Free Probability Theory
and
Random Matrices

Roland Speicher
Universität des Saarlandes
Saarbrücken

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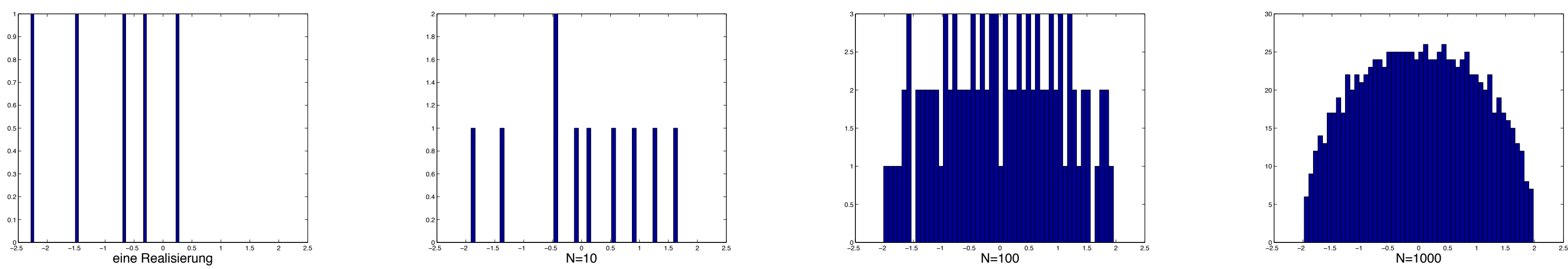
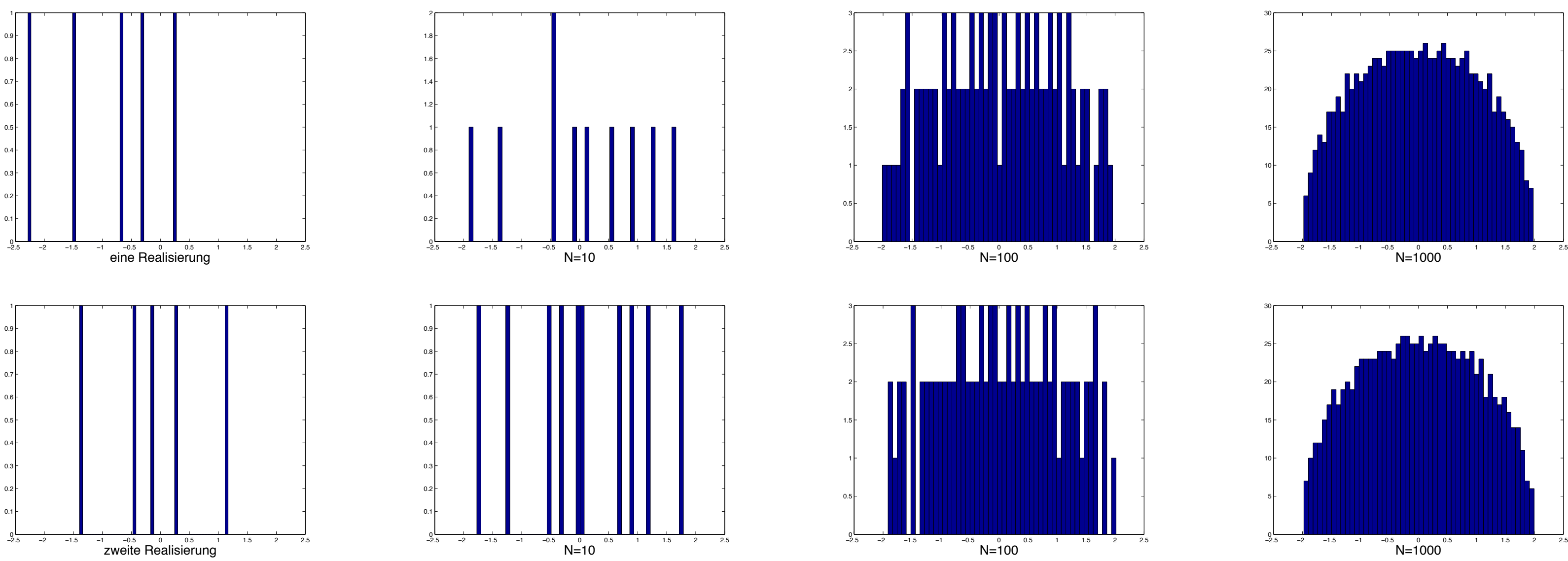
Typical phenomena for basic random matrix ensembles:

- almost sure convergence to a deterministic limit eigenvalue distribution
- this limit distribution can be effectively calculated


eine Realisierung
N=10
N=100
N=1000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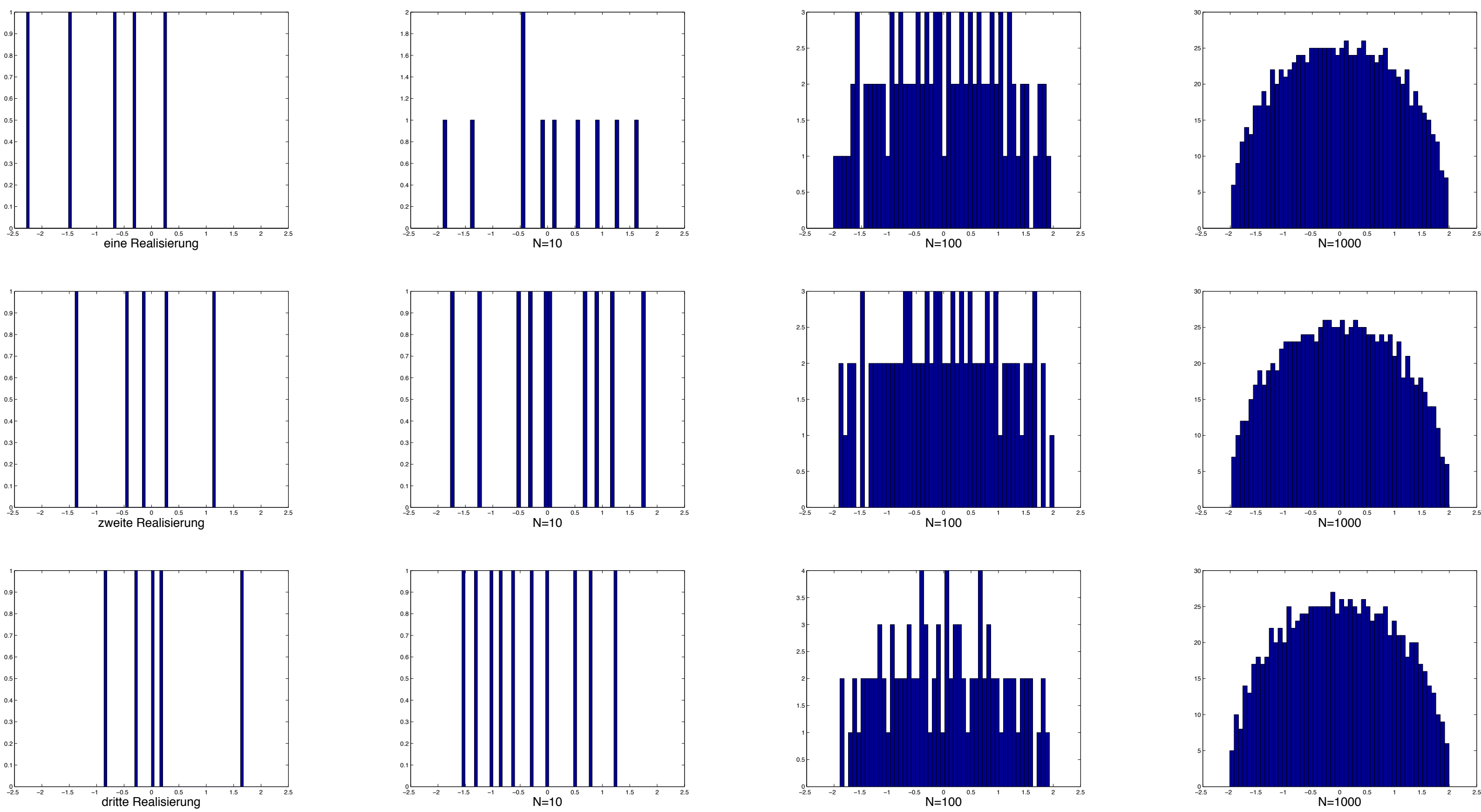
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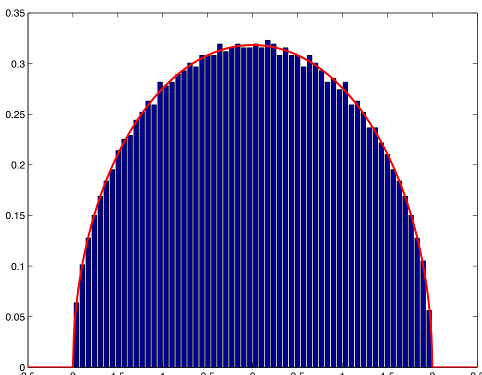
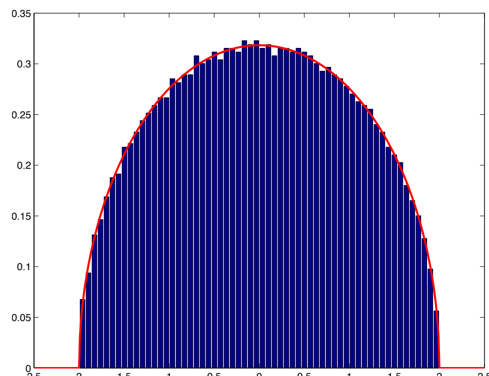
Consider selfadjoint **Gaussian $N \times N$ random matrix**.

We have almost sure convergence (convergence of "typical" realization) of its eigenvalue distribution to

Wigner's semicircle.



... one realization ...

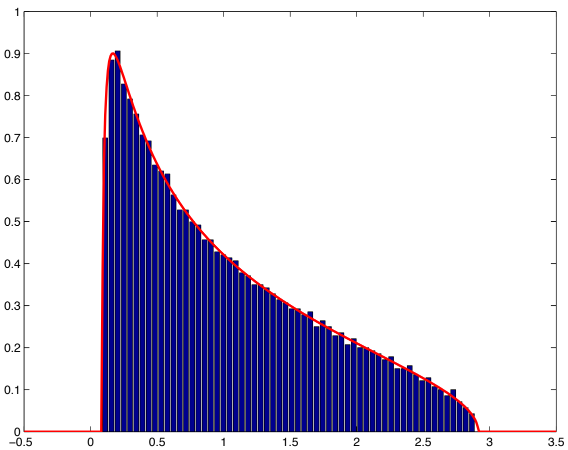


... another realization ...

$$N = 4000$$

Consider **Wishart random matrix** $A = XX^*$, where X is $N \times M$ random matrix with independent Gaussian entries.
Its eigenvalue distribution converges almost surely to

Marchenko-Pastur distribution.



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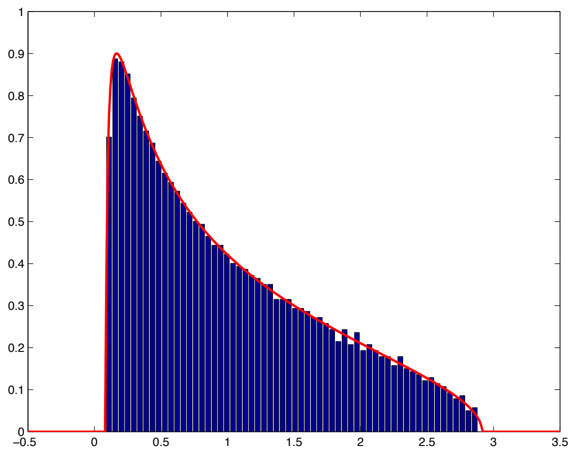

... another realization ...

$$N = 3000, M = 6000$$

We want to consider more complicated situations, built out of simple cases (like Gaussian or Wishart) by doing operations like

- taking the sum of two matrices
- taking the product of two matrices
- taking corners of matrices

Note: If several $N \times N$ random matrices A and B are involved then the eigenvalue distribution of non-trivial functions $f(A,B)$ (like $A+B$ or AB) will of course depend on the relation between the eigenspaces of A and of B.

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- if $\boldsymbol{N \to \infty}$ and

- if the eigenspaces are almost surely in a **"typical" or "generic" position**.

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- if $\boldsymbol{N \to \infty}$ and
- if the eigenspaces are almost surely in a **"typical" or "generic" position**.

This is the realm of **free probability theory**.

Consider $N \times N$ random matrices A and B such that

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  B has an asymptotic eigenvalue distribution for $N \to \infty$

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Then, almost surely, eigenspaces of A and of B are in generic position.

In such a generic case we expect that the asymptotic eigenvalue distribution of functions of A and B should almost surely depend in a deterministic way on the asymptotic eigenvalue distribution of A and of B the asymptotic eigenvalue distribution.

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Basic examples for such functions:

- the sum

$$A + B$$

- the product

$$AB$$

- corners of the unitarily invariant matrix B

Example: sum of independent Gaussian and Wishart ($M = 2N$) random matrices, for $N = 3000$

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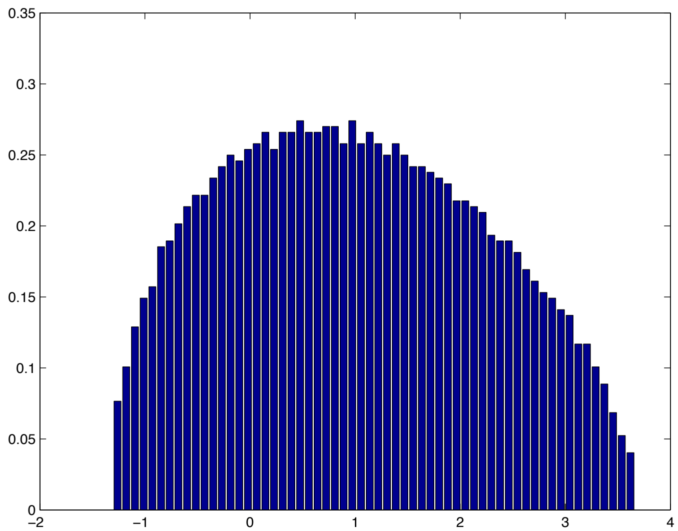

... one realization ...

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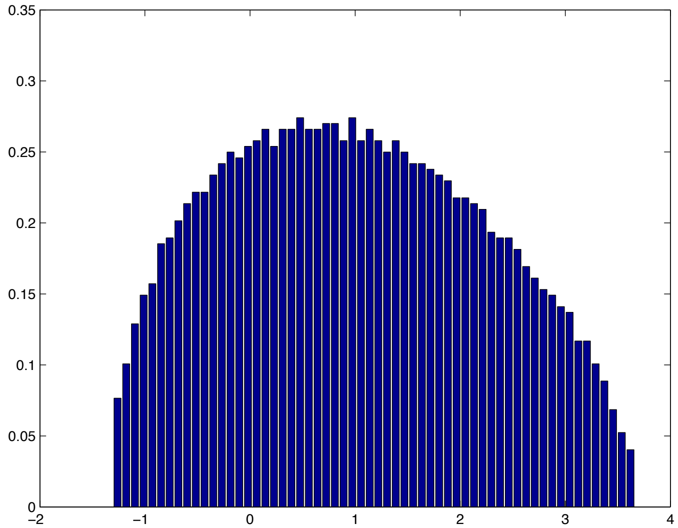

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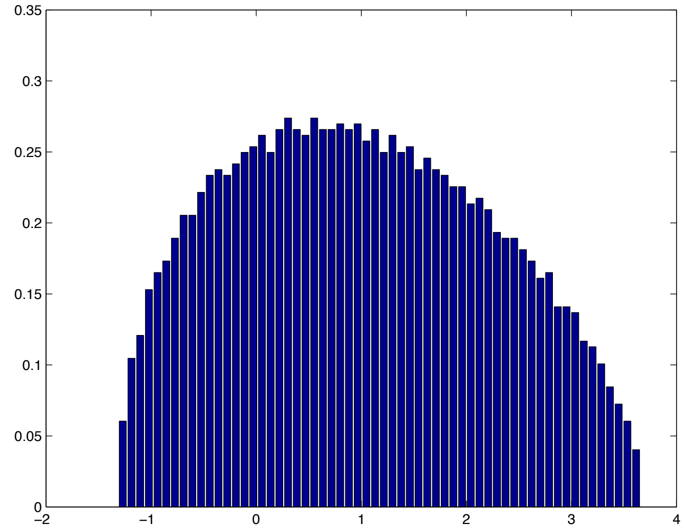

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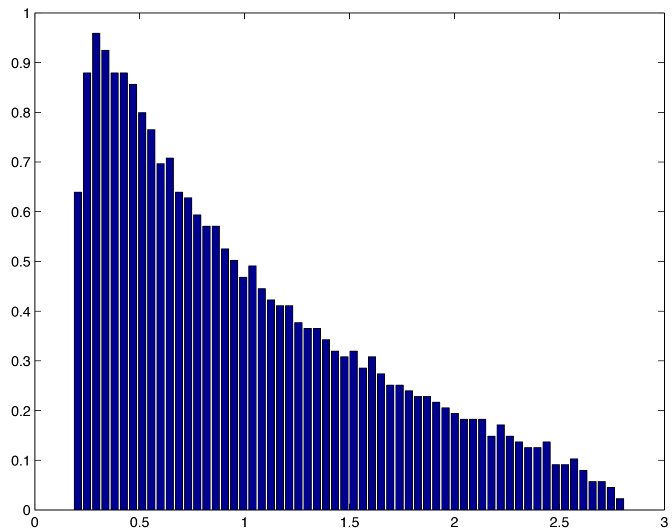

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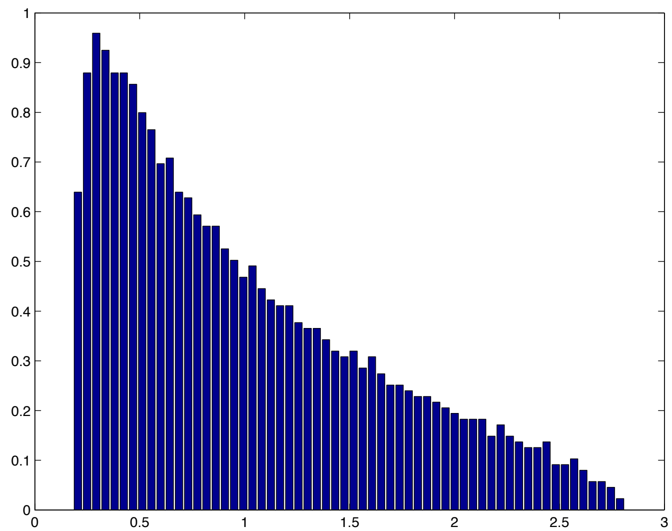

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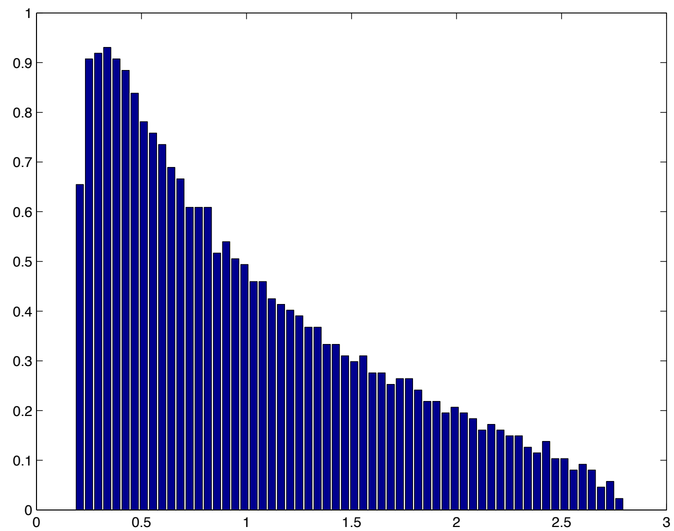

... another realization ...

Example: upper left corner of size $N/2 \times N/2$ of a randomly rotated $N \times N$ projection matrix,
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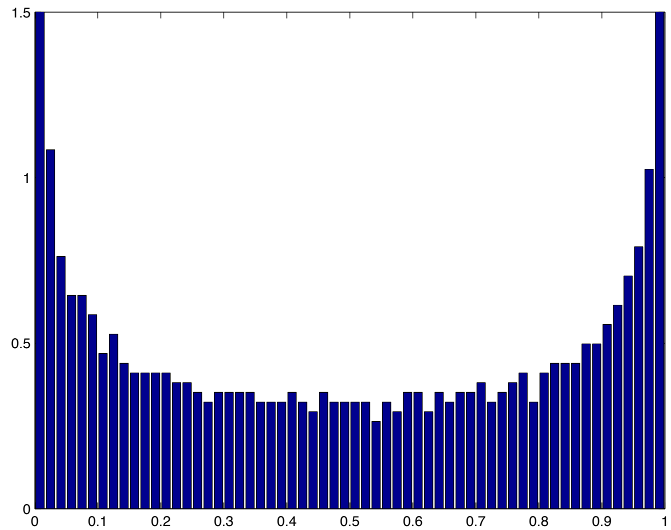

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$N = 2048$

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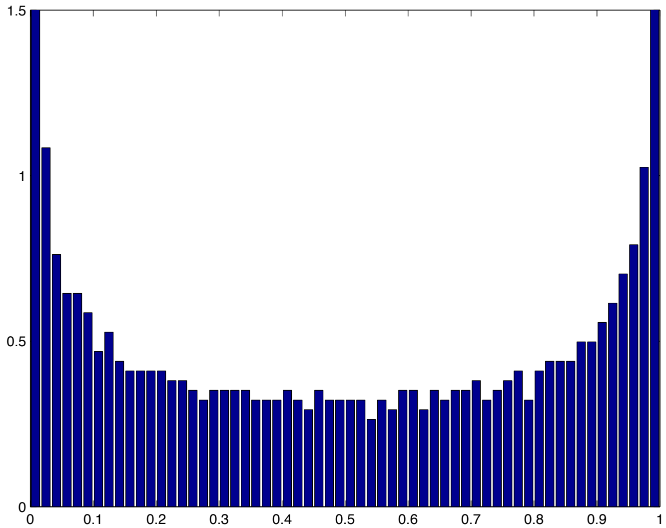

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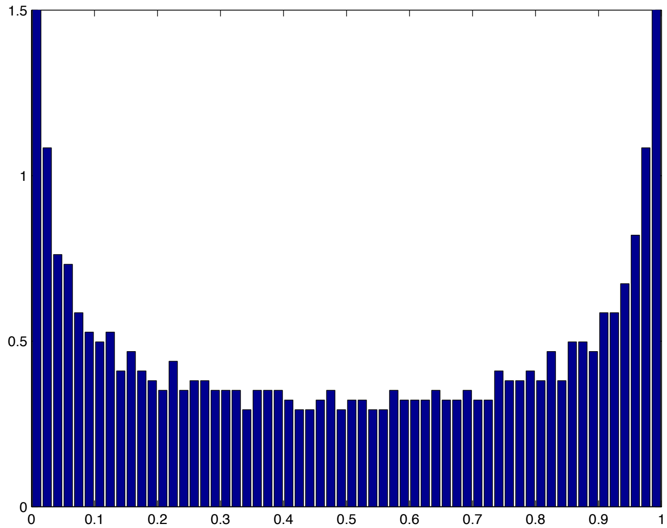

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Problems:

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- Is there an **algorithm for actually calculating** the corresponding asymptotic eigenvalue distributions?

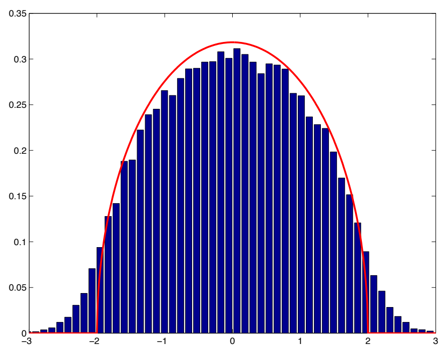
Instead of eigenvalue distribution of typical realization we will now look at eigenvalue distribution averaged over ensemble.

This has the advantages:

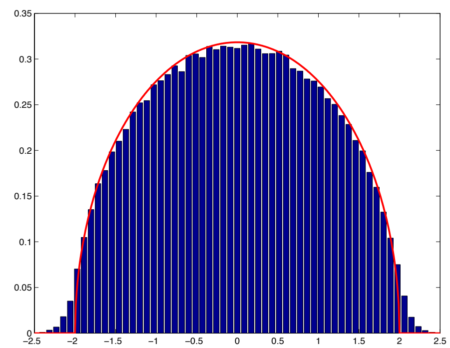
- convergence to asymptotic eigenvalue distribution happens much faster; very good agreement with asymptotic limit for moderate N

- theoretically easier to deal with averaged situation than with almost sure one (note however, this is just for convenience; the following can also be justified for typical realizations)

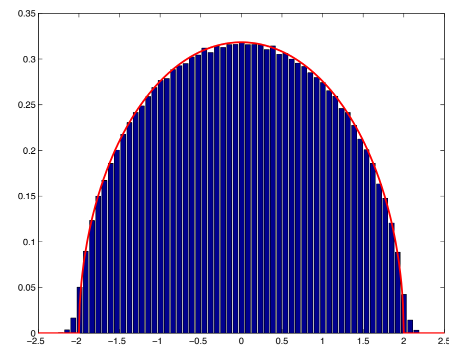
Example: Convergence of averaged eigenvalue distribution of $N \times N$ Gaussian random matrix to **semicircle**



N=5



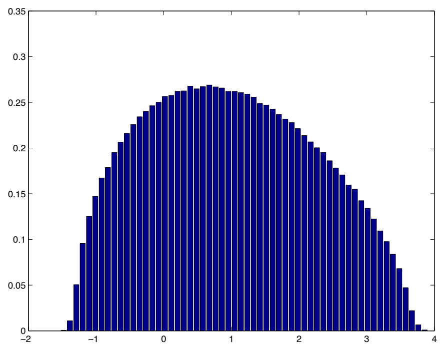
N=20



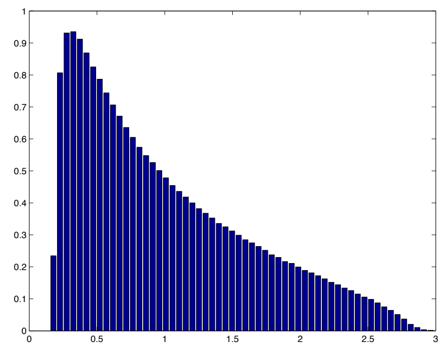
N=50

trials=10000

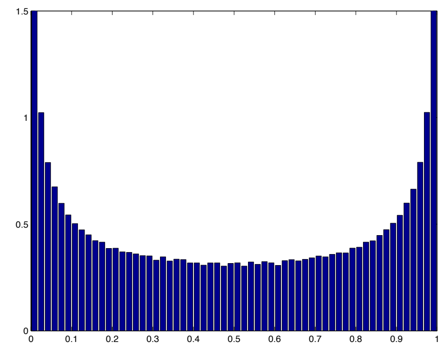
Examples: averaged sums, products, corners for moderate N



averaged Wigner + Wishart; N=100



averaged Wishart x Wishart; N=100



averaged upper corner; N=64

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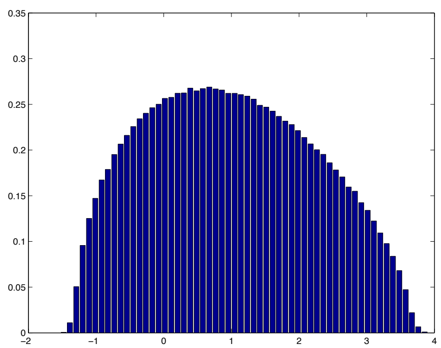
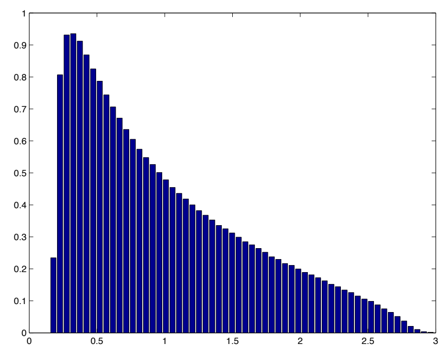
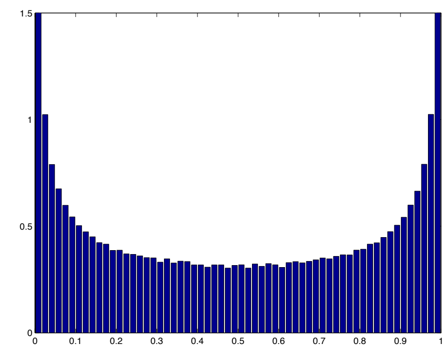


averaged Wigner + Wishart; N=100



averaged Wishart x Wishart; N=100



averaged upper corner; N=64

What is the asymptotic eigenvalue distribution in these cases?

How does one analyze asymptotic eigenvalue distributions?

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- analytical

- combinatorial

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- combinatorial: moment method
  try to calculate moments of the limit distribution
  advantage: can, in principle, deal directly with several matrices A, B; by looking on **mixed moments**

Moment Method

eigenvalue distribution of matrix A	$\widehat{=}$	knowledge of traces of powers, $\mathrm{tr}(A^k)$
$\frac{1}{N}\left(\lambda_1^k + \cdots + \lambda_N^k\right)$	$=$	$\mathrm{tr}(A^k)$
averaged eigenvalue distribution of random matrix A	$\widehat{=}$	knowledge of expectations of traces of powers, $E[\mathrm{tr}(A^k)]$

Moment Method

Consider random matrices A and B in generic position.

We want to understand $f(A, B)$ in a uniform way for many f!

We have to understand for all $k \in \mathbb{N}$ the moments

$$E\left[\mathrm{tr}\Big(f(A,B)^k\Big)\right].$$

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Thus we need to understand as basic objects

mixed moments $\qquad E\left[\mathrm{tr}\left(A^{n_1}B^{m_1}A^{n_2}B^{m_2}\cdots\right)\right]$

Use following notation:

$$\varphi(A) := \lim_{N\to\infty} E[\mathrm{tr}(A)].$$

Question: If A and B are in generic position, can we understand

$$\varphi\left(A^{n_1}B^{m_1}A^{n_2}B^{m_2}\cdots\right)$$

in terms of

$$\Big(\varphi(A^k)\Big)_{k\in\mathbb{N}} \qquad \text{and} \qquad \Big(\varphi(B^k)\Big)_{k\in\mathbb{N}}$$

Example: independent Gaussian random matrices

Consider two independent Gaussian random matrices A and B

Then, in the limit $N \to \infty$, the moments

$$\varphi\left(A^{n_1} B^{m_1} A^{n_2} B^{m_2} \cdots\right)$$

are given by

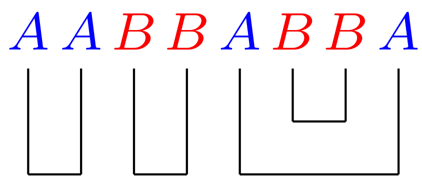
$$\#\Big\{\text{non-crossing/planar pairings of pattern}$$

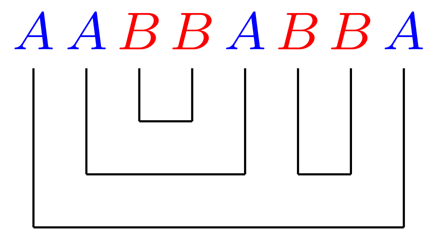
$$\underbrace{A \cdot A \cdots A}_{n_1\text{-times}} \cdot \underbrace{B \cdot B \cdots B}_{m_1\text{-times}} \cdot \underbrace{A \cdot A \cdots A}_{n_2\text{-times}} \cdot \underbrace{B \cdot B \cdots B}_{m_2\text{-times}} \cdots,$$

$$\text{which do not pair } A \text{ with } B\Big\}$$

Example: $\varphi(AABBABBA) = 2$

because there are two such non-crossing pairings:

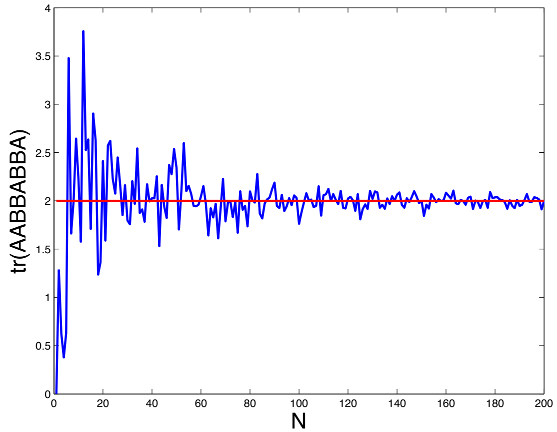


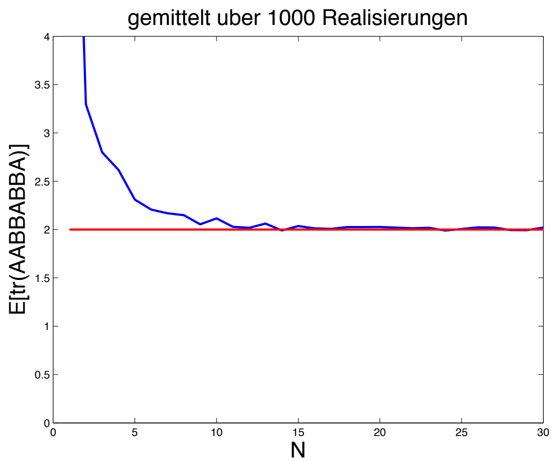


Example: $\varphi(AABBABBA) = 2$

one realization

averaged over 1000 realizations





$$\varphi\left(A^{n_1} B^{m_1} A^{n_2} B^{m_2} \cdots\right)$$

$$= \#\Big\{\text{non-crossing pairings which do not pair } A \text{ with } B\Big\}$$

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implies

$$\varphi\Big(\big(A^{n_1}-\varphi(A^{n_1})\cdot 1\big)\cdot\big(B^{m_1}-\varphi(B^{m_1})\cdot 1\big)\cdot\big(A^{n_2}-\varphi(A^{n_2})\cdot 1\big)\cdots\Big)$$

$$= \#\Big\{\text{non-crossing pairings which do not pair } A \text{ with } B, \text{ and for which each blue group and each red group is connected with some other group}\Big\}$$

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$$\varphi\Big(\big(A^{n_1} - \varphi(A^{n_1})\cdot 1\big)\cdot\big(B^{m_1} - \varphi(B^{m_1})\cdot 1\big)\cdot\big(A^{n_2} - \varphi(A^{n_2})\cdot 1\big)\cdots\Big)$$

$$= 0$$

Actual equation for the calculation of the mixed moments

$$\varphi_1\left(A^{n_1}B^{m_1}A^{n_2}B^{m_2}\cdots\right)$$

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However, the relation between the mixed moments,

$$\varphi\Big(\big(A^{n_1}-\varphi(A^{n_1})\cdot 1\big)\cdot\big(B^{m_1}-\varphi(B^{m_1})\cdot 1\big)\cdots\Big)=0$$

remains the same for matrix ensembles in generic position and constitutes the **definition of freeness**.

Definition [Voiculescu 1985]: A and B are **free** (with respect to φ) if we have for all $n_1, m_1, n_2, \dots \geq 1$ that

$$\varphi\Big(\big(A^{n_1} - \varphi(A^{n_1})\cdot 1\big)\cdot\big(B^{m_1} - \varphi(B^{m_1})\cdot 1\big)\cdot\big(A^{n_2} - \varphi(A^{n_2})\cdot 1\big)\cdots\Big) = 0$$

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$$\varphi\Big(\text{alternating product in centered words in } A \text{ and in } B\Big) = 0$$

Theorem [Voiculescu 1991]: Consider $N \times N$ random matrices A and B such that

- A has an asymptotic eigenvalue distribution for $N \to \infty$
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- A and B are independent
  (i.e., entries of A are independent from entries of C)

- B is a unitarily invariant ensemble
  (i.e., the joint distribution of its entries does not change under unitary conjugation)

Then, for $N \to \infty$, A and B are free.

Definition of Freeness

Let $(\mathcal{A}, \varphi)$ be **non-commutative probability space**, i.e., $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is unital linear functional (i.e., $\varphi(1) = 1$)

Unital subalgebras $\mathcal{A}_i$ $(i \in I)$ are **free** or **freely independent**, if $\varphi(a_1 \cdots a_n) = 0$ whenever

- $a_i \in \mathcal{A}_{j(i)}, \qquad j(i) \in I \quad \forall i, \qquad j(1) \neq j(2) \neq \cdots \neq j(n)$

- $\varphi(a_i) = 0 \quad \forall i$

Random variables $x_1, \dots, x_n \in \mathcal{A}$ are free, if their generated unital subalgebras $\mathcal{A}_i :=$ algebra$(1, x_i)$ are so.

What is Freeness?

Freeness between A and B is an infinite set of equations relating various moments in A and B:

$$\varphi\Big(p_1(A)q_1(B)p_2(A)q_2(B)\cdots\Big) = 0$$

Basic observation: freeness between A and B is actually a **rule for calculating mixed moments** in A and B from the moments of A and the moments of B:

$$\varphi\Big(A^{n_1}B^{m_1}A^{n_2}B^{m_2}\cdots\Big) = \text{polynomial}\Big(\varphi(A^i), \varphi(B^j)\Big)$$

Example:

$$\varphi\Big(\big(A^n - \varphi(A^n)1\big)\big(B^m - \varphi(B^m)1\big)\Big) = 0,$$

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$$\varphi\Big(\big(A^n - \varphi(A^n)1\big)\big(B^m - \varphi(B^m)1\big)\Big) = 0,$$

thus

$$\varphi(A^nB^m) - \varphi(A^n \cdot 1)\varphi(B^m) - \varphi(A^n)\varphi(1 \cdot B^m) + \varphi(A^n)\varphi(B^m)\varphi(1 \cdot 1) = 0,$$

Example:

$$\varphi\Big(\big(A^n - \varphi(A^n)1\big)\big(B^m - \varphi(B^m)1\big)\Big) = 0,$$

thus

$$\varphi(A^nB^m)-\varphi(A^n\cdot 1)\varphi(B^m)-\varphi(A^n)\varphi(1\cdot B^m)+\varphi(A^n)\varphi(B^m)\varphi(1\cdot 1) = 0,$$

and hence

$$\boldsymbol{\varphi(A^nB^m) = \varphi(A^n)\cdot\varphi(B^m)}$$

Freeness is a rule for calculating mixed moments, analogous to the concept of independence for random variables.

Thus freeness is also called **free independence**

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Note: free independence is a different rule from classical independence; free independence occurs typically for **non-commuting random variables**, like operators on Hilbert spaces or (random) matrices

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Example:

$$\varphi\Big(\big(A-\varphi(A)1\big)\cdot\big(B-\varphi(B)1\big)\cdot\big(A-\varphi(A)1\big)\cdot\big(B-\varphi(B)1\big)\Big)=0,$$

which results in

$$\begin{aligned}\boldsymbol{\varphi(ABAB)=\varphi(AA)\cdot\varphi(B)\cdot\varphi(B)+\varphi(A)\cdot\varphi(A)\cdot\varphi(BB)}\\ \boldsymbol{-\varphi(A)\cdot\varphi(B)\cdot\varphi(A)\cdot\varphi(B)}\end{aligned}$$

Motivation for the combinatorics of freeness: the free (and classical) CLT

Consider $a_1, a_2, \dots \in (\mathcal{A}, \varphi)$ which are

- identically distributed
- centered and normalized: $\varphi(a_i) = 0$ and $\varphi(a_i^2) = 1$
- either classically independent or freely independent

What can we say about

$$S_n := \frac{a_1 + \cdots + a_n}{\sqrt{n}} \quad \overset{n\to\infty}{\longrightarrow} \quad ???$$

We say that S_n converges (in distribution) to s if

$$\varphi(S_n^m) = \varphi(s^m) \qquad \forall m \in \mathbb{N}$$

We have

$$\varphi(S_n^m) = \frac{1}{n^{m/2}} \varphi\Big[(a_1 + \cdots a_n)^m\Big]$$

$$= \frac{1}{n^{m/2}} \sum_{i(1),\dots,i(m)=1}^{n} \varphi[a_{i(1)} \cdots a_{i(m)}]$$

Note:

$$\varphi[a_{i(1)} \cdots a_{i(m)}] = \varphi[a_{j(1)} \cdots a_{j(m)}]$$

whenever

$$\ker i = \ker j$$

For example, $i = (1, 3, 1, 5, 3)$ and $j = (3, 4, 3, 6, 4)$:

$$\varphi[a_1 a_3 a_1 a_5 a_3] = \varphi[a_3 a_4 a_3 a_6 a_4]$$

because independence/freeness allows to express

$$\varphi[a_1 a_3 a_1 a_5 a_3] = \text{polynomial}\Big(\varphi(a_1), \varphi(a_1^2), \varphi(a_3), \varphi(a_3^2), \varphi(a_5)\Big)$$
$$\varphi[a_3 a_4 a_3 a_6 a_4] = \text{polynomial}\Big(\varphi(a_3), \varphi(a_3^2), \varphi(a_4), \varphi(a_4^2), \varphi(a_6)\Big)$$

and

$$\varphi(a_1) = \varphi(a_3), \qquad \varphi(a_1^2) = \varphi(a_3^2)$$

$$\varphi(a_3) = \varphi(a_4), \qquad \varphi(a_3^2) = \varphi(a_4^2), \qquad \varphi(a_5) = \varphi(a_6)$$

We put

$\kappa_\pi := \varphi[a_1 a_3 a_1 a_5 a_3]$ where $\pi := \ker i = \ker j = \{\{1, 3\}, \{2, 5\}, \{4\}\}$

$\pi \in \mathcal{P}(5)$ is a partition of $\{1, 2, 3, 4, 5\}$.

Thus

$$\varphi(S_n^m) = \frac{1}{n^{m/2}} \sum_{i(1),\dots,i(m)=1}^{n} \varphi[a_{i(1)} \cdots a_{i(m)}]$$

$$= \frac{1}{n^{m/2}} \sum_{\pi \in \mathcal{P}(m)} \kappa_\pi \cdot \#\{i : \ker i = \pi\}$$

Note:

$$\#\{i : \ker i = \pi\} \quad = \qquad n(n-1)\cdots(n-\#\pi-1) \sim n^{\#\pi}$$

So

$$\varphi(S_n^m) \sim \sum_{\pi \in \mathcal{P}(m)} \kappa_\pi \cdot n^{\#\pi - m/2}$$

No singletons in the limit

Consider $\pi \in \mathcal{P}(m)$ with singleton:

$$\pi = \{\dots, \{k\}, \dots\},$$

thus

$$\begin{aligned}\kappa_\pi &= \varphi(a_{i(1)} \cdots a_{i(k)} \cdots a_{i(m)}) \\ &= \varphi(a_{i(1)} \cdots a_{i(k-1)} a_{i(k+1)} \cdots a_{i(m)}) \cdot \underbrace{\varphi(a_{i(k)})}_{=0}\end{aligned}$$

Thus: $\kappa_\pi = 0$ if π has singleton; i.e.,

$$\begin{aligned}\kappa_\pi \neq 0 \quad &\Longrightarrow \quad \pi = \{V_1, \dots, V_r\} \text{ with } \#V_j \geq 2 \, \forall j \\ &\Longrightarrow \quad r = \#\pi \leq \frac{m}{2}\end{aligned}$$

So in

$$\varphi(S_n^m) \sim \sum_{\pi \in \mathcal{P}(m)} \kappa_\pi \cdot n^{\#\pi - m/2}$$

only those π survive for $n \to \infty$ with

- π has no singleton, i.e., no block of size 1

- π has exactly $m/2$ blocks

Such π are exactly those, where each block has size 2, i.e.,

$$\pi \in \mathcal{P}_2(m) := \{\pi \in \mathcal{P}(m) \mid \pi \text{ is pairing}\}$$

Thus we have:

$$\lim_{n\to\infty} \varphi(S_n^m) = \sum_{\pi\in\mathcal{P}_2(m)} \kappa_\pi$$

In particular: odd moments are zero (because no pairings of odd number of elements), thus limit distribution is symmetric

Question: What are the even moments?

This depends on the κ_π's.

The actual value of those is now different for the classical and the free case!

Classical CLT: assume a_i are independent

If the a_i commute and are independent, then

$$\kappa_\pi = \varphi(a_{i(1)} \cdots a_{i(2k)}) = 1 \qquad \forall\, \pi \in \mathcal{P}_2(2k)$$

Thus

$$\lim_{n\to\infty} \varphi(S_n^m) = \#\mathcal{P}_2(m) = \begin{cases} 0, & m \text{ odd} \\ (m-1)(m-3)\cdots 5\cdot 3\cdot 1, & m \text{ even} \end{cases}$$

Those limit moments are the moments of a Gaussian distribution of variance 1.

Free CLT: assume a_i are free

If the a_i are free, then, for $\pi \in \mathcal{P}_2(2k)$,

$$\kappa_\pi = \begin{cases} 0, & \pi \text{ is crossing} \\ 1, & \pi \text{ is non-crossing} \end{cases}$$

E.g.,

$$\begin{aligned} \kappa_{\{1,6\},\{2,5\},\{3,4\}} &= \varphi(a_1 a_2 a_3 a_3 a_2 a_1) \\ &= \varphi(a_3 a_3) \cdot \varphi(a_1 a_2 a_2 a_1) \\ &= \varphi(a_3 a_3) \cdot \varphi(a_2 a_2) \cdot \varphi(a_1 a_1) \\ &= 1 \end{aligned}$$

but

$$\begin{aligned} \kappa_{\{1,5\},\{2,3\},\{4,6\}} &= \varphi(a_1 a_2 a_2 a_3 a_1 a_3) \\ &= \varphi(a_2 a_2) \cdot \underbrace{\varphi(a_1 a_3 a_1 a_3)}_{0} \end{aligned}$$

Free CLT: assume a_i are free

Put

$$NC_2(m) := \{\pi \in \mathcal{P}_2(m) \mid \pi \text{ is non-crossing}\}$$

Thus

$$\lim_{n\to\infty} \varphi(S_n^m) = \#NC_2(m) = \begin{cases} 0, & m \text{ odd} \\ c_k = \frac{1}{k+1}\binom{2k}{k}, & m = 2k \text{ even} \end{cases}$$

Those limit moments are the moments of a semicircular distribution of variance 1,

$$\lim_{n\to\infty} \varphi(S_n^m) = \frac{1}{2\pi}\int_{-2}^{2} t^m \sqrt{4-t^2}\,dt$$

How to recognize the Catalan numbers c_k

Put

$$c_k := \#NC_2(2k).$$

We have

$$c_k = \sum_{\pi \in NC(2k)} 1 = \sum_{i=1}^{k} \sum_{\pi = \{1,2i\} \cup \pi_1 \cup \pi_2} 1 = \sum_{i=1}^{k} c_{i-1} c_{k-i}$$

This recursion, together with $c_0 = 1, c_1 = 1$, determines the sequence of Catalan numbers:

$$\{c_k\} = 1, 1, 2, 5, 14, 42, 132, 429, \ldots.$$

Understanding the freeness rule: the idea of cumulants

- write moments in terms of other quantities, which we call **free cumulants**

- freeness is much easier to describe on the level of free cumulants: **vanishing of mixed cumulants**

- relation between moments and cumulants is given by summing over **non-crossing or planar partitions**

Non-crossing partitions

A **partition** of $\{1, \dots, n\}$ is a decomposition $\pi = \{V_1, \dots, V_r\}$ with

$$V_i \neq \emptyset, \qquad V_i \cap V_j = \emptyset \quad (i \neq y), \qquad \bigcup_i V_i = \{1, \dots, n\}$$

The V_i are the **blocks** of $\pi \in \mathcal{P}(n)$.

π is **non-crossing** if we do not have

$$p_1 < q_1 < p_2 < q_2$$

such that p_1, p_2 are in same block, q_1, q_2 are in same block, but those two blocks are different.

$$\mathbf{NC(n)} := \{\text{non-crossing partitions of } \{1,\dots,n\}\}$$

$NC(n)$ is actually a lattice with refinement order.

Moments and cumulants

For unital linear functional

$$\varphi : \mathcal{A} \to \mathbb{C}$$

we define **cumulant functionals** κ_n (for all $n \geq 1$)

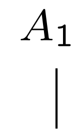
$$\kappa_n : \mathcal{A}^n \to \mathbb{C}$$

as multi-linear functionals by moment-cumulant relation

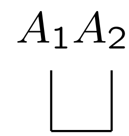
$$\varphi(A_1 \cdots A_n) = \sum_{\pi \in NC(n)} \kappa_\pi[A_1, \ldots, A_n]$$

Note: classical cumulants are defined by a similar formula, where only $NC(n)$ is replaced by $\mathcal{P}(n)$

$$\varphi(A_1) = \kappa_1(A_1)$$



$$\varphi(A_1 A_2) = \kappa_2(A_1, A_2)$$



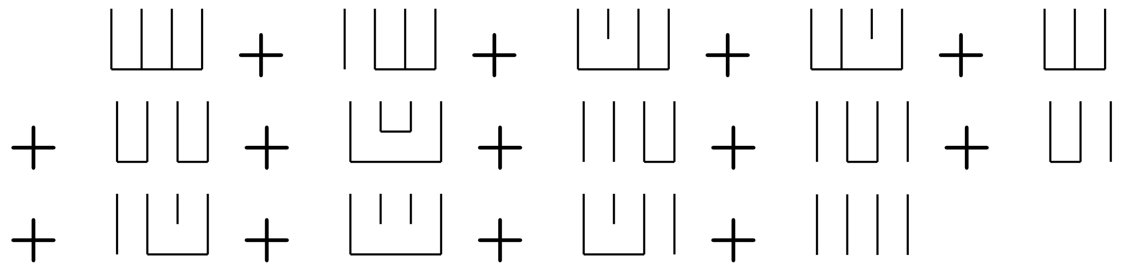
$$+ \kappa_1(A_1)\kappa_1(A_2)$$

thus

$$\kappa_2(A_1, A_2) = \varphi(A_1 A_2) - \varphi(A_1)\varphi(A_2)$$

$$
\begin{aligned}
\varphi(A_1 A_2 A_3) = {} & \kappa_3(A_1, A_2, A_3) \\
& + \kappa_1(A_1)\kappa_2(A_2, A_3) \\
& + \kappa_2(A_1, A_2)\kappa_1(A_3) \\
& + \kappa_2(A_1, A_3)\kappa_1(A_2) \\
& + \kappa_1(A_1)\kappa_1(A_2)\kappa_1(A_3)
\end{aligned}
$$

$$\varphi(A_1A_2A_3A_4) =$$



$$
\begin{aligned}
= \;& \kappa_4(A_1,A_2,A_3,A_4) + \kappa_1(A_1)\kappa_3(A_2,A_3,A_4) \\
&+ \kappa_1(A_2)\kappa_3(A_1,A_3,A_4) + \kappa_1(A_3)\kappa_3(A_1,A_2,A_4) \\
&+ \kappa_3(A_1,A_2,A_3)\kappa_1(A_4) + \kappa_2(A_1,A_2)\kappa_2(A_3,A_4) \\
&+ \kappa_2(A_1,A_4)\kappa_2(A_2,A_3) + \kappa_1(A_1)\kappa_1(A_2)\kappa_2(A_3,A_4) \\
&+ \kappa_1(A_1)\kappa_2(A_2,A_3)\kappa_1(A_4) + \kappa_2(A_1,A_2)\kappa_1(A_3)\kappa_1(A_4) \\
&+ \kappa_1(A_1)\kappa_2(A_2,A_4)\kappa_1(A_3) + \kappa_2(A_1,A_4)\kappa_1(A_2)\kappa_1(A_3) \\
&+ \kappa_2(A_1,A_3)\kappa_1(A_2)\kappa_1(A_4) + \kappa_1(A_1)\kappa_1(A_2)\kappa_1(A_3)\kappa_1(A_4)
\end{aligned}
$$

Freeness $\hat{=}$ vanishing of mixed cumulants

Theorem [Speicher 1994]: The fact that A and B are free is equivalent to the fact that

$$\kappa_n(C_1, \ldots, C_n) = 0$$

whenever

- $n \geq 2$
- $C_i \in \{A, B\}$ for all i
- there are i, j such that $C_i = A$, $C_j = B$

Freeness $\hat{=}$ vanishing of mixed cumulants

free product $\hat{=}$ direct sum of cumulants

$\varphi(\boldsymbol{A}^{\boldsymbol{n}})$ given by sum over **blue** planar diagrams

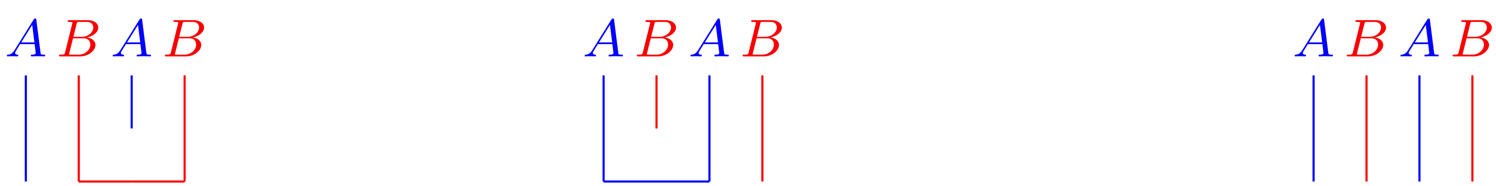
$\varphi(\boldsymbol{B}^{\boldsymbol{m}})$ given by sum over **red** planar diagrams

then: for A and B free, $\varphi(\boldsymbol{A}^{\boldsymbol{n_1}}\boldsymbol{B}^{\boldsymbol{m_1}}\boldsymbol{A}^{\boldsymbol{n_2}}\cdots)$ is given by sum over planar diagrams with monochromatic (blue or red) blocks

Vanishing of Mixed Cumulants

$$\varphi(ABAB) =$$

$$\kappa_1(A)\kappa_1(A)\kappa_2(B,B)+\kappa_2(A,A)\kappa_1(B)\kappa_1(B)+\kappa_1(A)\kappa_1(B)\kappa_1(A)\kappa_1(B)$$



Sum of Free Variables

Consider A, B free.

Then, by freeness, the moments of $A+B$ are uniquely determined by the moments of A and the moments of B.

Notation: We say the distribution of $A+B$ is the

free convolution

of the distribution of A and the distribution of B,

$$\boldsymbol{\mu_{A+B} = \mu_A \boxplus \mu_B}.$$

Sum of Free Variables

In principle, freeness determines this, but the concrete nature of this rule on the level of moments is not apriori clear.

Example:

$$\varphi\Big((A+B)^1\Big) = \varphi(A) + \varphi(B)$$

$$\varphi\Big((A+B)^2\Big) = \varphi(A^2) + 2\varphi(A)\varphi(B) + \varphi(B^2)$$

$$\varphi\Big((A+B)^3\Big) = \varphi(A^3) + 3\varphi(A^2)\varphi(B) + 3\varphi(A)\varphi(B^2) + \varphi(B^3)$$

$$\begin{aligned}\varphi\Big((A+B)^4\Big) = \varphi(A^4) &+ 4\varphi(A^3)\varphi(B) + 4\varphi(A^2)\varphi(B^2) \\ &+ 2\Big(\varphi(A^2)\varphi(B)\varphi(B) + \varphi(A)\varphi(A)\varphi(B^2) \\ &\quad - \varphi(A)\varphi(B)\varphi(A)\varphi(B)\Big) + 4\varphi(A)\varphi(B^3) + \varphi(B^4)\end{aligned}$$

Sum of Free Variables

Corresponding rule on level of free cumulants is easy: If A and B are free then

$$\begin{aligned}\kappa_n(A+B, A+B, \dots, A+B) = &\kappa_n(A, A, \dots, A) + \kappa_n(B, B, \dots, B)\\ &+\kappa_n(\dots, A, B, \dots) + \cdots\end{aligned}$$

Sum of Free Variables

Corresponding rule on level of free cumulants is easy: If A and B are free then

$$\kappa_n(A+B, A+B, \ldots, A+B) = \kappa_n(A, A, \ldots, A) + \kappa_n(B, B, \ldots, B)$$

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Corresponding rule on level of free cumulants is easy: If A and B are free then

$$\kappa_n(A+B, A+B, \dots, A+B) = \kappa_n(A, A, \dots, A) + \kappa_n(B, B, \dots, B)$$

i.e., we have **additivity of cumulants for free variables**

$$\kappa_n^{A+B} = \kappa_n^A + \kappa_n^B$$

Sum of Free Variables

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i.e., we have additivity of cumulants for free variables

$$\kappa_n^{A+B} = \kappa_n^A + \kappa_n^B$$

Also: Combinatorial relation between moments and cumulants can be rewritten easily as a relation between corresponding formal power series.

Sum of Free Variables

Consider one random variable $A \in \mathcal{A}$ and define their **Cauchy transform G** and their **$\mathcal{R}$-transform $\mathcal{R}$** by

$$G(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\varphi(A^n)}{z^{n+1}}, \qquad \mathcal{R}(z) = \sum_{n=1}^{\infty} \kappa_n(A, \dots, A) z^{n-1}$$

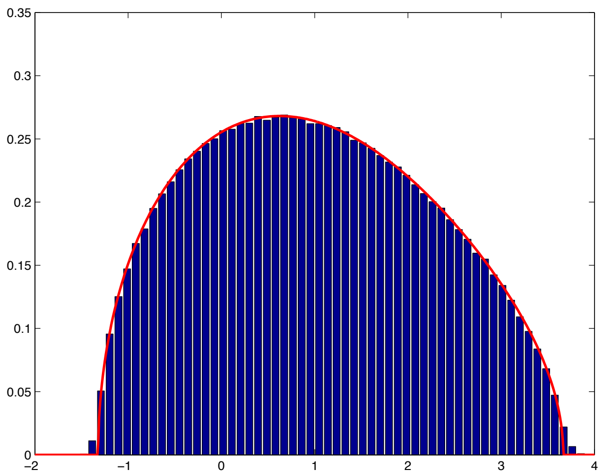
Theorem [Voiculescu 1986, Speicher 1994]: Then we have

- $\frac{1}{G(z)} + \mathcal{R}(G(z)) = z$
- $\mathcal{R}^{A+B}(z) = \mathcal{R}^{A}(z) + \mathcal{R}^{B}(z)$ if A and B are free

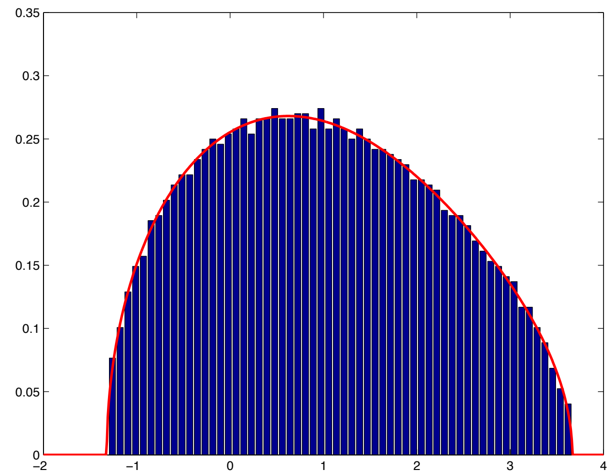
This, together with the relation between Cauchy transform and $\mathcal{R}$-transform and with the Stieltjes inversion formula, gives an effective algorithm for calculating free convolutions, i.e., the asymptotic eigenvalue distribution **of sums of random matrices in generic position**:

$$
\begin{array}{ccccccccccc}
A & \rightsquigarrow & G^A & \rightsquigarrow & R^A & & & & & & \\
 & & & & \downarrow & & & & & & \\
 & & & & R^A & +R^B = R^{A+B} & & \rightsquigarrow & G^{A+B} & \rightsquigarrow & A+B \\
 & & & & \uparrow & & & & & & \\
B & \rightsquigarrow & G^B & \rightsquigarrow & R^B & & & & & &
\end{array}
$$

Example: Wigner + Wishart $(M = 2N)$



averaged Wigner + Wishart; N=100



... one realization ...

trials=4000

N=3000

Sum of Free Variables

If A and B are free, then the

free (additive) convolution

of its distributions is given by

$$\mu_{A+B} = \mu_A \boxplus \mu_B.$$

How can we describe this???

On the level of moments this is getting complicated ....

$$\varphi\Big((A+B)^1\Big) = \varphi(A) + \varphi(B)$$

$$\varphi\Big((A+B)^2\Big) = \varphi(A^2) + 2\varphi(A)\varphi(B) + \varphi(B^2)$$

$$\varphi\Big((A+B)^3\Big) = \varphi(A^3) + 3\varphi(A^2)\varphi(B) + 3\varphi(A)\varphi(B^2) + \varphi(B^3)$$

$$\begin{aligned}\varphi\Big((A+B)^4\Big) = \varphi(A^4) &+ 4\varphi(A^3)\varphi(B) + 4\varphi(A^2)\varphi(B^2)\\ &+ 2\Big(\varphi(A^2)\varphi(B)\varphi(B) + \varphi(A)\varphi(A)\varphi(B^2)\\ &\quad - \varphi(A)\varphi(B)\varphi(A)\varphi(B)\Big) + 4\varphi(A)\varphi(B^3) + \varphi(B^4)\end{aligned}$$

... and its better to consider cumulants.

Sum of Free Variables

Consider free cumulants

$$\kappa_n^A := \kappa_n(A, A, \dots, A)$$

Then we have, for A and B free:

$$\begin{aligned}\kappa_n(A+B, A+B, \dots, A+B) = &\kappa_n(A, A, \dots, A) + \kappa_n(B, B, \dots, B)\\ &+\kappa_n(\dots, A, B, \dots) + \cdots\end{aligned}$$

Sum of Free Variables

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$$\kappa_n^A := \kappa_n(A, A, \dots, A)$$

Then we have, for A and B free:

$$\kappa_n(A+B, A+B, \dots, A+B) = \kappa_n(A, A, \dots, A) + \kappa_n(B, B, \dots, B)$$

Sum of Free Variables

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$$\kappa_n(A+B, A+B, \ldots, A+B) = \kappa_n(A, A, \ldots, A) + \kappa_n(B, B, \ldots, B)$$

i.e., we have **additivity of cumulants for free variables**

$$\kappa_n^{A+B} = \kappa_n^A + \kappa_n^B$$

Sum of Free Variables

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i.e., we have additivity of cumulants for free variables

$$\kappa_n^{A+B} = \kappa_n^A + \kappa_n^B$$

But: how good do we understand the relation between moments and free cumulants???

Relation between moments and free cumulants

We have

$$m_n := \varphi(A^n) \qquad \text{moments}$$

and

$$\kappa_n := \kappa_n(A, A, \dots, A) \qquad \text{free cumulants}$$

Combinatorially, the relation between them is given by

$$m_n = \varphi(A^n) = \sum_{\pi \in NC(n)} \kappa_\pi$$

Example:

$$m_1 = \kappa_1, \quad m_2 = \kappa_2 + \kappa_1^2, \quad m_3 = \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3$$

$$m_3 = \kappa + \kappa + \kappa + \kappa + \kappa = \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3$$

Theorem [Speicher 1994]: Consider formal power series

$$M(z) = 1 + \sum_{k=1}^{\infty} m_n z^n, \qquad C(z) = 1 + \sum_{k=1}^{\infty} \kappa_n z^n$$

Then the relation

$$m_n = \sum_{\pi \in NC(n)} \kappa_\pi$$

between the coefficients is equivalent to the relation

$$M(z) = C[zM(z)]$$

Proof

First we get the following recursive relation between cumulants and moments

$$m_n = \sum_{\pi \in NC(n)} \kappa_\pi$$

$$= \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s \geq 0 \\ i_1+\cdots+i_s+s=n}} \sum_{\pi_1 \in NC(i_1)} \cdots \sum_{\pi_s \in NC(i_s)} \kappa_s \kappa_{\pi_1} \cdots \kappa_{\pi_s}$$

$$= \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s \geq 0 \\ i_1+\cdots+i_s+s=n}} \kappa_s m_{i_1} \cdots m_{i_s}$$

$$m_n = \sum_{s=1}^{n} \sum_{\substack{i_1,\ldots,i_s \geq 0 \\ i_1+\cdots+i_s+s=n}} \kappa_s m_{i_1} \cdots m_{i_s}$$

Plugging this into the formal power series $M(z)$ gives

$$M(z) = 1 + \sum_n m_n z^n$$

$$= 1 + \sum_n \sum_{s=1}^{n} \sum_{\substack{i_1,\ldots,i_s \geq 0 \\ i_1+\cdots+i_s+s=n}} k_s z^s m_{i_1} z^{i_1} \cdots m_{i_s} z^{i_s}$$

$$= 1 + \sum_{s=1}^{\infty} \kappa_s z^s \Big(M(z)\Big)^s = C[zM(z)]$$

Remark on classical cumulants

Classical cumulants c_k are combinatorially defined by

$$m_n = \sum_{\pi \in \mathcal{P}(n)} c_\pi$$

In terms of generating power series

$$\tilde{M}(z) = 1 + \sum_{n=1}^{\infty} \frac{m_n}{n!} z^n, \qquad \tilde{C}(z) = \sum_{n=1}^{\infty} \frac{c_n}{n!} z^n$$

this is equivalent to

$$\tilde{C}(z) = \log \tilde{M}(z)$$

From moment series to Cauchy transform

Instead of $M(z)$ we consider **Cauchy transform**

$$G(z) := \varphi(\frac{1}{z-A}) = \int \frac{1}{z-t} d\mu_A(t) = \sum \frac{\varphi(A^n)}{z^{n+1}} = \frac{1}{z} M(1/z)$$

and instead of $C(z)$ we consider **R-transform**

$$R(z) := \sum_{n \geq 0} \kappa_{n+1} z^n = \frac{C(z) - 1}{z}$$

Then $M(z) = C[zM(z)]$ becomes

$$R[G(z)] + \frac{1}{G(z)} = z \qquad \text{or} \qquad G[R(z) + 1/z] = z$$

What is advantage of $G(z)$ over $M(z)$?

For any probability measure μ is its Cauchy transform

$$G(z) := \int \frac{1}{z-t} d\mu(t)$$

an analytic function $G : \mathbb{C}^+ \to \mathbb{C}^-$ and we can recover μ from G by **Stieltjes inversion formula**

$$d\mu(t) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \Im G(t + i\varepsilon) dt$$

Example: semicircular distribution μ_s

μ_s has moments given by the Catalan numbers or, equivalently, has cumulants

$$\kappa_n = \begin{cases} 0, & n \neq 2 \\ 1, & n = 2 \end{cases}$$

(because $m_n = \sum_{\pi \in NC_2(n)} \kappa_\pi$ says that $\kappa_\pi = 0$ for $\pi \in NC(n)$ which is not a pairing), thus

$$R(z) = \sum_{n \geq 0} \kappa_{n+1} z^n = \kappa_2 \cdot z = z$$

and hence

$$z = R[G(z)] + \frac{1}{G(z)} = G(z) + \frac{1}{G(z)}$$

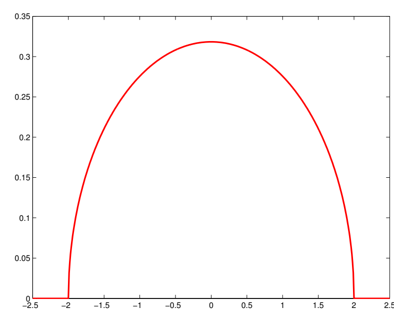
or $$G(z)^2 + 1 = zG(z)$$

$$G(z)^2 + 1 = zG(z) \qquad \text{thus} \qquad G(z) = \frac{z \pm \sqrt{z^4 - 4}}{2}$$

We have "-", because $G(z) \sim 1/z$ for $z \to \infty$; then

$$d\mu_s(t) = -\frac{1}{\pi}\Im\Big(\frac{t - \sqrt{t^2 - 4}}{2}\Big)dt$$

$$= \begin{cases} \frac{1}{2\pi}\sqrt{4 - t^2}dt, & \text{if } t \in [-2, 2] \\ 0, & \text{otherwise} \end{cases}$$



What is the free Binomial $\left(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}\right)^{\boxplus 2}$

$$\mu := \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}, \qquad \nu := \mu \boxplus \mu$$

Then $$G_\mu(z) = \int \frac{1}{z-t} d\mu(t) = \frac{1}{2}\Big(\frac{1}{z+1} + \frac{1}{z-1}\Big) = \frac{z}{z^2-1}$$

and so $$z = G_\mu[R_\mu(z) + 1/z] = \frac{R_\mu(z) + 1/z}{(R_\mu(z) + 1/z)^2 - 1}$$

thus $$R_\mu(z) = \frac{\sqrt{1+4z^2} - 1}{2z}$$

and so $$R_\nu(z) = 2R_\mu(z) = \frac{\sqrt{1+4z^2} - 1}{z}$$

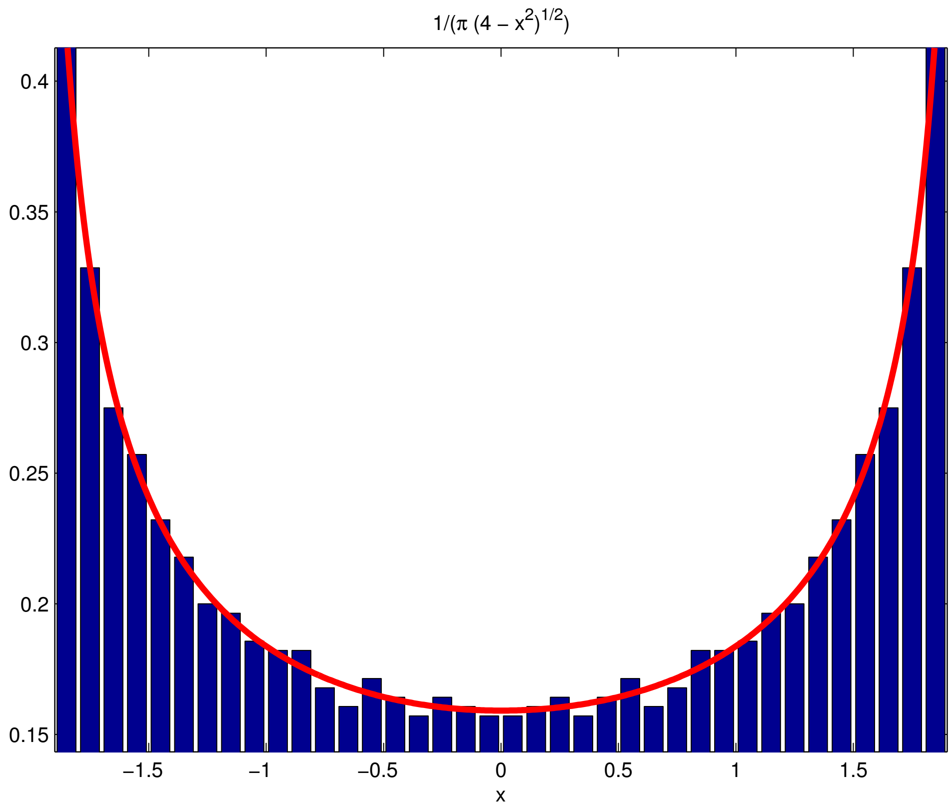
$$R_\nu(z) = \frac{\sqrt{1+4z^2}-1}{z} \qquad \text{gives} \qquad G_\nu(z) = \frac{1}{\sqrt{z^2-4}}$$

and thus

$$d\nu(t) = -\frac{1}{\pi}\Im\frac{1}{\sqrt{t^2-4}}dt = \begin{cases} \frac{1}{\pi\sqrt{4-t^2}}, & |t| \le 2 \\ 0, & \text{otherwise} \end{cases}$$

So

$$\left(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}\right)^{\boxplus 2} = \nu = \text{arcsine-distribution}$$



2800 eigenvalues of $A + UBU^*$, where A and B are diagonal matrices with 1400 eigenvalues $+1$ and 1400 eigenvalues -1, and U is a randomly chosen unitary matrix

Lessons to learn

Free convolution of discrete distributions is in general non discrete

Since it is true that

$$\delta_x \boxplus \delta_y = \delta_{x+y}$$

we see, in particular, that $\boxplus$ is not linear, i.e, for $\alpha + \beta = 1$

$$(\alpha\mu_1 + \beta\mu_2) \boxplus \nu \neq \alpha(\mu_1 \boxplus \nu) + \beta(\mu_2 \boxplus \nu)$$

Non-commutativity matters: the sum of two commuting projections is a quite easy object, the sum of two non-commuting projections is much harder to grasp

Product of Free Variables

Consider A, B free.

Then, by freeness, the moments of AB are uniquely determined by the moments of A and the moments of B.

Notation: We say the distribution of AB is the

free multiplicative convolution

of the distribution of A and the distribution of B,

$$\mu_{AB} = \mu_A \boxtimes \mu_B.$$

Caveat: AB not selfadjoint

Note: $\boxtimes$ is in general not operation on probability measures on $\mathbb{R}$. Even if both A and B are selfadjoint, AB is only selfadjoint if A and B commute (which they don't, if they are free)

But: if B is positive, then we can consider $B^{1/2}AB^{1/2}$ instead of AB. Since A and B are free it follows that both have the same moments; e.g.,

$$\varphi(B^{1/2}AB^{1/2}) = \varphi(B^{1/2}B^{1/2})\varphi(A) = \varphi(B)\varphi(A) = \varphi(AB)$$

So the "right" definition is

$$\mu_A \boxtimes \mu_B = \mu_{B^{1/2}AB^{1/2}}.$$

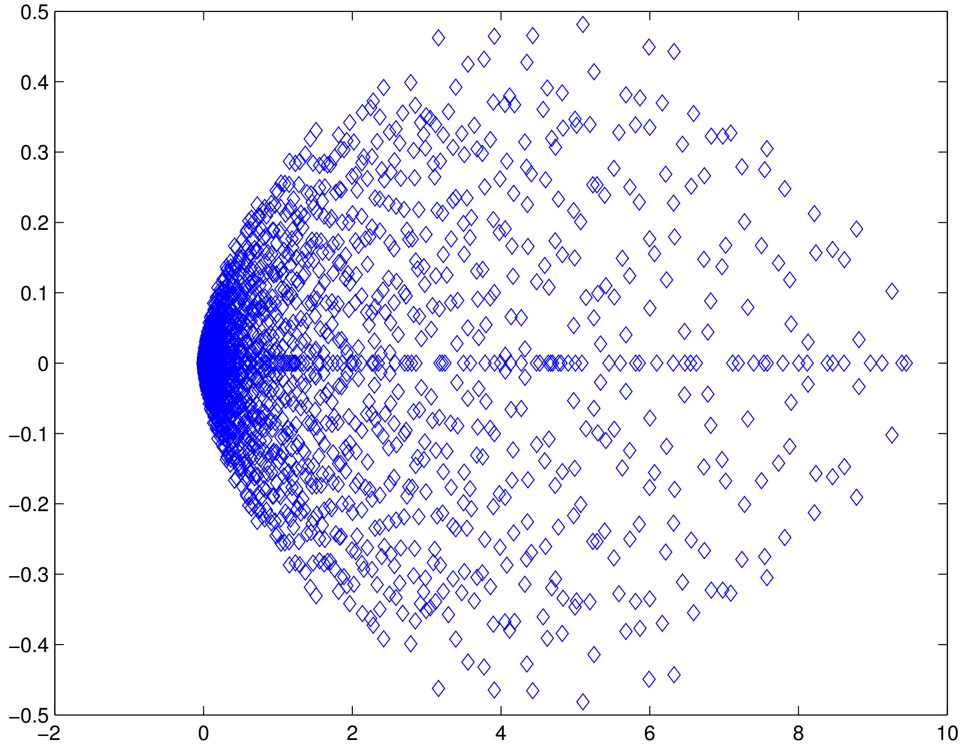
If we also restrict A to be positive, then this gives a binary operation on probability measures on $\mathbb{R}^+$

Meaning of $\boxtimes$ for random matrices

If A and B are symmetric matrices with positive eigenvalues, then the eigenvalues of $AB = (AB^{1/2})B^{1/2}$ are the same as the eigenvalues of $B^{1/2}(AB^{1/2})$ and thus are necessarily also real and positive. If A and B are asymptotically free, the eigenvalues of AB are given by $\mu_A \boxtimes \mu_B$.

However: this is not true any more for three or more matrices!!!
If A, B, C are symmetric matrices with positive eigenvalues, then there is no reason that the eigenvalues of ABC are real.

If A, B, C are asymptotically free, then still the moments of ABC are the same as the moments of $C^{1/2}B^{1/2}AB^{1/2}C^{1/2}$, i.e., the same as the moments of $\mu_A \boxtimes \mu_B \boxtimes \mu_C$. But since the eigenvalues of ABC are not real, the knowledge of the moments is not enough to determine them.



3000 complex eigenvalues of the product of three independent 3000×3000 Wishart matrices

Back to the general theory of ⊠

In principle, the moments of AB are, for A and B free, determined by the moments of A and the moments of B; but again the concrete nature of this rule on the level of moments is not clear...

$$\varphi\Big((AB)^1\Big) = \varphi(A)\varphi(B)$$

$$\varphi\Big((AB)^2\Big) = \varphi(A^2)\varphi(B)^2 + \varphi(A)^2\varphi(B^2) - \varphi(A)^2\varphi(B)^2$$

$$\begin{aligned}\varphi\Big((AB)^3\Big) = {} & \varphi(A^3)\varphi(B)^3 + \varphi(A)^3\varphi(B^3) + 3\varphi(A)\varphi(A^2)\varphi(B)\varphi(B^2) \\ & - 3\varphi(A)\varphi(A^2)\varphi(B)^3 - 3\varphi(A)^3\varphi(B)\varphi(B^2) + 2\varphi(A)^3\varphi(B)^3\end{aligned}$$

... so let's again look on cumulants.

Product of Free Variables

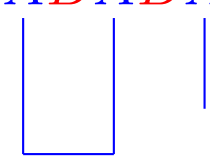
Corresponding rule on level of free cumulants is relatively easy (at least conceptually): If A and B are free then

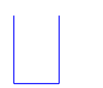
$$\kappa_n(AB, AB, \dots, AB) = \sum_{\pi \in NC(n)} \kappa_\pi[A, A, \dots, A] \cdot \kappa_{K(\pi)}[B, B, \dots, B],$$

where $K(\pi)$ is the **Kreweras complement** of π: $K(\pi)$ is the maximal σ with

$$\pi \in NC(\text{blue}), \quad \sigma \in NC(\text{red}), \quad \pi \cup \sigma \in NC(\text{blue} \cup \text{red})$$

$ABABAB$



$K($  $) =$

Product of Free Variables

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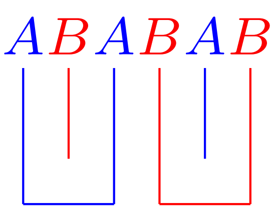
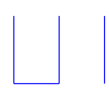
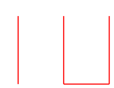
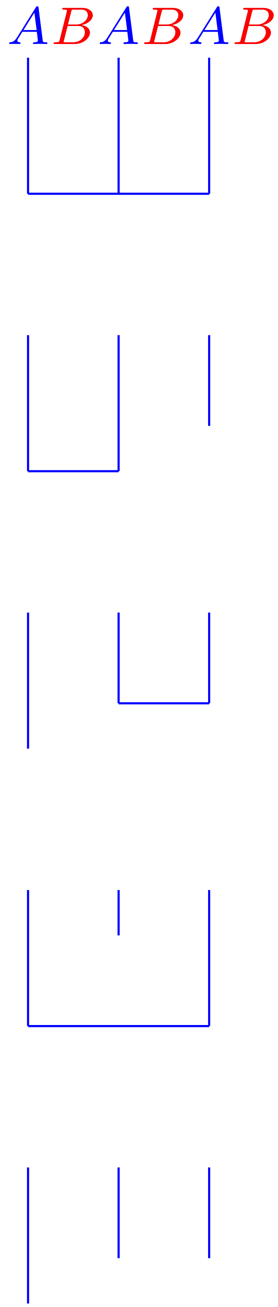


$K($  $) =$ 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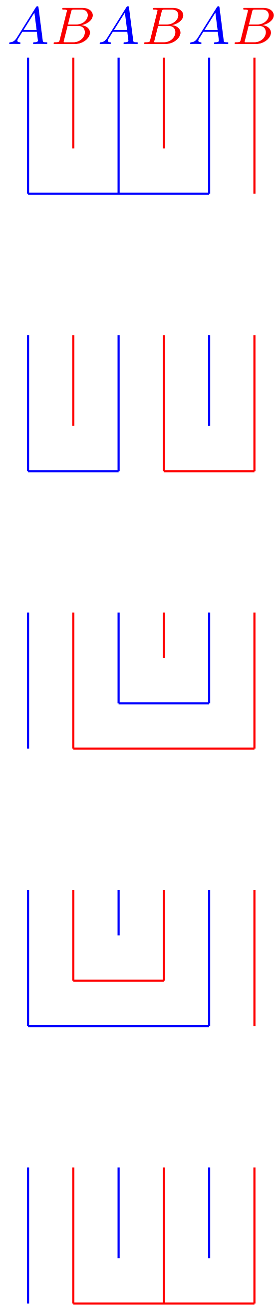

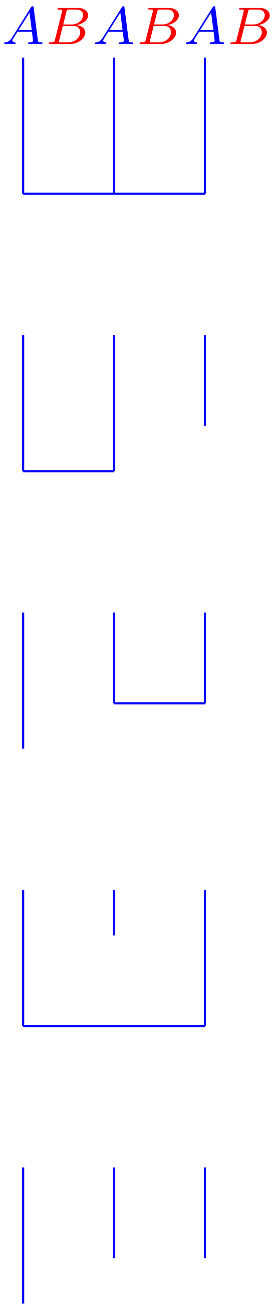

$$\kappa_3(A, A, A)$$

$$\kappa_2(A, A)\kappa_1(A)$$

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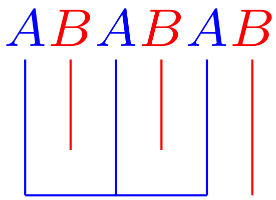
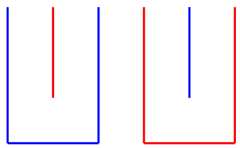
$$\kappa_2(A, A)\kappa_1(A)$$

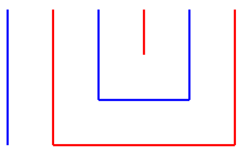
$$\kappa_1(A)\kappa_1(A)\kappa_1(A)$$



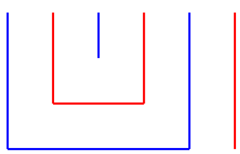
$$\kappa_3(A,A,A)\ \kappa_1(B)\kappa_1(B)\kappa_1(B)$$



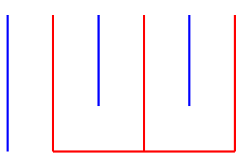
$$\kappa_2(A,A)\kappa_1(A)\ \kappa_2(B,B)\kappa_1(B)$$



$$\kappa_2(A,A)\kappa_1(A)\ \kappa_2(B,B)\kappa_1(B)$$



$$\kappa_2(A,A)\kappa_1(A)\ \kappa_2(B,B)\kappa_1(B)$$



$$\kappa_1(A)\kappa_1(A)\kappa_1(A)\ \kappa_3(B,B,B)$$

Product of Free Variables

Theorem [Voiculescu 1987; Haagerup 1997; Nica, Speicher 1997]:
Put

$$M_A(z) := \sum_{m=1}^{\infty} \varphi(A^m) z^m$$

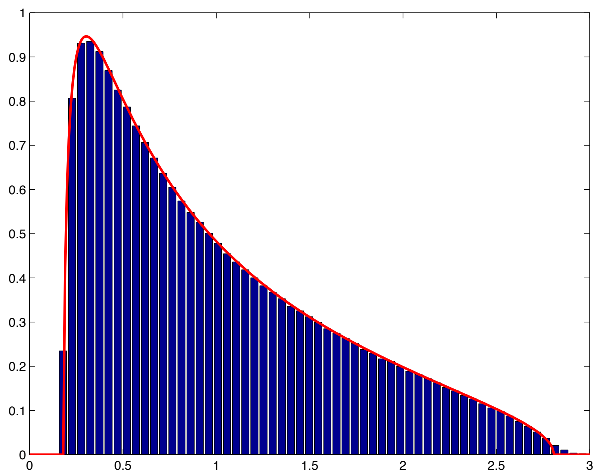
and define

$$S_A(z) := \frac{1+z}{z} M_A^{<-1>}(z) \qquad \textbf{\textit{S}-transform of } \boldsymbol{A}$$

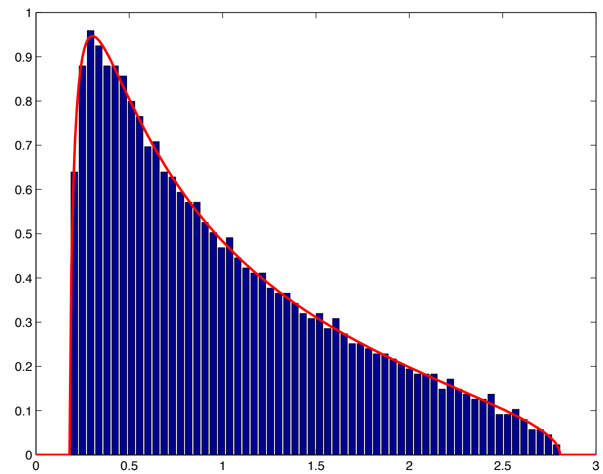
Then: If A and B are free, we have

$$S_{AB}(z) = S_A(z) \cdot S_B(z).$$

Example: Wishart x Wishart ($M = 5N$)



averaged Wishart x Wishart; N=100



... one realization ...

trials=10000

N=2000

Corners of Random Matrices

Theorem [Nica, Speicher 1996]: The asymptotic eigenvalue distribution of a corner B of ratio α of a unitarily invariant random matrix A is given by

$$\mu_B = \mu_{\alpha A}^{\boxplus 1/\alpha}$$

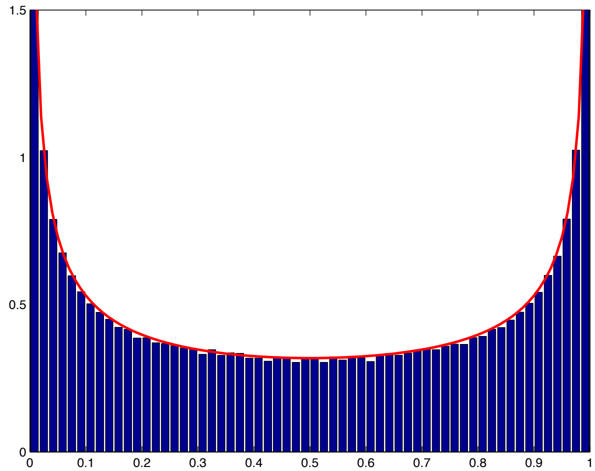
Corners of Random Matrices

Theorem [Nica, Speicher 1996]: The asymptotic eigenvalue distribution of a corner B of ratio α of a unitarily invariant random matrix A is given by

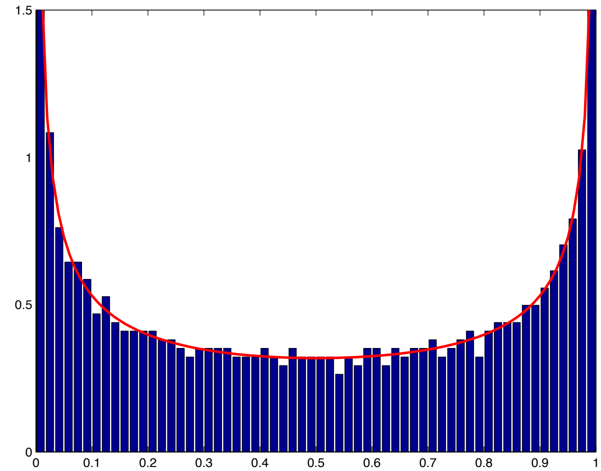
$$\mu_B = \mu_{\alpha A}^{\boxplus 1/\alpha}$$

In particular, a corner of size $\alpha = 1/2$, has up to rescaling the same distribution as $\mu_A \boxplus \mu_A$.

Upper left corner of size $N/2 \times N/2$ of a projection matrix, with $N/2$ eigenvalues 0 and $N/2$ eigenvalues 1 is, up to rescaling, the same as a free Bernoulli, i.e., the arcsine distribution



averaged upper corner; N=64



... one realization ...

trials=5000

N=2048

This actually shows

Theorem [Nica, Speicher 1996]: For any probability measure μ on $\mathbb{R}$, there exists a semigroup $(\mu^{\boxplus t})_{\mathbf{t \geq 1}}$ of probability measures, such that $\mu^{\boxplus 1} = \mu$ and

$$\mu^{\boxplus s} \boxplus \mu^{\boxplus t} = \mu^{\boxplus (s+t)} \qquad (s, t \geq 1)$$

(Note: if this $\mu^{\boxplus t}$ exists for all $t \geq 0$, then μ is freely infinitely divisible.)

In the classical case, such a statement is not true!

Polynomials of independent Gaussian rm

Consider m independent Gaussian random matrices

$$X_N^{(1)}, \ldots, X_N^{(m)}.$$

They converge, for $N \to \infty$, to m free semicircular elements

$$s_1, \ldots, s_m.$$

In particular, for any selfadjoint polynomial in m non-commuting variables

$$p(X_N^{(1)}, \ldots, X_N^{(m)}) \to p(s_1, \ldots, s_m)$$

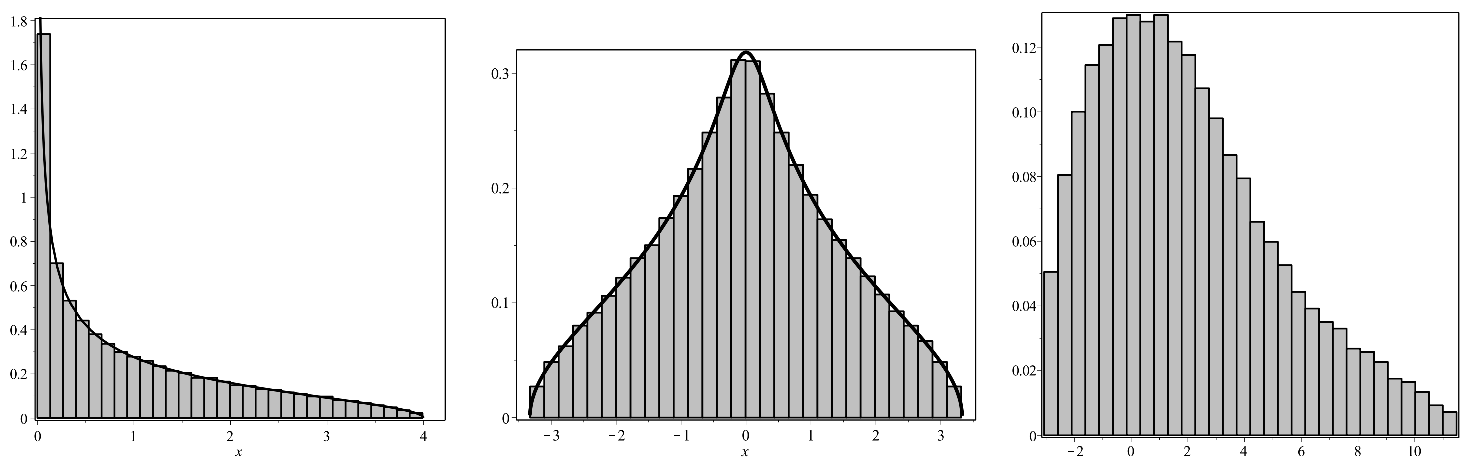
Question: What can we say about the distribution of $p(s_1, \ldots, s_m)$?

Eigenvalue distributions of

$$p(x_1) = x_1^2, \qquad p(x_1, x_2) = x_1 x_2 + x_2 x_1$$

$$p(x_1, x_2, x_3) = x_1 x_2^2 x_1 + x_1 x_3^2 x_1 + 2x_2$$

where x_1, x_2, and x_3 are independent 4000×4000 Gaussian random matrices; in the first two cases the asymptotic eigenvalue distribution can be calculated by free probability tools as the solid curve, for the third case no explicit solution exists.



What are qualitative features of such distributions?

What can we say about distribution of

$$\lim_{N\to\infty} p(X_N^{(1)},\ldots,X_N^{(m)}) = p(s_1,\ldots,s_m)$$

for arbitrary polynomials p?

Conjectures:

- It has no atoms.
- It has a density with respect to Lebesgue measure.
- This density has some (to be specified) regularity properties.

How to attack such questions?

... maybe with ...

- free stochastic analysis
- free Malliavin calculus

A **free Brownian motion** is given by a family $\big(S(t)\big)_{t\geq 0} \subset (\mathcal{A}, \varphi)$ of random variables ($\mathcal{A}$ von Neumann algebra, φ faithful trace), such that

- $S(0) = 0$

- each increment $S(t) - S(s)$ $(s < t)$ is semicircular with mean $= 0$ and variance $= t - s$, i.e.,

$$d\mu_{S(t)-S(s)}(x) = \frac{1}{2\pi(t-s)}\sqrt{4(t-s) - x^2}\,dx$$

- disjoint increments are free: for $0 < t_1 < t_2 < \cdots < t_n$,

$$S(t_1), \quad S(t_2) - S(t_1), \quad \ldots, \quad S(t_n) - S(t_{n-1}) \qquad \text{are free}$$

A **free Brownian motion** is given

- abstractly, by a family $\big(S(t)\big)_{t\geq 0}$ of random variables with
  - $S(0) = 0$
  - each $S(t) - S(s)$ $(s < t)$ is $(0, t-s)$-semicircular
  - disjoint increments are free
- concretely, by the sum of creation and annihilation operators on the full Fock space
- asymptotically, as the limit of matrix-valued (Dyson) Brownian motions

Free Brownian motions as matrix limits

Let $(X_N(t))_{t\geq 0}$ be a symmetric $N \times N$-matrix-valued Brownian motion, i.e.,

$$X_N(t) = \begin{pmatrix} B_{11}(t) & \dots & B_{1N}(t) \\ \vdots & \ddots & \vdots \\ B_{N1}(t) & \dots & B_{NN}(t) \end{pmatrix}, \qquad \text{where}$$

- B_{ij} are, for $i \geq j$, independent classical Brownian motions
- $B_{ij}(t) = B_{ji}(t)$.

Then, $\boldsymbol{(X_N(t))_{t\geq 0}} \xrightarrow{\text{distr}} \boldsymbol{(S(t))_{t\geq 0}}$, in the sense that almost surely

$$\lim_{N\to\infty} \mathrm{tr}\Big(X_N(t_1)\cdots X_N(t_n)\Big) = \varphi\Big(S(t_1)\cdots S(t_n)\Big) \qquad \forall 0 \leq t_1, t_2, \dots, t_n$$

Intermezzo on realisations on Fock spaces

Classical Brownian motion can be realized quite canonically by operators on the symmetric Fock space.

Similarly, free Brownian motion can be realized quite canonically by operators on the full Fock space

First: symmetric Fock space ...

For Hilbert space $\mathcal{H}$ put

$$\mathcal{F}_s(\mathcal{H}) := \bigoplus_{n\geq 0}^{\infty} \mathcal{H}^{\otimes \mathrm{sym}\, n}, \qquad \text{where} \qquad \mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$$

with inner product

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_{\mathrm{sym}} = \delta_{nm} \sum_{\pi \in S_n} \prod_{i=1}^{n} \langle f_i, g_{\pi(i)} \rangle.$$

Define creation and annihilation operators

$$\begin{aligned} a^*(f) f_1 \otimes \cdots \otimes f_n &= f \otimes f_1 \otimes \cdots \otimes f_n \\ a(f) f_1 \otimes \cdots \otimes f_n &= \sum_{i=1}^{n} \langle f, f_i \rangle f_1 \otimes \cdots \otimes \check{f}_i \otimes \cdots \otimes f_n \\ a(f)\Omega &= 0 \end{aligned}$$

... and classical Brownian motion

Put $\qquad \varphi(\cdot) := \langle \Omega, \cdot \Omega \rangle, \qquad x(f) := a(f) + a^*(f)$

then $(x(f))_{f \in \mathcal{H}, f \text{ real}}$ is Gaussian family with covariance

$$\varphi\Big(x(f)x(g)\Big) = \langle f, g \rangle.$$

In particular, choose $\mathcal{H} := L^2(\mathbb{R}_+)$, $f_t := 1_{[0,t)}$, then

$B_t := a(1_{[0,t)}) + a^*(1_{[0,t)})$ $\qquad$ is classical Brownian motion, meaning

$$\varphi(B_{t_1} \cdots B_{t_n}) = E[W_{t_1} \cdots W_{t_n}] \qquad \forall 0 \le t_1, \dots, t_n$$

Now: full Fock space ...

For Hilbert space $\mathcal{H}$ put

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n\geq 0}^{\infty} \mathcal{H}^{\otimes n}, \qquad \text{where} \qquad \mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$$

with usual inner product

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle = \delta_{nm} \langle f_1, g_1 \rangle \cdots \langle f_n, g_n \rangle.$$

Define creation and annihilation operators

$$b^*(f) f_1 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n$$

$$b(f) f_1 \otimes \cdots \otimes f_n = \langle f, f_1 \rangle f_2 \otimes \cdots \otimes f_n$$

$$b(f)\Omega = 0$$

... and free Brownian motion

Put $\quad \varphi(\cdot) := \langle \Omega, \cdot \Omega \rangle, \qquad x(f) := b(f) + b^*(f)$

then $(x(f))_{f \in \mathcal{H}, f \text{ real}}$ is semicircular family with covariance

$$\varphi\Big(x(f)x(g)\Big) = \langle f, g \rangle.$$

In particular, choose $\mathcal{H} := L^2(\mathbb{R}_+)$, $f_t := 1_{[0,t)}$, then

$S_t := b(1_{[0,t)}) + b^*(1_{[0,t)})$ is free Brownian motion.

Semicircle as real part of one-sided shift

Consider case of one-dimensional $\mathcal{H}$. Then this reduces to

$$\text{basis:} \quad \Omega = e_0, \quad e_1, \quad e_2, \quad e_3, \quad \dots$$

and one-sided shift l

$$le_n = e_{n+1}, \qquad l^* e_n = \begin{cases} e_{n-1}, & n \geq 1 \\ 0, & n = 0 \end{cases}$$

One-sided shift is canonical non-unitary isometry

$$l^* l = 1, \qquad l^* l \neq 1 \quad (= 1 - \text{projection on } \Omega)$$

With $\varphi(a) := \langle \Omega, a\Omega \rangle$ we claim: distribution of $l + l^*$ is semicircle.

Moments of $l + l^*$

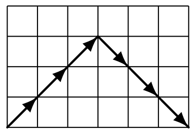
In the calculation of $\langle \Omega, (l + l^*)^n \Omega \rangle$ only such products in creation and annihilation contribute, where we never annihilate the vacuum, and where we start and end at the vacuum. So odd moments are zero.

Examples

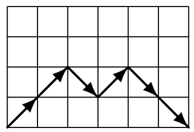
$$
\begin{aligned}
&\varphi\Big((l+l^*)^2\Big): \quad && l^*l \\
&\varphi\Big((l+l^*)^4\Big): \quad && l^*l^*ll, \quad l^*ll^*l \\
&\varphi\Big((l+l^*)^6\Big): \quad && l^*l^*l^*lll, \quad l^*ll^*l^*ll, \quad l^*ll^*ll^*l, \quad l^*l^*ll^*ll, \quad l^*l^*lll^*l
\end{aligned}
$$

Those contributing terms are in clear bijection with non-crossing pairings (or with Dyck paths).

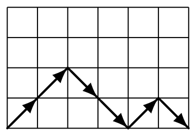
(l^*, l^*, l^*, l, l, l)



(l^*, l^*, l, l^*, l, l)



(l^*, l, l^*, l^*, l, l)



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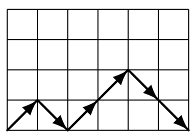

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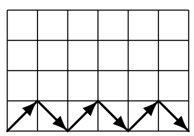


Stochastic Analysis on "Wigner" space

Starting from a free Brownian motion $\big(S(t)\big)_{t\geq 0}$ we define multiple "Wigner" integrals

$$I(f) = \int \cdots \int f(t_1, \dots, t_n) dS(t_1) \dots dS(t_n)$$

for scalar-valued functions $f \in L^2(\mathbb{R}^n_+)$, by avoiding the diagonals, i.e. we understand this as

$$I(f) = \underset{\text{all } t_i \text{ distinct}}{\int \cdots \int} f(t_1, \dots, t_n) dS(t_1) \dots dS(t_n)$$

Definition of Wigner integrals

More precisely: for f of form

$$f = 1_{[s_1,t_1]\times\cdots\times[s_n,t_n]}$$

for pairwisely disjoint intervals $[s_1, t_1], \ldots, [s_n, t_n]$ we put

$$I(f) := (S_{t_1} - S_{s_1}) \cdots (S_{t_n} - S_{s_n})$$

Extend $I(\cdot)$ linearly over set of all off-diagonal step functions (which is dense in $L^2(\mathbb{R}_+^n)$. Then observe Ito-isometry

$$\varphi[I(g)^* I(f)] = \langle f, g \rangle_{L^2(\mathbb{R}_+^n)}$$

and extend I to closure of off-diagonal step functions, i.e., to $L^2(\mathbb{R}_+^n)$.

Note: free stochastic integrals are usually bounded operators

Free Haagerup Inequality [Bozejko 1991; Biane, Speicher 1998]:

$$\left\| \int \cdots \int f(t_1, \ldots, t_n) dS(t_1) \ldots dS(t_n) \right\| \le (n+1) \|f\|_{L^2(\mathbb{R}_+^n)}$$

Intermezzo: combinatorics and norms

Consider free semicirculars $s_1, s_2, \ldots$ of variance 1. Since $(s_1 + \cdots + s_n)/\sqrt{n}$ is again a semicircular element of variance 1 (and thus of norm 2), we have

$$\left\| \left(\frac{s_1 + \cdots + c_n}{\sqrt{n}} \right)^k \right\| = 2^k$$

The free Haagerup inequality says that this is drastically reduced if we subtract the diagonals, i.e.,

$$\lim_{n\to\infty} \left\| \frac{1}{n^{k/2}} \sum_{\substack{i(1),\ldots,\imath(k)=1 \\ \text{all } i(.) \text{ different}}}^{n} s_{i(1)} \cdots s_{i(k)} \right\| = k+1$$

Intermezzo: combinatorics and norms

Note: one can calculate norms from asymptotic knowledge of moments!

If x is selfadjoint and φ faithful (as our φ for the free Brownian motion is) then one has

$$\|x\| = \lim_{p\to\infty} \|x\|_p = \lim_{p\to\infty} \sqrt[p]{\varphi(|x|^p)} = \lim_{m\to\infty} \sqrt[2m]{\varphi(x^{2m})}$$

So, if s is a semicircular element, then

$$\varphi(s^{2m}) = c_m = \frac{1}{1+m}\binom{2m}{m} \sim 4^m,$$

thus

$$\|s\| = \lim_{m\to\infty} \sqrt[2m]{c_m} \sim \sqrt[2m]{4^m} = 2$$

Exercise: combinatorics and norms

Consider free semicirculars $s_1, s_2, \ldots$ of variance 1. Prove by considering moments that

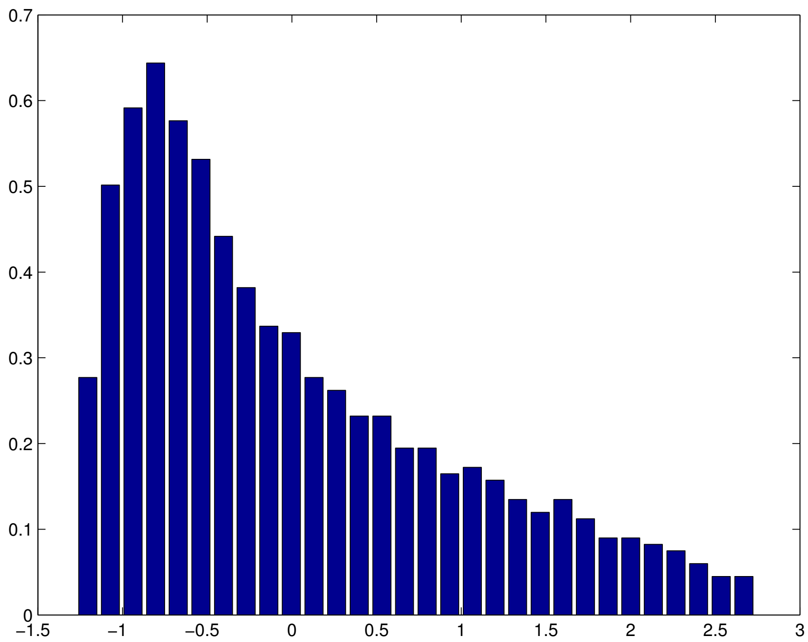
$$\left\| \frac{1}{n} \sum_{i,j=1}^{n} s_i s_j \right\| = 4, \qquad \lim_{n \to \infty} \left\| \frac{1}{n} \sum_{i,j=1, i \neq j}^{n} s_i s_j \right\| = 3$$

Realize that the involved NC pairings are in bijection with NC partitions and NC partitions without singletons, respectively.

2000 eigenvalues of the matrix

$$\frac{1}{12}\sum_{ij=1,i\neq j}^{12} X_i X_j,$$

where the X_i are independent Gaussian random matrices



Multiplication of Multiple Wigner Integrals

The Ito formula shows up in multiplicaton of two multiple Wigner integrals

$$\int f(t_1)dS(t_1) \cdot \int g(t_2)dS(t_2)$$

$$= \iint f(t_1)g(t_2)dS(t_1)dS(t_2) +$$

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Multiplication of Multiple Wigner Integrals

$$\iint f(t_1, t_2) dS(t_1) dS(t_2) \cdot \int g(t_3) dS(t_3)$$

$$= \iiint f(t_1, t_2) g(t_3) dS(t_1) dS(t_2) dS(t_3)$$

$$+ \iint f(t_1, t) g(t) dS(t_1) \underbrace{dS(t) dS(t)}_{dt}$$

$$+ \iint f(t, t_2) g(t) \underbrace{dS(t) dS(t_2) dS(t)}$$

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$$+ \iint f(t, t_2) g(t) \underbrace{dS(t) dS(t_2) dS(t)}_{dt \varphi[dS(t_2)] = 0}$$

Multiplication of Multiple Wigner Integrals

Free Ito Formula [Biane. Speicher 1998]:

$$dS(t)AdS(t) = \varphi(A)dt \quad \text{for } A \text{ adapted}$$

Multiplication of Multiple Wigner Integrals

Consider $f \in L^2(\mathbb{R}_+^n)$, $\qquad g \in L^2(\mathbb{R}_+^m)$

For $0 \le p \le \min(n,m)$, define

$$f \overset{p}{\frown} g \in L^2(\mathbb{R}_+^{n+m-2p})$$

by

$$f \overset{p}{\frown} g(t_1, \dots, t_{m+n-2p})$$

$$= \int f(t_1, \dots, t_{n-p}, s_1, \dots, s_p) g(s_p, \dots, s_1, t_{n-p+1}, \dots, t_{n+m-2p}) ds_1 \cdots ds_p$$

Then we have

$$I(f) \cdot I(g) = \sum_{p=0}^{\min(n,m)} I(f \overset{p}{\frown} g)$$

Free Chaos Decomposition

One has the canonical isomorphism

$$L^2(\{S(t) \mid t \geq 0\}) \mathrel{\widehat{=}} \bigoplus_{n=1}^{\infty} L^2(\mathbb{R}_+^n), \qquad f \mathrel{\widehat{=}} \bigoplus_{n=0}^{\infty} f_n,$$

via

$$f = \sum_{n=0}^{\infty} I(f_n) = \sum_{n=0}^{\infty} \int \cdots \int f_n(t_1, \dots, t_n) dS(t_1) \dots dS(t_n).$$

The set

$$\{I(f_n) \mid f_n \in L^2(\mathbb{R}_+^n)\}$$

is called n**-th chaos**.

Note: Polynomials in free semicircular elements

$$p(s_1, \ldots, s_m)$$

can be realized as elements from **finite chaos**

$$\{I(f) \mid f \in \bigoplus_{\text{finite}} L^2(\mathbb{R}_+^n)\}$$

Problems

We are interested in properties of selfadjoint elements from fixed (or finite) chaos, like:

- regularity properties of distribution of $I(f)$ from fixed (or finite) chaos
  - atoms or density ????
- distinguish distributions of $I(f)$ and $I(g)$ from different chaos

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Looking on elements in fixed chaos gives some quite unexpected constraints

We have analogue of classical result of Nualart and Peccati 2005.

Theorem (Kemp, Nourdin, Peccati, Speicher 2012): Consider, for fixed n, a sequence $f_1, f_2, \dots \in L^2(\mathbb{R}_+^n)$ with $f_k^* = f_k$ and $\|f_k\|_2 = 1$ for all $k \in \mathbb{N}$. Then the following statements are equivalent.

(i) We have $\lim_{k\to\infty} \varphi[I(f_k)^4] = 2$.

(ii) We have for all $p = 1, 2, \dots, n-1$ that

$$\lim_{k\to\infty} f_k \overset{p}{\frown} f_k = 0 \qquad \text{in } L^2(\mathbb{R}_+^{2n-2p}).$$

(iii) The selfadjoint variable $I(f_k)$ converges in distribution to a semicircular variable of variance 2.

Corollary: For $n \geq 2$ and $f \in L^2(\mathbb{R}_+^n)$, the law of $I(f)$ is not semicircular

Quantitative Estimates ...

Given two self-adjoint random variables X, Y, define the distance

$$d_{\mathcal{C}_2}(X,Y) := \sup\{|\varphi[h(X)] - \varphi[h(Y)]| \,:\, \mathcal{I}_2(h) \leq 1\};$$

where

$$\mathcal{I}_2(h) \quad \hat{=} \quad \|\partial h'\| \qquad \text{and} \qquad \partial X^n = \sum_{k=0}^{n-1} X^k \otimes X^{n-1-k}$$

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Rigorously: If h is the Fourier transform of a complex measure ν on $\mathbb{R}$,

$$h(x) = \widehat{\nu}(x) = \int_{\mathbb{R}} e^{ix\xi}\, \nu(d\xi)$$

then we define

$$\mathcal{I}_2(h) = \int_{\mathbb{R}} \xi^2\, |\nu|(d\xi)$$

... in Terms of the Free Gradient Operator

Define the free Malliavin gradient operator by

$$\nabla_t \left(\int f(t_1, \ldots, t_n) \, dS_{t_1} \cdots dS_{t_n} \right)$$
$$:= \sum_{k=1}^{n} \int f(t_1, \ldots, t_{k-1}, t, t_{k+1}, \ldots, t_n)$$
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Theorem (Kemp, Nourdin, Peccati, Speicher):

$$d_{\mathcal{C}_2}(F, S) \leq \frac{1}{2} \varphi \otimes \varphi \left(\left| \int \nabla_s (N^{-1} F) \sharp (\nabla_s F)^* \, ds - 1 \otimes 1 \right| \right)$$

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But no estimate against Fourth Moment in this case!

In the classical case one can estimate the corresponding expression of the above gradient, for F living in some n-th chaos, in terms of the fourth moment of the considered variable, thus giving a quantitative estimate for the distance between the considered variable (from a fixed chaos) and a normal variable in terms of the difference between their fourth moments. In the free case such a general estimate does not seem to exist; at the moment we are only able to do this for elements F from the second chaos.

Corollary: Let $F = I(f) = I(f)^*$ ($f \in L^2(\mathbb{R}_+^2)$) be an element from the second chaos with variance 1, i.e., $\|f\|_2 = 1$, and let S be a semiciruclar variable with mean 0 and variance 1. Then we have

$$d_{\mathcal{C}_2}(F, S) \leq \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{\varphi(F^4) - 2}.$$

Some general literature

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- A. Nica, R. Speicher: Lectures on the Combinatorics of Free Probability. *London Mathematical Society Lecture Note Series*, vol. 335, Cambridge University Press, 2006

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Biane, Speicher: Stochastic Calculus with Respect to Free Brownian Motion and Analysis on Wigner Space. *Prob. Theory Rel. Fields*, 1998

Kemp, Nourdin, Peccati, Speicher: Wigner Chaos and the Fourth Moment. *Ann. Prob.* (to appear)

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