Y_{d-2} \hookrightarrow \ldots \hookrightarrow Y_r \hookrightarrow \ldots \hookrightarrow Y_0 = X_d$$

as follows: For $d > r \geq 0$, let Y_r be the smallest simplicial subset of $N(\mathcal{B}\coprod_{\mathcal{A}}\mathcal{C})$ containing Y_{r+1}, all simplices of dimension d and suspect index r as well as all suspect simplices of dimension $d+1$ and suspect index $r+1$. Note that Y_d contains all simplices of dimension d and suspect index d as well as all suspect simplices of dimension $d+1$ and suspect index $d+1$. Note moreover that Lemma 1.5 implies that no further d-simplices are added in each step.

Next, we claim that the inclusion $Y_{r+1} \hookrightarrow Y_r$ is inner anodyne. First, we claim that we have a bijection between the non-degenerate non-suspect simplices of $Y_r \setminus Y_{r+1}$ and the non-degenerate suspect simplices of $Y_r \setminus Y_{r+1}$. If $\widetilde{\sigma}$ is a non-degenerate suspect simplex in $Y_r \setminus Y_{r+1}$, we claim that $d_{r+1}\widetilde{\sigma}$ is also a simplex of $Y_r \setminus Y_{r+1}$ which is not suspect. Indeed, to be a suspect, it would require the map $\widetilde{\sigma}(r) \to \widetilde{\sigma}(r+2)$ being $\varepsilon_{\widetilde{\sigma}(r+2)}$. This would require $\varepsilon_{\widetilde{\sigma}(r+2)} \circ \widetilde{\sigma}(r(r+1)) = \varepsilon_{\widetilde{\sigma}(r+2)}$ and thus given the description of morphismsin $\mathcal{D}$, the map $\widetilde{\sigma}(r) \to \widetilde{\sigma}(r+1)$, would have been the identity, meaning that $\widetilde{\sigma}$ is degenerate.

Now we proceed in the opposite direction: Let σ be a non-degenerate non-suspect simplex of $Y_r \setminus Y_{r+1}$. We recall that there is a unique decomposition of the morphism $\sigma(r) \to \sigma(r+1) =: z$ into $\gamma\colon \sigma(r) \to Fr(z)$ followed by $\varepsilon_z\colon r(z) \to z$. (Note that $\gamma \neq \mathrm{id}$ since σ would have been suspect otherwise.) Define $\widetilde{\sigma}\colon [d+1] \to \mathcal{D}$ as

$$\sigma(0) \to \sigma(1) \to \ldots \to \sigma(r) \xrightarrow{\gamma} Fr(z) \xrightarrow{\varepsilon_z} z = \sigma(r{+}1) \to \sigma(r{+}2) \to \ldots \to \sigma(d).$$

This $(d+1)$-simplex is a non-degenerate suspect simplex of suspect index $r+1$ and thus in $Y_r \setminus Y_{r+1}$. Moreover, it is immediate that applying d_{r+1} gives σ again.

So to prove the bijection, we only have to check that this construction recovers any non-degenerate suspect $\widetilde{\sigma}$ of dimension $(d+1)$ and suspect index $(r+1)$ if we apply it to $d_{r+1}\widetilde{\sigma}$. This follows again from the uniqueness property in the description of $\mathcal{D}$.

Now we claim that we can add all non-degenerate simplices of $Y_r \setminus Y_{r+1}$ by filling the $(r+1)$-st horn of the suspect simplices in question, and that all these horn-fillings can be done at once. Given the bijection we established, we only need to check that for any non-degenerate suspect simplex $\widetilde{\sigma}$ in $Y_r \setminus Y_{r+1}$, its $(r+1)$-st horn is contained in Y_{r+1}. This is an immediate consequence of Lemma 1.5. $\square$

References

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