MODERATE DEVIATIONS FOR TRACES OF WORDS IN A MULTI-MATRIX MODEL

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Abstract: We prove a moderate deviation principle for traces of words of weakly interacting random matrices defined by a multi-matrix model with a potential being a small perturbation of the GUE. The remarkable strength of high order expansions of the matrix model recently found by Guionnet and Maurel-Segala is the key fact that allows us to develop our result and provides also an alternative proof for a special case of the central limit theorem for traces of words, studied in [11].

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1. Introduction

1.1 The multi-matrix model In this note we study Hermitian matrices whose distribution is given by a small convex perturbation of the Gaussian Unitary Ensemble, denoted by GUE. We fix $m \in \mathbb{N}$ the number of random matrices we shall consider. Let $\mathcal{H}_N(\mathbb{C})^m$ be the set of m-tuples $A = (A_1, \ldots, A_m)$ of $N \times N$ hermitian matrices, such that $\mathcal{R}eA_i(kl), k < l,\ \mathcal{I}mA_i(kl), k < l,\ 2^{-\frac{1}{2}}A_i(kk)$ is a family of independent real Gaussian variables of variance $(2N)^{-1}$. We will consider the following perturbation of the GUE:

$$\mu_{V_t}^N(dA) = \frac{1}{Z_{V_t}^N} \exp\{-N\mathrm{tr}\,(V_t(A_1, \ldots, A_m))\}\,\mu^N(d\,A), \tag{1.1}$$

where $Z_{V_t}^N$ is the normalizing constant, $V_t(A_1, \ldots, A_m) := \sum_{i=1}^n t_i q_i(A_1, \ldots, A_m)$ a polynomial potential with $n \in \mathbb{N}$, $t = (t_1, \ldots, t_n) \in \mathbb{C}^n$ and monomials $(q_j)_{1 \le j \le n}$ fixed, and

$$\mu^N(d\,A) = \frac{1}{Z^N} \exp\left\{-\frac{N}{2}\mathrm{tr}\left(\sum_{i=1}^m A_i^2\right)\right\} d^N\,A,$$

the law of the m-dimensional GUE with $d^N\,A$ the Lebesgue measure on $\mathcal{H}_N(\mathbb{C})^m$. In this note we will prove a moderate deviation principle (MDP) and a central limit theorem (CLT) for normalized traces of a non-commutative monomial, $\mathrm{tr}(q_l(A_1, \ldots, A_m))$, under $\mu_{V_t}^N$.

For $m = 1$ and $V_t(A) = 0$ we recover the classical GUE. In this case, a MDP for the difference of the cumulative distribution function (cdf) of the semicircle distribution and the cdf of the eigenvalues was proved in [4].

Matrix models, in which general polynomials $V_t(A)$ were allowed in (1.1) while still keeping $m = 1$, have been studied intensively in physics. The choice $V_t(A) = t\,A^4, t \in \mathbb{C}$, was a commonly studied object, see [1, 2]. What was really striking, is the connection their analysis established between matrix integrals like $Z_{V_t}^N$ and map enumeration. A genus expansion of $\frac{Z_{V_t}^N}{Z^N}$ and $\log \frac{Z_{V_t}^N}{Z^N}$ respectively in powers of N was given for $V_t(x) = \frac{1}{2}x^2 + \sum_{k=1}^{\nu} t_k x^k$ with suitable chosen parameters $(t_1, \ldots, t_\nu)$ and it was placed on solid mathematical grounds by Ercolani and McLaughlin [6], who proved their results via a Riemann-Hilbert approach. It turned out, that the coefficients of the powers of N in the expansion [6, Theorem 1.1] are closely related to map enumeration.

Moreover, for $m = 1$ and for a polynomial potential V_t of even degree and with a positive leading coefficient, Johansson [13] proved a CLT for $\sum_{i=1}^N f(\lambda_i)$, where λ_i are the eigenvalues of the hermitian matrix, for functions f fulfilling some kind of regularity condition, being pointwise bounded by $C(V_t+1)$ for a constant $C > 0$ and its derivative being pointwise bounded by a polynomial.

Since many interesting models like the Ising model on random graphs, the q-Potts model on random graphs, the Chain model and the so-called induced QCD model (quantum chromodynamics, see [15]) are of form (1.1) and have $m > 1$, the question arose, whether a genus expansion for this multi-matrix model with general polynomial potentials V_t could be obtained. A first order expansion was obtained by Guionnet [7], a paper that relied on [12]. Guionnet and Maurel-Segala refined the expansion up to second order [11] until Maurel-Segala [16] gave the full genus expansion for the multi-matrix model of form (1.1). In [11], the authors also derived a CLT for N times the difference of the empirical measure and its limit, the solution of a Schwinger-Dyson equation. It was also shown that the CLT still holds, when the limit of the empirical measure is replaced by its expectation. For a special case of that CLT, we will give an alternative proof. Finally, notice that Cabanal-Duvillard [3] introduced a stochastic calculus approach and proved a CLT for traces of non-commutative polynomials of Gaussian Wigner and Wishart matrices, as well as for traces of non-commutative polynomials of pairs ($m = 2$) of independent Gaussian Wigner matrices.

After introducing a technical assumption for the polynomial potential V_t in (1.1), we will state our main result in the next subsection. Since the results of the paper [16] are the crucial foundation of this note, we will give the two main theorems of [16] in section 2, in which we finally prove our main result.

In the last section, we compare the two different admissible regions of parameters of the polynomial V_t as they appear in [6] and [16] and finally, we will apply our result to the Ising model and the q-Potts model.

1.2 Main results Before stating our main result, we introduce the notion of c-convexity for the potential V_t.

Definition 1.1. *If there exists $c > 0$, such that for all N in $\mathbb{N}$, the function*

$$\varphi_{V_t}^N : \begin{array}{ccc} \mathcal{H}_N(\mathbb{C})^m = (\mathbb{R}^{N^2})^m & \longrightarrow & \mathbb{C} \\ (A_1, \ldots, A_m) & \longrightarrow & \operatorname{tr}\left(V_t(A_1, \ldots, A_m) + \frac{1-c}{2} \sum_{i=1}^m A_i^2\right) \end{array}$$

is real and convex as a function of the entries of the matrices, we say that V_t is c-convex.

If V_t is c-convex, the Hessian of the trace of $V_t(A_1, \ldots, A_m) + \frac{1}{2} \sum_i^m A_i^2$ is symmetric and positive definite with eigenvalues bigger than c for any $N \in \mathbb{N}$. Remark, that the condition implies that $Z_{V_t}^N$ is automatically finite. An example of a c-convex V_t is

$$V_t = \sum_{i=1}^n P_i \left(\sum_{k=1}^m \alpha_k^i A_k \right) + \sum_{k,l} \beta_{k,l} A_k A_l$$

with convex real polynomials P_i in one unknown and for all l, $\sum_k |\beta_{k,l}| \leq (1 - c)$. This is due to Klein's Lemma [9, Lemma 6.2] which states that the trace of a real convex function of a self-adjoint matrix is a convex function as function of the entries of the matrix. Hence, a special class of examples are V_t of the form

$$V_t = \sum_{i=1}^{n} t_i \Big(\sum_{k=1}^{m} \alpha_k^i A_k \Big)^{2\,p_i}$$

with non-negative t_i's, integers p_i's and real α's (see the potentials V_t considered in [6] in the one matrix case $m = 1$).

Taking the potential $V_t := V_t(A) := V_t(A_1, \ldots, A_m)$ of form $\sum_{i=1}^n t_i q_i(A_1, \ldots, A_m)$, $n \in \mathbb{N}$, with complex $t = (t_1, \ldots, t_n) \in \mathbb{C}^n$ and non-commutative monomials $(q_j)_{1 \le j \le n}$, we next define for any $\eta > 0$ and $c > 0$

$$B_{\eta,c} = \Big\{ t \in \mathbb{C}^n \big| \ |t| = \max_i |t_i| \le \eta, V_t \text{ is } c\text{−convex} \Big\}.$$

Let $\mathbb{E}$ denote the expectation with respect to the probability measure $\mu_{V_t}^N$, defined in (1.1). Besides being c-convex, we require the potential function to also be self-adjoint in terms of the following definition.

Definition 1.2. *We say that the potential V_t is self-adjoint, if $V_t = V_t^\dagger$ holds with respect to the involution † that is given for all $z \in \mathbb{C}$ and all monomials $q_l(H_1, \ldots, H_m) = H_{l_1} \ldots H_{l_p}$ by*

$$(zq_l)^\dagger = (zH_{l_1} \ldots H_{l_p})^\dagger = \overline{z} H_{l_p} \ldots H_{l_1}. \tag{1.2}$$

Thus, if the potential V_t is self-adjoint, all the appearing monomials q_l are also self-adjoint. Note that a self-adjoint potential V_t or monomial q_l always has a real trace.

We say that a sequence of probability measures $(\mu_n)_{n \in \mathbb{N}}$, on some topological space X obeys a large deviation principle with speed a_n and good rate function $I(\cdot) : X \to \mathbb{R}_0^+$ if

- I is lower semi-continuous and has compact level sets $N_L := \{x \in X : I(x) \le L\}$, for every $L \in [0, \infty)$.
- For every open set $G \subseteq X$ it holds
$$\liminf_{n \to \infty} \frac{1}{a_n} \log \mu_n(G) \ge - \inf_{x \in G} I(x).$$
- For every closed set $A \subseteq X$ it holds
$$\limsup_{n \to \infty} \frac{1}{a_n} \log \mu_n(A) \le - \inf_{x \in A} I(x).$$

Similarly, we will say that a sequence of random variables $(Y_n)_{n \in \mathbb{N}}$ with topological state space X obeys a large deviation principle with speed a_n and good rate function $I(\cdot) : X \to \mathbb{R}_0^+$ if the sequence of their distributions does. Formally a moderate deviation principle is nothing but an LDP. However, we will speak about a moderate deviation principle (MDP) for a sequence of random variables, whenever the scaling of the corresponding random variables is between that of an ordinary Law of Large Numbers and that of a Central Limit Theorem.

Now our result is as follows. For a non-commutative monomial q_l, we define $\overline{\phi}_l := \operatorname{tr}(q_l(A_1,\dots,A_m)) - \mathbb{E}[\operatorname{tr}(q_l(A_1,\dots,A_m))]$, where the expectation is with respect to $\mu^N_{V_t}$.

Theorem 1.3 (A Moderate Deviation Principle (MDP))**.** *Let $t=(t_1,\dots,t_n)\in\mathbb{R}^n, c>0$ and $V_t = \sum_{i=1}^n t_i q_i(A_1,\dots,A_m)$ be self-adjoint with $t_l \neq 0$. Then there exists $\eta>0$, such that for $n\in\mathbb{N}$, with $t\in B_{\eta,c}\cap\mathbb{R}^n$ the sequence of distributions $\left(\mu^N_{V_t}\circ\left(\frac{1}{N^\gamma}\overline{\phi}_l\right)^{-1}\right)_N$ obeys a MDP with speed $N^{2\gamma}$ and rate function*

$$I(x) = \frac{x^2}{2}\left(\frac{\partial^2}{\partial {t_l}^2}F^0(t)\right)^{-1}$$

for any $0<\gamma<1$. The function $F^0(\cdot)$ is given by (1.3).

Since $F^0(t)$ is the connection between matrix models and map enumeration, we will give some more details (for a detailed explanation of that connection see [19]). A *map* is a connected graph drawn on a compact oriented connected *surface*, such that the edges are not intersecting (do not cross each other), the number of holes in the surface in order to avoid intersections is minimal, which is equivalent to obtaining a disjoint union of sets, where each set (or *face*) is homeomorphic to open disks after cutting the surfaces along the edges. That minimal number of holes is called the *genus* g of the surface. Thus a planar graph can be drawn on a surface of genus 0. We consider maps that are colored in m colors, i.e. each edge has a color $c\in\{1,\dots,m\}$.

For a non-commutative monomial in m indeterminants,

$$q_l(H_1,\dots,H_m) = H_{l_1}H_{l_2}\cdots H_{l_{k_l}}, \quad \text{with } k_l\in\mathbb{N}, l_j\in\{1,\dots,m\}, 1\le j\le k_l,$$

we define vertices, that are of type q_l as follows: We say that a vertex is of type q_l, if it has k_l colored half-edges, one marked half edge and an orientation, such that the marked half-edge is of color l_1, the next one with respect to the orientation is of color l_2 and so forth, until the last half-edge is colored with l_{k_l}. Thus, we obtain a bijection between monomials and stars. Moreover, the graphical interpretation of the involution † as defined in (1.2) is quite simple. Just shift the marker of the first half-edge towards the next neighboring half-edge against the orientation of the vertex and afterwards reverse the orientation of the vertex.

The function $F^0(t)$, which appears in the rate function of the MDP, is a generating function for maps of genus 0 associated with V_t,

$$F^0(t) = \sum_{k\in\mathbb{N}^n}\frac{(-t)^k}{k!}\mathcal{C}_0^k, \tag{1.3}$$

where $k! = \prod_{i=1}^n k_i!$, $(-t)^k = \prod_{i=1}^n (-t_i)^{k_i}$ and $\mathcal{C}_0^k$ being the number of maps on a surface of genus 0 with k_i vertices of type q_i.

It is not quite obvious, how the second derivative of $F^0(t)$ with respect to t_l looks like and we will see below (in Theorem 2.2) that it can be regarded as a generating function for maps of genus 0, which have two fixed stars q_l:

$$\frac{\partial^2}{\partial t_l^2} F^0(t) = \sum_{k \in \mathbb{N}^n} \frac{(-t)^k}{k!} \mathcal{C}_0^{k+j}, \tag{1.4}$$

where $j = (0, \dots, 0, j_l = 2, 0, \dots, 0))$ and everything else as above.

Now, we also state a special case of the CLT [11, Theorem 4.7] of which we will give an alternative proof.

Theorem 1.4 (A Central Limit Theorem (CLT))**.** *Let $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ and $c > 0$. Then there exists $\eta > 0$, such that for self-adjoint $V_t = \sum_{i=1}^n t_i q_i(A_1, \dots, A_m), t_l \neq 0, n \in \mathbb{N}$, with $t \in B_{\eta,c} \cap \mathbb{R}^n$ and any fixed $l \in \{1, \dots, n\}$ the distribution of the random variable $\overline{\phi}_l := \mathrm{tr}(q_l(A_1, \dots, A_m)) - \mathbb{E}[\mathrm{tr}(q_l(A_1, \dots, A_m))]$ with respect to $\mu_{V_t}^N$ converges towards the normal distribution with expectation 0 and variance $\frac{\partial^2}{\partial {t_l}^2} F^0(t)$.*

2. Proof of main results

Before we give the proofs, we state the two main theorems of [16], on which the proofs build upon. In physics it is known that the perturbed GUE is related to the enumeration of graphs on surfaces. The main result in [16] is, that for small convex perturbations, the moments of the empirical measure can be developed into a series whose g-th term is a generating function of graphs on a surface of genus g:

Theorem 2.1 (Theorem 1.1 in [16])**.** *Let $V_t = \sum_{i=1}^n t_i q_i(A_1, \dots, A_m)$ and $c > 0$. For all $g \in \mathbb{N}$, there exists $\eta_g > 0$, such that for all $t \in B_{\eta_g, c}$, $Z_{V_t}^N$ has the following expansion*

$$F_{V_t}^N := \frac{1}{N^2} \log Z_{V_t}^N = F^0(t) + \frac{1}{N^2} F^1(t) + \cdots + \frac{1}{N^{2g}} F^g(t) + o\left(\frac{1}{N^{2g}}\right), \tag{2.1}$$

where F^g is the generating function for maps of genus g associated with V_t,

$$F^g(t) = \sum_{k \in \mathbb{N}^n} \frac{(-t)^k}{k!} \mathcal{C}_g^k,$$

and $k! = \prod_i k_i!$, $(-t)^k = \prod_i (-t_i)^{k_i}$ and $\mathcal{C}_g^k$ is the number of maps on a surface of genus g with k_i vertices of type q_i.

Now it is obvious, why such an expansion is called genus expansion and why the leading term $F^0(t)$ was called the planar approximation. For our purposes, we will only apply that theorem for $g = 1$. That the expansion in (2.1) can be derived term by term is the content of the next theorem. In fact, this means that the asymptotics of much more observables are available. For j=$(j_1, \dots, j_n) \in \mathbb{N}^n$, we introduce the operator of derivation

$$\mathcal{D}_j = \frac{\partial^{\sum_i j_i}}{\partial t_1^{j_1} \cdots \partial t_n^{j_n}}.$$

Theorem 2.2 (Theorem 1.3 in [16]). *Let V_t as above, $c > 0$. For all $j = (j_1, \dots, j_n) \in \mathbb{N}^n$, for all $g \in \mathbb{N}$, there exists $\eta_g > 0$, such that for all $t \in B_{\eta_g, c}$,*

$$\mathcal{D}_j F_{V_t}^N = \mathcal{D}_j F^0(t) + \cdots + \frac{1}{N^{2g}} \mathcal{D}_j F^g(t) + o\left(\frac{1}{N^{2g}}\right), \tag{2.2}$$

where $\mathcal{D}_j F^g(t)$ is the generating function for rooted maps of genus g associated with V_t with some fixed vertices:

$$\mathcal{D}_j F^g(t) = \sum_{k \in \mathbb{N}^n} \frac{(-t)^k}{k!} \mathcal{C}_g^{k+j},$$

where $\mathcal{C}_g^k$ is again the number of maps on a surface of genus g with k_i vertices of type q_i. For details see [16, Theorem 7.4].

Thus, we find for example for $m = 2, V_t = t_1 H_1^2 H_2^2 H_1^2 + t_2 H_2 H_1 H_2$ and $j = (2, 0)$, that $D_j F^0$ counts all planar maps with $k_1 + 2$ vertices of type $q_1 = H_1^2 H_2^2 H_1^2$ and k_2 vertices of type $q_2 = H_2 H_1 H_2$. Moreover, because of Theorem 2.2 the rate function I from Theorem 1.3 is the same as

$$I(x) = \frac{x^2}{2} \left(\sum_{k \in \mathbb{N}^n} \frac{(-t)^k}{k!} \mathcal{C}_0^{k+2e_l} \right)^{-1}.$$

Theorems 2.1 and 2.1 yield asymptotic information concerning the statistics of words of the multi-matrix models. For example, by differentiating $\log Z_{V_t}^N$, one obtains

$$\frac{\partial}{\partial t_l} \log Z_{V_t}^N = -N\, \mathbb{E}\big[\mathrm{tr}(q_l(A_1, \dots, A_m))\big]. \tag{2.3}$$

In conjunction with (2.2), one learns the following:

$$\lim_{N \to \infty} \mathbb{E}\left(\frac{1}{N} \mathrm{tr}(q_l(A_1, \dots, A_m)) \right) = \frac{\partial}{\partial t_l} F^0(t).$$

Out of these two theorems, we deduce an asymptotic expansion for the moment generating function of $\phi_l := \phi_l(A) := \mathrm{tr}(q_l(A_1, \dots, A_m))$ and $\overline{\phi}_l := \overline{\phi}_l(A) := \phi_l(A) - \mathbb{E}[\phi_l(A)]$. Remember, that $\mathbb{E}$ denotes the expectation with respect to the probability measure $\mu_{V_t}^N$.

Lemma 2.3. *Let $t \in \mathbb{R}^n$ and $c > 0$. Then there exists $\eta > 0$, such that for self-adjoint $V_t = \sum_{i=1}^n t_i q_i(A_1, \ldots, A_m)$, $n \in \mathbb{N}$, with t in $B_{\eta,c} \cap \mathbb{R}^n$, and any fixed $l \in \{1, \ldots, n\}$ we have for any $s \in \mathbb{R}$ that*

$$\begin{aligned} \mathbb{E}\left(\exp\bigl(s\ \overline{\phi}_l\bigr)\right) &= \\ \exp\Bigl\{N^2\Bigl[F^0\Bigl(t - \frac{s}{N}e_l\Bigr) - F^0(t) + \frac{s}{N}\frac{\partial}{\partial t_l}F^0(t)\Bigr] &+ F^1\Bigl(t - \frac{s}{N}e_l\Bigr) - F^1(t) + \frac{s}{N}\frac{\partial}{\partial t_l}F^1(t) + o(1)\Bigr\}. \end{aligned}$$

In the non-centered case, we find

$$\mathbb{E}\left(\exp\bigl(s\ \phi_l\bigr)\right) = \exp\Bigl\{N^2\Bigl[F^0\Bigl(t - \frac{s}{N}e_l\Bigr) - F^0(t)\Bigr] + F^1\Bigl(t - \frac{s}{N}e_l\Bigr) - F^1(t) + o(1)\Bigr\}.$$

Proof. Consider for any $s \in \mathbb{R}$ and $l \in \{1, \ldots, n\}$ the potential $V_{t - \frac{s}{N} e_l}$. We easily see that

$$V_{t-\frac{s}{N}e_l} = \sum_{\substack{i=1 \\ i \neq l}}^n t_i\, q_i(H_1, \ldots, H_m) + \Bigl(t_l - \frac{s}{N}\Bigr) q_l(H_1, \ldots, H_m) = V_t - \frac{s}{N}\, q_l(H_1, \ldots, H_m).$$

We abbreviate $\widetilde{V} := \widetilde{V}(t, s, l) := V_{t - \frac{s}{N} e_l}$ and obtain

$$\begin{aligned} Z^N_{\widetilde{V}} &= \int_{\mathcal{H}_N(\mathbb{C})^m} \exp\Bigl(-N\Bigl(\mathrm{tr}(\widetilde{V}(H))\Bigr)\Bigr)\ \mu^N(d\,H) \\ &= \int_{\mathcal{H}_N(\mathbb{C})^m} \exp\bigl(-N(\mathrm{tr}(V_t))\bigr)\ \exp(s\,\mathrm{tr}(q_l(H)))\,\mu^N(d\,H)) \\ &= \mathbb{E}[\exp(s\,\mathrm{tr}(q_l(H)))]\, Z^N_{V_t}. \end{aligned}$$

Hence we get

$$\mathbb{E}\left[\exp(s\,\mathrm{tr}(q_l(H)))\right] = \frac{Z^N_{\widetilde{V}}}{Z^N_{V_t}}. \tag{2.4}$$

We want to apply Theorem 2.1 for both terms on the right hand side of the last equality. Fix $c > 0$. Then by Theorem 2.1 there exists a $\eta := \eta(c) > 0$ such that the expansion (2.1) holds true for $Z^N_{V_t}$ for all $t \in B_{\eta,c}$.

Now for fixed $s \in \mathbb{R}$ we choose N sufficiently large such that $|s/N| < \varepsilon$ for a $\varepsilon > 0$. We abbreviate $\widetilde{t} := t - \frac{s}{N}\, e_l$. For any $t \in B_{\eta,c}$ the polynomial V_t is c-convex, thus for N sufficiently large we can find a $c' > 0$ such that $V_{\widetilde{t}}$ is c'-convex. For this $c' > 0$ we can find a $\eta' := \eta'(c') > 0$ such that (2.1) holds true for any $t \in B_{\eta',c'}$, where V_t is c'-convex. Now we choose $\varepsilon < \eta'$ and $\eta'' := \min(\eta' - \varepsilon, \eta)$.

Since $\eta'' \leq \eta$, the condition $|t| \leq \eta''$ induces $|t| \leq \eta$ and therefore, the relevant set to consider is $B_{\eta'',c}$. Summarizing, for fixed $s \in \mathbb{R}$ and N sufficiently large, for any $t \in B_{\eta'',c}$ we can apply Theorem 2.1 for both terms in (2.4).

Remark that we apply Theorem 2.1 taking $g = 1$. Now we obtain the following asymptotic expansion in the non-centered case:

$$\begin{aligned}\frac{Z^N_{\widetilde{V}}}{Z^N_{V_t}} &= \exp \quad \left\{N^2 F^0\left(t - \frac{s}{N}e_l\right) + F^1\left(t - \frac{s}{N}e_l\right) + o(1)\right\}\\ &\qquad \times \exp\left\{-N^2F^0(t) - F^1(t) + o(1)\right\}\\ &= \exp \quad \left\{N^2\left[F^0\left(t - \frac{s}{N}e_l\right) - F^0(t)\right] + F^1\left(t - \frac{s}{N}e_l\right) - F^1(t) + o(1)\right\}.\end{aligned}$$

In the centered case, we also have to consider the contribution of $\mathbb{E}\left[\exp(-s\,\mathbb{E}[\phi_l(H)])\right]$ to the asymptotic expansion. As it is well known in statistical mechanics and as we have seen in (2.3), differentiating the free energy yields the expectation of the observables and hence we find

$$\begin{aligned} & s\mathbb{E}\left[\operatorname{tr}(q_l)\right] = -\frac{s}{N}\frac{\partial}{\partial t_l}\log Z^N_{V_t}\\ \iff \quad & \exp\left(-s\mathbb{E}\left[\operatorname{tr}(q_l)\right]\right) = \exp\left(\frac{s}{N}\frac{\partial}{\partial t_l}\log Z^N_{V_t}\right). \end{aligned} \tag{2.5}$$

Finally, (2.4), (2.5) and Theorem 2.2 yield

$$\begin{aligned}\mathbb{E}(\exp(s\,\bar{\phi}_l)) &= \frac{Z^N_{\widetilde{V}}}{Z^N_{V_t}} \times \exp\left(\frac{s}{N}\frac{\partial}{\partial t_l}\log Z^N_{V_t}\right)\\ &= \exp\left\{N^2\left[F^0\left(t - \frac{s}{N}e_l\right) - F^0(t)\right]\right.\\ &\qquad \left. + F^1\left(t - \frac{s}{N}e_l\right) - F^1(t) + o(1)\right\}\\ &\qquad \times \exp\left\{\frac{s}{N}\left(N^2\frac{\partial}{\partial t_l}F^0(t) + \frac{\partial}{\partial t_l}F^1(t) + o(1)\right)\right\},\end{aligned}$$

and therefore we obtain

$$\begin{aligned}\mathbb{E}(\exp(s\,\bar{\phi}_l)) &= \exp\left\{N^2\left[F^0\left(t - \frac{s}{N}e_l\right) - F^0(t) + \frac{s}{N}\frac{\partial}{\partial t_l}F^0(t)\right]\right.\\ &\qquad \left. + F^1\left(t - \frac{s}{N}e_l\right) - F^1(t) + \frac{s}{N}\frac{\partial}{\partial t_l}F^1(t) + o(1)\right\}.\end{aligned}$$

□

This lemma is now the crucial ingredient for the proofs of our main results.

Proof. (Proof of the central limit theorem, Theorem 1.4) By Taylors theorem, we have

$$F^i\left(t - \frac{s}{N}e_l\right) - F^i(t) + \frac{s}{N}\frac{\partial}{\partial t_l}F^i(t) = \frac{s^2}{2N^2}\frac{\partial^2}{\partial t_l^2}F^i(\xi^i_N), \quad i = 0, 1, \tag{2.6}$$

where $\xi^i_N \in \left(t \wedge \left(t - \frac{s}{N}e_l\right), t \vee \left(t - \frac{s}{N}e_l\right)\right)$.

Combining this with Lemma 2.3, we find the limit of the moment generating function of $\overline{\phi}_l$ to be that of a centered random variable having a normal distribution with variance $\frac{\partial^2}{\partial t_l^2}F^0(t)$:

$\lim_{N\to\infty} \mathbb{E}\left[\exp(s\,\overline{\phi}_l)\right] = \lim_{N\to\infty} \exp\left(\frac{s^2}{2}\frac{\partial^2}{\partial t_l^2}F^0(\xi_N^0) + \frac{s^2}{2N^2}\frac{\partial^2}{\partial t_l^2}F^1(\xi_N^1) + o(1)\right) = \exp\left(\frac{s^2}{2}\frac{\partial^2}{\partial t_l^2}F^0(t)\right)$, since $\xi_N^0 \to t$, for $N \to \infty$, and F^0, F^1 being differentiable functions of any order around the origin by Theorem 2.2 (they are even analytic functions, see [16, Lemma 4.4]). By Levy's continuity theorem the CLT is established. □

Proof. (Proof of the moderate deviation principle, Theorem 1.3) In order to apply the Gärtner-Ellis theorem (cf. Theorem 2.3.6 in [5]), we calculate the limit of the properly scaled cumulant generating function of the random variable $\frac{1}{N^\gamma}\overline{\phi}_l$, where $0 < \gamma < 1$: let $s \in \mathbb{R}$, we obtain

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N^{2\gamma}} \log \mathbb{E}\left[\exp\left(s\,N^{2\gamma}\frac{1}{N^\gamma}\overline{\phi}_l\right)\right] = \lim_{N\to\infty} \frac{1}{N^{2\gamma}} \log \mathbb{E}\left[\exp\left(s\,N^{\gamma}\overline{\phi}_l\right)\right] \\
= \;& \lim_{N\to\infty} \frac{1}{N^{2\gamma}} \log\left(\exp\left\{N^2\left[F^0\left(t - \frac{sN^\gamma}{N}e_l\right) - F^0(t) + \frac{sN^\gamma}{N}\frac{\partial}{\partial t_l}F^0(t)\right]\right.\right. \\
& \left.\left. + F^1\left(t - \frac{sN^\gamma}{N}e_l\right) - F^1(t) + \frac{sN^\gamma}{N}\frac{\partial}{\partial t_l}F^1(t) + o(1)\right\}\right) \\
= \;& \lim_{N\to\infty} \frac{1}{N^{2\gamma}} \left[\frac{N^2}{2}\left(\frac{sN^\gamma}{N}\right)^2 \frac{\partial^2}{\partial {t_l}^2}F^0(\xi_N^0) + \frac{1}{2}\left(\frac{sN^\gamma}{N}\right)^2 \frac{\partial^2}{\partial {t_l}^2}F^1(\xi_N^1) + o(1)\right] \\
= \;& \frac{s^2}{2}\frac{\partial^2}{\partial {t_l}^2}F^0(t). \qquad (2.7)
\end{aligned}$$

The second equality is due to Lemma 2.3. Although we cannot simply replace s by sN^γ and assume the lemma to hold, we can carefully go through the proof of Lemma 2.3 when the new scaling is applied. It turns out, that the lemma still holds, which basically relies on N^γ/N going to 0. The third equality is due to Taylor's theorem (2.6) and the last one follows by F^0 being differentiable of any order around the origin and the fact that $\xi_N^0 \in \left(t \wedge (t - \frac{sN^\gamma}{N}e_l), t \vee (t - \frac{sN^\gamma}{N}e_l)\right)$, which yields $\xi_N^0 \to t$, for $N \to \infty$, since $\gamma \in (0,1)$. In particular the right hand side of (2.7) is finite for all $s \in \mathbb{R}$, is everywhere differentiable in s and steep, since

$$\left|\frac{\partial}{\partial s}\frac{s^2}{2}\frac{\partial^2}{\partial {t_l}^2}F^0(t)\right| = \left|s\frac{\partial^2}{\partial {t_l}^2}F^0(t)\right| \longrightarrow \infty \quad \text{for} \quad s \to \infty.$$

Thus, we can apply the Gärtner-Ellis theorem, and finish the proof by calculating the Legendre-Fenchel transform of $\frac{s}{2}\frac{\partial^2}{\partial t_l^2}F^0(t)$, which is

$$I(x) = \sup_{s\in\mathbb{R}}\left\{sx - \frac{s^2}{2}\frac{\partial^2}{\partial t_l^2}F^0(t)\right\} = \frac{x^2}{2}\left(\frac{\partial^2}{\partial t_l^2}F^0(t)\right)^{-1}.$$

□

Remarks:

(1) For all the calculations carried out above, the second order expansion (g=1) in [11] could have been sufficient, while we needed Theorem 2.2 to establish our Lemma 2.3, which was crucial for proving

our result.

(2) Instead of using the asymptotic expansion of $\log Z^N_{V_t}$ up to order $g = 1$ in Lemma 2.3 and applying Taylor's theorem, we could have only used that expansion up to order $g = 2$ for proving the MDP and the CLT.

(3) In case of $m = 1$, we would find e.g. the limiting variance of $\overline{\phi}_l$ to be $\frac{\partial^2}{\partial t_l^2} F^0(t) = \sum_{k\in\mathbb{N}^n} \frac{(-t)^k}{k!}\kappa_0^{k+2e_l}$, where $\kappa_0^{k+2e_l}$ is the number of maps on a surface of genus 0 with k_i vertices of valence i, $i = \{1,\ldots,n\}$, and two vertices of valence l. That corresponds to a non-colored graph, respectively a graph in which all edges are colored with one color. Notice moreover that $F^0(t)$ corresponds to $e_0(t_1,\ldots,t_n)$ in the notation of [6].

(4) What about a LDP for $\left(\mu^N_{V_t} \circ \left(\frac{1}{N}\phi_l\right)^{-1}\right)_N$? Along the proof of (2.7) we obtain that

$$\lim_{N\to\infty} \frac{1}{N^2} \log \mathbb{E}[\exp\{sN\phi_l\}] = F^0(t - se_l) - F^0(t),$$

for $s \in \mathbb{R}$, such that $t - se_l \in B_{\eta,c}$. Hence we are only able to apply the Gärtner-Ellis theorem locally, for sufficient small s, obtaining a LDP only locally for sufficiently small intervals $(-\epsilon,\epsilon)$ with a implicit rate function

$$\widetilde{I}(x) = \sup_{\substack{s\in\mathbb{R},\\ t-se_l\in B_{\eta,c}}} \left\{sx - F^0(t - se_l) + F^0(t)\right\}.$$

In other words, the expansions in [16] are not strong enough to obtain a full LDP. For a similar discussion in case of $m = 1$, see [17].

Generalization:

All the results above can be generalized from monomials to polynomials, $\sum_{i=1}^n \alpha_i \operatorname{tr}(q_i)$, provided that each q_i has a non-vanishing coefficient t_i in the potential V_t. We will briefly state the crucial steps of the calculations and only look at $\psi = \alpha_1 \operatorname{tr}(q_{l_1}) + \alpha_2 \operatorname{tr}(q_{l_2})$, where $l_1, l_2 \in \{1,\ldots,n\}$, $l_1 \neq l_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$ fix, and $\overline{\psi} = \alpha_1 \operatorname{tr}(q_{l_1}) + \alpha_2 \operatorname{tr}(q_{l_2}) - \mathbb{E}[\alpha_1 \operatorname{tr}(q_{l_1}) + \alpha_2 \operatorname{tr}(q_{l_2})]$ respectively. In order to be able to apply Theorem 2.1 and a Taylor expansion in two variables, we argue as in the proof of Lemma 2.3 that for $c > 0$ we can find a $\eta = \eta(c) > 0$ such that $t - \kappa\frac{s}{N}(\alpha_1 e_{l_1} + \alpha_2 e_{l_2}) \in B_{\eta,c}$ for every $\kappa \in [0,1]$ and N sufficiently large. With the new potential $V_{t-\frac{s}{N}(\alpha_1 e_{l_1}+\alpha_2 e_{l_2})} = V_t - \frac{s}{N}(\alpha_1 q_{l_1} + \alpha_2 q_{l_2})$, we find that

$$\mathbb{E}\left[e^{s(\alpha_1 \mathrm{tr} q_{l_1} + \alpha_2 \mathrm{tr} q_{l_2})}\right] = Z^N_{V_{t-\frac{s}{N}(\alpha_1 e_{l_1}+\alpha_2 e_{l_2})}} \left(Z^N_{V_t}\right)^{-1}. \tag{2.8}$$

Centering ψ, we need to calculate $\mathbb{E}[\alpha_1 \mathrm{tr} q_{l_1} + \alpha_2 \mathrm{tr} q_{l_2}]$. As above, the following holds,

$$\mathbb{E}[\alpha_1 \mathrm{tr} q_{l_1} + \alpha_2 \mathrm{tr} q_{l_2}] = -\frac{1}{N}\left(\alpha_1 \frac{\partial}{\partial t_{l_1}} \log Z^N_{V_t} + \alpha_2 \frac{\partial}{\partial t_{l_2}} \log Z^N_{V_t}\right). \tag{2.9}$$

The moment generating function now turns out to be (via Theorem 2.1)

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{s\overline{\psi}}\right] &\overset{(2.8)}{=} \frac{Z^N_{V_{t-\frac{s}{N}(\alpha_1 e_{l_1}+\alpha_2 e_{l_2})}}}{Z^N_{V_t}} \exp\left\{\frac{s}{N}\sum_{i=1}^{2}\alpha_i \frac{\partial}{\partial t_{l_i}} \log Z^N_{V_t}\right\} \\ &\overset{(2.9)}{=} \exp\left\{N^2\left(F^0\left(t-\frac{s}{N}(\alpha_1 e_{l_1}+\alpha_2 e_{l_2})\right) - F^0(t) + \frac{s}{N}\sum_{i=1}^{2}\alpha_i\frac{\partial}{\partial t_{l_i}}F^0(t)\right)\right. \\ &\qquad \left. + F^1\left(t-\frac{s}{N}(\alpha_1 e_{l_1}+\alpha_2 e_{l_2})\right) - F^1(t) + \frac{s}{N}\sum_{i=1}^{2}\alpha_i\frac{\partial}{\partial t_{l_i}}F^1(t) + o(1)\right\}.\end{aligned}$$

Hence we obtain

$$\mathbb{E}\left[\mathrm{e}^{s\overline{\psi}}\right] = \exp\left\{\frac{s^2}{2}\sum_{i,j=1}^{2}\alpha_i\alpha_j\frac{\partial^2}{\partial t_{l_i}\partial t_{l_j}}F^0(\xi_N) + \mathcal{O}\left(\frac{1}{N^2}\right)\right\}, \tag{2.10}$$

and therefore

$$\lim_{N\to\infty}\mathbb{E}\left[\mathrm{e}^{s\overline{\psi}}\right] = \exp\left\{\frac{s^2}{2}\sum_{i,j=1}^{2}\alpha_i\alpha_j\frac{\partial^2}{\partial t_{l_i}\partial t_{l_j}}F^0(t)\right\}.$$

Here, we used the Taylor expansion in two variables and $\xi_N = t - \kappa\frac{s}{N}(\alpha_1 e_{l_1} + \alpha_2 e_{l_2})$, for a $\kappa \in (0,1)$. Having thus obtained the CLT for $\overline{\psi}$ under $\mu^N_{V_t}$, we can use the expansion (2.10) to obtain the MDP for $\frac{1}{N^\gamma}\overline{\psi}$ under $\mu^N_{V_t}$, where $\gamma \in (0,1)$, with speed $N^{2\gamma}$ and rate function $I(x) = \frac{x^2}{2}\left(\sum_{i,j=1}^2 \alpha_i\alpha_j\frac{\partial^2}{\partial t_{l_i}\partial t_{l_j}}F^0(t)\right)^{-1}$ via the Gärtner-Ellis approach:

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N^{2\gamma}}\log\mathbb{E}\left[\exp\left\{sN^\gamma\overline{\psi}\right\}\right] &= \lim_{N\to\infty}\frac{1}{N^{2\gamma}}\left\{\frac{N^2}{2}\left(\frac{sN^\gamma}{N}\right)^2\left[\sum_{i,j=1}^{2}\alpha_i\alpha_j\frac{\partial^2}{\partial t_{l_i}\partial t_{l_j}}F^0(\xi_N) + o(1)\right]\right\} \\ &= \frac{s^2}{2}\sum_{i,j=1}^{2}\alpha_i\alpha_j\frac{\partial^2}{\partial t_{l_i}\partial t_{l_j}}F^0(t),\end{aligned}$$

since $\xi_N = t - \kappa\frac{sN^\gamma}{N}(\alpha_1 e_{l_1} + \alpha_2 e_{l_2})$, with $\kappa \in (0,1)$ and F^0 being differentiable of any order for t sufficiently small. Since $\frac{s^2}{2}\sum_{i,j=1}^2\alpha_i\alpha_j\frac{\partial^2}{\partial t_{l_i}\partial t_{l_j}}F^0(t)$ is finite for $s \in \mathbb{R}$, everywhere differentiable in s and steep, we have established the MDP.

3. Discussion and Applications

Let us consider the one-matrix model $m = 1$ first. In this case, the results of [16] are comparable to the expansions given in [6, Theorem 1.1]. The former results can be applied to V_t being a c-convex polynomial with $t \in B_{\eta,c}\cap\mathbb{R}^n$. The latter results hold true for polynomials V_t such that $t = (t_1, \dots, t_n)$ lies in the region

$$\mathbb{T}(T,\gamma) := \left\{t \in \mathbb{R}^n \mid |t| \leq T, t_n > \gamma\sum_{j=1}^{n-1}|t_j|\right\}. \tag{3.1}$$

To be more specific, there is a $T > 0$ and a $\gamma > 0$ such that for $t \in \mathbb{T}(T, \gamma)$ the asymptotic expansion of $\log Z_{V_t}^N$ given in [6, Theorem 1.1] is available. An analytic comparison between these two admissible regions of parameters prompts us to claim the following :

Claim 1: If n is even and $T > 0$ and $\gamma > 1$ are such that $T < \frac{1}{n(n-1)^2(1+\gamma)}$, it holds that

$$\mathbb{T}(T, \gamma) \subset B_{T,c(T,\gamma)} \quad \text{for some } c(T, \gamma) > 0.$$

Claim 2: If n is uneven, for V_t we can find $T > 0$ and $\gamma > 0$, such that $t \in \mathbb{T}(T, \gamma)$, whereas $\forall\, \eta > 0$ and $\forall\, c > 0$ we find $t \notin B_{\eta,c}$.

In the situation of claim 1, the expansion of $\log Z_{V_t}^N$ does not hold a priori as we did not choose T and γ with respect to that condition but rather arbitrary. But we can deduce that for our choice of T and γ as above, the expansion of $\log Z_{V_t}^N$ holds for any $t \in \mathbb{T}(T, \gamma)$, because the potential can be shown to be c-convex. Because of this observation, it is sometimes mentioned that the c-convexity encompasses the condition of Ercolani and McLaughlin for T sufficiently small and γ sufficiently large, see e.g. [9]. While the first claim only holds for even n, the second claim states that for uneven n, only the expansion of Ercolani and McLaughlin is applicable, since V_t cannot be c-convex in this situation.

Proof. (Proof of Claim 1) Let $T > 0$ and $\gamma > 1$ be as in claim 1. As $m = 1$, the potential V_t is of form $V_t(H) = \sum_{i=1}^n t_i H^i$, where H is a hermitian matrix. Therefore, $\mathrm{tr}(V_t + \frac{1-c}{2} H^2)$ is always real.
We will deduce the existence of a $c(T, \gamma)$ such that V_t is c-convex by an application of Klein's lemma and therefore we show that $f(x) := \sum_{i=1}^n t_i x^i + \frac{1-c}{2} x^2$ is a convex function. Thus, we need to establish the positivity of the second derivative, $g(x) := f''(x) = \sum_{i=2}^n t_i (i-1) i\ x^{i-2} + 1 - c \geq 0$ for any $x \in \mathbb{R}$. Observe, that we can find a $c_1(T, \gamma)$ such that $g(0) = 2t_2 + 1 - c > 0$, because of $\gamma > 1$ and $T < \frac{1}{2}$ for $n \geq 2$.
The next case to consider is that of $|x| \geq 1$. As $t \in \mathbb{T}(T, \gamma)$, it is obvious from (3.1) that

$$g(x) = \sum_{i=1}^{n} t_i (i-1) i\ x^{i-2} + 1 - c > \sum_{i=1}^{n-1} \left(t_i (i-1) i\ x^{i-2} + \gamma |t_i| (n-1) n\ x^{n-2} \right) + 1 - c.$$

Because n is even and $\gamma > 1$, it holds for every $i \leq n$ that

$$\gamma |t_i| (n-1) n\ x^{n-2} = \left| \gamma |t_i| (n-1) n\ x^{n-2} \right| > \left| t_i (i-1) i\ x^{i-2} \right|,$$

which gives $g(x) > 0$ for x with $|x| \geq 1$.
When $|x| < 1$, we start with the observation that

$$0 \leq \left| \sum_{i=1}^{n-1} \left(t_i (i-1) i\ x^{i-2} + \gamma |t_i| (n-1) n\ x^{n-2} \right) \right| < T(n-1)^2 n + \gamma T n (n-1)^2 < 1.$$

Thus, we can find a $c_2(T,\gamma) > 0$ such that

$$g(x) > \sum_{i=1}^{n-1} \left(t_i(i-1)i\ x^{i-2} + \gamma|t_i|(n-1)n\ x^{n-2}\right) + 1 - c_2(T,\gamma) > 0.$$

Hence, we choose $c(T,\gamma) = \min\{c_1(T,\gamma), c_2(T,\gamma)\}$ and applying Klein's lemma yields

$$\mathbb{T}(T,\gamma) \subset B_{T,c(T,\gamma)}.$$

$\square$

Proof. (Proof of Claim 2) In case of uneven n, take e.g. the potential $V_t = 2t_2H^2 + t_3H^3$. Obviously, this potential is not c-convex (take e.g. the case $N = 1$, in which the matrix H reduces to one real-valued unknown h and we immediately see that $\operatorname{tr}\left(V_t(h) + \frac{1-c}{2}h^2\right) = \frac{4t_2+1-c}{2}h^2 + t_3h^3$ is not a convex function in h for any $c > 0$, regardless of our choice of $\eta > 0$). Along the same lines, we see that any potential V_t with $t_n \neq 0$ for n uneven cannot be c-convex. $\square$

Remark:
When using the expansion (2.1), a CLT and a MDP for the centered and properly scaled random variable $\operatorname{tr}(q_l) = \operatorname{tr}(X^l)$ could only be obtained if $t_l \neq 0$. Considering the case of $V_t = t_nH^n, n \geq 2$, either even or uneven, we see that the expansion can be applied to the numerator and denominator in (2.4) even for t_l with $l < n$: Provided that $t \in \mathbb{T}^o(T,\gamma)$, where $\mathbb{T}^o(T,\gamma)$ denotes the interior of the set $\mathbb{T}(T,\gamma)$, it still holds for large N that $\max\{t_n, |\frac{s}{N}|\} < T$ and $t_n > |\gamma\frac{s}{N}|$, for $s \in \mathbb{N}$. And once this expansion was established, the proofs can be left unchanged to yield the CLT and MDP for the distribution of the centered and properly scaled random variable $\operatorname{tr}(X^l) = \sum_{i=1}^N \lambda_i^l$ with $l < n$, although $t_l = 0$.

Next, we consider an example for $m = 2$, the random Ising model on random graphs. The Gibbs measure of that model is given by

$$\mu_{Is}^N(dH_1, dH_2) = \frac{1}{Z_{Is}^N} \exp\left\{-N\left(\operatorname{tr}(V_{t_1}^1(H_1)) + \operatorname{tr}(V_{t_2}^2(H_2)) - t_3\operatorname{tr}(H_1H_2)\right)\right\}\mu^N(d\,H_1)\mu^N(d\,H_2).$$

Here, $V_t^i(H_i)$ are convex self-adjoint polynomials depending on the parameter $t_i, i = 1,2$, and $t_3 \in \mathbb{R}$. This model has been studied with regard to the first order asymptotic of the logarithmic partition function in [7] by using large deviations techniques and the first order could be given by a variational formula.

Choosing the parameters t_i, $i = 1,2,3$, small enough guarantees that the function $V_{t_1}^1(H_1) + V_{t_2}^2(H_2) - t_3H_1H_2$ is c-convex, see also [10], and provides that the free energy can also be expanded into a generation function for maps. which was done in [8].

Now, we take potential functions $V^j_{t_j}$ of type

$$V^j_{t_j} = \sum_{i=1}^{n_j} t_{j,i} H^i_j, \ \ j = 1, 2, \tag{3.2}$$

and denote the N real eigenvalues of the two $N \times N$ hermitian matrices H^i_j by $\lambda_{j,i}, j = 1, 2, i = 1, \ldots, N$. Thus, we can apply Theorem 1.3 and 1.4 for small enough t for $l \le n$ with $t_{j,l} \neq 0$ in $V^j_{t_j}$ to yield a CLT or a MDP for the sequence of distributions $(\mu^N_{V_t} \circ (\frac{1}{N^\gamma} \sum_{i=1}^n (\lambda^l_{j,i} - \mathbb{E}[\lambda^l_{j,i}]))^{-1})_N$ under μ^N_{Is}. The CLT and MDP can in case of small enough $t_{j,i}$ and t_3 also be extended to hold for traces of polynomials P of type

$$P(H_1, H_2) = \sum_k \alpha_k H_1^k + \sum_i \beta_i H_2^i + \delta H_1 H_2,$$

for $\alpha_k, \beta_i, \delta \in \mathbb{R}$, where we only have to take care, that any monomial appearing in P also has a non-vanishing parameter in its original potential function, i.e. $t_{1,k}, t_{2,i}, t_3 \neq 0$.

Finally, let us mention two models for general m, which are the chain model (see [14]) and the q-Potts model (see e.g. [18]), for which the MDP and CLT can be applied. The potentials V are given by

$$V(H_1, \ldots, H_m) = \sum_{i=1}^m V^i_{t_{1,i}}(H_i) - \sum_{i=2}^m t_{2,i}(H_{i-1} H_i),$$

and

$$V(H_1, \ldots, H_m) = \sum_{i=1}^m V^i_{t_{1,i}}(H_i) - \sum_{i=2}^m t_{2,i}(H_1 H_i),$$

where the V^i_t are convex self-adjoint polynomials with parameters $t_{1,i}$, $i = 1, \ldots, m$ small enough. As in the Ising model on random graphs, a MDP and a CLT can be established for polynomials, which consist of monomials appearing in the corresponding potentials.

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