Optimal designs for estimating pairs of coefficients in Fourier regression models

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Abstract

In the common Fourier regression model we investigate the optimal design problem for estimating pairs of the coefficients, where the explanatory variable varies in the interval $[-\pi, \pi]$. L-optimal designs are considered and for many important cases L-optimal designs can be found explicitly, where the complexity of the solution depends on the degree of the trigonometric regression model and the order of the terms for which the pair of the coefficients has to be estimated.

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1 Introduction

Fourier regression models of the form

$$(1.1) \qquad y = \beta_0 + \sum_{j=1}^{m} \beta_{2j-1} \sin(jt) + \sum_{j=1}^{m} \beta_{2j} \cos(jt) + \varepsilon,\ \ t \in [-\pi, \pi].$$

are widely used in applications to describe periodic phenomena. Typical subject areas include engineering [see e.g. McCool (1979)], medicine [see e.g. Kitsos, Titterington and Torsney (1988)], agronomy [see e.g. Weber and Liebig (1981)] or more generally biology [see the recent collection of research papers edited by Lestrel (1997)]. Applications of trigonometric regression models also appear in two dimensional shape analysis [see e.g. Young and Ehrlich (1977) and Currie et al. (2000) among many others]. An optimal design can improve the efficiency of the statistical analysis substantially and therefore the problem of designing experiments for Fourier regression models has been discussed by several authors. Optimal designs with respect to Kiefer's ϕ_p-optimality criteria have been studied by Karlin and Studden (1966), page 347, Hill (1978) or Wu (2002) among others [see also Pukelsheim (1993), p. 241], while Lau and Studden (1985) discuss the problem of constructing robust designs if the degree m in model (1.1) is not exactly known. Designs for identifying the degree m have been determined by Biedermann, Dette and Hoffmann (2007), Dette and Haller (1998) and Zen and Tsai (2004). More recent work discussed the problem of constructing optimal designs for the estimation of a particular coefficient in the Fourier regression model (1.1) [see Dette and Melas (2003) and Dette, Melas and Shpilev (2007)]. These authors also demonstrated that uniform designs are optimal for estimating a subset of the coefficients $\{\beta_{2i_1-1}, \beta_{2i_1}, \ldots, \beta_{2i_r-1}, \beta_{2i_r}\}$, where $1 \leq i_1 < \ldots < i_r \leq m,\ r \in \{1, \ldots, m\}$.
The main purpose of the present paper is to obtain further insight into the construction of optimal designs for estimating parameter subsystems in the Fourier regression model (1.1). In particular we are interested in the L-optimal design problem for estimating pairs of the coefficients $\{\beta_{2i_1}, \beta_{2i_2}\}$ and $\{\beta_{2j_1-1}, \beta_{2j_2-1}\}$, where $i_1, i_2 \in \{0, \ldots, m\},\ j_1, j_2 \in \{1, \ldots, m\}$. The precise estimation of specific pairs of coefficients is of particular interest because in many biological applications, such as two dimensional shape analysis, one or two coefficients have a concrete biological interpretation [see e.g. Young and Ehrlich (1977), Currie et al. (2000)]. In Section 2, we introduce the general notation and state several preliminary results. We formulate and prove a particular version of the equivalence theorem for L-optimal designs, which is an important tool for the determination of the optimal designs in Section 3. Here L-optimal designs are found explicitly for several cases. Finally, several examples are presented in Section 4 in order to illustrate the theoretical results.

2 L-optimal designs

Consider the trigonometric regression model (1.1), define $\beta = (\beta_0, \beta_1, ...\beta_{2m})^T$ as the vector of unknown parameters and

$$f(t) = (f_0(t), \dots, f_{2m}(t))^T = (1, \sin t, \cos t, \dots, \sin(mt), \cos(mt))^T.$$

as the vector of regression functions. Following Kiefer (1974) we call any probability measure ξ on the design space $[-\pi, \pi]$ with finite support an (approximate) design. The support points of the design ξ give the locations, where observations are taken, while the weights give the corresponding proportions of the total number of observations to be taken at these points. If the designs ξ puts masses ξ_i at the points t_i $(i = 1, \dots, k)$ and n uncorrelated observations can be taken, then the the quantities $\xi_i n$ are rounded to integers such that $\sum_{i=1}^k n_i = n$ [for a rounding procedure - see e.g. Pukelsheim and Rieder (1992)] and the experimenter takes n_i observations at each t_i $(i = 1, \dots, k)$. In this case the covariance matrix of the least squares estimate for the parameter β in the trigonometric regression model (1.1) is approximately given by

$$\frac{\sigma^2}{n} M^{-1}(\xi),$$

where

$$M(\xi) = \left(\int_{-\pi}^{\pi} f(t) f^T(t) d\xi(t) \right) \in R^{2m+1 \times 2m+1} \tag{2.1}$$

denotes the information matrix of the design ξ [see Pukelsheim (1993)]. Note, that for a symmetric design ξ after an appropriate permutation $P \in R^{2m+1 \times 2m+1}$ of the order of the regression functions the information matrix (2.1) will be block diagonal, that is

$$\widetilde{M}(\xi) = PM(\xi)P = \begin{pmatrix} M_c(\xi) & 0 \\ 0 & M_s(\xi) \end{pmatrix} \tag{2.2}$$

with blocks given by

$$M_c(\xi) = \left(\int_{-\pi}^{\pi} \cos(it) \cos(jt) d\xi(t) \right)_{i,j=0}^{m}, \tag{2.3}$$

$$M_s(\xi) = \left(\int_{-\pi}^{\pi} \sin(it) \sin(jt) d\xi(t) \right)_{i,j=1}^{m}. \tag{2.4}$$

For a given matrix

$$L = \sum_{i=0}^{k} l_i l_i^T, \; k \leq 2m \tag{2.5}$$

with vectors $l_i \in \mathbb{R}^{2m+1}$ the class Ξ_L is defined as the set of all approximate designs for which the linear combinations of the parameters $l_i^T\beta, i = 0,\dots,k$ are estimable, that is $l_i \in$Range$(M(\xi))$; $i = 0,\dots,k$. Similarly, the sets Ξ_s and Ξ_c are defined as the sets of all designs for which the matrices $M_s(\xi)$ and $M_c(\xi)$ are nonsingular, respectively. Finally a designs ξ^* is called L-optimal if

$$\xi^* = \arg\min_{\xi\in\Xi_L} trLM^+(\xi),$$

where L is a fixed and nonnegative definite matrix and for a given matrix A the matrix A^+ is the pseudo-inverse which is characterized by the four conditions

$$(a) \quad AA^+A = A \qquad\qquad (b) \quad A^+AA^+ = A^+$$

$$(c) \quad (AA^+)^T = AA^+ \qquad\qquad (d) \quad (A^+A)^T = A^+A.$$

[see Rao (1968)]. The following result gives a characterization of L-optimal designs, which is particularly useful for designs with a singular information matrix. For a nonsingular information matrix this theorem was formulated and proved in Ermakov and Zhigljavsky (1987). The theorem is stated for a general regression model $y = \beta^T f(t) + \varepsilon$ with $2m+1$ regression functions and a general design space χ.

Theorem 2.1 *Let $L \in \mathbb{R}^{(2m+1)\times(2m+1)}$ denote a given and nonnegative definite matrix of the form (2.5) and assume that the set of information matrices is compact.*

1) The design ξ is an element of the class Ξ_L if and only if

$$l_i^T M^-(\xi)M(\xi) = l_i^T, \; i = 0,\dots,k.$$

2) The design ξ is L-optimal if and only if

$$\max_{t\in\chi} \varphi(t,\xi^*) = trLM^+(\xi^*), \tag{2.6}$$

where $\varphi(t,\xi) = f^T(t)M^+(\xi)LM^+(\xi)f(t)$ Moreover, the equality

$$\varphi(t_i,\xi^*) = trLM^+(\xi^*) \tag{2.7}$$

holds for any $t_i \in supp(\xi^*)$.

Proof. The first part of this theorem was proved in Rao (1968). For a proof of the second part we use the following lemma.

Lemma 2.1 *If A is a nonnegative definite matrix, then $A+A^+ \geq 2A^+A$ and there is equality if and only only if $A = A^+A$.*

Proof of Lemma 2.1. Let $A = B^T B$, where $B \in \mathbb{R}^{n\times n}$. By definition of the pseudo-inverse matrix we have $A^+ = B^+ B^{+T}$, so that

$$\begin{aligned} A + A^+ - 2A^+A &= B^TB + B^+B^{+T} - 2B^+B^{+T}B^TB \\ &= (B^T - B^+)(B - B^{+T}) = (B - B^{+T})^T(B - B^{+T}) \geq 0\,, \end{aligned}$$

where the third equality is obtained from the identity

$$B^+B^{+T}B^TB = B^+(BB^+)^TB = B^+BB^+B = B^+B.$$

□

Now we return to the proof of Theorem 2.1. In the first part we show that a L-optimal design satisfies the conditions (2.6) and (2.7) For any $\alpha \in (0,1)$ define $\xi_\alpha = (1-\alpha)\xi^* + \alpha\xi_t$, where ξ_t denotes the Dirac measure at the point t and assume that the design ξ^* is L-optimal. In this case the inequality

$$trLM^+(\xi_\alpha) \geq trLM^+(\xi^*)$$

is satisfied for all $\alpha \in (0,1)$, which implies

$$\frac{\partial trLM^+(\xi_\alpha)}{\partial\alpha}\ |_{\alpha=0+} \geq 0.$$

On the other hand we obtain observing the identity

$$\frac{\partial LM^+(\xi_\alpha)M(\xi_\alpha)}{\partial\alpha} = 0 = L\frac{\partial M^+(\xi_\alpha)}{\partial\alpha}M(\xi_\alpha) + LM^+(\xi_\alpha)\frac{\partial M(\xi_\alpha)}{\partial\alpha}$$

the inequality

$$\begin{aligned} \frac{\partial trLM^+(\xi_\alpha)}{\partial\alpha}\ |_{\alpha=0+} &= tr\left\{L\frac{\partial M^+(\xi_\alpha)}{\partial\alpha}\right\}|_{\alpha=0+} = tr\left\{-LM^+(\xi_\alpha)\frac{\partial M(\xi_\alpha)}{\partial\alpha}M^+(\xi_\alpha)\right\}|_{\alpha=0+} \\ &= trLM^+(\xi^*) - trLM^+(\xi^*)M(\xi_t)M^+(\xi^*) \\ &= trLM^+(\xi^*) - \int f^T(s)M^+(\xi^*)LM^+(\xi^*)f(s)\xi_t(ds) \\ &= trLM^+(\xi^*) - \varphi(t,\xi^*) \geq 0. \end{aligned}$$

Therefore we have

$$\varphi(t,\xi^*) \leq trLM^+(\xi^*)$$

for all t. Moreover, the equality

$$\begin{aligned} \int \varphi(t,\xi)\xi(dt) &= \int trLM^+(\xi)f(t)f^T(t)M^+(\xi)\xi(dt) \\ &= trLM^+(\xi)M(\xi)M^+(\xi) = trLM^+(\xi) \end{aligned}$$

is obviously fulfilled for any design $\xi \in \Xi_L$. It now follows that statements (2.6) and (2.7) are satisfied.

In order to prove the converse implication we assume that the design ξ^* satisfies (2.6) but is not L-optimal. We denote an L-optimal design by ξ_1 and define

$$\xi_\alpha = (1-\alpha)\xi^* + \alpha\xi_1.$$

In the following discussion we use the notation $M(\xi_\alpha) = M_\alpha,\ M(\xi_1) = M_1,\ M(\xi^*) = M_*$ and obtain from Lemma 2.1 the inequality

$$\begin{aligned}
0 &\leq \alpha(1-\alpha) trL(M_1M_*^+ + M_*M_1^+ - 2M_1M_*^+M_*M_1^+)M_\alpha^+ \\
&= trL(\alpha M_1M_*^+ - \alpha^2 M_1M_*^+ + \alpha M_*M_1^+ - \alpha^2 M_*M_1^+)M_\alpha^+ \\
&+ trL(-2\alpha M_1M_*^+M_*M_1^+ + 2\alpha^2 M_1M_*^+M_*M_1^+)M_\alpha^+ \\
&= trL(\alpha M_1M_*^+ - \alpha^2 M_1M_*^+ + \alpha^2 M_1M_*^+M_*M_1^+)M_\alpha^+ \\
&+ trL(\alpha M_*M_1^+ - \alpha^2 M_*M_1^+ - 2\alpha M_1M_*^+M_*M_1^+ + \alpha^2 M_1M_*^+M_*M_1^+)M_\alpha^+ \\
&= trL(\alpha M_1((1-\alpha)M_*^+ + \alpha M_1^+))M_\alpha^+ \\
&+ trL(\alpha M_*M_1^+ - \alpha^2 M_*M_1^+ - 2\alpha M_*M_*^+ + \alpha^2 M_*M_*^+ + M_*M_*^+ - M_*M_*^+)M_\alpha^+ \\
&= trL(\alpha M_1((1-\alpha)M_*^+ + \alpha M_1^+))M_\alpha^+ \\
&+ trL((1-\alpha)M_*(\alpha M_1^+ + (1-\alpha)M_*^+) - I)M_\alpha^+ \\
&= trL((\alpha M_1^+ + (1-\alpha)M_*^+)M_\alpha - \mathrm{I})M_\alpha^+ = trL(\alpha M_1^+ + (1-\alpha)M_*^+ - M_\alpha^+)\,,
\end{aligned}$$

which implies

$$trL(M_\alpha^+) \leq trL(\alpha M_1^+ + (1-\alpha)M_*^+).$$

In other words, we have proved that the functional $\xi \to trLM^+(\xi)$ is convex on the set Ξ_L, and consequently it follows that

$$0 > trLM^+(\xi_1) - trLM^+(\xi^*) \geq \frac{\partial trLM^+(\xi_\alpha)}{\partial\alpha}\ |_{\alpha=0+} = trLM^+(\xi^*) - \int \varphi(t,\xi^*)\xi_1(dt).$$

On the other hand, we have by statement (2.6) the inequality $trLM^+(\xi^*) \geq \varphi(t,\xi^*)$, which yields

$$trLM^+(\xi^*) \geq \int \varphi(t,\xi^*)\xi_1(dt).$$

i.e. a contradiction to the previous inequality. Theorem 2.1 is proved.

□

In general an analytical determination of L-optimal designs is very difficult. Theorem 2.1 can be used to check the optimality of a given design numerically. Moreover we will use this characterization in the following section for an explicit construction of L-optimal designs for special parameter subsystems in the Fourier regression model (1.1).

3 Analytical solutions of the $L-$optimal design problem

In the present section we develop explicit solutions of the L-optimal design problem in the Fourier regression model (1.1) for several parameter subsystems. We begin with the problem of constructing optimal designs for estimating two coefficients corresponding to the sinus- or cosinus functions. More precisely, define $e_j \in \mathbb{R}^{2m+1}$ as the jth unit vector and consider the matrices

$$\begin{aligned} L_{(2\lfloor \frac{m}{2} \rfloor -1, 4\lfloor \frac{m}{2} \rfloor -1)} &= e_{2\lfloor \frac{m}{2} \rfloor -1} e^T_{2\lfloor \frac{m}{2} \rfloor -1} + e_{4\lfloor \frac{m}{2} \rfloor -1} e^T_{4\lfloor \frac{m}{2} \rfloor -1} \\ L_{(2\lfloor \frac{m}{2} \rfloor,\ 4\lfloor \frac{m}{2} \rfloor)} &= e_{2\lfloor \frac{m}{2} \rfloor} e^T_{2\lfloor \frac{m}{2} \rfloor} + e_{4\lfloor \frac{m}{2} \rfloor} e^T_{4\lfloor \frac{m}{2} \rfloor} \end{aligned}$$

We call an L-optimal design with the matrix $L_{(2\lfloor \frac{m}{2} \rfloor -1, 4\lfloor \frac{m}{2} \rfloor -1)}$ and $L_{(2\lfloor \frac{m}{2} \rfloor,\ 4\lfloor \frac{m}{2} \rfloor)}$ L-optimal design for estimating the pair of coefficients $\beta_{2\lfloor \frac{m}{2} \rfloor -1}, \beta_{4\lfloor \frac{m}{2} \rfloor -1}$ and $\beta_{2\lfloor \frac{m}{2} \rfloor}, \beta_{4\lfloor \frac{m}{2} \rfloor}$, respectively.

Theorem 3.1 *Consider the trigonometric regression model (1.1) with $m > 3$.*

1) The design

$$\xi^*_{(2\lfloor \frac{m}{2} \rfloor -1,\ 4\lfloor \frac{m}{2} \rfloor -1)} = \begin{pmatrix} -t_n & -t_{n-1} & \dots & -t_1 & t_1 & \dots & t_n \\ \frac{1}{2n} & \frac{1}{2n} & \dots & \frac{1}{2n} & \frac{1}{2n} & \dots & \frac{1}{2n} \end{pmatrix},$$

with

$$n = 2\left\lfloor \frac{m}{2} \right\rfloor, \quad t_i = 2\left\lfloor \frac{i}{2} \right\rfloor \frac{\pi}{n} + (-1)^{(i-1)} x, \quad x = \frac{2\arctan(\sqrt[4]{5})}{n}$$

is L-optimal for estimating the pair of coefficients $\beta_{2\lfloor \frac{m}{2} \rfloor -1}, \beta_{4\lfloor \frac{m}{2} \rfloor -1}$.
Moreover,

$$trLM^+(\xi^*_{(2\lfloor \frac{m}{2} \rfloor -1,\ 4\lfloor \frac{m}{2} \rfloor -1)}) = \frac{\sqrt{5}}{2} + \frac{3}{2}.$$

2) For any $\alpha \in [0, \omega_n]$ the design

$$\xi^*_{(2\lfloor \frac{m}{2} \rfloor,\ 4\lfloor \frac{m}{2} \rfloor)} = \begin{pmatrix} -\pi & -t_{n-1} & \dots & -t_1 & 0 & t_1 & \dots & t_{n-1} & \pi \\ \omega_n - \alpha & \omega_{n-1} & \dots & \omega_1 & \omega_0 & \omega_1 & \dots & \omega_{n-1} & \alpha \end{pmatrix},$$

with

$$\begin{aligned} n &= 2\left\lfloor \frac{m}{2} \right\rfloor, \quad t_i = \frac{(i-1)\pi}{n}, \ i = 2, \dots, n, \\ \omega_0 &= \sqrt{5}\omega_1, \ \omega_1 = \frac{\sqrt{5}-1}{4n}, \quad \omega_i = \omega_{i-2}, \ i = 2, \dots, n \end{aligned}$$

is L-optimal for estimating the pair of coefficients $\beta_{2\lfloor \frac{m}{2} \rfloor}, \beta_{4\lfloor \frac{m}{2} \rfloor}$.

Moreover,

$$trLM^+(\xi^*_{(2\lfloor \frac{m}{2} \rfloor,\ 4\lfloor \frac{m}{2} \rfloor)}) = trLM^+(\xi^*_{(0,\ 2\lfloor \frac{m}{2} \rfloor)}) = \frac{\sqrt{5}}{2} + \frac{3}{2}.$$

3) The L-optimal design $\xi^*_{(0,\ 2\lfloor \frac{m}{2} \rfloor)}$ *for estimating the pair of coefficients* $\beta_0, \beta_{2\lfloor \frac{m}{2} \rfloor}$ *coincides with the design* $\xi^*_{(2\lfloor \frac{m}{2} \rfloor,\ 4\lfloor \frac{m}{2} \rfloor)}$ *defined in part 2).*

Proof. We will only prove the first part of the theorem for the case where the degree m of the Fourier regression model is even. All other statements of the theorem are treated similarly. In this case $\lfloor \frac{m}{2} \rfloor = \frac{m}{2}$ and the design in part 1) of Theorem 2.1 can be rewritten as

$$\xi^*_{(m-1,\ 2m-1)} = \begin{pmatrix} -t_m & -t_{m-1} & \dots & -t_1 & t_1 & \dots & t_m \\ \frac{1}{2m} & \frac{1}{2m} & \dots & \frac{1}{2m} & \frac{1}{2m} & \dots & \frac{1}{2m} \end{pmatrix},$$

with

$$t_i = \left\lfloor \frac{i}{2} \right\rfloor \frac{2\pi}{m} + (-1)^{(i-1)} x, \quad x = \frac{2}{m} \arctan(\sqrt[4]{5}).$$

The proof is based on the characterization of L-optimal designs given in Theorem 2.1. Recall the definition of the matrix $M_s = M_s(\xi) \in \mathbb{R}^{m \times m}$ in (2.4) and let $m_{s[i,j]} = m_{s[i,j]}(\xi)$ denote the element of the matrix M_s in the position (i, j). We consider the system of equations

$$m_{s[j, \frac{m}{2}]} = 0, \quad j = 1, 2, \dots, m, \quad j \neq \frac{m}{2}, \tag{3.1}$$

$$m_{s[j,m]} = 0, \quad j = 1, 2, \dots, m-1, \tag{3.2}$$

$$m_{s[\frac{m}{2}, \frac{m}{2}]} = \sin^2(\frac{m}{2} x), \tag{3.3}$$

$$m_{s[m,m]} = \sin^2(mx). \tag{3.4}$$

We will show below that the design $\xi^*_{(m-1,\ 2m-1)}$ satisfies these equations. Under this assumption we obtain for the function $\varphi(t, \xi^*)$ in Theorem 2.1 the representation

$$\varphi(t, \xi^*) = f^T(t) M^+(\xi^*) L M^+(\xi^*) f(t) = \frac{\sin^2(\frac{m}{2} t)}{\sin^4(\frac{m}{2} x)} + \frac{\sin^2(mt)}{\sin^4(mx)}.$$

Now a straightforward calculation shows

$$trLM^+(\xi^*) = \frac{\sqrt{5}}{2} + \frac{3}{2},$$

and it is easy to prove that the condition (2.6) is satisfied calculating the solution of the equation $\frac{\partial\varphi(t,\xi^*)}{\partial t} = 0$. Moreover, a similar calculation shows that the equalities

$$trLM^+(\xi^*) = \varphi(t_i, \xi^*) = \frac{1}{\sin^2(\frac{m}{2}x)} + \frac{1}{\sin^2(mx)} = \frac{\sqrt{5}}{2} + \frac{3}{2}$$

are satisfied, and it remains to show the identities (3.1) - (3.4). For a proof of these identities we note that

$$m_{s[j,\frac{m}{2}]} = 2\sum_{k=1}^{m} \sin(jt_k)\sin(\frac{m}{2}t_k)\omega_k = \frac{1}{m}\sum_{k=1}^{m} \sin(jt_k)\sin(\frac{m}{2}t_k) \ ,$$

and that $\sin(\frac{m}{2}t_k) = (-1)^{\lfloor\frac{k-1}{2}\rfloor}\sin(\frac{m}{2}x)$. Consequently, we obtain

$$m_{s[j,\frac{m}{2}]} = \frac{\sin(\frac{m}{2}x)}{m}\sum_{k=1}^{m}(-1)^{\lfloor\frac{k-1}{2}\rfloor}\sin(jt_k).$$

Next we prove the identity $\sum_{k=1}^{m}(-1)^{\lfloor\frac{k-1}{2}\rfloor}\sin(jt_k) = 0$ if $j \neq \frac{m}{2}$. For this purpose we use the definition of t_k and obtain

$$\begin{aligned}
\sum_{k=1}^{m}(-1)^{\lfloor\frac{k-1}{2}\rfloor}\sin(jt_k) &= \sin(jx) + \sin\big(\frac{2\pi j}{m}\big)\cos(jx) - \cos\big(\frac{2\pi j}{m}\big)\sin(jx) \\
&+ (-1)\Big(\sin\big(\frac{2\pi j}{m}\big)\cos(jx) + \cos\big(\frac{2\pi j}{m}\big)\sin(jx)\Big) + \dots \\
&+ (-1)^{(\frac{m}{2}-1)}\Big(\sin\big((\frac{m}{2}-1)\frac{2\pi j}{m}\big)\cos(jx) + \cos\big((\frac{m}{2}-1)\frac{2\pi j}{m}\big)\sin(jx)\Big) \\
&\quad + (-1)^{(j+1)+(\frac{m}{2}-1)}\sin(jx) \\
&= \sin(jx)\Big[2\sum_{k=0}^{\frac{m}{2}-1}(-1)^k\cos(\frac{2\pi kj}{m}) + (-1)^{j+\frac{m}{2}} - 1\Big].
\end{aligned}$$

Now applying standard trigonometric formulae it is easy to calculate the sum in the last expression if $j \neq \frac{m}{2}$, i.e.

$$\begin{aligned}
2\sum_{k=0}^{\frac{m}{2}-1}(-1)^k\cos(\frac{2\pi kj}{m}) &= \frac{2}{\cos(\frac{j\pi}{m})}\sum_{k=0}^{\frac{m}{2}-1}(-1)^k\cos(\frac{2\pi kj}{m})\cos(\frac{j\pi}{m}) \\
&= \frac{1}{\cos(\frac{j\pi}{m})}((\cos(\frac{j\pi}{m}) + \cos(\frac{3j\pi}{m})) - (\cos(\frac{3j\pi}{m}) + \cos(\frac{5j\pi}{m})) + \dots \\
&\quad + (-1)^{\frac{m}{2}-1}(\cos(\frac{(m-3)j\pi}{m}) + \cos(\frac{(m-1)j\pi}{m}))) \\
&= 1 + (-1)^{j+\frac{m}{2}-1}.
\end{aligned}$$

Therefore we have proved that

$$\sum_{k=1}^{m}(-1)^{\lfloor\frac{k-1}{2}\rfloor}\sin(jt_k)=\sin(jx)\left(1+(-1)^{\frac{m}{2}-1+j}+(-1)^{j+\frac{m}{2}}-1\right)=0,$$

which shows that the first equation in (3.1) is satisfied. The equality $m_{s[j,m]}=0$ $(j\neq m)$ follows by similar arguments, that is

$$\begin{aligned}
m_{s[j,m]} &= \frac{1}{m}\sum_{k=1}^{m}\sin(jt_k)\sin(mt_k)=\frac{\sin(mx)}{m}\sum_{k=1}^{m}(-1)^k\sin(jt_k)\\
&= \sin(jx)\Big(2\sum_{k=0}^{\frac{m}{2}-1}\cos(\frac{2\pi kj}{m})+(-1)^j-1\Big)\\
&= \sin(jx)\left(1-(-1)^j+(-1)^j-1\right)\\
&= 0.
\end{aligned}$$

Finally we obtain by a direct calculation that

$$\begin{aligned}
m_{s[\frac{m}{2},\frac{m}{2}]} &= \sin^2(\frac{m}{2}x)\frac{1}{m}\sum_{k=1}^{m}(-1)^{2\lfloor\frac{k-1}{2}\rfloor}=\sin^2(\frac{m}{2}x),\\
m_{s[m,m]} &= \sin^2(mx)\frac{1}{m}\sum_{k=1}^{m}(-1)^{2k}=\sin^2(mx),
\end{aligned}$$

which completes the proof of Theorem 3.1. □

Remark 3.1 Note that Theorem 3.1 is only correct for trigonometric regression models of degree $m>3$. For example, if $m=2$ the L-optimal design $\xi^*_{(1,3)}$ is given by

$$\xi^*_{(1,3)}=\begin{pmatrix}-\pi+x & -x & x & \pi-x\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\end{pmatrix},$$

where $x=\arctan(\sqrt[4]{5})$, and $trLM_s^{-1}(\xi^*_{(1,3)})=\frac{\sqrt{5}}{2}+\frac{3}{2}$. For any $\alpha\in\left[0,\frac{5-\sqrt{5}}{8}\right]$ the L-optimal design $\xi^*_{(0,2)}$ is given by

$$\xi^*_{(0,2)}=\begin{pmatrix}-\pi & -\frac{\pi}{2} & 0 & \frac{\pi}{2} & \pi\\ \frac{5-\sqrt{5}}{8}-\alpha & \frac{\sqrt{5}-1}{8} & \frac{5-\sqrt{5}}{8} & \frac{\sqrt{5}-1}{8} & \alpha\end{pmatrix},$$

and $trLM_c^{-1}(\xi^*_{(0,2)})=\frac{\sqrt{5}}{2}+\frac{3}{2}$. If $m=3$ the L-optimal design $\xi^*_{(1,3)}$ is given by

$$\xi^*_{(1,3)}=\begin{pmatrix}-\pi+x & -x & x & \pi-x\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\end{pmatrix},$$

where $x = \arctan(\sqrt[4]{5})$ and $trLM_s^+(\xi^*) = \frac{3+\sqrt{5}}{2}$, while the L-optimal design $\xi^*_{(0,2)}$ is given by

$$\xi^*_{(0,2)} = \begin{pmatrix} -\pi & -\pi + x & -x & 0 & x & \pi - x & \pi \\ \frac{1}{4} - z - \alpha & z & z & \frac{1}{4} - z & z & z & \alpha \end{pmatrix},$$

for any $\alpha \in \left[0, \frac{1}{4} - z\right]$, where $z \approx 0.15195067$, $x \approx 0.932928804$ and $trLM_c^{-1}(\xi^*_{(0,2)}) \approx 2.77004565$. The optimality of the designs can be easily checked by an application of Theorem 2.1.

Remark 3.2 Note that by Theorem 3.1 the sum of variances of the estimates for the corresponding coefficients in a Fourier regression model of degree $m > 3$ is given by $\frac{\sqrt{5}}{2} + \frac{3}{2}$ for the L-optimal design, while the D-optimal design yields 4 for this sum. Thus the sum of variances obtained by the L-optimal design is approximately 65% smaller than the sum obtained by the D-optimal design.

There are two other cases, where L-optimal designs for the trigonometric regression model (1.1) can be constructed explicitly. They are stated in the following two theorems. For the sake of brevity only a proof of Theorem 3.3 is given here.

Theorem 3.2 *Consider the trigonometric regression model (1.1) with $m = 3k$. The design*

$$\xi^* = \begin{pmatrix} -t_m & -t_{m-1} & \dots & -t_1 & t_1 & \dots & t_m \\ \omega_m & \omega_{m-1} & \dots & \omega_1 & \omega_1 & \dots & \omega_m \end{pmatrix},$$

with

$$\begin{aligned} t_1 &= \frac{\pi}{2k} - \frac{x}{k},\ t_2 = \frac{\pi}{2k},\ t_3 = \frac{\pi}{2k} + \frac{x}{k},\ t_i = t_{i-3} + \frac{\pi}{k},\ i = 4, 5 \dots, m, \\ \omega_1 &= \frac{z}{k},\ \ \omega_2 = \frac{1-4z}{2k},\ \ \omega_3 = \frac{z}{k},\ \ \omega_i = \omega_{i-3},\ i = 4, 5 \dots, m \end{aligned}$$

is L-optimal for estimating one of the pairs of the coefficients $\{\beta_{2k-1}, \beta_{4k-1}\}$, $\{\beta_{2k-1}, \beta_{6k-1}\}$, $\{\beta_{4k-1}, \beta_{6k-1}\}$. Here only the values x and z depend on the particular pair under consideration and are defined as the solution of the system

$$(3.5) \qquad \begin{aligned} \frac{\partial trLM_c^{-1}(\xi^*)}{\partial x} &= 0 \\ \frac{\partial trLM_c^{-1}(\xi^*)}{\partial z} &= 0\ . \end{aligned}$$

Similarly, for any $\alpha \in [0, \omega_m]$ the design

$$\xi^* = \begin{pmatrix} -\pi & -t_{m-1} & \dots & -t_1 & 0 & t_1 & \dots & t_{m-1} & \pi \\ \omega_m - \alpha & \omega_{m-1} & \dots & \omega_1 & \omega_0 & \omega_1 & \dots & \omega_{m-1} & \alpha \end{pmatrix},$$

$m=3k$	$\{2k-1,4k-1\}$	$\{2k-1,6k-1\}$	$\{4k-1,6k-1\}$
x	$\pi/3$	0.6476	$3\pi/10$
z	$1/4$	0.14	$(3-\sqrt{5})/4$
$trLM_s^{-1}$	$8/3$	2.7044	$(\sqrt{5}+3)/2$

Table 1: *The solutions of x and z of the the system (3.5). The L-optimal design for estimating the specific pair of coefficients in the Fourier regression model (1.1) is specified in the first part of Theorem 3.2.*

$m=3k$	$\{0,2k\}$	$\{0,4k\}$	$\{0,6k\}$	$\{2k,4k\}$	$\{2k,6k\}$	$\{4k,6k\}$
x	0.9329	$\pi/2$	$\pi/3$	1.1177	0.9232	1.1668
z	0.1519	$1/4$	$1/6$	0.1258	0.14	0.1478
$trLM_c^{-1}$	2.77	2	2	3.4826	2.7044	$(\sqrt{5}+3)/2$

Table 2: *The solutions of x and z of the the system (3.6). The L-optimal design for estimating the specific pair of coefficients in the Fourier regression model (1.1) is specified in the second part of Theorem 3.2.*

with

$$\begin{aligned} t_0 &= 0,\ t_1 = \frac{x}{k},\ t_2 = \frac{\pi - x}{k},\ t_i = t_{i-3} + \frac{\pi}{k},\ i = 3,4\ldots,m-1, \\ \omega_0 &= \frac{1-4z}{2k},\quad \omega_1 = \frac{z}{k},\quad \omega_2 = \frac{z}{k},\quad \omega_i = \omega_{i-3},\ i = 3,4\ldots,m \end{aligned}$$

is L-optimal for estimating one of the pairs $\{\beta_0,\beta_{2k}\}$, $\{\beta_0,\beta_{4k}\}$, $\{\beta_0,\beta_{6k}\}$, $\{\beta_{2k},\beta_{4k}\}$, $\{\beta_{2k},\beta_{6k}\}$, $\{\beta_{4k},\beta_{6k}\}$. Here only the values x and z depend on the particular pair under consideration and are defined as the solution of the system

$$(3.6) \qquad \begin{aligned} \frac{\partial trLM_s^{-1}(\xi^*)}{\partial x} &= 0 \\ \frac{\partial trLM_s^{-1}(\xi^*)}{\partial z} &= 0. \end{aligned}$$

Note that Theorem 3.2 shows that all designs for estimating one of the considered pairs have the same structure. Only the values x and z depend on the particular pair under consideration. Some numerical values for the parameters x and z in Theorem 3.2. the Tables 1 and 2.

Theorem 3.3 *Consider the trigonometric regression model (1.1) with $m = 4k$. The design*

$$\xi^* = \begin{pmatrix} -t_m & -t_{m-1} & \dots & -t_1 & t_1 & \dots & t_m \\ \omega_m & \omega_{m-1} & \dots & \omega_1 & \omega_1 & \dots & \omega_m \end{pmatrix},$$

with

$$\begin{aligned} t_1 &= \frac{x_1}{k},\ t_2 = \frac{x_2}{k},\ t_3 = \frac{\pi - x_2}{k},\ t_4 = \frac{\pi - x_1}{k},\ t_i = t_{i-4} + \frac{\pi}{k},\ i = 5, 6 \dots, m, \\ \omega_1 &= \frac{z_1}{k},\ \ \omega_2 = \frac{1 - 4z_1}{m},\ \ \omega_3 = \frac{1 - 4z_1}{m},\ \ \omega_4 = \frac{z_1}{k},\ \ \omega_i = \omega_{i-4},\ i = 5, 6 \dots, m \end{aligned}$$

is L-optimal for estimating one of the pairs of coefficients $\{\beta_{2k-1}, \beta_{4k-1}\}$, $\{\beta_{2k-1}, \beta_{6k-1}\}$, $\{\beta_{2k-1}, \beta_{8k-1}\}$, $\{\beta_{4k-1}, \beta_{6k-1}\}$, $\{\beta_{4k-1}, \beta_{8k-1}\}$, $\{\beta_{6k-1}, \beta_{8k-1}\}$. Here only the values x_1, x_2 and z_1 depend on the particular pair under consideration and are defined as the solution of the system

$$\begin{aligned} \frac{\partial trLM_s^{-1}(\xi^*)}{\partial x_1} &= 0\ ,\quad \frac{\partial trLM_s^{-1}(\xi^*)}{\partial x_2} = 0 \\ \frac{\partial trLM_s^{-1}(\xi^*)}{\partial z_1} &= 0 \end{aligned} \tag{3.7}$$

Similarly, if $n = \frac{5m}{4}$ then for any $\alpha \in [0, \omega_n]$ the design

$$\xi^* = \begin{pmatrix} -\pi & -t_{n-1} & \dots & -t_1 & 0 & t_1 & \dots & t_{n-1} & \pi \\ \omega_n - \alpha & \omega_{n-1} & \dots & \omega_1 & \omega_0 & \omega_1 & \dots & \omega_{n-1} & \alpha \end{pmatrix},\ ,$$

with

$$\begin{aligned} t_0 &= 0,\ t_1 = \frac{x_1}{k},\ t_2 = \frac{x_2}{k},\ t_3 = \frac{\pi - x_2}{k},\ t_4 = \frac{\pi - x_1}{k},\ t_i = t_{i-5} + \frac{\pi}{k},\ i = 5, 6 \dots, n-1, \\ \omega_0 &= \frac{1 - 4z_1 - 4z_2}{2k},\ \omega_1 = \omega_4 = \frac{z_1}{k},\ \omega_2 = \omega_3 = \frac{z_2}{k},\ \omega_i = \omega_{i-5},\ i = 5, 6, \dots, n \end{aligned}$$

is optimal for estimating any of the pairs of the coefficients $\{\beta_0, \beta_{4k}\}$, $\{\beta_0, \beta_{6k}\}$, $\{\beta_0, \beta_{8k}\}$, $\{\beta_{2k}, \beta_{4k}\}$, $\{\beta_{2k}, \beta_{6k}\}$, $\{\beta_{2k}, \beta_{8k}\}$, $\{\beta_{4k}, \beta_{6k}\}$, $\{\beta_{4k}, \beta_{8k}\}$, $\{\beta_{6k}, \beta_{8k}\}$. Here only the values x_1, x_2, z_1 and z_2 depend on the particular pair under consideration and are defined as the solution of the system

$$\begin{aligned} \frac{\partial trLM_c^{-1}(\xi^*)}{\partial x_1} &= 0\ ,\quad \frac{\partial trLM_c^{-1}(\xi^*)}{\partial x_2} = 0 \\ \frac{\partial trLM_c^{-1}(\xi^*)}{\partial z_1} &= 0\ ,\quad \frac{\partial trLM_c^{-1}(\xi^*)}{\partial z_2} = 0. \end{aligned} \tag{3.8}$$

The numerical values of the quantities x_1, x_2, z_1 and z_2 are listed in the Table 3 and 4.

$m=4k$	$\{2k-1,4k-1\}$	$\{2k-1,6k-1\}$	$\{2k-1,8k-1\}$
x_1	$\pi/4$	0.6476	0.4845
x_2	$\pi/2$	$\pi/2$	1.1912
z_1	$(6-\sqrt{6})/20$	0.14	0.0909
$trLM_s^{-1}$	$(7+2\sqrt{6})/4$	2.7044	2.731
$m=4k$	$\{4k-1,6k-1\}$	$\{4k-1,8k-1\}$	$\{6k-1,8k-1\}$
x_1	0.7338	$\arctan(\sqrt[4]{5})/2$	0.4523
x_2	1.3884	$(\pi-\arctan(\sqrt[4]{5}))/2$	1.2566
z_1	0.168	1/8	0.1417
$trLM_s^{-1}$	2.96	$(\sqrt{5}+3)/2$	$(\sqrt{5}+3)/2$

Table 3: *The solutions x_1, x_2 and z_1 of the the system (3.7). The L-optimal design for estimating the specific pair of coefficients in the Fourier regression model (1.1) is specified in the first part of Theorem 3.3.*

$m=4k$	{0,4k}	{0,6k}	{0,8k}	{2k,4k}	{2k,6k}
x_1	$\pi/4$	$\pi/3$	$\pi/4$	$\pi/4$	0.9232
x_2	$\pi/2$	$\pi/3$	$\pi/2$	$\pi/2$	0.9232
z_1	0.0863	1/6	1/8	$\frac{2\sqrt{2}-1}{28}$	0.07
z_2	0.0773	1/6	1/8	$\frac{4-\sqrt{2}}{28}$	0.07
$trLM_c^{-1}$	$(\sqrt{5}+3)/2$	2	2	$\sqrt{2}+9/4$	2.7044
$m=4k$	{2k,8k}	{4k,6k}	{4k,8k}	{6k,8k}	
x_1	0.7132	1.0157	$\pi/4$	0.8814	
x_2	$\pi/2$	$\pi/2$	$\pi/2$	$\pi/2$	
z_1	0.033	0.0708	0.0863	0.0477	
z_2	0.14	0.093	0.0773	0.1297	
$trLM_c^{-1}$	2.731	3.1149	$(\sqrt{5}+3)/2$	$(\sqrt{5}+3)/2$	

Table 4: *The solutions of x_1, x_2, z_1 and z_2 of the the system (3.8). The L-optimal design for estimating the specific pair of coefficients in the Fourier regression model (1.1) is specified in the second part of Theorem 3.3.*

Proof of Theorem 3.3. We will only prove the first part of Theorem 3.3, the second part is treated similarly. We begin with the case $k = 1$, i.e. $m = 4$, for which it is easy to check by direct calculations that the design ξ^* defined in the first part of Theorem 3.3 is L-optimal. The corresponding numerical values of the quantities x_1, x_2 and z_1 can be found as the solution of the system of equations defined by (3.7). Now let $k \geq 2$ and $m = 4k$. We consider the system of equations

$$\begin{cases} m_{s[k,j]} = 0, \quad j = 1, 2, \dots, 4k, \quad j \neq k, \ j \neq 3k \\ m_{s[4k,j]} = 0, \quad j = 1, 2, \dots, 4k-1, \quad j \neq 2k, \\ m_{s[k,k]} = 4\left(z_1 \sin^2(x_1) + (1/4 - z_1)\sin^2(x_2)\right) \\ m_{s[k,3k]} = 4\left(z_1 \sin(x_1)\sin(3x_1) + (1/4 - z_1)\sin(x_2)\sin(3x_2)\right) \\ m_{s[4k,2k]} = 4\left(z_1 \sin(2x_1)\sin(4x_1) + (1/4 - z_1)\sin(2x_2)\sin(4x_2)\right) \\ m_{s[4k,4k]} = 4\left(z_1 \sin^2(4x_1) + (1/4 - z_1)\sin^2(4x_2)\right) \end{cases} \tag{3.9}$$

where $m_{s[i,j]} = m_{s[i,j]}(\xi^*)$ is the element of the matrix $M_s(\xi^*) \in R^{m\times m}$ in the i-th row and j-th column. We will prove below that these equalities are satisfied for the design ξ^*. In this case it follows that the quantities $trLM_s^{-1}(\xi^*)$ and $\varphi(t,\xi^*) = f^T(t)M^+(\xi^*)LM^+(\xi^*)f(t)$ in Theorem 2.1 are independent of the value k (note that the matrix L is a given diagonal matrix where the non vanishing entries depend on the particular pair of parameters under consideration). Consequently it is sufficient to prove Theorem 3.3 in the case $k = 1$, which has been done in the previous paragraph
In order to prove that the equalities (3.9) are satisfied we note that for $i = 1, \dots, 4k-1, \ i \neq k, 2k, 3k$ we have

$$\begin{aligned} m_{s[k,i]} &= 2\sum_{j=1}^{4k} \sin(kt_j)\sin(it_j)\omega_j \\ &= 2\sum_{j=1}^{k} \left(\sin(kt_{4j-3})\sin(it_{4j-3})\omega_{4j-3} + \sin(kt_{4j-2})\sin(it_{4j-2})\omega_{4j-2}\right) \\ &+ 2\sum_{j=1}^{k} \left(\sin(kt_{4j-1})\sin(it_{4j-1})\omega_{4j-1} + \sin(kt_{4j})\sin(it_{4j})\omega_{4j}\right) \\ &= 2\sin(x_1)\frac{z_1}{k}\sum_{j=1}^{k}(-1)^{j-1}\left(\sin(it_{4j-3}) + \sin(it_{4j})\right) \\ &+ 2\sin(x_2)\Big(\frac{1}{4} - z_1\Big)\frac{1}{k}\sum_{j=1}^{k}(-1)^{j-1}\left(\sin(it_{4j-2}) + \sin(it_{4j-1})\right). \end{aligned}$$

For the first sum in the last line we obtain

$$
\begin{aligned}
&\sum_{j=1}^{k}(-1)^{j-1}\left(\sin(it_{4j-3})+\sin(it_{4j})\right)\\
=&\sum_{j=1}^{k}(-1)^{j-1}\Big(\sin\big(\frac{i}{k}(x_1+(s-1)\pi)\big)+\sin\big(\frac{i}{k}(\pi-x_1+(s-1)\pi)\big)\Big)\\
=&\frac{(-1)^k}{2\sin\left(\frac{i\pi}{k}\right)}\Big(\cos\big(\frac{ix_1-i\pi}{k}-i\pi\big)-\cos\big(\frac{ix_1-i\pi}{k}+i\pi\big)\Big)\\
+&\frac{(-1)^k}{2\sin\left(\frac{i\pi}{k}\right)}\Big(\cos\big(\frac{ix_1}{k}+i\pi\big)-\cos\big(\frac{ix_1}{k}-i\pi\big)\Big)=0.
\end{aligned}
$$

Similarly, it follows (substituting x_1 for x_2) that the second sum also vanishes, which implies

$$m_{s[k,i]}=0,\ \ for\ i=1,\dots,4k-1,\ \ i\neq k,2k,3k.$$

For the element $m_{s[4k,i]}$ we find for $i=1,\dots,4k-1,\ \ i\neq k,2k,3k$

$$
\begin{aligned}
m_{s[4k,i]} &= 2\sum_{j=1}^{k}\left(\sin(kt_{4j-3})\sin(it_{4j-3})\omega_{4j-3}+\sin(kt_{4j-2})\sin(it_{4j-2})\omega_{4j-2}\right)\\
&+ 2\sum_{j=1}^{k}\left(\sin(kt_{4j-1})\sin(it_{4j-1})\omega_{4j-1}+\sin(kt_{4j})\sin(it_{4j})\omega_{4j}\right)\\
&= 2\sin(x_1)\frac{z_1}{k}\sum_{j=1}^{k}\left(\sin(it_{4j-3})-\sin(it_{4j})\right)\\
&+ 2\sin(x_2)\Big(\frac{1}{4}-z_1\Big)\frac{1}{k}\sum_{j=1}^{k}\Big(\sin(it_{4j-2})-\sin(it_{4j-1})\Big).
\end{aligned}
$$

A straightforward calculation now yields

$$
\begin{aligned}
&\sum_{j=1}^{k}\left(\sin(it_{4j-3})-\sin(it_{4j})\right)\\
=&\sum_{j=1}^{k}\Big(\sin\big(\frac{i}{k}(x_1+(s-1)\pi)\big)-\sin\big(\frac{i}{k}(\pi-x_1+(s-1)\pi)\big)\Big)\\
=&\frac{\left(\cos\left(\frac{ix_1-i\pi}{k}-i\pi\right)-\cos\left(\frac{ix_1-i\pi}{k}+i\pi\right)+\cos\left(\frac{ix_1}{k}-i\pi\right)-\cos\left(\frac{ix_1}{k}+i\pi\right)\right)}{2\sin\left(\frac{i\pi}{k}\right)}=0,
\end{aligned}
$$

and the same arguments show that the second sum also vanishes, which implies $m_{s[4k,i]}=0$. To conclude the proof it remains to calculate the quantities $m_{s[ik,jk]},\ \ i,j=1,\dots,4$, for which we obtain

$$
\begin{aligned}
m_{s[ik,jk]} &= 2\sum_{r=1}^{k}\left(\sin(ikt_{4r-3})\sin(jkt_{4r-3})\omega_{4r-3}+\sin(ikt_{4r-2})\sin(jkt_{4r-2})\omega_{4r-2}\right)\\
&+ 2\sum_{r=1}^{k}\left(\sin(ikt_{4r-1})\sin(jkt_{4r-1})\omega_{4r-1}+\sin(ikt_{4r})\sin(jkt_{4r})\omega_{4r}\right)\\
&= \frac{2z_1}{k}\sum_{r=1}^{k}(\sin(i(x_1+(r-1)\pi))\sin(j(x_1+(r-1)\pi))\\
&+ \sin(i(-x_1+r\pi))\sin(j(-x_1+r\pi)))\\
&+ \Big(\frac{1}{4}-z_1\Big)\frac{2\sin(ix_2)\sin(jx_2)}{k}\sum_{r=1}^{k}\big((-1)^{(i+j)(r-1)}+(-1)^{(i+j)r}\big)\\
&= \Big(\frac{2\sin(ix_1)\sin(jx_1)z_1}{k}+\Big(\frac{1}{4}-z_1\Big)\frac{2\sin(ix_2)\sin(jx_2)}{k}\Big)\\
&\quad\times\sum_{r=1}^{k}\big((-1)^{(i+j)(r-1)}+(-1)^{(i+j)r}\big).
\end{aligned}
$$

From this representation it is obvious that the quantities $m_{s[ik,jk]},\ i,j=1,\dots,4$ have the values specified by the system of equations and the theorem has been proved. □

Remark 3.3 Note that for any k with $m/2<k\le m$ and for any $\beta\in[0,\frac{1}{2k}]$ the design

$$
\xi^*_{(0,2k)}=\begin{pmatrix} -\pi & -\pi+\frac{\pi}{k} & \dots & -\pi+\frac{2k-1}{k}\pi & \pi\\ \frac{1}{2k}-\beta & \frac{1}{2k} & \dots & \frac{1}{2k} & \beta \end{pmatrix}
$$

is L-optimal for estimating the pair of coefficients $\{\beta_0,\,\beta_{2k}\}$. Moreover, in this case it follows that

$$
trLM_c^{-1}(\xi^*_{(0,2k)})=2
$$

[see Dette and Melas (2003), Lemma 2.3]

4 Examples

In this section we present several examples, which illustrate the theoretical results obtained in Section 3.

4.1 L-optimal design for estimating the coefficients of $\sin(2t)$ and $\sin(4t)$ in the Fourier regression model of degree 4.

In this example we present the L-optimal design for estimating the coefficients β_3, β_7 (i.e. the coefficients at $\sin(2t)$ and $\sin(4t)$) in the trigonometric regression model of degree 4. The L-optimal design is directly obtained from Theorem 3.1 and given by

$$\xi^*_{(3,7)} = \begin{pmatrix} -\pi - x & -\frac{\pi}{2} - x & -\frac{\pi}{2} + x & -x & x & \frac{\pi}{2} - x & \frac{\pi}{2} + x & \pi - x \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{pmatrix},$$

where $x = \frac{1}{2}\arctan\left(\sqrt[4]{5}\right) \approx 0.49068$. The corresponding support points of the design $\xi^*_{(3,7)}$ are depicted in Figure 1.

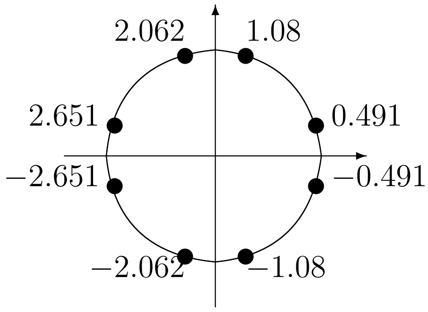

Figure 1: *Support points of the L-optimal design for estimating the coefficients β_3, β_7 in the Fourier regression model of degree* 4.

The matrices $M_s(\xi^*_{(3,7)})$ and $M_s^{-1}(\xi^*_{(3,7)})$ for this design are given by

$$M_s(\xi^*_{(3,7)}) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{3-\sqrt{5}}{4} & 0 \\ 0 & \frac{5-\sqrt{5}}{4} & 0 & 0 \\ \frac{3-\sqrt{5}}{4} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{3\sqrt{5}-5}{2} \end{bmatrix}, \quad M_s^{-1}(\xi^*_{(3,7)}) = \begin{bmatrix} \frac{5+3\sqrt{5}}{5} & 0 & -\frac{2\sqrt{5}}{5} & 0 \\ 0 & \frac{5+\sqrt{5}}{5} & 0 & 0 \\ -\frac{2\sqrt{5}}{5} & 0 & \frac{5+3\sqrt{5}}{5} & 0 \\ 0 & 0 & 0 & \frac{5+3\sqrt{5}}{10} \end{bmatrix},$$

respectively. The matrix corresponding to the cosine terms is of no interest, but is readily seen that (using an appropriate permutation of the regression functions)

$$M^+(\xi^*_{(3,7)}) = \begin{pmatrix} M_c^+(\xi^*_{(3,7)}) & 0 \\ 0 & M_s^{-1}(\xi^*_{(3,7)}) \end{pmatrix}.$$

A straightforward calculation shows that the matrix $M^+(\xi^*_{(3,7)})LM^+(\xi^*_{(3,7)})$ is given by

$$M^+(\xi^*_{(3,7)})LM^+(\xi^*_{(3,7)}) = \begin{pmatrix} 0 & 0 \\ 0 & M_s^{-1}(\xi^*_{(3,7)})L_s M_s^{-1}(\xi^*_{(3,7)}) \end{pmatrix},$$

where L_s is the corresponding block of the matrix L. Therefore the function $\varphi(t, \xi^*_{(3,7)})$ is given explicitly by

$$\varphi(t, \xi^*_{(3,7)}) = f^T(t)M^+(\xi^*_{(3,7)})LM^+(\xi^*_{(3,7)})f(t) = \frac{(1+\sqrt{5})^2}{5}\sin^2(2t) + \frac{(3+\sqrt{5})^2}{20}\sin^2(4t)\,.$$

The equalities (2.7) are checked by a direct calculation and the function $\varphi(t, \xi^*_{(3,7)})$ is depicted in Figure 2.

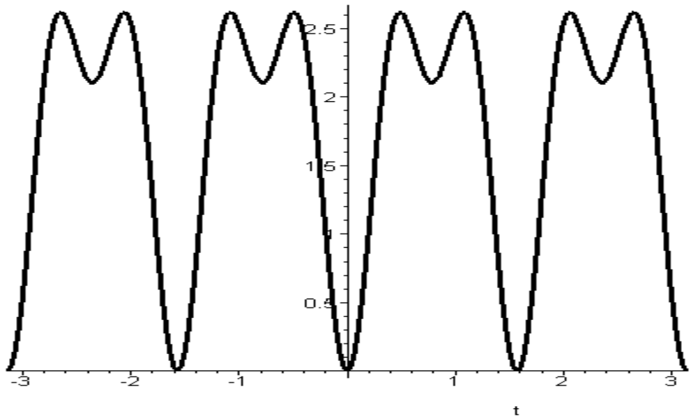

Figure 2: *The function $\varphi(t, \xi^*_{(3,7)})$ defined in Example 4.1.*

4.2 L-optimal design for estimating the coefficients of $\cos(2t)$ and $\cos(3t)$ in the Fourier regression model of degree $m = 4$.

We consider again the trigonometric regression model of degree $m = 4$ and use Theorem 3.3 to determine the L-optimal design for estimating the pair of coefficients β_4 and β_6, which correspond to the terms $\cos(2t)$ and $\cos(3t)$. The L-optimal design for estimating these coefficients is given by

$$\xi^*_{(4,6)} = \begin{pmatrix} -\pi & -2.13 & -\frac{\pi}{2} & -1.02 & 0 & 1.02 & \frac{\pi}{2} & 2.13 & \pi \\ 0.175-\alpha & 0.09 & 0.145 & 0.09 & 0.175 & 0.09 & 0.145 & 0.09 & \alpha \end{pmatrix},$$

and the support points of the optimal design are depicted in Figure 3. We finally note that a straightforward calculation yields for the function $\varphi(t, \xi^*_{(4,6)})$ in Theorem 2.1 $\varphi(t, \xi^*_{(4,6)}) =$

$$f^T(t)M^+(\xi^*_{(4,6)})LM^+(\xi^*_{(4,6)})f(t) =$$

$$= 2.851 - 0.262\cos(2t) + 0.116\cos(4t) + 0.262\cos(6t) + 0.147\cos(8t).$$

This function is depicted in Figure 4.

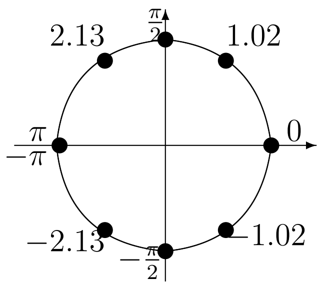

Figure 3: *The support points of the L-optimal design for estimating pair of coefficients* β_4, β_6 *in the Fourier regression model of degree* $m = 4$.

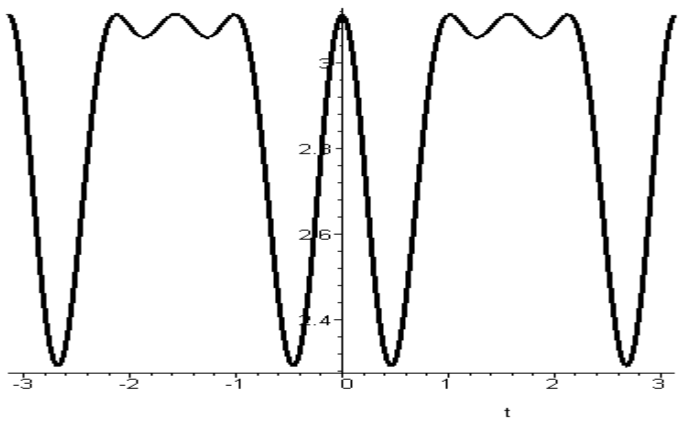

Figure 4: *The function* $\varphi(t, \xi^*_{(4,6)})$ *defined in Example 4.2.*

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