ALIASING EFFECTS FOR RANDOM FIELDS OVER SPHERES OF ARBITRARY DIMENSION

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Abstract. In this paper, aliasing effects are investigated for random fields defined on the d-dimensional sphere $\mathbb{S}^d$, and reconstructed from discrete samples. First, we introduce the concept of an aliasing function on $\mathbb{S}^d$. The aliasing function allows to identify explicitly the aliases of a given harmonic coefficient in the Fourier decomposition. Then, we exploit this tool to establish the aliases of the harmonic coefficients approximated by means of the quadrature procedure named spherical uniform sampling. Subsequently, we study the consequences of the aliasing errors in the approximation of the angular power spectrum of an isotropic random field, the harmonic decomposition of its covariance function. Finally, we show that band-limited random fields are aliases-free, under the assumption of a sufficiently large amount of nodes in the quadrature rule.

1. Introduction

1.1. **Motivations.** We are concerned with the study of the aliasing effects for the harmonic expansion of a random field defined on the d-dimensional sphere $\mathbb{S}^d$. The analysis of spherical random fields over $\mathbb{S}^d$ is strongly motivated by a growing set of applications in several scientific disciplines, such as Cosmology and Astrophysics for $d = 2$ (see, for example, [BM07, MP10]), as well as in Medical Image Analysis ([HCW+13, HCK+15]), Material Physics ([MS08]), and Nuclear Physics ([AA18]) for $d > 2$. For example, in Medical Image Analysis the statistical representation of the shape of a brain region is commonly modelled as the realization of a Gaussian random field, defined across the entire surface of the region (see for example [BSX+07]). Many shape modelling frameworks in computational anatomy apply shape parametrization techniques for cortical structures based on the spherical harmonic representation, to encode global shape features into a small number of coefficients (see [HCW+13]). This data reduction technique, however, can not provide a proper representation with a single parametrization of multiple disconnected subcortical structures, specifically the left and right hippocampus and amygdala. The so-called 4D-hyperspherical harmonic representation of surface anatomy aims to solve this issue by means of a stereographic projection of an entire collection of disjoint 3-dimensional objects onto the hypersphere of dimension 4. Indeed, a stereographic projection embeds a 3-dimensional volume onto the surface of a 4-dimensional hypersphere, avoiding thus, the issues related to flatten 3-dimensional surfaces to the 3-dimensional sphere. Subsequently, any disconnected objects of dimension 3 can be projected onto a connected surface in $\mathbb{S}^4$, and, thus, represented as the linear combination of hyperspherical harmonics of dimension 4 (see [HCK+15]).

A spherical random field T is a stochastic process defined over the unit sphere $\mathbb{S}^d$ and thus depending on the location $x = (\boldsymbol{\vartheta}, \varphi) = \left(\vartheta^{(1)}, \dots, \vartheta^{(d-1)}, \varphi\right) \in \mathbb{S}^d$, where $\vartheta^{(i)} \in [0, \pi\ )$, for $i = 1, \dots, d-1$, and $\varphi \in [0, 2\pi]$. The harmonic analysis has been proved to be an insightful tool to study several issues related to the random fields on the sphere and the development of spherical random fields in a series of spherical harmonics has many uses in several branches of probability and statistics. We are referring, for example, to the study of the asymptotic behaviour of the bispectrum of spherical random fields (see [Mar06]), their Euler-Poincaré characteristic (see [CM18]), the estimation of their spectral parameters ([DLM14]), and the development

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of quantitative central limit theorems for nonlinear functional of corresponding random eigenfunctions (see [MR15]). Under some integrability conditions (see Section 2.2), the following harmonic expansion holds:

$$T(\boldsymbol{\vartheta}, \varphi) = \sum_{\ell, \mathbf{m}} a_{\ell,\mathbf{m}} Y_{\ell,\mathbf{m}}(\boldsymbol{\vartheta}, \varphi),$$

where $\ell \in \mathbb{N}$ and $\mathbf{m} = (m_1, \ldots, m_{d-1}) \in \mathbb{N}^{d-2} \otimes \mathbb{Z}$ are the harmonic (or wave) numbers.
The set of spherical harmonics $Y_{\ell,\mathbf{m}} = Y_{\ell,m_1,\ldots,m_{d-1}} : \mathbb{S}^d \to \mathbb{C}$ provides an orthonormal basis for the space $L^2(\mathbb{S}^d) = L^2(\mathbb{S}^d, \mathrm{d}x)$, where $\mathrm{d}x$ is the uniform Lebesgue measure over $\mathbb{S}^d$ (see Section 2.1). The harmonic coefficients $a_{\ell,\mathbf{m}} = a_{\ell,m_1,\ldots,m_{d-1}}$, given by

$$a_{\ell,\mathbf{m}} = \langle T, Y_{\ell,\mathbf{m}} \rangle_{L^2(\mathbb{S}^d)} = \int_{\mathbb{S}^d} T(x) \overline{Y_{\ell,\mathbf{m}}}(x) \, \mathrm{d}x, \tag{1}$$

contain all the stochastic information of $T(\boldsymbol{\vartheta}, \varphi)$.
Nevertheless, the explicit computation of the integral (1) is an unachievable target in many experimental situations. Indeed, the measurements of $T(\boldsymbol{\vartheta}, \varphi)$ can be in practise collected only over a finite sample of locations $\{x_i : i = 1 \ldots N\}$. As a consequence, for any choice of ℓ and $\mathbf{m}$ the integral producing the harmonic coefficient $a_{\ell,\mathbf{m}}$ is approximated by the sum of finitely many elements $T(x_i)$, $i = 1 \ldots, n$, the samples of the random field. This discretization produces aliasing errors, that is, different coefficients become indistinguishable - aliases - of one another. The set of coefficients, acting as aliases each other, depends specifically on the chosen sampling procedure.

The concept of aliasing comes from signal processing and related disciplines. In general, aliasing makes different signals to become indistinguishable when sampled, and it can be produced when the reconstruction of the signal from samples is different from the original continuous one (see, for example, [PM96, Chapter 1]).
The aliasing phenomenon arising in the harmonic expansion of a 2-dimensional spherical random field has been investigated by [LN97]. On the one hand, it is there proved that band-limited random fields over $\mathbb{S}^2$, which can be roughly viewed as linear combinations of finitely many spherical harmonics, can be uniquely reconstructed with a sufficiently large sample size. On the other, an explicit definition of the aliasing function, a crucial tool to identify the aliases of a given harmonic coefficient, is developed when the sampling is based on the combination of a Gauss-Legendre quadrature formula and a trapezoidal rule (see Section 4 for further details). In many practical applications, this sampling procedure is the most convenient scheme to perform numerical analysis over the sphere (see, for example, [AH12, SB93, Sze75]). Further reasons of interest to study the aliasing effects in $\mathbb{S}^2$ have arisen in the field of optimal design of experiments. In [DMP05], designs over $\mathbb{S}^2$ based on this sampling scheme have been proved to be optimal with respect to the whole set of Kiefer's Φ_p-criteria, presented in [Kie74], that is, they are the most efficient among all the approximate designs for regression problems with spherical predictors.

Recently, interest has occurred in regression problems in spherical frameworks of arbitrary dimension and the related discretization problems (see, for example, [LS15]). In particular, in [DKSG18], the experimental designs, obtained by the discretization of the uniform distribution over $\mathbb{S}^d$ by means of the combination of the so-called Gegenbauer-Gauss quadrature rules (see Section 3.2 for further details) and a trapezoidal rule, have been proved to be optimal with respect not only to the aforementioned Kiefer's Φ_p-criteria, but also to another class of orthogonally invariant information criteria, the Φ_{E_s}-criteria. Given the improved interest for spheres of dimension larger than 2, it is therefore pivotal to carry out further investigations into the aliasing effects for random fields sampled over $\mathbb{S}^d$, $d > 2$. On the one hand, this research improves the understanding of the behaviour of the approximated harmonic coefficients when computed over discrete samplings, in particular over a spherical uniform sampling (see Section 3.3). On the other hand, our investigations make large use of the properties of the hyperspherical harmonics, providing thus a deeper insight on their structure, carrying on with the results presented in [DKSG18].

We work under the following assumption: a spherical random field T is observed over the a finite set of locations $\{x_i = (\boldsymbol{\vartheta}_i, \varphi_i) : i = 1, \ldots, N\}$, the so-called sampling points. Thus, for any set of harmonic numbers

ℓ and $\mathbf{m}$, the approximated - or aliased - harmonic coefficient is given by

$$\tilde{a}_{\ell,\mathbf{m}} = \sum_{\ell',\mathbf{m}'} \tau\left(\ell,\mathbf{m};\ell',\mathbf{m}'\right) a_{\ell',\mathbf{m}'},$$

where $\tau\left(\ell,\mathbf{m};\ell',\mathbf{m}'\right)$, defined in Section 4.1 by (29), is the aforementioned aliasing function. The coefficient $a_{\ell',\mathbf{m}'}$ is said to be an alias of $a_{\ell,\mathbf{m}}$ with intensity $\left|\tau\left(\ell,\mathbf{m};\ell',\mathbf{m}'\right)\right|$ if $\tau\left(\ell,\mathbf{m};\ell',\mathbf{m}'\right)\neq 0$.

First, we study the general structure of the aliasing function under the very mild assumption that the sampling is separable with respect to the angular coordinates, that is, the sampling points $\{x_i : i = 1,\dots,N\}$ can be written as follows

$$\left\{\left(\vartheta^{(1)}_{k_0},\dots,\vartheta^{(d-1)}_{k_{d-2}},\varphi_{k_{d-1}}\right) : k_{j-1} = 0,\dots,Q_{j-1}-1 \quad \text{for } j=1,\dots,d\right\},$$

where $Q_0, Q_1, \dots, Q_{d-1} \in \mathbb{N}$ are defined so that $\prod_{j=0}^{d-1} Q_j = N$ (see Section 3.1). Then, we investigate on the explicit structure of such a function and, consequently, on the identification of aliases assuming a spherical uniform design as the sampling procedure.

Second, under the assumption of isotropy, we consider the aliasing effects for the angular power spectrum of a random field, which describes the decomposition of the covariance function in terms of the frequency $\ell \geq 0$ (see Section 2.2), providing information on the dependence structure of the random field.

Third, we investigate also on the aliasing effects for band-limited random fields. More specifically, we establish suitable conditions on the sample size in order to guarantee the annihilation of the aliasing phenomenon.

1.2. **Plan of the paper.** This paper is structured as follows. In Section 2, we introduce some fundamental background results on the harmonic analysis over the d-dimensional sphere as well as a short review on spherical random fields. Section 3 includes also a short overview on the so-called Gegenbauer-Gauss quadrature formula, crucial to build a spherical uniform sampling, and provides some auxiliary results. In Section 4, we present the main findings of this work. In particular, Theorem 4.1 describes the construction of the aliasing function $\tau\left(\ell,\mathbf{m};\ell',\mathbf{m}'\right)$ under the assumption of the separability of the sampling with respect to the angular components, while Theorem 4.3 identifies the aliases for any harmonic coefficient $a_{\ell,\mathbf{m}}$ when the sampling is uniform. In Section 5, we study the aliasing effects for the angular power spectrum of an isotropic random field (see Theorem 5.1), while in Section 6 we provide an algorithm to remove the aliasing effects for a band-limited random field sampled over a spherical uniform design (see Theorem 6.1). Finally, Section 7 collects all the proofs.

2. Preliminaries

This section collects some introductory results, concerning harmonic analysis and its application to spherical random fields. It also includes a quick overview on the Gegenbauer-Gauss formula. The reader is referred to [SW71, AH12, VK91] for further details about the harmonic analysis on the sphere, to [AT07] for a detailed description of random fields and their properties, while [MP11] provides an extended description of spherical random fields over $\mathbb{S}^2$. Further details concerning the Gegenbauer-Gauss quadrature rule can be found in [AS64, AH12, SB93, Sze75].

2.1. **Harmonic analysis on the sphere.** Let $\vartheta^{(i)} \in [0,\pi]$, for $i = 1,\dots,d-1$, and $\varphi \in [0, 2\pi)$ be the spherical polar coordinates over $\mathbb{S}^d$. Since now on, we will denote by $x = (\boldsymbol{\vartheta},\varphi) = \left(\vartheta^{(1)},\dots,\vartheta^{(d-1)},\varphi\right)$ the generic spherical coordinate, that is, the direction of a point on $\mathbb{S}^d$. Let the function $f : [0,\pi]^{d-1} \to [-1,1]$ be defined by

$$f\left(\boldsymbol{\vartheta}\right) = f\left(\vartheta^{(1)},\dots,\vartheta^{(d-1)}\right) = \prod_{j=1}^{d-1}\left(\sin\vartheta^{(j)}\right)^{d-j}. \tag{2}$$

Thus, the uniform Lebesgue measure $\mathrm{d}x$ over $\mathbb{S}^d$, namely, the element of the solid angle, is defined by

$$\begin{aligned}\mathrm{d}x =& \left(\sin\vartheta^{(1)}\right)^{d-1}\mathrm{d}\vartheta^{(1)}\left(\sin\vartheta^{(2)}\right)^{d-2}\mathrm{d}\vartheta^{(2)}\dots\sin\vartheta^{(d-1)}\,\mathrm{d}\vartheta^{(d-1)}\,\mathrm{d}\varphi\\ =& f\left(\vartheta^{(1)},\dots,\vartheta^{(d-1)}\right)\mathrm{d}\vartheta^{(1)}\dots\mathrm{d}\vartheta^{(d-1)}\,\mathrm{d}\varphi,\end{aligned}$$

such that the surface area of the hypersphere corresponds to

$$\int_{\mathbb{S}^d}\mathrm{d}x = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}.$$

Let us denote by $\mathcal{H}_\ell$ the restriction of the space of harmonic homogeneous polynomials of order ℓ to $\mathbb{S}^d$. As well-known in the literature (see, for example, [AH12, SW71]), the space of square-integrable functions over $\mathbb{S}^d$ can be described as the direct sum of the spaces $\mathcal{H}_\ell$, that is,

$$L^2\left(\mathbb{S}^d\right) = \bigoplus_{\ell\geq 0}\mathcal{H}_\ell.$$

For any integer $\ell \geq 0$, since now on called frequency, we define the following set

(3)
$$\mathcal{M}_\ell = \left\{\mathbf{m}\in\mathbb{Z}^{d-1} : m_1 = 0,\dots,\ell; m_2 = 0,\dots,m_1;\dots;m_{d-2} = 0,\dots,m_{d-3}; m_{d-1} = -m_{d-2},\dots,m_{d-2}\right\}.$$

Following [AW82, AH12, VK91], for any $\ell \geq 0$, it holds that

$$\mathcal{H}_\ell = \mathrm{Span}\{Y_{\ell,\mathbf{m}} : \mathbf{m}\in\mathcal{M}_\ell\},$$

where, for $x \in \mathbb{S}^d$, $Y_{\ell,\mathbf{m}} = Y_{\ell,m_1,\dots,m_{d-1}} : \mathbb{S}^d \to \mathbb{C}$ denotes the so-called spherical - or hyperspherical - harmonic of degree ℓ and order $\mathbf{m}$. In other words, fixed $\ell \geq 0$, $\mathcal{M}_\ell$ appoints the finitely many vectors $\mathbf{m}$ which identify the spherical harmonics spanning the space $\mathcal{H}_\ell$.

Another common approach to introduce spherical harmonics exploits the so-called d-spherical Laplace-Beltrami operator $\Delta_{\mathbb{S}^d}$ (see, for example, [MP11]). Fixed $\ell \geq 0$, the spherical harmonics $Y_{\ell,\mathbf{m}}(x)$ corresponding to any $m \in \mathcal{M}_\ell$ are the eigenfunctions of $\Delta_{\mathbb{S}^d}$ with eigenvalue $\varepsilon_{\ell;d} = \ell(\ell + d - 1)$, that is,

$$\left(\Delta_{\mathbb{S}^d} + \varepsilon_{\ell;d}\right)Y_{\ell,\mathbf{m}}(x) = 0, \text{ for } x\in\mathbb{S}^d.$$

As proved for example in [AW82], for any $\ell \geq 0$, the size of $\{Y_{\ell,\mathbf{m}} : \mathbf{m}\in\mathcal{M}_\ell\}$, namely, the multiplicity of the set of spherical harmonics with eigenvalue $\varepsilon_{\ell;d}$, is given by

$$\Xi_d(\ell) = \frac{(2\ell + d - 1)(\ell + d - 2)!}{\ell!\,(d-1)!}. \tag{4}$$

The set $\{Y_{\ell,\mathbf{m}}(x) : \ell \geq 0; \mathbf{m}\in\mathcal{M}_\ell\}$ provides therefore an orthonormal basis for $L^2\left(\mathbb{S}^d\right)$. For any $g \in L^2\left(\mathbb{S}^d\right)$, the following Fourier - or harmonic - expansion holds

$$g(x) = \sum_{\ell\geq 0}\sum_{\mathbf{m}\in\mathcal{M}_\ell} a_{\ell,\mathbf{m}}Y_{\ell,\mathbf{m}}(x), \text{ for } x\in\mathbb{S}^d,$$

where $\{a_{\ell,\mathbf{m}} : \ell \geq 0; \mathbf{m}\in\mathcal{M}_\ell\}$ are the so-called harmonic coefficients, given by the integral

$$a_{\ell,\mathbf{m}} = \langle g, Y_{\ell,\mathbf{m}}\rangle_{L^2(\mathbb{S}^d)} = \int_{\mathbb{S}^d} g(x)\,\bar{Y}_{\ell,\mathbf{m}}(x)\,\mathrm{d}x.$$

Since now on, for the sake of notational simplicity, we fix $m_0 = \ell$. Furthermore, we will use indifferently the two equivalent short and long notations $Y_{\ell,\mathbf{m}}(x)$ and $Y_{\ell,m_1,\dots,m_{d-1}}\left(\vartheta^{(1)},\dots,\vartheta^{(d-1)},\varphi\right)$. Following [AW82], the hyperspherical harmonics are defined by

$$Y_{\ell,\mathbf{m}}(x) = \frac{1}{\sqrt{2\pi}}\prod_{j=1}^{d-1}\left(h_{m_{j-1},m_j;j}C_{m_{j-1}-m_j}^{\left(m_j+\frac{d-j}{2}\right)}\left(\cos\vartheta^{(j)}\right)\left(\sin\vartheta^{(j)}\right)^{m_j}\right)e^{im_{d-1}\varphi}, \tag{5}$$

where $h_{m_{k-1},m_k;k}$ is a normalizing constant, given by

$$h_{m_{j-1},m_j;j} = \left(\frac{2^{2m_j+d-j-2} (m_{j-1}-m_j)! \, (2m_{j-1}+d-j) \, \Gamma^2 \left(m_j + \frac{d-j}{2} \right)}{\pi \, (m_{j-1}+m_j+d-j-1)!} \right)^{\frac{1}{2}} . \tag{6}$$

The function $C_n^{(\alpha)} : [-1,1] \to \mathbb{R}$, $\alpha \in [-1/2, \infty) \setminus \{0\}$, is the Gegenbauer (or ultraspherical) polynomial of degree n and parameter α. Following for example [AS64, Sze75], they are orthogonal with respect to the measure

$$\nu_\alpha (t) = \left(1-t^2\right)^{\alpha-\frac{1}{2}} \mathbb{1}_{[-1,1]} (t) ,$$

that is,

$$\int_{-1}^{1} C_n^{(\alpha)} (t) \, C_{n'}^{(\alpha)} (t) \, \nu_\alpha (t) \, \mathrm{d}t = \frac{\pi 2^{1-2\alpha} \Gamma (n+2\alpha)}{n! \, (n+\alpha) \, \Gamma^2 (\alpha)} \delta_n^{n'} , \tag{7}$$

see, for example, [Sze75, Formula 4.7.15].

Roughly speaking, each hyperspherical harmonic in (5) can be viewed as product of a complex exponential function and a set of Gegenbauer polynomials, whose orders and parameters are properly nested and normalized to guarantee orthonormality, that is,

$$\int_{\mathbb{S}^d} Y_{\ell,\mathbf{m}} (x) \, \bar{Y}_{\ell',\mathbf{m}'} (x) \, \mathrm{d}x = \delta_\ell^{\ell'} \prod_{k=1}^{d-1} \delta_{m_k}^{m'_k} .$$

Hyperspherical harmonics feature also the following property, known as addition formula (see, for example, [AW82]):

$$\sum_{\mathbf{m}\in\mathcal{M}_\ell} Y_{\ell,\mathbf{m}} (x) \, \bar{Y}_{\ell',\mathbf{m}'} (x') = \frac{(2\ell+d-1) \, \Gamma\left(\frac{d+1}{2}\right) (\ell+d-2)!}{2\pi^{\frac{d+1}{2}} (d-1)! \ell!} C_\ell^{\left(\frac{d-1}{2}\right)} (\langle x, x' \rangle) =: K_\ell (x, x') , \tag{8}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $L^2\left(\mathbb{R}^{d+1}\right)$. Note that K_ℓ can be viewed as the kernel of the projector over the harmonic space $\mathcal{H}_\ell$, the restriction to the sphere of the space of homogeneous and harmonic polynomials of order ℓ. The projection $\mathcal{P}_\ell$ of $g \in L^2\left(\mathbb{S}^d\right)$ onto $\mathcal{H}_\ell$ is given by

$$\mathcal{P}_\ell [g] (x) = \int_{\mathbb{S}^d} g (y) \, K_\ell (x,y) \, \mathrm{d}y, \quad x \in \mathbb{S}^d .$$

It follows that

$$\mathcal{P}_\ell [g] (x) = \sum_{\mathbf{m}\in\mathcal{M}_\ell} a_{\ell,\mathbf{m}} Y_{\ell,\mathbf{m}} (x) , \quad \text{for } x \in \mathbb{S}^d ,$$

and that any function $g \in L^2\left(\mathbb{S}^d\right)$ can be rewritten as the sum of projections over the spaces $\mathcal{H}_\ell$,

$$g (x) = \sum_{\ell\geq 0} \mathcal{P}_\ell [g] (x) , \text{ for } x \in \mathbb{S}^d .$$

2.2. **Spherical random fields.** Given a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$, a spherical random field $T_\omega (x)$, $\omega \in \Omega$ and $x \in \mathbb{S}^d$, describes a stochastic process defined the sphere $\mathbb{S}^d$. Since now on, the dependence on $\omega \in \Omega$ will be omitted and the random field will be denoted by $T (x)$, $x \in \mathbb{S}^d$, for the sake of the simplicity (see also [AT07]).

If T has a finite second moment, that is, $\mathbb{E}\left[|T (x)|^2\right] < \infty$ for all $x \in \mathbb{S}^d$, a spherical random field can be decomposed in terms of the projections over the space $\mathcal{H}_\ell$, $\ell \geq 0$, so that

$$T (x) = \sum_{\ell\geq 0} T_\ell (x) , \quad x \in \mathbb{S}^d , \tag{9}$$

where $T_\ell (x) = \mathcal{P}_\ell [T] (x)$. Each projector onto $\mathcal{H}_\ell$ can be described as a linear combination of finitely many hyperspherical harmonics,

$$T_\ell (x) = \sum_{\mathbf{m}\in\mathcal{M}_\ell} a_{\ell,\mathbf{m}} Y_{\ell,\mathbf{m}} (x) , \quad x \in \mathbb{S}^d . \tag{10}$$

As in the deterministic case described in Section 2.1, for any $\ell \geq 0$ and $\mathbf{m} \in \mathcal{M}_\ell$, the random harmonic coefficient is defined by

$$a_{\ell,\mathbf{m}} = \int_{\mathbb{S}^d} T(x)\, \bar{Y}_{\ell,\mathbf{m}}(x)\, \mathrm{d}x. \tag{11}$$

The random harmonic coefficients contain all the stochastic information of the random field T, namely, $a_{\ell,\mathbf{m}} = a_{\ell,\mathbf{m}}(\omega)$, for $\omega \in \Omega$, $\ell \geq 0$ and $\mathbf{m} \in \mathcal{M}_\ell$.

A random field is said to be band-limited if there exists a bandwidth $L_0 \in \mathbb{N}$, so that $a_{\ell,\mathbf{m}} = 0$ for any $\ell > L_0$, whenever $m \in \mathcal{M}_\ell$. In this case, it holds that

$$T(x) = \sum_{\ell=0}^{L_0} \sum_{\mathbf{m}\in\mathcal{M}_\ell} a_{\ell,\mathbf{m}} Y_{\ell,\mathbf{m}}(x), \quad x \in \mathbb{S}^d. \tag{12}$$

By the practical point of view, band-limited random fields provide a useful approximation of fields with harmonic coefficients decaying fast enough as the frequency ℓ grows.

Let us define the expectation $\mu(x) = \mathbb{E}[T(x)]$; the covariance function $\Gamma : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ of the random field T is given by

$$\Gamma(x, x') = \mathbb{E}\left[(T(x) - \mu(x))\left(\bar{T}(x') - \bar{\mu}(x')\right)\right], \tag{13}$$

where, for $z \in \mathbb{C}$, $\bar{z}$ denotes its complex conjugate. Without losing any generality, assume that T is centered, so that, for $x, x' \in \mathbb{S}^d$, it holds that

$$\mu(x) = 0$$
$$\Gamma(x; x') = \mathbb{E}\left[T(x)\, \bar{T}(x')\right].$$

Let $\gamma : \mathbb{S}^d \times \mathbb{S}^d \to [0, \pi], \gamma(x, x') = \arccos\langle x, x'\rangle_{\mathbb{R}^{d+1}}$ be the geodesic distance between $x, x' \in \mathbb{S}^d$. A spherical random field is said to be isotropic if it is invariant in distribution with respect to rotations of the coordinate system or, more precisely,

$$T(x) \overset{\mathrm{d}}{=} T(Rx), \text{ for } x \in \mathbb{S}^d, R \in SO(d+1),$$

where $\overset{\mathrm{d}}{=}$ denotes equality in distribution, and $SO(d+1)$ is the so-called special group of rotations in $\mathbb{R}^{d+1}$. Following [BKMP09, BM07, MP11], if the random field is isotropic, then Γ depends only on γ and its variance $\sigma^2(x) = \Gamma(x, x)$ does not depend on the location $x \in \mathbb{S}^d$, so that it holds that

$$\sigma^2(x) = \mathbb{E}\left[|T(x)|^2\right] = \sigma^2, \quad \text{for all } x \in \mathbb{S}^d,$$

where $\sigma^2 \in \mathbb{R}^+$. The covariance function itself can be therefore rewritten in terms of its dependence on the distance between x and x', so that

$$\Gamma(x, x') = \Gamma(\gamma(x, x')).$$

Let us finally define the correlation function $\rho : [-1, 1] \to [-1, 1]$, which is invariant with respect to rotations when the random field is isotropic, that is

$$\rho(\cos\gamma(x, x')) = \frac{\Gamma(x, x')}{\sqrt{\Gamma(x, x)\, \Gamma(x', x')}} = \frac{\Gamma(\gamma(x, x'))}{\sigma^2}, \quad x, x' = \mathbb{S}^d \tag{14}$$

As far as the random harmonic coefficients $\{a_{\ell,\mathbf{m}} : \ell \geq 0, \mathbf{m} \in \mathcal{M}_\ell\}$ are concerned, since $\mu(x) = 0$ for $x \in \mathbb{S}^d$, we have that $\mathbb{E}[a_{\ell,\mathbf{m}}] = 0$. Furthermore, the spectral representation of the covariance function yields

$$\mathrm{Cov}(a_{\ell,\mathbf{m}}, a_{\ell',\mathbf{m}'}) = \mathbb{E}[a_{\ell,\mathbf{m}} \bar{a}_{\ell',\mathbf{m}'}] = C_\ell \delta_\ell^{\ell'} \prod_{k=1}^{d-1} \delta_{m_k}^{m'_k}, \tag{15}$$

where $\{C_\ell : \ell \geq 0\}$ is the so-called angular power spectrum of T. The angular power spectrum of a random field can be viewed as the harmonic decomposition of its covariance function and can be rewritten as the average

$$C_\ell = \frac{1}{\Xi_d(\ell)} \sum_{\mathbf{m}\in\mathcal{M}_\ell} \mathrm{Var}(a_{\ell,\mathbf{m}}), \tag{16}$$

where $\Xi_d(\ell)$ is given by (4), see, for example, [Mar06] for $d = 2$.
The Fourier expansion of T can be read as a decomposition of the field into a sequence of uncorrelated random variables, preserving its spectral characteristics. Combining (8), (13) and (15) yields

$$\Gamma(x, x') = \sum_{\ell \geq 0} C_\ell K_\ell(x, x'),$$

where we rewrite the covariance function in terms of the projection kernel corresponding to the frequency level ℓ.

3. The Gauss-Gegenbauer quadrature formula and the spherical uniform design

This section includes a quick overview on the Gegenbauer-Gauss formula. We also introduce the spherical uniform sampling and two related auxiliary results. Further details concerning the Gegenbauer-Gauss quadrature rule can be found in [AS64, AH12, SB93, Sze75], while the spherical uniform sampling is presented by [DKSG18].

3.1. Separability of the sampling.

Separability of the sampling. We first introduce a very mild condition on the sampling procedure. Generalizing the proposal introduced by [LN97] on $\mathbb{S}^2$ to $\mathbb{S}^d$, $d > 2$, here we consider a discretization scheme produced by the combination of d one-dimensional quadrature rules, with respect to the coordinates $\vartheta^{(j)}$, $j = 1, \dots, d-1$, and φ.

More specifically, we introduce the following condition on the sampling points and weights.

Condition 3.1 (Separability of the sampling scheme). Fix $Q_0, Q_1, \dots, Q_{d-1} \in \mathbb{N}$, so that $N = \prod_{j=0}^{d-1} Q_j$. For any $j = 1, \dots, d$, there exists a finite sequence of positive real-valued weights

$$\left\{ w^{(j)}_{k_{j-1}} : k_{j-1} = 0, \dots, Q_{j-1} - 1 \right\}, \tag{17}$$

so that

$$\sum_{k_{j-1}=0}^{Q_{j-1}-1} w^{(j)}_{k_{j-1}} = 1.$$

The sampling points $\{x_i : i = 1, \dots, N\}$ are component-wise given by

$$\left\{ \left(\vartheta^{(1)}_{k_0}, \dots, \vartheta^{(d-1)}_{k_{d-2}}, \varphi_{k_{d-1}} \right) : k_{j-1} = 0, \dots, Q_{j-1} - 1 \quad \text{for } j = 1, \dots, d \right\}. \tag{18}$$

Roughly speaking, each sequence in (17) corresponds to the set of weights for a quadrature formula with respect to the j-th angular component of the angle vector $x = \left(\vartheta^{(1)}, \dots, \vartheta^{(d-1)}, \varphi\right)$. The subscript index is related to the harmonic numbers $\ell = m_0, m_1, \dots, m_{d-1}$.

Each value of the index $i^* \in \{1, \dots, N\}$ corresponds uniquely to a suitable choice of values $\left\{k^*_0, \dots, k^*_{d-1}\right\}$, while the related weight w_{i^*} is given by

$$w_{i^*} = \prod_{j=1}^{d} w^{(j)}_{k^*_{j-1}}.$$

3.2. The Gauss-Gegenbauer quadrature formula.

The Gauss-Gegenbauer quadrature formula. In general, a quadrature rule denotes an approximation of a definite integral of a function by means of a weighted sum of function values, estimated at specified points within the domain of integration (see, for example, [SB93]). In particular, a r-point Gaussian quadrature rule is a formula specifically built to yield an exact result for polynomials of degree smaller or equal to $2r - 1$, after a suitable choice of the points and weights $\{t_k, \omega_k : k = 0, \dots, r-1\}$. For this reason, it is also called quadrature formula of degree $2r - 1$. The domain of integration is conventionally taken as $[-1, 1]$, and the choice of points and weights usually depends on the so-called weight function a, whereas the integral can be written in the form $\int_{-1}^{1} p(t)\, a(t)\, \mathrm{d}t$. Here $p(t)$ is approximately polynomial, and $a(t) \in L^1([-1,1])$ is a well-known function. In this case, a proper selection of $\{t_k, \omega_k : k = 0, \dots, r-1\}$ yields

$$\int_{-1}^{1} p(t)\, a(t)\, \mathrm{d}t = \sum_{k=0}^{r-1} \omega_k p(t_k).$$

Following for example [SB93], it can be shown that the quadrature points can be chosen as the roots of some polynomial belonging to some suitable class of orthogonal polynomials, depending on the function a. When $a(t) = 1$ for all $t \in [-1,1]$, the associated polynomials are the Legendre polynomials. In this case, the method is then known as Gauss-Legendre quadrature (see [AS64, Formula 25.4.29]). Such a method is widely used in the 2-dimensional spherical framework (see, for example, [AH12]), and the aliases produced by this formula were largely investigated in [LN97]).
More in general, as stated in [AS64, Formula 25.4.33], when $a(t) = a_{\alpha,\beta}(t) = (1-t)^{\alpha}(1+t)^{\beta}$, the method is known as the Gauss-Jacobi quadrature formula, since it makes use of the Jacobi polynomials (see also [Sze75, p.47]). Since it is well-known that Jacobi polynomials reduce to Gegenbauer polynomials when $\alpha = \beta$ (see, for example, [Sze75, Formula 4.1.5]), we refer to the quadrature rule denoted by a weight function $\nu_\alpha(t)$ (equal to $a_{\alpha,\beta}(t)$ for $\alpha = \beta$) as the Gauss-Gegenbauer quadrature (see, for example, [ESM14]).

Subsequently, the discrete uniform sampling over the sphere is obtained by combining a trapezoidal rule for the angle φ and $(d-1)$ Gauss-Gegenbauer quadrature rules for the coordinates $\vartheta^{(j)}$, for $j = 1, \dots, d-1$, with weight function $a_j(t) = \nu_{\alpha(j)}(t)$, $\alpha(j) = d-1-j$.
This method has been described in details by [DKSG18, Lemma 3.1] in the framework of optimal design for regression problems with spherical predictors. Indeed, by the theoretical point of view, the (continuous) uniform distribution on the sphere provides an optimal design for experiments on the unit sphere, but this distribution is not implementable as a design in real experiments (for more details, see [DKSG18, Theorem 3.1]). Thus, a set of equivalent discrete designs is established by means of the combination of the following quadrature formulas over the sphere, written as in [DKSG18, Lemma 3.1]), to which we refer to for a proof.

Definition 3.2 (Gauss-Gegenbauer quadrature)**.** Let $a \in L^1([-1,1])$ be a positive weight function so that $\bar{a} = \int_{-1}^{1} a(t)\, \mathrm{d}t$. Consider also the set of $r \in \mathbb{N}$ points $-1 \leq t_0 < \dots < t_{r-1} \leq 1$, associated to the positive weights $\omega_0, \dots, \omega_{r-1}$ such that $\sum_{k=0}^{r-1} \omega_k = 1$. Then the set of points and weights$\{t_k, \omega_k : k = 0, \dots, r-1\}$ generates a quadrature formula of degree $z \geq r$, namely,

$$\int_{-1}^{1} a(t)\, t^p\, \mathrm{d}t = \bar{a} \sum_{k=0}^{r-1} \omega_k t_k^p, \quad \text{for } p = 0, \dots, z, \tag{19}$$

if and only if the following conditions are satisfied:

(1) The polynomial $\prod_{k=0}^{r-1}(t - t_k)$ is orthogonal to all polynomials of degree smaller or equal to $z - r$ with respect to $a(t)$,

$$\int_{-1}^{1} \prod_{k=0}^{r-1} (t - t_k)\, a(t)\, t^p\, \mathrm{d}t = 0, \quad \text{for } p = 0, \dots, z - r;$$

(2) the weights ω_k are given by

$$\omega_k = \frac{1}{\bar{a}} \int_{-1}^{1} a(t)\, \lambda_k(t)\, \mathrm{d}t, \quad \text{for } k = 0, \dots, r-1, \tag{20}$$

where $\lambda_k(t)$ is the k-th Lagrange interpolation formula with nodes $t_0, \dots, t_{r-1}$, given by

$$\lambda_k(t) = \prod_{i=0, i \neq k}^{r-1} \frac{t - t_i}{t_i - t_k}.$$

3.3. **The spherical uniform sampling.** Assume now $z = 2Q_0$ in Definition 3.2. Following [Sze75, Formula 4.7.15] (see also (7)), the Gegenbauer polynomials $C_n^{(\alpha)}$ are orthogonal with respect to $a(t) = \nu_\alpha(t)$. Fixed n, the real-valued n roots of $C_n^{(\alpha)}$ have multiplicity 1 and are located in the interval $[-1, 1]$. Thus, it follows that for any $r \in \{Q_0 + 1, \dots, 2Q_0\}$, there exists at least one set of points and weights $\left\{t_k^{(j)}, \omega_k^{(j)} : k = 0, \dots, r-1\right\}$, $j = 1 \dots, d-1$, generating a quadrature formula (19) with $a(t) = a_j(t) = \nu_{\alpha(j)}(t)$, and $\alpha(j) = d - 1 - j$.

The following Condition exploits properly these quadrature formulas for $\boldsymbol{\vartheta}$, combined with a trapezoidal rule for φ, to establish a well-defined uniform distribution over the sphere of arbitrary dimension d (see also, for example, [AH12, DKSG18]).

Condition 3.3 (Spherical uniform sampling). Assume that Condition 3.1 holds and fix $M \in \mathbb{N}$ so that $Q_{d-1} = 2M$. The sampling with respect to φ is uniform, so that for any $k_{d-1} = 0, \dots, 2M-1$, it holds that

$$\varphi_{k_{d-1}} = \frac{k_{d-1}\pi}{M}; \tag{21}$$

$$w_{k_{d-1}}^{(d)} = \frac{\pi}{M}. \tag{22}$$

The sampling with respect to each component $\vartheta^{(j)}$, $j = 1, \dots, d-1$ has the form

$$\vartheta_{k_{j-1}}^{(j)} = \arccos\left(t_{k_{j-1}}^{(j)}\right); \tag{23}$$

$$w_{k_{j-1}}^{(j)} = \frac{\omega_{k_{j-1}}^{(j)}}{\left(\sin \vartheta_{k_{j-1}}^{(j)}\right)^{d-j}}, \tag{24}$$

where, for any $j = 1, \dots, d-1$, $\left\{t_{k_{j-1}} : k_{j-1} = 0, \dots, Q_{j-1} - 1\right\}$ in (23) are the zeros of $C_{Q_{j-1}}^{\left(\frac{d-j}{2}\right)}$, while $\left\{\omega_{k_{j-1}} : k_{j-1} = 0, \dots, Q_{j-1} - 1\right\}$ in (24) are the corresponding weights in the Gauss-Gegenbauer framework, given by (20) in Definition (3.2).

We present now two auxiliary results crucial to prove Theorem 4.3, referring to the aliasing effects under Condition 3.3. Their proofs can be found in Section 7.2

The first Lemma establishes the parity properties of the cubature points and weights for each angular component $\vartheta^{(j)}$ with respect to $\vartheta^{(j)} = \pi/2$, for $j = 1, \dots, d-1$. Indeed, due to the parity formula $C_r^{(\alpha)}(-t) = (-1)^r C_r^{(\alpha)}(t)$ (see [Sze75, Formula 4.7.4]), the roots of $C_r^{(\alpha)}(t)$, $t_1, \dots, t_r$, are symmetric with respect to 0, namely, $t_k = -t_{r-k-1}$ for $k = 0, \dots, [r/2]$. As a consequence, the following lemma holds.

Lemma 3.4. *Let the cubature points and weights be given by (23) and (24) respectively in the framework described by Definition 3.2. Hence, for any $j = 1, \dots, d-1$, it holds that*

$$\vartheta_{k_{j-1}}^{(j)} = \pi - \vartheta_{Q_{j-1}-k_{j-1}-1}^{(j)};$$
$$w_{k_{j-1}}^{(j)} = w_{Q_{j-1}-k_{j-1}-1}^{(j)}.$$

The next result exploits Lemma 3.4 to develop parity properties on the Gauss-Gegenbauer quadrature formula.

Lemma 3.5. *Let $\psi \in [0, \pi]$, and $j = 1, \dots, d-1$. Let $m_i \in \mathbf{m}$, with $m_0 = \ell$ and $m_i' \in \mathbf{m}'$, with $m_0' = \ell'$ and define, for $j = 1, \dots, d-1$,*

$$G_j(\psi) = C_{m_{j-1}-m_j}^{\left(m_j + \frac{d-j}{2}\right)}(\cos\psi)\, C_{m_{j-1}'-m_j'}^{\left(m_j' + \frac{d-j}{2}\right)}(\cos\psi)\,(\sin\psi)^{d-j}.$$

Then it holds that

$$G_j(\pi - \psi) = (-1)^{m_{j-1}+m_{j-1}'-m_j-m_j'}\, G(\psi). \tag{25}$$

Furthermore, for $Q \in \mathbb{N}$, let $\{\psi_k : k = 0, \dots, Q-1\}$ and $\{w_k : k = 0, \dots, Q-1\}$ be samples of points and weights in $[-1, 1]$ so that for $k = 0, \dots, [Q/2]$

$$\psi_k = \psi_{Q-1-k},$$
$$w_k = w_{Q-1-k},$$

where $[\cdot]$, $t \in \mathbb{R}$ denotes the floor function. Then, if $\left(m_{j-1} + m'_{j-1} - m_j - m'_j\right) = 2c+1$, $c \in \mathbb{N}$, it holds that

$$\sum_{k=0}^{Q-1} w_k G_j\left(\psi_k\right) = 0. \tag{26}$$

4. Aliasing effects on the sphere

This section presents our main results concerning the aliasing phenomenon for d-dimensional spherical random fields. First, we define the aliasing function, the key tool to determine explicitly the aliases for any given harmonic coefficient. Then, we study the aliasing function and, more in general, the set of harmonic numbers identifying the aliases for any given coefficient $a_{\ell,\mathbf{m}}$ in two different cases. The proof of the theorems presented in this section are collected in Section 7.1.

As a first step, we just assume that the aliasing function is separable with respect to the angular components. This assumption is very mild, as it reflects both the separability of the spherical harmonics and the practical convenience of choosing separable sampling points, with respect to the angular coordinates.
As a second step, we study the aliasing effects under the assumption that the sample comes from a spherical uniform design.

4.1. The aliasing function.

In practical applications, the measurements of the random fields can be sampled only over a finite number of locations on $\mathbb{S}^d$. As a straightforward consequence, the integral (11) can not be explicitly computed, but it has to be replaced by a sum of finitely many samples of T.

Fixed a sample size $N \in \mathbb{N}$ and given a set of sampling points over $\mathbb{S}^d$ $\{x_i = (\boldsymbol{\vartheta}_i, \varphi_i) : i = 1, \dots, N\}$, the measurements of the spherical random field T are collected in the sample $\{T(x_i) : i = 1, \dots, N\}$. For any $\ell \geq 0$ and $\mathbf{m} \in \mathcal{M}_\ell$, the approximated harmonic coefficient is given by

$$\tilde{a}_{\ell,\mathbf{m}} = \sum_{i=1}^{N} w_i T\left(\boldsymbol{\vartheta}_i, \varphi_i\right) \bar{Y}_{\ell,\mathbf{m}}\left(\boldsymbol{\vartheta}_i, \varphi_i\right) f\left(\boldsymbol{\vartheta}_i\right), \tag{27}$$

where $f(\boldsymbol{\vartheta})$ is given by (2). Combining (9) and (10) with (27) yields

$$\begin{aligned}\tilde{a}_{\ell,\mathbf{m}} &= \sum_{i=1}^{N} w_i \left(\sum_{\ell' \geq 0} \sum_{\mathbf{m}' \in \mathcal{M}_{\ell'}} a_{\ell',\mathbf{m}'} Y_{\ell',\mathbf{m}'}\left(\boldsymbol{\vartheta}_i, \varphi_i\right)\right) \bar{Y}_{\ell,\mathbf{m}}\left(\boldsymbol{\vartheta}_i, \varphi_i\right) f\left(\vartheta_i\right) \\ &= \sum_{\ell' \geq 0} \sum_{\mathbf{m}' \in \mathcal{M}_{\ell'}} \tau\left(\ell, \mathbf{m}; \ell', \mathbf{m}'\right) a_{\ell',\mathbf{m}'}.\end{aligned} \tag{28}$$

where $\tau(\ell, \mathbf{m}; \ell', \mathbf{m}')$ is given by

$$\tau\left(\ell, \mathbf{m}; \ell', \mathbf{m}'\right) = \sum_{i=1}^{N} w_i Y_{\ell',\mathbf{m}'}\left(\boldsymbol{\vartheta}_i, \varphi_i\right) \bar{Y}_{\ell,\mathbf{m}}\left(\boldsymbol{\vartheta}_i, \varphi_i\right) f\left(\boldsymbol{\vartheta}_i\right). \tag{29}$$

Since now on, we will refer to $\tau(\ell, \mathbf{m}; \ell', \mathbf{m}')$ as the *aliasing function* and to $\tilde{a}_{\ell,\mathbf{m}}$ as the *aliased coefficient*. For $\ell' \neq \ell$ and $m' \neq m$, the coefficients $a_{\ell',\mathbf{m}'}$ in (28) are called *aliases* of $a_{\ell,\mathbf{m}}$ if $\tau(\ell, \mathbf{m}; \ell', \mathbf{m}') \neq 0$. As stated by [LN97] for the case $d = 2$, on the one hand, the following equality

$$\tau\left(\ell, \mathbf{m}; \ell', \mathbf{m}'\right) = \delta_{\ell'}^{\ell} \prod_{i=1}^{d-1} \delta_{m'_i}^{m_i},$$

is a necessary and sufficient condition to identify $a_{\ell,\mathbf{m}}$ and $\tilde{a}_{\ell,\mathbf{m}}$. This equality does not hold in general (see Section 6). On the other hand, fixed $\ell, \ell', \mathbf{m}$ and $\mathbf{m}'$, if $\tau(\ell, \mathbf{m}; \ell', \mathbf{m}') \neq 0$, that is, $a_{\ell',\mathbf{m}'}$ is an alias of $a_{\ell,\mathbf{m}}$, its intensity, denoting how large is the contribution of this alias, is given by $|\tau(\ell, \mathbf{m}; \ell', \mathbf{m}')|$.

The total amount of aliases in (28) and the corresponding intensity depends specifically on the choice of the sampling points $\{x_i : i = 1, \ldots, N\}$ over $\mathbb{S}^d$, which characterizes entirely the subsequent structure of (29). In other words, every setting chosen for the sampling points leads to a specific set of aliases, described by the corresponding aliasing function.

Here we study the aliasing function $\tau(\ell, \mathbf{m}; \ell', \mathbf{m}')$ first in a more general framework, under the assumption of a separable sampling with respect to the angular coordinates in Section 4.2, and then for a discrete version of the spherical uniform distribution in Section 4.3.

4.2. **The separability of the aliasing function.** Let us assume now that the assumptions of Condition 3.1 hold. Thus, given $Q_0, Q_1, \ldots, Q_{d-1} \in \mathbb{N}$, so that $N = \prod_{j=0}^{d-1} Q_j$, for $j = 1, \ldots, d-1$, the corresponding set of quadrature points and weights is given by

$$\left\{\left(\vartheta^{(j)}_{k_{j-1}}, w^{(j)}_{k_{j-1}}\right) \in [0,\pi] \times [0,1] : k_{j-1} = 0, \ldots, Q_{j-1} - 1\right\},$$

while, for $j = d$, we have that

$$\left\{\left(\varphi_{k_{d-1}}, w^{(d)}_{k_{d-1}}\right) \in [0,2\pi] \times [0,1] : k_{d-1} = 0, \ldots, Q_{d-1} - 1\right\},$$

so that

$$\sum_{k_{j-1}=0}^{Q_{j-1}-1} w^{(j)}_{k_{j-1}} = 1 \quad \text{for } j = 1, \ldots, d.$$

As a straightforward consequence, the following result holds.

Theorem 4.1. *Let Condition 3.1 hold. Then it holds that*

$$\tau(\ell, \mathbf{m}; \ell', \mathbf{m}') = \frac{1}{2\pi} \prod_{j=1}^{d-1} h_{m_{j-1},m_j;j} h_{m'_{j-1},m'_j;j} I^{Q_{j-1}}_{m_{j-1},m_j}\left(m'_{j-1}, m'_j\right) J^{Q_{d-1}}_{m_{d-1}}\left(m'_{d-1}\right), \tag{30}$$

where $h_{m_{j-1},m_j;j}$ is given by (6) and

(31)
$$J^{Q_{d-1}}_{m_{d-1}}\left(m'_{d-1}\right) = \sum_{k_{d-1}=0}^{Q_{d-1}-1} w^{(d)}_{k_{d-1}} e^{i\left(m'_{d-1} - m_{d-1}\right)\varphi_{k_{d-1}}};$$

(32)
$$I^{Q_{j-1}}_{m_{j-1},m_j}\left(m'_{j-1}, m'_j\right) = \sum_{k_{j-1}=0}^{Q_{j-1}-1} w^{(j)}_{k_{j-1}} \left(\sin \vartheta^{(j)}_{k_{j-1}}\right)^{m_j + m'_j + d - j} C^{\left(m_j + \frac{d-j}{2}\right)}_{m_{j-1}-m_j}\left(\cos \vartheta^{(j)}_{k_{j-1}}\right) C^{\left(m'_j + \frac{d-j}{2}\right)}_{m'_{j-1}-m'_j}\left(\cos \vartheta^{(j)}_{k_{j-1}}\right).$$

Remark 4.2. Loosely speaking, the function $\tau(\ell, \mathbf{m}; \ell', \mathbf{m}')$ can be rewritten as a chain of products of functions, pairwise coupled by two indexes m_j, m'_j, $j = 1, \ldots, d-2$. Indeed, as shown by (5), each angular component $\vartheta^{(j)}$ is related to two harmonic numbers m_{j-1} and m_j. While $J^{Q_{d-1}}_{m_{d-1}}\left(m'_{d-1}\right)$ is concerned with the discretization of components along the azimuthal angle φ, the factors $I^{Q_{j-1}}_{m_{j-1},m_j}\left(m'_{j-1}, m'_j\right)$, $j = 1, \ldots, d-1$, represent the discretization along the j-th component of the vector $\boldsymbol{\vartheta}$. Finally, the multiplicative factor $h_{m_{j-1},m_j;j}$ comes from the normalization of hyperspherical harmonics in (5).
Since now on, we will refer to $I^{Q_{j-1}}_{m_{j-1},m_j}\left(m'_{j-1}, m'_j\right)$, for $j = 1, \ldots, d-1$, and $J^{Q_{d-1}}_{m_{d-1}}\left(m'_{d-1}\right)$ as the aliasing (function) j-th and d-th factors respectively.

4.3. **Aliasing and spherical uniform designs.** As already mentioned in Section 1.1, the motivations behind the study of this particular setting come from two different sources. On the one hand, the uniform design is largely used in the framework on numerical analysis over the sphere (see [AH12, SB93, Sze75]). On the other hand, in the field of mathematical statistics, the spherical uniform sampling has be proved to be the the most efficient design with respect to a large set of optimality criteria such as the Kiefer's Φ_p- as well as the Φ_{E_s}-criteria, in the framework of optimal designs of experiments (see [DKSG18]). Furthermore, in Remark 4.5, we show that our findings align with the results established [LN97]) for the two-dimensional case. Example 4.6 establishes explicitly the set of aliases of a given harmonic coefficient.

The main results of this section, stated in the forthcoming Theorem 4.3, require some further notation, produced in Remark 4.4.

Theorem 4.3. *Assuming that Condition 3.3 holds, for any $\ell \geq 0$ and $\mathbf{m} \in \mathcal{M}_\ell$, the aliased harmonic coefficient defined in (28) is given by*

$$\tilde{a}_{\ell,\mathbf{m}} = a_{\ell,\mathbf{m}} + \sum_{s_0 \in D_0(\ell)} \sum_{\mathbf{s} \in Z^{\mathbf{Q}}_{\ell,\mathbf{m}}} \eta\left(\ell,\mathbf{m};\ell+2s_0,\mathbf{m}+2\mathbf{s}\right) a_{\ell+2s_0,\mathbf{m}+2\mathbf{s}}, \tag{33}$$

where $\eta\left(\ell,\mathbf{m};\ell+2s_0,\mathbf{m}+2\mathbf{s}\right)$ is defined by (48), while the sets $D_0(\ell)$ and $Z^{\mathbf{Q}}_{\ell,\mathbf{m}}$ are given by (34) and (47).

Remark 4.4. Let us fix preliminarily $m_0 = \ell$. Since now on, $\mathbf{s} = (s_1, \dots, s_{d-1}) \in \mathbb{Z}^{d-1}$ will denote a $(d-1)$-vector of indices, while $\mathbf{Q} = (Q_0, Q_1, \dots, Q_{d-1})$ is a d-vector collecting the cardinality of the quadrature nodes for each angular component in $(\boldsymbol{\vartheta},\varphi)$. Following Lemmas 3.4 and 3.5, for $\ell \geq 0$ and $\mathbf{m} \in \mathcal{M}_\ell$, Theorem 4.3 establishes that the aliases for $a_{\ell,\mathbf{m}}$ are identified by the harmonic numbers $(\ell',\mathbf{m}')$, so that $\left|m_j - m'_j\right| = 2s_j$, $j = 0,\dots,d-1$. The aliases of $a_{\ell,\mathbf{m}}$ take thus the form

$$a_{\ell+2s_0,\mathbf{m}+2\mathbf{s}} = a_{\ell+2s_0,m_1+2s_1,\dots,m_{d-2}+2s_{d-2},m_{d-1}+2rM},$$

where the indices $s_0,\dots,s_{d-1}$ belong to suitable sets defined as follows. For the index s_0, we define

$$D_0 = D_0(\ell) = \left\{ s_0 \in \mathbb{Z} : s_0 \geq -\frac{\ell}{2} \right\}. \tag{34}$$

Then, for $j = 1,\dots,d-2$, we have that

$$H^{(j)}_{m_j}\left(m_{j-1}+2s_{j-1}\right) = \left\{ s_j \in \mathbb{Z} : -\frac{m_j}{2} \leq s_j \leq \frac{(m_{j-1}+2s_{j-1}) - m_j}{2} \right\}. \tag{35}$$

Finally, the last index s_{d-1}, characterizing the trapezoidal rule on φ, depends on the constant M given in Condition 3.3, so that $s_{d-1} = rM$, where r belongs to the following set,

$$R^M_{m_{d-1}}\left(m_{d-2}+2s_{d-2}\right) := \left\{ r \in \mathbb{Z} : -\frac{(m_{d-2}+2s_{d-2}) + m_{d-1}}{2M} \leq r \leq \frac{(m_{d-2}+2s_{d-2}) - m_{d-1}}{2M} \right\}. \tag{36}$$

Notice that for $j = 1,\dots,d-1$ each index s_j, belongs to a set whose size depends on the value of s_{j-1}. Furthermore, while $D_0(\ell)$ provides just a lower bound for s_0, each $H^{(j)}_{m_j}\left(m_{j-1}+2s_{j-1}\right)$, $j = 1,\dots,d-1$, features only finitely many elements.
Let us now define the following sets,

$$A_0 = A_0\left(\ell,Q_0\right) = \left\{ s_0 \in \mathbb{Z} : -\frac{\ell}{2} \leq s_0 \leq Q_0 - \ell - 1 \right\}; \tag{37}$$

$$B_0 = B_0\left(\ell,Q_0\right) = \left\{ s_0 \in \mathbb{Z} : Q_0 - \ell \leq s_0 \leq \infty \right\}, \tag{38}$$

and, for $j = 1,\dots,d-2$,

$$A_j = A_j\left(m_j,Q_j\right) = \left\{ s_j \in \mathbb{Z} : -\frac{m_j}{2} \leq s_j \leq Q_j - m_j - 1 \right\}; \tag{39}$$

$$B_j = B_j\left(m_{j-1},m_j,s_{j-1},Q_j\right) = \left\{ s_j \in \mathbb{Z} : Q_j - m_j \leq s_j \leq \frac{m_{j-1}-m_j}{2} + s_{j-1} \right\}. \tag{40}$$

Observe that the definition of A_j and B_j is formally correct only if $Q_j - m_j < \frac{m_{j-1}-m_j}{2} + s_{j-1}$, that is, $s_{j-1} > Q_j - \frac{m_{j-1}+m_j}{2}$. Thus, since now on, for $s_{j-1} \leq Q_j - \frac{m_{j-1}+m_j}{2}$, we consider

$$A_j = \left\{ s_j \in \mathbb{Z} : -\frac{m_j}{2} \leq s_j \leq \frac{m_{j-1}-m_j}{2} + s_{j-1} \right\}; \tag{41}$$

$$B_j = \emptyset, \tag{42}$$

to take into account all the possible combinations of s_{j-1} and Q_j. It is straightforward to observe that

$$D_0 = A_0 \cup B_0, \quad H^{(j)}_{m_j}(m_{j-1} + 2s_{j-1}) = A_j \cup B_j, \quad \text{for } j = 1, \dots, d-2.$$

Define now the following sets

$$H^{(j);0}_{m_j}(m_{j-1} + 2s_{j-1}) = H^{(j)}_{m_j}(m_{j-1} + 2s_{j-1}) \cap \{s_j \neq 0\}; \tag{43}$$

$$R^{M;0}_{m_{d-1}}(m_{d-2} + 2s_{d-2}) \cap \{r \neq 0\}, \tag{44}$$

which are equal to $H^{(j)}_{m_{j-1},m_j}(s_{j-1})$ and $R^M_{m_{d-1}}(m_{d-2}] + 2s_{d-2})$ respectively, but omitting the null value. Finally, we define, for $j = 1, \dots, d-2$,

$$\begin{aligned} \Delta_j &= \Delta_j(m_{j-1} + 2s_{j-1}, m_j, Q_{j-1}, s_{j-1}) \\ &= \left\{ s_j \in \mathbb{Z} : s_j \in \left(H^{(j);0}_{m_j}(m_{j-1} + 2s_{j-1}) \mathbb{1}\{s_{j-1} \in A_{j-1}\} + H^{(j)}_{m_j}(m_{j-1} + 2s_{j-1}) \mathbb{1}\{s_{j-1} \in B_{j-1}\} \right) \right\}. \end{aligned} \tag{45}$$

while

$$\begin{aligned} \Delta_{d-1} =& \Delta_{d-1}(m_{d-2} + 2s_{d-2}, m_{d-1}, M, s_{d-2}) \\ =& \Big\{ s_{d-1} = Mr; M = Q_{d-1}/2, r \in \mathbb{Z} : r \in \Big(R^{M,0}_{m_{d-1}}(m_{d-2} + 2s_{d-2}) \mathbb{1}\{s_{d-2} \in A_{d-2}\} \\ &+ R^M_{m_{d-1}}(m_{d-2} + 2s_{d-2}) \mathbb{1}\{s_{d-2} \in B_{d-2}\} \Big) \Big\}, \end{aligned} \tag{46}$$

In other words, when $s_j \in \Delta_j$, it can take any value in $H^{(j)}_{m_{j-1}}(m_{j-1} + 2s_{j-1})$ if $s_{j-1} \in B_{j-1}$. Otherwise, if $s_{j-1} \in A_{j-1}$, it can take any value in $H^{(j)}_{m_{j-1}}(m_{j-1} + 2s_{j-1})$ except to the null value.
We collect these sets together with the notation

$$Z^{\mathbf{Q}}_{\ell,\mathbf{m}} = \{(s_1, \dots, s_{d-1}) : s_1 \in \Delta_1, \dots, s_{d-1} \in \Delta_{d-1}; s_1 \geq \dots \geq s_{d-1}\}. \tag{47}$$

Finally, we define

$$\eta(\ell, \mathbf{m}; \ell + 2s_0, \mathbf{m} + 2\mathbf{s}) = \prod_{j=1}^{d-1} h_{m_{j-1},m_j;j} h_{m_{j-1}+2s_{j-1},m_j+2s_j;j} I^{Q_{j-1}}_{m_{j-1},m_j}(m_{j-1} + 2s_{j-1}, m_j + 2s_j), \tag{48}$$

where $h_{m_{j-1},m_j;j}$ and $I^{Q_{j-1}}_{m_{j-1},m_j}(m_{j-1} + 2s_{j-1}, m_j + 2s_j)$ are defined by (6) and (32) respectively, and corresponding to $\tau(\ell, \mathbf{m}; \ell', \mathbf{m}')$ as given by (30), with $\ell' = \ell + 2s_0$, $\mathbf{m}' = \mathbf{m} + 2\mathbf{s}$ and $J^{Q_{d-1}}_{m_{d-1}}\left(m'_{d-1}\right) = 2\pi$.

Remark 4.5 (Comparison with the 2-dimensional case). The aliasing effects over $\mathbb{S}^2$ have been studied by [LN97], involving a trapezoidal rule for the coordinate ϑ and the Gauss-Laplace quadrature formula for the angle ϑ. More formally, fixed $Q \in \mathbb{N}$, a quadrature formula is obtained by a set of Q points and weights $\{\theta_k, w_k : k = 0, \dots, Q-1\}$, obtained as in Definition 3.2. The points $\{\theta_k : k = 0, \dots, Q-1\}$ are, in this case, the nodes of the Legendre polynomial of order Q. Recall that, for $d = 2$, m does not identify a vector of harmonic numbers, but just an integer, defined so that $-\ell \leq m \leq \ell$. Thus, the aliases of the harmonic coefficient $a_{\ell,m}$ are given by the following formula,

$$\begin{aligned} a_{\ell+2s,,m+2rM} =& \sum_{s=-\ell/2}^{Q-\ell-1} \sum_{r \in R^M_m(\ell+2s)} \zeta_{\ell,m} \zeta_{\ell+2s,m+2rM} I^Q_{\ell,m}(\ell + 2s, m + 2rM) a_{\ell+2s,m+2rM} \\ &+ \sum_{s \geq Q-\ell} \sum_{r \in R^{M;0}_m(\ell+2s)} \zeta_{\ell,m} \zeta_{\ell+2s,m+2rM} I^Q_{\ell,m}(\ell + 2s, m + 2rM) a_{\ell+2s,m+2rM}, \end{aligned}$$

where

$$\zeta_{\ell,m} = \left(\frac{2\ell+1}{2}\frac{(\ell-m)!}{(\ell+m)!}\right)^{\frac{1}{2}};$$

$$I^{Q}_{\ell,m}\left(\ell+2s, m+2rM\right) = \sum_{k=0}^{Q-1} w_k \sin\vartheta_k P_{\ell,m}\left(\cos\vartheta_k\right) P_{\ell+2s,m+2rM}\left(\cos\vartheta_k\right).$$

Simple algebraical manipulations show that this formula coincides with (33) claimed in Theorem 4.3 for $d=2$.

Before concluding this section, the reader is provided with a simple example, with the aim of giving a practical insight on the identification of the aliases of an harmonic coefficient.

Example 4.6. Let us fix $d=3$ and calculate the aliases of the harmonic coefficient $a_{0,0,0}$. Let us assume, furthermore, that $Q=Q_0=Q_1=Q_{d-2}=2M$. We have that

$$\tilde{a}_{0,0,0} = a_{0,0,0} + \sum_{s_0\in D_0}\sum_{(s_1,s_2)\in Z^{Q}_{0,0,0}} h_{0,0;1}h_{2s_0,2s_1;1}I^{Q}_{0,0}\left(2s_0,2s_1\right)h_{0,0;2}h_{2s_1,2s_2;2}I^{Q}_{0,0}\left(2s_1,2s_2\right)a_{2s_0,2s_1,2s_2}.$$

On the one hand, using (6) yields

$$h_{0,0;1} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}; \qquad h_{0,0;2} = \frac{1}{\sqrt{2}};$$

$$h_{2s_0,2s_1;1} = \left(\frac{2^{4s_1+1}\left(2s_0-2s_1\right)!\left(2s_0+1\right)\Gamma^2\left(2s_1+1\right)}{\pi\left(2s_0+2s_1+1\right)!}\right)^{\frac{1}{2}} = \left(\frac{2^{4s_1+1}\left(2s_0-2s_1\right)!\left(2s_0+1\right)\left(\left(2s_1\right)!\right)^2}{\pi\left(2s_0+2s_1+1\right)!}\right)^{\frac{1}{2}};$$

$$h_{2s_1,2s_2;2} = \left(\frac{2^{4s_2-1}\left(2s_1-2s_2\right)!\left(4s_1+1\right)\Gamma^2\left(2s_2+\frac{1}{2}\right)}{\pi\left(2s_1+2s_2\right)!}\right)^{\frac{1}{2}} = \left(\frac{\left(2s_1-2s_2\right)!\left(4s_1+1\right)\left(\left(4s_2\right)!\right)^2}{2^{4s_2+1}\left(2s_1+2s_2\right)!\left(\left(2s_2\right)!\right)^2}\right)^{\frac{1}{2}},$$

so that we can define

$$\begin{aligned}\epsilon_{s_0,s_1,s_2} &= h_{0,0;1}h_{2s_0,2s_1;1}h_{0,0;2}h_{2s_1,2s_2;2}\\ &= \left(\frac{\left(2s_0-2s_1\right)!\left(2s_1-2s_2\right)!\left(2s_0+1\right)\left(4s_1+1\right)}{\left(2s_0+2s_1+1\right)!\left(2s_1+2s_2\right)!}\right)^{\frac{1}{2}}\frac{2^{2(s_1-s_2)}\left(2s_1\right)!\left(4s_2\right)!}{\pi\left(2s_2\right)!}.\end{aligned}$$

On the other hand, we obtain from (34), (37),(38),(39), and (40) that

$$D_0 = \left\{s_0\in\mathbb{Z}: s_0\geq 0\right\}, A_0 = \left\{s_0\in\mathbb{Z}: 0\leq s_0\leq Q-1\right\}, B_0 = \left\{s_0\in\mathbb{Z}: s_0\geq Q-1\right\},$$
$$H^{(1)}_0\left(2s_0\right) = \left\{s_1\in\mathbb{Z}: 0\leq s_1\leq s_0\right\}, A_1 = \left\{s_1\in\mathbb{Z}: 0\leq s_1\leq Q-1\right\}, B_1 = \left\{s_1\in\mathbb{Z}: Q-1\leq s_1\leq s_0\right\}.$$
$$R^{Q}_{m_2}\left(2s_1\right) = \left\{r\in\mathbb{Z}: -\frac{s_1}{Q}\leq r\leq\frac{s_1}{Q}\right\},$$

Hence, from (47) we have that

$$\begin{aligned}Z^{Q}_{0,0,0} = \Big\{\left(s_1,r\right): s_1 &\in \left(H^{(1);0}_0\left(2s_0\right)\mathbb{1}\{s_0\in A_0\} + H^{(1)}_0\left(2s_0\right)\mathbb{1}\{s_0\in B_0\}\right),\\ r &\in \left(R^{Q,0}_0\left(2s_1\right)\mathbb{1}\{s_1\in A_1\} + R^{Q}_0\left(2s_1\right)\mathbb{1}\{s_1\in B_1\}\right)\Big\}.\end{aligned}$$

We can then rewrite

$$\begin{aligned}\tilde{a}_{0,0,0} = a_{0,0,0} &+ \sum_{s_0=0}^{Q-1}\sum_{s_1=1}^{s_0}\sum_{\substack{s_2=-s_1\\ s_2\neq 0}}^{s_1} \epsilon_{s_0,s_1,s_2} I_{0,0}^{Q}\left(2s_0,2s_1\right) I_{0,0}^{Q}\left(2s_1,2s_2\right) a_{2s_0,2s_1,2s_2}\\ &+ \sum_{s_0\geq Q}\left(\sum_{s_1=0}^{Q-1}\sum_{\substack{s_2=-s_1\\ s_2\neq 0}}^{\frac{s_1}{Q}} \epsilon_{s_0,s_1,s_2} I_{0,0}^{Q}\left(2s_0,2s_1\right) I_{0,0}^{Q}\left(2s_1,2s_2\right)\right.\\ &\left.+\sum_{s_1=Q}^{s_0}\sum_{s_2=-s_1}^{s_1} \epsilon_{s_0,s_1,s_2} I_{0,0}^{Q}\left(2s_0,2s_1\right) I_{0,0}^{Q}\left(2s_1,2s_2\right)\right) a_{2s_0,2s_1,2s_2}. \end{aligned} \tag{49}$$

Observe that the first line in (49) describes the aliases obtained for $s_0 \in A_0$, while the other two lines contain the aliases corresponding to $s_0 \in B_0$. Notice that if $s_0 \in A_0$, then $B_1 = \emptyset$. As a consequence, it follows that both the indexes s_1 and s_2 can not take the null-value. When $s_0 \in B_0$, we have that $A_1 = \{0, \ldots, Q-1\}$ and $B_1 = \{Q, \ldots, s_0\}$. Hence, we obtain the second and the third sums in (49).

5. Aliasing for angular power spectrum

In this section, our purpose is to investigate on the aliasing effects as far as the spectral approximation of an isotropic random field is concerned. More specifically, we establish a method to identify the aliases of each element of the power spectrum $\{C_\ell : \ell \geq 0\}$.

Assume to have an isotropic random field on $\mathbb{S}^d$, so that (14) and (15) hold. When the integral (11) is replaced with the sum (28) under the Condition 3.3, we want to study how the aliasing errors arising in (28), affect the estimation of $C_\ell = \operatorname{Var}\left(a_{\ell,\mathbf{m}}\right)$ (see (15)). In particular we are interested on developing the presence of aliases when C_ℓ is approximated by the average

$$\tilde{C}_\ell = \frac{1}{\Xi_d\left(\ell\right)} \sum_{\mathbf{m}\in\mathcal{M}_\ell} \operatorname{Var}\left(\tilde{a}_{\ell,\mathbf{m}}\right), \tag{50}$$

where $\Xi_d\left(\ell\right)$ is given by (4) (cf, for example, (16)). Let us recall that $D_0\left(\ell\right)$ is given by (34), and let $V_{\ell,'\mathbf{m}}^{Q}\left(\ell'\right)$ be defined by

$$V_{\ell,\mathbf{m}}^{\mathbf{Q}}\left(\ell'\right) = \sum_{\mathbf{s}\in Z_{\ell,\mathbf{m}}^{\mathbf{Q}}} \prod_{j=1}^{d-1} h_{m_{j-1},m_j;j}^2 h_{m_{j-1}+2s_{j-1},m_j+2s_j;j}^2 \left(I_{m_{j-1},m_j}^{Q_{j-1}}\left(m_{j-1}+2s_{j-1},m_j+2s_j\right)\right)^2.$$

Our findings, which extend to the d-dimensional sphere the outcomes of [LN97, Theorem 3.1] (cf. Remark 4.5), are produced in the following theorem.

Theorem 5.1. *Let T be an isotropic random field on $\mathbb{S}^d$ with angular power spectrum given by (15). Under the assumption given in Condition 3.3, it holds that*

$$\tilde{C}_\ell = \sum_{s_0\in D_0(\ell)} \Lambda_\ell^{Q}\left(\ell+2s_0\right) C_{\ell+2s_0},$$

where

$$\Lambda_\ell^{Q}\left(\ell+2s_0\right) = \frac{1}{\Xi_d\left(\ell\right)} \sum_{\mathbf{m}\in\mathcal{M}_\ell} V_{\ell,'\mathbf{m}}^{Q}\left(\ell+2s_0\right).$$

The proof of Theorem 5.1 can be found in Section 7.1.

6. Band-limited random fields

In this section, we establish the condition on the sample size, leading to an exact reconstruction of the harmonic coefficients $a_{\ell,\mathbf{m}}$ for band-limited random fields, in the paradigm of the spherical uniform design. In other words, for band-limited random fields and for a suitable choice of $\mathbf{Q}$, the approximation of the integral (11) by the sum (27) is exact and, then, there are no aliases, analogously to the findings described in [LN97, Section 4] for $d = 2$. The reader is referred to Section 7.1 for the proofs of the theorems collected in this section.

If the number of sampling points is sufficiently large with respect to the band-width characterizing the random field, we obtain two crucial results, stated in the next theorem. On the one hand, the band-limited random fields are alias-free in $\tilde{a}_{\ell,\mathbf{m}}$ and, on the other, they are exactly reconstructed by means of the Gaussian quadrature procedure described above.

Theorem 6.1. *Assume that $T(x)$ is band-limited with bandwidth L_0, that is, the harmonic expansion given by (12) holds. If also Condition 3.3 holds, with $Q = Q_0 = \ldots = Q_{d-2} > L_0$ and $M > L_0$. Then, it holds that*

$$\tilde{a}_{\ell,\mathbf{m}} = a_{\ell,\mathbf{m}} \quad \text{for } \ell \leq L_0, \mathbf{m} \in \mathcal{M}_\ell. \tag{51}$$

Furthermore, for any $L \in \mathbb{N}$ satisfying $Q \geq L \geq L_0$, the following reconstruction holds exactly:

$$T(x) = \sum_{k_0=0}^{Q_0-1} \cdots \sum_{k_{d-1}=0}^{Q_{d-1}-1} \left(\prod_{j=0}^{d-1} w_{k_j}^{(j+1)} \right) \left(\prod_{j=1}^{d-1} \left(\sin \vartheta_{k_{j-1}}^{(j)} \right)^{d-j} \right) T\left(\vartheta_{k_0}^{(1)}, \ldots, \vartheta_{k_{d-2}}^{(d-1)}, \varphi_{k_{d-1}} \right)$$
$$\cdot \sum_{\ell=0}^{L} K_\ell(x, x_{\mathbf{k}}), \tag{52}$$

where $x_{\mathbf{k}} = \left(\boldsymbol{\vartheta}_{k_0,\ldots,k_{d-2}}, \varphi_{k_{d-1}} \right)$ and K_ℓ is given by (8).

A random field has a band-limited power spectrum with bandwidth P_L if $C_\ell = 0$ for any $\ell > P_L$. The following theorem shows that these random fields are aliases-free in $\tilde{C}_\ell$, employing a Gauss sampling under Condition 3.3 and given a suitable sample size.

Theorem 6.2. *Let T be a random field with a band-limited power spectrum with bandwidth P_L, sampled by means of a Gauss scheme under Condition 3.3, so that $Q = Q_0 = \ldots = Q_{d-2} \geq M > P_L$. Thus, it holds that*

$$\mathrm{Var}(\tilde{a}_{\ell,\mathbf{m}}) = \mathrm{Var}(a_{\ell,\mathbf{m}}) = C_\ell.$$

7. Proofs

In this section, we provide proofs for the main and auxiliary results.

7.1. Proofs of the main results.

Proof of Theorem 4.1. Using (2), (5), (17) and (18) in (29) yields

$$\tau(\ell,\mathbf{m};\ell',\mathbf{m}') = \sum_{k_0=0}^{Q_0-1}\cdots\sum_{k_{d-1}=0}^{Q_{d-1}-1}\left(\prod_{j=1}^{d} w_{k_{j-1}}^{(j)}\right)\left(\prod_{j=1}^{d-1}\left(\sin\vartheta_{k_{j-1}}^{(j)}\right)^{d-j}\right)$$
$$\cdot\left(\frac{1}{\sqrt{2\pi}}\prod_{j=1}^{d-1}\left(h_{m'_{j-1},m'_j;j}C_{m'_{j-1}-m'_j}^{\left(m'_j+\frac{d-j}{2}\right)}\left(\cos\vartheta_{k_{j-1}}^{(j)}\right)\left(\sin\vartheta_{k_{j-1}}^{(j)}\right)^{m'_j}\right)e^{im'_{d-1}\varphi_{k_{d-1}}}\right)$$
$$\cdot\left(\frac{1}{\sqrt{2\pi}}\prod_{j=1}^{d-1}\left(h_{m_{j-1},m_j;j}C_{m_{j-1}-m_j}^{\left(m_j+\frac{d-j}{2}\right)}\left(\cos\vartheta_{k_{j-1}}^{(j)}\right)\left(\sin\vartheta_{k_{j-1}}^{(j)}\right)^{m_j}\right)e^{im_{d-1}\varphi_{k_{d-1}}}\right)$$
$$=\frac{1}{2\pi}\prod_{j=1}^{d-1}\left(\sum_{k_{j-1}=0}^{Q_{j-1}-1} w_{k_{j-1}}^{(j)}\left(\sin\vartheta_{k_{j-1}}^{(j)}\right)^{m_j+m'_j+d-j} h_{m_{j-1},m_j;j}h_{m'_{j-1},m'_j;j}C_{m_{j-1}-m_j}^{\left(m_j+\frac{d-j}{2}\right)}\left(\cos\vartheta_{k_{j-1}}^{(j)}\right)\right.$$
$$\left.\cdot C_{m'_{j-1}-m'_j}^{\left(m'_j+\frac{d-j}{2}\right)}\left(\cos\vartheta_{k_{j-1}}^{(j)}\right)\right)\left(\sum_{k_{d-1}=0}^{Q_{d-1}-1} w_{k_{d-1}}^{(d)} e^{i\left(m'_{d-1}-m_{d-1}\right)\varphi_{k_{d-1}}}\right),$$

as claimed. □

Proof of Theorem 4.3. We divide this proof in two parts. The first part establishes explicit bounds for the indices $s_0,\ldots,s_{d-2},r$ by means of

(1) the parity properties of the Gegenbauer polynomials (see Lemma 3.5);
(2) the definition of $\mathcal{M}_\ell$ (cf. (3)), which exploits the definition of spherical harmonics in (5).

The second part of the proof detects then some sets of indices $s_0,\ldots,s_{d-2},r$ for which $\tau(\ell,\mathbf{m};\ell',\mathbf{m}')=0$ as a consequence of

(1) the order of the quadrature formula (see (19)).
(2) the orthogonality of the Gegenbauer polynomials (see (7));

For both the cases, we follow a backward induction step, studying first the aliasing effects due to the trapezoidal sampling for coordinate $j=d$, using the results holding for the j-th component to prove the statement for the $j-1$-th component, until we reach $j=1$.

Part 1 - Here our purpose it to exploit either properties due to the uniform sampling and the ones related to the harmonic numbers of spherical harmonics, to establish lower and, where possible, upper bounds for the indices $s_0,\ldots,s_{d-2},r$. These indices identify the aliases of the harmonic coefficient $a_{\ell,\mathbf{m}}$, given in the form $a_{\ell+2s_0,\mathbf{m}+2\mathbf{s}}$.
Let us consider initially $j=d$ and apply to the coordinate φ the standard trapezoidal rule. As well as in [LN97] (see also [DKSG18]), using (21) and (22) in (31) yields

$$J_{m_{d-1}}^{2M}\left(m'_{d-1}\right)=\frac{\pi}{M}\sum_{q=0}^{2M-1}e^{i\left(m'_{d-1}-m_{d-1}\right)\frac{q\pi}{M}}=2\pi\delta_{m_{d-1}+2rM}^{m'_{d-1}}, \tag{53}$$

where $r\in\mathbb{Z}$ is such that $|m_{d-1}+2rM|\le m'_{d-2}$. Indeed, from (5) it follows that $Y_{\ell',\mathbf{m}'}(x)$ is well-defined only for $\left|m'_{d-1}\right|\le m'_{d-2}$. Thus, it holds that $r\in R_{m_{d-1}}^{M}\left(m'_{d-2}\right)$, where

$$R_{m_{d-1}}^{M}\left(m'_{d-2}\right):=\left\{r\in\mathbb{Z}:-\frac{m'_{d-2}+m_{d-1}}{2M}\le r\le\frac{m'_{d-2}-m_{d-1}}{2M}\right\}.$$

Consider now $j=d-1$. The component $\vartheta^{(d-1)}$ is subject to the aforementioned Gauss-Legendre quadrature formula (cf. the case $d=2$ in [LN97]). Indeed, by using (53) jointly with the definition of the sampling points and weights given by (23) and (24) respectively with $j=d-1$, the $(d-1)$-th aliasing factor is given by

$$I^{Q_{d-2}}_{m_{d-2},m_{d-1}}\left(m'_{d-2},m_{d-1}+2rM\right)=\sum_{k_{d-2}=0}^{Q_{d-2}-1}w^{(d-1)}_{k_{d-2}}\left(\sin\vartheta^{(d-1)}_{k_{d-2}}\right)^{2(m_{d-1}+rM)+1}$$
$$\cdot C^{\left(m_{d-1}+\frac{1}{2}\right)}_{m_{d-2}-m_{d-1}}\left(\cos\vartheta^{(d-1)}_{k_{d-2}}\right)C^{\left(m_{d-1}+2rM+\frac{1}{2}\right)}_{m'_{d-2}-m_{d-1}-2rM}\left(\cos\vartheta^{(d-1)}_{k_{d-2}}\right). \tag{54}$$

Observe now that the Legendre polynomials can be expressed in terms of a Gegenbauer polynomial by means of the formula

$$\frac{(2m_{d-1})!}{2^{m_{d-1}}\left(m_{d-1}\right)!}\left(\sin\vartheta^{(d-1)}_{k_{d-2}}\right)^{m_{d-1}}C^{\left(m_{d-1}+\frac{1}{2}\right)}_{m_{d-2}-m_{d-1}}\left(\cos\vartheta^{(d-1)}_{k_{d-2}}\right)=P_{m_{d-2},m_{d-1}}\left(\cos\vartheta^{(d-1)}_{k_{d-2}}\right),$$

see for example [Sze75, Formula 4.7.35]. Hence, we obtain that

$$I^{Q_{d-2}}_{m_{d-2},m_{d-1}}\left(m'_{d-2},m_{d-1}+2rM\right)$$
$$=c_{m_{d-1}}c_{m_{d-1}+2rM}\sum_{k_{d-2}=0}^{Q_{d-2}-1}w^{(d-1)}_{k_{d-2}}\sin\vartheta_{k_{d-2}}P_{m_{d-2},m_{d-1}}\left(\cos\vartheta^{(d-1)}_{k_{d-2}}\right)P_{m'_{d-2},m_{d-1}+2rM}\left(\cos\vartheta^{(d-1)}_{k_{d-2}}\right), \tag{55}$$

where

$$c_m=\left(\frac{(2m)!}{2^m\left(m\right)!}\right)^{-1}.$$

In analogy to [LN97, Theorem 2.1], using (25), given in Lemma 3.5, for $j=d-1$, in (55) leads to

$$I^{Q_{d-2}}_{m_{d-2},m_{d-1}}\left(m'_{d-2},m_{d-1}+2rM\right)=0\quad\text{for any } m'_{d-1}=m_{d-2}+2s_{d-2}+1,s_{d-2}\in\mathbb{N}_0.$$

In other words, the $d-1$-th aliasing factor is not null only for even values of $\left|m'_{d-2}-m_{d-2}\right|$, that is,

$$m'_{d-2}=m_{d-2}+2s_{d-2},$$

where $s_{d-2}\in D_{m_{d-2}}$, given by

$$D_{m_{d-2}}=\left\{s_{d-2}\in\mathbb{Z}:s_{d-2}\geq-\frac{m_{d-2}}{2}\right\},$$

which guarantees that $m'_{d-2}\geq 0$ and, thus, a well-defined aliasing factor in (54).
On the one hand, using $m'_{d-2}=m_{d-2}+2s_{d-2}$ in the set concerning the d-th aliasing factor, we have that $r\in R^M_{m_{d-1}}\left(m_{d-2}+2s_{d-2}\right)$, as given by (36).
On the other hand, following (3) and (5), it holds that $m'_{d-2}=m_{d-2}+2s_{d-2}\leq m'_{d-3}$. Thus, $s_{d-2}\in R_{m_{d-2}}\left(m'_{d-3}\right)$, where

$$R_{m_{d-2}}\left(m'_{d-3}\right)=\left\{s_{d-2}\in\mathbb{Z}:s_{d-2}\leq\frac{m'_{d-3}-m_{d-2}}{2}\right\}.$$

Therefore we obtain that $s_{d-2}\in H^{(d-2)}_{m_{d-2}}\left(m'_{d-3}\right)$, where

$$H^{(d-2)}_{m_{d-2}}\left(m'_{d-3}\right)=D_{m_{d-2}}\cap R_{m_{d-2}}\left(m'_{d-3}\right).$$

Consider now $2\leq j\leq d-2$. For each component, we use a suitable Gauss-Gegenbauer quadrature rule described above (see also [DKSG18, Lemma 3.1]). Using Lemma 3.5 yields to the following outcome. If $I^{Q_j}_{m_j,m_{j+1}}\left(m'_j,m'_{j+1}\right)\neq 0$ only when $m'_j=m_j+2s_j$, for $s_j\in H^{(j+1)}_{m_j}\left(m'_{j-1}\right)$, then $I^{Q_{j-1}}_{m_{j-1},m_j}\left(m'_{j-1},m'_j\right)\neq 0$ only when $m'_{j-1}=m_{j-1}+2s_{j-1}$, $s_{j-1}\in H^{(j)}_{m_{j-1}}\left(m'_{j-2}\right)$.
On the one hand, Formula (26) in Lemma 3.5 with $m'_j=m_j+2s_j$ yields $I^{Q_{j-1}}_{m_{j-1},m_j}\left(m'_{j-1},m_j+2s_j\right)\neq 0$ only for $m'_{j-1}=m_{j-1}+2s_{j-1}$, so that the aliases with respect to the j-th component are identified by the function

$$I^{Q_{j-1}}_{m_{j-1},m_j}\left(m_{j-1}+2s_{j-1},m_j+2s_j\right)$$
$$=\sum_{k_{j-1}=0}^{Q_{j-1}-1}w^{(j)}_{k_{j-1}}\left(\sin\vartheta^{(j)}_{k_{j-1}}\right)^{2(m_j+s_j)+d-j}C^{\left(m_j+\frac{d-j}{2}\right)}_{m_{j-1}-m_j}\left(\cos\vartheta^{(j)}_{k_{j-1}}\right)C^{\left(m_j+2s_j+\frac{d-j}{2}\right)}_{m_{j-1}+2s_{j-1}-(m_j+2s_j)}\left(\cos\vartheta^{(j)}_{k_{j-1}}\right).$$

It is straightforward to set $s_{j-1} \in D_{m_{j-1}}$, where

$$D_{m_{j-1}} = \left\{ s_{j-1} \in \mathbb{Z} : s_{j-1} \geq -\frac{m_{j-1}}{2} \right\},$$

so that the polynomials in $I^{Q_{j-1}}_{m_{j-1},m_j}\left(m_{j-1}+2s_{j-1}, m_j+2s_j\right)$,

$$w^{(j)}_{k_{j-1}} \left(\sin \vartheta^{(j)}_{k_{j-1}}\right)^{2(m_j+s_j)+d-j} C^{\left(m_j+\frac{d-j}{2}\right)}_{m_{j-1}-m_j}\left(\cos \vartheta^{(j)}_{k_{j-1}}\right) C^{\left(m_j+2s_j+\frac{d-j}{2}\right)}_{m_{j-1}+2s_{j-1}-(m_j+2s_j)}\left(\cos \vartheta^{(j)}_{k_{j-1}}\right)$$
$$= \omega^{(j)}_{k_{j-1}}\left(1-t^{(j)}_{k_{j-1}}\right)^{(m_j+s_j)} C^{\left(m_j+\frac{d-j}{2}\right)}_{m_{j-1}-m_j}\left(t^{j}_{k_{j-1}}\right) C^{\left(m_j+2s_j+\frac{d-j}{2}\right)}_{m_{j-1}+2s_{j-1}-(m_j+2s_j)}\left(t^{j}_{k_{j-1}}\right),$$

is of degree $m_{j-1}+2s_{j-1} \geq 0$.
On the other hand, taking into account (3) and (5), it follows that $m'_{j-1} = m_{j-1}+2s_{j-1} \leq m'_{j-2}$. Thus we obtain that $s_{j-1} \in R_{m_{j-1}}\left(m'_{j-2}\right)$, where

$$R_{m_{j-1}}\left(m'_{j-2}\right) = \left\{ s_{j-1} \in \mathbb{Z} : s_{j-1} \leq \frac{m'_{j-2}-m_{j-1}}{2} \right\},$$

with $m'_{j-2} = m_{j-2}+2s_{j-2}$. Combining these two results and recalling (35), for $j = 2, \ldots, d-1$, it holds that

$$s_{j-1} \in H^{(j-1)}_{m_{j-1}}\left(m'_{j-2}\right), \quad \text{where } H^{(j-1)}_{m_{j-1}}\left(m'_{j-2}\right) = D_{m_{j-1}} \cap R_{m_{j-1}}\left(m'_{j-2}\right).$$

Furthermore, the following step of the backward procedure yields $m'_{j-2} = m_{j-2}+2s_{j-2}$, so that

$$s_{j-1} \in H^{(j-1)}_{m_{j-1}}\left(m_{j-2}+2s_{j-2}\right),$$

for $j = 2, \ldots, d-1$. Consider, finally, the case $j = 1$. This aliasing factor is given by

$$I^{Q_0}_{\ell,m_1}\left(\ell', m_1+2s_1\right) \quad \text{for } s_1 \in H^{(1)}_{m_1}\left(\ell'\right).$$

Here we can thus select $\ell' = \ell + 2s_0$, $s_0 \in D_0(\ell)$, where $D_0(\ell)$ is given by (34). Note that s_0 is the only index that is not selected from a set of finitely many elements.

Part 2 - Here our aim is to use the order of the used quadrature formula to convert, when possible, the sums of $I^{Q_{j-1}}_{m_{j-1},m_j}\left(m'_{j-1}, m'_j\right)$ to integrals. Then, we exploit the orthogonality of the Gegenbauer polynomials (see Section 2) to establish further combinations of indices $s_0, \ldots, s_{d-1}, r$ which lead to a null aliasing function.
First of all, for any $j = 1, \ldots, d-1$, as stated in Remark 4.4, the following decomposition holds

$$D_0(\ell) = A_0 \cup B_0,$$
$$H^{(j)}_{m_j}\left(m_{j-1}+2s_{j-1}\right) = A_j \cup B_j,$$

where A_0, B_0, A_j, and B_j are given by (37), (38), (39), and (40) respectively. Recall also that A_j and B_j are defined by (41), and (42) if $s_{j-1} \leq Q_j - \frac{m_{j-1}+m_j}{2}$.
Now, let $h_{d-2} : [-1,1] \to \mathbb{R}$ be a polynomial function of degree strictly smaller than $2Q_{d-2}$; hence, by using the aforementioned Gauss-Legendre quadrature formula (of order $2Q_{d-2}$) we obtain that

$$\sum_{k_{d-2}=0}^{Q_{d-2}-1} w^{(d-1)}_{k_{d-2}} \sin \vartheta^{(d-1)}_{k_{d-2}} h_{d-2}\left(\cos \vartheta^{(d-1)}_{k_{d-2}}\right) = \sum_{k_{d-2}=0}^{Q_{d-2}-1} \omega^{(d-1)}_{k_{d-2}} h_{d-2}\left(t_p\right) = \int_{-1}^{1} h_{d-2}(t)\, \mathrm{d}t. \tag{56}$$

As a straightforward consequence, (cf. [LN97, Section 2.2]), for $0 \leq m_{d-2} \leq (Q_{d-2}-1)$ and $s_{d-2} \in \mathbb{Z} \cap [-m_{d-2}/2, Q_{d-2}-m_{d-2}-1]$, (56) holds with $h_{d-2}(t) = P_{m_{d-2},m_{d-1}}(t)\, P_{m_{d-2}+2s_{d-2},m_{d-1}}(t)$, a polynomial of degree smaller than $2Q_{d-2}$. Hence, we obtain that

$$I^{Q_{d-2}}_{m_{d-2},m_{d-1}}\left(m_{d-2}+2s_{d-2}, m_{d-1}\right) = \int_{-1}^{1} P_{m_{d-2},m_{d-1}}(t)\, P_{m_{d-2}+2s_{d-2},m_{d-1}}(t)\, \mathrm{d}t$$
$$= \left(\frac{(m_{d-2}-m_{d-1})!}{(m_{d-2}+m_{d-1})}\frac{(2m_{d-2}+1)}{2}\right)^{-1} .\delta^0_{s_{d-2}}$$

Hence, in the uniform sampling approach, all the aliases of $a_{\ell,\mathbf{m}}$ corresponding to the values $r = 0$ and $-m_{d-2}/2 \leq s_{d-2} \leq Q_{d-2}-m_{d-2}$, $s_{d-2} \neq 0$, are annihilated. Aliases of $a_{\ell,\mathbf{m}}$ exist for the following combinations of the indices s_{d-2}, r:

- $s_{d-2} \in A_{d-2}$ and $r \in R^{M;0}_{m_{d-1}}(m_{d-2}+2s_{d-2})$;
- $s_{d-2} \in B_{d-2}$ and $r \in R^{M}_{m_{d-1}}(m_{d-2}+2s_{d-2})$,

where $R^{M;0}_{m_{d-1}}(m_{d-2}+2s_{d-2})$ is given by (44). Thus, if we define $s_{d-1} = rM$, it holds that $s_{d-1} \in \Delta_{d-1}$, where Δ_{d-1} is defined by (46).
Take now $1 \le j \le d-2$ and let $h_{j-1} : [-1,1] \to \mathbb{R}$ be a polynomial function of degree strictly smaller than $2Q_{j-1}$. The Gauss-Gegenbauer quadrature rule leads thus to

$$\sum_{k_{j-1}=0}^{Q_{j-1}-1} w^{(j)}_{k_{j-1}} \left(\sin\vartheta^{(j)}_{k_{j-1}}\right)^{d-j} h_{j-1}\left(\cos\vartheta^{(j)}_{k_{j-1}}\right) = \sum_{k_{j-1}=0}^{Q_{j-1}-1} \omega^{(j)}_{k_{j-1}} h_{j-1}\left(t^{(j)}_{k_{j-1}}\right) = \int_{-1}^{1} h_{j-1}(t)\,\mathrm{d}t. \tag{57}$$

Then, for $0 \le m_{j-1} \le (Q_{j-1}-1)$ and $s_{j-1} \in \mathbb{Z} \cap [-m_{j-1}/2, Q_{j-1}-m_{j-1})$, (57) holds with

$$h_{j-1}(t) = \left(1-t^2\right)^{(m_j+s_j)} C^{\left(m_j+\frac{d-j}{2}\right)}_{m_{j-1}-m_j}(t)\, C^{\left(m_j+2s_j+\frac{d-j}{2}\right)}_{m_{j-1}+2s_{j-1}-(m_j+2s_j)}(t),$$

a polynomial of degree $2(m_{j-1}+s_{j-1}) < 2Q_{j-1}$. Hence, from the orthogonality of the Gegenbauer polynomials (cf. (7)), it follows that

$$\begin{aligned} I^{Q_{j-1}}_{m_{j-1},m_j}(m_{j-1}+2s_{j-1}, m_j) &= \int_{-1}^{1} C^{\left(m_j+\frac{d-j}{2}\right)}_{m_{j-1}-m_j}(t)\, C^{\left(m_j+\frac{d-j}{2}\right)}_{m_{j-1}+2s_{j-1}-m_j}(t) \left(1-t^2\right)^{m_j+\frac{d-j-1}{2}} \\ &= \frac{\pi 2^{1-2\left(m_j+\frac{d-j}{2}\right)} \Gamma(m_{j-1}+m_j+d-j)}{(m_{j-1}-m_j)!\left(m_{j-1}+\frac{d-j}{2}\right)\Gamma^2\left(\left(m_j+\frac{d-j}{2}\right)\right)} \delta^0_{s_{j-1}}. \end{aligned} \tag{58}$$

Thus, $I^{Q_{j-1}}_{m_{j-1},m_j}(m_{j-1}+2s_{j-1}, m_j)$ is annihilated for $s_j = 0$ and $-m_{j-1}/2 \le s_{j-1} \le Q_{j-1}-m_{j-1}$, $s_{j-1} \neq 0$.
For any $j = 1, \ldots, d-2$, aliases $a_{\ell+s_0,\mathbf{m}+\mathbf{s}}$ exist for

- $s_{j-1} \in A_{j-1}$ and $s_j \in H^{(j);0}_{m_j}(m_{j-1}+2s_{j-1})$;
- $s_{j-1} \in B_j$ and $s_j \in H^{(j)}_{m_j}(m_{j-1}+2s_{j-1})$,

where $H^{(j);0}_{m_j}(m_{j-1}+2s_{j-1})$ is given by (43). In other words, for any $j = 1, \ldots, d-2$, it holds that $s_j \in \Delta_j$, where Δ_j is defined by (45).
Recombining all these results for $j = 1, \ldots, d$ yields to the fact that the aliases $a_{\ell+2s_0,\mathbf{m}+2\mathbf{s}}$ exist for $\mathbf{s} \in Z^{\mathbf{Q}}_{\ell,\mathbf{m}}$, where $Z^{\mathbf{Q}}_{\ell,\mathbf{m}}$ is defined by (47), as well as for $s_0 \in D_0(\ell)$ (cf. Part 1), as claimed. □

Proof of Theorem 5.1. Let us fix $\ell \ge 0$ and $\mathbf{m} \in \mathcal{M}_\ell$, and recall furthermore that the random variables $\left\{a_{\ell+2s_0,\mathbf{m}+\mathbf{s}}, s_0 \in D_0(\ell), \mathbf{s} \in Z^{\mathbf{Q}}_{\ell,m}\right\}$ are uncorrelated with variance $C_{\ell+2s_0}$. The variance of $\tilde{a}_{\ell,\mathbf{m}}$ is, thus, given by

$$\begin{aligned} \operatorname{Var}(\tilde{a}_{\ell,\mathbf{m}}) &= \sum_{s_0 \in D_0(\ell)} \sum_{\mathbf{s} \in Z^{\mathbf{Q}}_{\ell,\mathbf{m}}} \left(\prod_{j=1}^{d-1} h^2_{m_{j-1},m_j;j} h^2_{m_{j-1}+2s_{j-1},m_j+2s_j;j} \left(I^{Q_{j-1}}_{m_{j-1},m_j}(m_{j-1}+2s_{j-1}, m_j+2s_j)\right)^2 \right) \\ &\quad \cdot \operatorname{Var}\left(a_{\ell+2s_0,m_1+2s_1,\ldots,m_{d-1}+2s_{d-1}}\right) \\ &= \sum_{s_0 \in D_0(\ell)} \sum_{\mathbf{s} \in Z^{\mathbf{Q}}_{\ell,\mathbf{m}}} \left(\prod_{j=1}^{d-1} h^2_{m_{j-1},m_j;j} h^2_{m_{j-1}+2s_{j-1},m_j+2s_j;j} \left(I^{Q_{j-1}}_{m_{j-1},m_j}(m_{j-1}+2s_{j-1}, m_j+2s_j)\right)^2 \right) C_{\ell+2s_0} \\ &= \sum_{s_0 \in D_0(\ell)} V^{Q}_{\ell,'\mathbf{m}}(\ell')\, C_{\ell+2s_0}. \end{aligned}$$

Using this result in (50) completes the proof. □

Proof of Theorem 6.1. First of all, let us consider the harmonic coefficient $a_{\ell,\mathbf{m}}$ and study its aliases, denoted by $a_{\ell',\mathbf{m}'}$, under Condition 3.3, with $Q = Q_0 = \ldots = Q_{d-2} > L_0$ and $M > L_0$. For any $\ell' \ge m'_1 \ge \ldots \ge m'_{d-2}$, note that

$$a_{\ell',\mathbf{m}'} = a_{\ell',m'_1,\ldots,m'_{d-2},m'_{d-1}} = 0, \quad \text{for any } m'_{d-1} > M > L_0.$$

Thus $a_{\ell,m_1,\ldots,m_{d-2},m_{d-1}+2rM}=0$ for any $r\neq 0$. Recalling that

$$a_{\ell',m'_1,\ldots,m'_{d-2},m_{d-1}} \quad \text{for any } m'_{d-2}\geq Q>L_0,$$

we obtain that

$$a_{\ell',m'_1,\ldots,m_{d-2}+2s_{d-2},m_{d-1}}=0 \quad \text{for any } s_{d-2}\geq Q-m_{d-2}.$$

Using now (58) leads to $s_{d-2}=0$. Reiterating this backward procedure for the other harmonic numbers m'_j, $j=d-3,\ldots,1$ and ℓ' yields (51).
To prove (52), it suffices to use the band-width in the expansion (10), that is,

$$T(x)=\sum_{\ell=0}^{L}\sum_{m\in\mathcal{M}_\ell}\tilde{a}_{\ell,\mathbf{m}}Y_{\ell,\mathbf{m}}(x).$$

Using now in the equation above (28), (33), and (51) yields the claimed result. □

Proof of Theorem 6.2. First, since the power spectrum is band-limited, it holds that $C_{\ell+2s_0}=0$ for $s_0\geq(Q-\ell)/2$. Furthermore, for $0\leq\ell\leq Q$ and $\mathbf{m}\in\mathcal{M}_\ell$, if $s_0\in[-\ell/2,(Q-\ell)/2-1]$, we obtain that

$$s_1\in\left[-m_1/2,\frac{\ell-m_1}{2}+s_0\right]\subseteq\left[-m_1/2,\frac{Q-m_1}{2}-1\right].$$

Consequently, simple algebraical manipulations leads to

$$s_j\in\left[-m_j/2,\frac{\ell-m_j}{2}+s_{j-1}\right]\subseteq\left[-m_j/2,\frac{Q-m_j}{2}-1\right],$$

for any $j=1,\ldots,d-2$.
Thus, it follows that, for $s_{d-2}\in\left[-m_d-2/2,\frac{Q-m_d}{2}-1\right]$ and $Q\geq M>P_L$, $R^M_{m_{d-1}}(m_{d-2}+s_{d-2})=\{0\}$, and, then, $r=0$. Then, by using (58) backward from $j=d-2$ to $j=1$ with any element of the product in (32) yields $s_j=0$ for $j=0,\ldots,d-2$. It follows that $V^{\mathbf{Q}}_{\ell,\mathbf{m}}(\ell')=0$ and $\mathrm{Var}(\tilde{a}_{\ell,\mathbf{m}})=C_\ell=\mathrm{Var}(a_{\ell,\mathbf{m}})$, as claimed. □

7.2. **Proofs of the auxiliary results.**

Proof of Lemma 3.4. The symmetry of the sampling angles follow the symmetry of the roots of the Gegenbauer polynomials. Furthermore, note that

$$\sin\vartheta^{(j)}_{Q_{j-1}-k_{j-1}-1}=\sin\left(\pi-\vartheta^{(j)}_{k_{j-1}}\right)=\sin\vartheta^{(j)}_{k_{j-1}}.$$

Then, we have that

$$\begin{aligned}
\omega^{(j)}_{Q_{j-1}-k_{j-1}-1}=&\frac{1}{\int_{-1}^{1}(1-t^2)^{d-1-j}\,\mathrm{d}t}\int_{-1}^{1}\left(1-t^2\right)^{d-1-j}\lambda_{Q_{j-1}-k_{j-1}-1}(t)\,\mathrm{d}t\\
=&\frac{1}{\int_{-1}^{1}(1-t^2)^{d-1-j}\,\mathrm{d}t}\int_{-1}^{1}\left(1-t^2\right)^{d-1-j}\prod_{i=0,i\neq(Q_{j-1}-k_{j-1}-1)}^{r-1}\frac{t-t_i}{t_i-t_{Q_{j-1}-k_{j-1}-1}}\,\mathrm{d}t\\
=&\frac{1}{\int_{-1}^{1}(1-t^2)^{d-1-j}\,\mathrm{d}t}\int_{-1}^{1}\left(1-t^2\right)^{d-1-j}\prod_{i=0,i\neq(k_{j-1})}^{r-1}\frac{t-t_i}{t_i-t_{k_{j-1}}}\,\mathrm{d}t\\
=&\omega^{(j)}_{k_{j-1}}.
\end{aligned}$$

so that $w^{(j)}_{k_{j-1}}=w^{(j)}_{Q_{j-1}-k_{j-1}-1}$, as claimed. □

Proof of Lemma 3.5. First of all, note that this result for $d=2$, involving thus Legendre polynomials, has been already claimed in [LN97, Theorem 2.1].

As far as $d>2$ is concerned, let us preliminarily recall that, for $t\in[-1,1]$, $C_n^{(\alpha)}(-t)=(-1)^n C_n^{(\alpha)}(t)$ (see, for example, [Sze75, Formula 4.7.4]). Thus, simple trigonometric identities yield

$$\begin{aligned} G_j(\pi-\psi) =& C_{m_{j-1}-m_j}^{\left(m_j+\frac{d-j}{2}\right)}(\cos(\pi-\psi))\, C_{m'_{j-1}-m'_j}^{\left(m'_j+\frac{d-j}{2}\right)}(\cos(\pi-\psi))\sin(\pi-\psi)^{d-j} \\ =& C_{m_{j-1}-m_j}^{\left(m_j+\frac{d-j}{2}\right)}(-\cos\psi)\, C_{m'_{j-1}-m'_j}^{\left(m'_j+\frac{d-j}{2}\right)}(-\cos\psi)(\sin\psi)^{d-j} \\ =& (-1)^{m_{j-1}+m'_{j-1}-m_j-m'_j} C_{m_{j-1}-m_j}^{\left(m_j+\frac{d-j}{2}\right)}(\cos\psi)\, C_{m'_{j-1}-m'_j}^{\left(m'_j+\frac{d-j}{2}\right)}(\cos\psi)\left(\sin C_{m'_{j-1}-m'_j}^{\left(m'_j+\frac{d-j}{2}\right)}\right)^{d-j} \\ =& (-1)^{m_{j-1}+m'_{j-1}-m_j-m'_j} G_j(\psi), \end{aligned}$$

as claimed.

In order to prove (26), consider initially only even values of Q. Hence, by means of Lemma 3.4, we have that

$$\begin{aligned} \sum_{k=0}^{Q-1} w_k G_j(\psi_k) &= \sum_{k=0}^{[Q/2]} \left(w_k G_j(\psi_k) + w_{Q-k-1} G_j(\psi_{Q-k-1})\right) \\ &= \sum_{k=0}^{[Q/2]} w_k \left(G_j(\psi_k) + G_j(\pi-\psi_k)\right) \\ &= \sum_{k=0}^{[Q/2]} w_k \left(G_j(\psi_k) + (-1)^{2c+1} G_j(\psi_k)\right) = 0. \end{aligned}$$

Moreover, if Q is odd, since sampling points have to be symmetric with respect to $\pi/2$, the additional point with respect to the previous case has to coincide with $\pi/2$. Thus $G(\pi/2)=0$ and (26) holds, as claimed. □

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