Nonparametric analysis of covariance using quantile curves

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Abstract

We consider the problem of testing the equality of J quantile curves from independent samples. A test statistic based on an L^2-distance between non-crossing nonparametric estimates of the quantile curves from the individual samples is proposed. Asymptotic normality of this statistic is established under the null hypothesis, local and fixed alternatives, and the finite sample properties of a bootstrap based version of this test statistic are investigated by means of a simulation study.

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1 Introduction

In recent years quantile regression models have found considerable applications, in particular in medicine, economics and environment modelling [see Yu et. al. (2003) or Koenker (2005)], because in contrast to mean regression quantile regression models are robust to outliers and require weaker assumptions on the data generating process. Following Koenker and Bassett (1978), quantile regression can be considered as a supplement to least squares methods and yield a great extension of parametric and nonparametric regression methods. Many authors propose parametric quantile regression models because of their simplicity and – in some cases – interpretability of the parameters. On the other hand, if a parametric model is not appropriate, nonparametric estimation methods have also been proposed in the recent literature [see e.g. Yu and Jones (1997, 1998), De Gooijer and Zerom (2003) or Horowitz and Lee (2005) among others]. Because a "correct" parametric specification of the quantile regression can increase the efficiency of the statistical analysis substantially, several authors have proposed specification tests for the hypothesis of a parametric form of quantile regression models [see e.g. Zheng (1998), Bierens and Ginther (2001) or Horowitz and Spokoiny (2001) among others].

The present paper is devoted to the analysis of a response, say Y, across several groups in the presence of covariates. More precisely, we will investigate the problem of comparing $J \geq 2$ independent samples, say $\{(X_{i1}, Y_{i1})\}_{i=1}^{n_1}, \ldots, \{(X_{iJ}, Y_{iJ})\}_{i=1}^{n_J}$ using nonparametric quantile regression techniques. An important question in this context is whether the data are poolable. Some effort has been spent on nonparametric analysis of covariance using mean regression [see Hall and Hart (1990), Härdle and Marron (1990), King, Hart and Wehrly (1991), Delgado (1993), Dette and Munk (1998), Dette and Neumeyer (2001), Kulasekera (1995), Young and Bowman (1995) among others]. Our work in this area is motivated by the fact that the methods for nonparametric analysis of covariance based on mean regression are usually not robust with respect to outliers. Consider for example the problem of testing the equality of two nonparametric regression functions $H_0 : g_1 = g_2$ from independent samples. If $g_1(t) = \cos(\pi t),\ g_2(t) = \cos(\pi t) + t$ the wild bootstrap test proposed by Dette and Neumeyer (2001) yields the rejection probabilities

$$91.6\% \qquad 96.1\% \qquad 98.3\%$$

for the level $5\%, 10\%$ and 20%, respectively, where the the sample size of each sample is 50 and the errors are centered normally distributed errors with variance $\sigma^2 = 0.5$. However, if 20% of the errors are replaced by Cauchy distributed random variables multiplied by σ, the power of the test drops dramatically and is given by

$$36.0\% \qquad 42.9\% \qquad 49.7\%.$$

This example indicates that it is necessary to use more robust methods for the nonparametric analysis of covariance, and quantile regression offers an interesting alternative. Despite these observations the problem of comparing different samples using quantile regression has not found much attention in the statistical literature. To our knowledge, the problem of comparing conditional median regression has been considered by Batalgi, Hidalgo and Li (1996) and Lavergne (2001). A test for comparing other conditional quantile curves than the median curves has been investigated by Sun (2006), who proposed a test generalizing ideas of Zheng (1996).

In the present paper we present an alternative approach to the problem of comparing nonparametric conditional quantile curves. Our work is motivated by several observations. First, the approach of Sun (2006) requires the choice of d additional bandwidths (where d is the dimension of the predictor), which are not used directly for the estimation of the conditional quantile curves. Second, the tests proposed in the references are based on estimates of the conditional quantile curves which may cross, and it is not clear how the power is affected by this crossing. Third, it is known that the tests based on the approach of Zheng (1996) are usually less efficient than tests based on the L^2-distance [see e.g. Dette and van Lieres and Wilkau (2001)]. A further difference between the cited references and the present work is that we also investigate the asymptotic distribution of the proposed test statistic under fixed alternatives. Results of this type are important for studying the power of the test and for the construction of tests of precise hypotheses in the sense of Berger and Delampady (1987) as demonstrated in Sections 3 and 4. The paper will be organized as follows. In Section 2 we introduce the model, the testing problem and the test statistic considered in this paper. In Section 3 we discuss the asymptotic theory. The finite sample properties of a bootstrap version of the proposed test are investigated by means of a simulation study in Section 4. Finally, all technical details are deferred to an appendix.

2 An L^2-distance between non-crossing quantile curves

We consider J independent samples, say

$$\{(X_{i1}, Y_{i1})_{i=1}^{n_1}\}, \dots, \{(X_{iJ}, Y_{iJ})_{i=1}^{n_J}\}, \tag{2.1}$$

where for each $j = 1, \dots, J$ the random variables $(X_{1j}, Y_{1j}), \dots, (X_{n_j j}, Y_{n_j j})$ are independent identically distributed. We assume that the explanatory variable X_{ij} has a continuous and positive density, say f_j, on the interval $[0, 1]$. The restriction to a one dimensional predictor is made for the sake of a transparent presentation, and the general case will be briefly mentioned in Remark 3.6. Throughout this paper let $F_j(y|x) = P(Y_{1j} \leq y | X_{1j} = x)$ denote the conditional distribution function of Y_{ij} given $X_{ij} = x$, and assume that it has a density, say $f_{j,Y}(y|x)$, which is continuous in both arguments. For fixed $p \in (0, 1)$ let $F_j^{-1}(p|x)$ denote the corresponding conditional quantile function ($j = 1, \dots, J$). We are interested in the hypothesis that the data can be pooled for the estimation of the conditional $p-$quantile curve, that is

$$H_0 : F_1^{-1}(p|\cdot) = \dots = F_J^{-1}(p|\cdot) \quad \text{versus} \quad H_1 : F_i^{-1}(p|\cdot) \neq F_j^{-1}(p|\cdot) \text{ for some } i \neq j. \tag{2.2}$$

The test statistic proposed in this paper will be based on an appropriate estimate of the quantity

$$M^2 := \sum_{j=1}^{J} \sum_{i=1}^{j-1} \int (F_i^{-1}(p|t) - F_j^{-1}(p|t))^2 w_{ij}(t) dt, \tag{2.3}$$

where $w_{ij}(\cdot)$ denote strictly positive weight functions. Note that the null hypothesis is satisfied if and only if $M^2 = 0$, and as a consequence it is reasonable to reject the null hypothesis if an estimator of M^2 attains a large value.

Note that estimating M^2 requires appropriate nonparametric estimates of the conditional quantile functions. Several such estimators have been proposed in the literature [see e.g. Yu and Jones (1997, 1998), Takeuchi, Le, Sears and Smola (2006) or Dette and Volgushev (2008) among others]. In this paper we follow the last-named authors who proposed non-crossing estimates of quantile curves using a simultaneous inversion and isotonization of an estimate of the conditional distribution function. To be precise, let

$$\widehat{F}_j(y|x) := \sum_{k=1}^{n_j} \tilde{w}_{kj}(x) I\{Y_{kj} \leq y\} \tag{2.4}$$

denote a nonparametric estimate of the conditional distribution function, where the quantities $\tilde{w}_{kj}$ are either the Nadaraya-Watson weights, i.e.

$$\tilde{w}_{kj}(x) := \frac{K_r\left(\frac{X_{kj}-x}{h_r}\right)}{\sum_{l=1}^{n_j} K_r\left(\frac{X_{lj}-x}{h_r}\right)} \tag{2.5}$$

or the local linear weights, i.e.

$$\begin{aligned} \tilde{w}_{kj}(x) &:= \frac{K_r\left(\frac{X_{kj}-x}{h_r}\right)(S_{j,2}(x)-(x-X_{kj})S_{j,1}(x))}{S_{j,2}(x)S_{j,0}(x)-S_{j,1}^2(x)}, \\ S_{j,i}(x) &:= \sum_{l=1}^{n_j} K_r\Big(\frac{x-X_{lj}}{h_r}\Big)(x-X_{lj})^i \quad i=0,1,2. \end{aligned} \tag{2.6}$$

In (2.5) and (2.6) K_r denotes a nonnegative kernel and h_r is a bandwidth converging to 0 with increasing sample sizes. Following Dette and Volgushev (2008) we consider a strictly increasing distribution function $G : \mathbb{R} \to (0,1)$, a nonnegative kernel K_d with bandwidth h_d, and define for $j = 1, \dots, J$

$$\widehat{H}_j^{-1}(p|x) := \frac{1}{N_j h_d} \sum_{k=1}^{N_j} \int_{-\infty}^{p} K_d\left(\frac{\widehat{F}_j\left(G^{-1}(\frac{k}{N_j})|x\right)-u}{h_d}\right) du\,, \tag{2.7}$$

where $N_j \in \mathbb{N}$ and $\hat{F}_j$ is the Nadaraya-Watson or local linear estimate of the conditional distribution function from the jth sample defined by (2.4). Note that it is intuitively clear that $\hat{H}_j^{-1}(p|x)$ is a consistent estimate of

$$H_{h_d,j}^{-1}(p|x) := \frac{1}{h_d}\int_0^1 \int_{-\infty}^{p} K_d\left(\frac{F_j(G^{-1}(v|x))-u}{h_d}\right) du dv\,. \tag{2.8}$$

If $h_d \to 0$, the right hand side of this equation can be approximated as follows

$$\begin{aligned} H_{h_d,j}^{-1}(p|x) &\approx H_j^{-1}(p|x) := \int_{\mathbb{R}} I\{F_j(y|x) \le p\} dG(y) \\ &= \int_0^1 I\{F_j(G^{-1}(v|x)) \le p\} dv \;=\; G \circ F_j^{-1}(p|x), \end{aligned} \tag{2.9}$$

and as a consequence an estimate of the conditional quantile function can be defined by

$$\widehat{F}_j^{-1}(p|x) := G^{-1}(\widehat{H}_j^{-1}(p|x)). \tag{2.10}$$

For the two bandwidths h_d and h_r we assume throughout this paper

$$nh_r^5 \to c\,; \qquad h_d = o(h_r) \tag{2.11}$$

for $n \to \infty$ and a given constant $c \geq 0$. Moreover, if $n = \sum_{j=1}^{J} n_j$ denotes the total sample size, we assume for the relative sample sizes of the different groups

$$\lim_{n\to\infty} \frac{n_j}{n} = a_j \in (0,1); \qquad j = 1, \ldots, J\ ; \tag{2.12}$$

and $n_j = O(N_j)$ for each $j = 1, \ldots, J$.

The estimate of the quantity M^2 is now defined in an obvious manner, that is

$$T_n = \int \sum_{j=1}^{J} \sum_{i=1}^{j-1} (\hat{F}_j^{-1}(p|t) - \hat{F}_i^{-1}(p|t))^2 \hat{w}_{ij}(t) dt\ . \tag{2.13}$$

Here $\hat{F}_j^{-1}(p|t)$ corresponds either to the Nadaraya-Watson estimator with the quantity $\hat{w}_{ij}$ defined by

$$\hat{w}_{ij}(x) = \left(\widehat{f}_i(x)\widehat{f}_j(x)\right)^2 \tag{2.14}$$

with $\widehat{f}_i$ being a kernel density estimator of f_i or to the local linear estimator, where we set

$$\hat{w}_{ij}(x) = \frac{1}{n_i^4 n_j^4 h_r^{16}} \left((S_{i,2}S_{i,0} - S_{i,1}^2)(S_{j,2}S_{j,0} - S_{j,1}^2)\right)^2 (x)\ . \tag{2.15}$$

The two statistics corresponding to (2.14) and (2.15) will be denoted by T_n^{NW} and T_n^{LL} throughout this paper. It is intuitively clear that T_n^{NW} and T_n^{LL} are consistent estimates of

$$M_{NW}^2 = \sum_{j=1}^{J} \sum_{i=1}^{j-1} \int (F_i^{-1}(p|t) - F_j^{-1}(p|t))^2 f_i^2(t) f_j^2(t) dt\ , \tag{2.16}$$

and

$$M_{LL}^2 = \mu_2^4(K_r) \sum_{j=1}^{J} \sum_{i=1}^{j-1} \int (F_i^{-1}(p|t) - F_j^{-1}(p|t))^2 f_i^4(t) f_j^4(t) dt\ , \tag{2.17}$$

respectively. In the following section we investigate the asymptotic properties of the statistics T_n^{NW} and T_n^{LL}.

3 Weak convergence under H_0 and H_1

For the investigations of the asymptotic properties of the statistics T_n^{NW} and T_n^{LL} we require, besides the assumptions stated in Section 2, the following basic assumptions:

(A) The function G is strictly increasing, twice continuously differentiable and the second derivative of G^{-1} is bounded on every interval $[a, b]$ with $0 < a \leq b < 1$.

(B) The density f_j of X_{1j} is twice differentiable and f_j'' is bounded.

(C) The conditional distribution function $F_j(y|x)$ is three times differentiable with respect to both arguments. The kth partial derivatives with respect to y or x are denoted by ∂_1^k or ∂_2^k, respectively, and we assume that the derivatives $\partial_2(F_j(y|x))$, $\partial_2^2(F_j(y|x))$, $\partial_2^3(F_j(y|x))$ and $\partial_1^3(F_j(y|x))$. Moreover, we assume that $\inf_x f_{j,Y}(F_j^{-1}(p|x)|x) > 0$.

(D) The conditional quantile function $F_j^{-1}(y|x)$ is twice differentiable with respect to x with bounded second derivative.

(E) The kernels K_r and K_d are symmetric, bounded, nonnegative and their support is given by the interval $[-1, 1]$. Additionally, K_d is twice continuously differentiable on the interval $(-1, 1)$ and K_d'' is Lipschitz continuous. For $i, j \in \mathbb{N}$ we use the notation

$$\mu_i(K) = \int K(u)u^i du \quad \text{and} \quad \mu_i^{(j)}(K) = \int K^j(u)u^i du$$

and assume

$$\mu_0(K_r) = \mu_0(K_d) = 1.$$

(F) The bandwidths h_d and h_r satisfy

$$h_r = o(h_d^{3/4}), \quad h_d = o(h_r^{5/4}). \tag{3.1}$$

To illustrate assumption (3.1), consider the case where h_r is proportional to the optimal bandwidth, i.e. $h_r \sim n^{-1/5}$. In this case one could use $h_d = b_n h_r^{5/4}$, where b_n is a sequence converging to 0 such that $b_n n^{1/80} \to \infty$. Our first result states the asymptotic distribution of the test statistic T_n in the two sample case (i.e. $J = 2$) under the null hypothesis H_0 of equal quantile curves.

Theorem 3.1. *If $J = 2$ and the assumptions stated in Section 2 as well as assumptions (A) - (F) are satisfied, then under the null hypothesis we have*

$$n\sqrt{h_r}\left(T_n^{NW} - h_r^4 B_1^{NW} - \frac{1}{nh_r} B_2^{NW}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, V^{NW}), \tag{3.2}$$

where the terms B_1^{NW} and B_2^{NW} are defined as

$$\begin{aligned} B_1^{NW} &= \int_0^1 (f_2(x)C_1(x) - f_1(x)C_2(x))^2 dx, \\ B_2^{NW} &= p(1-p)\mu_0^{(2)}(K_r) \int_0^1 \left(\left(\partial_1(F_2^{-1}(p|x))\right)^2 \frac{f_1^2(x)f_2(x)}{a_2} + \left(\partial_1(F_1^{-1}(p|x))\right)^2 \frac{f_1(x)f_2^2(x)}{a_1} \right) dx, \\ C_j(x) &= \partial_1(F_j^{-1}(p|x))\mu_2(K_r) \left(\partial_2(F_j(F_j^{-1}(p|x)|x))f_j'(x) + \frac{1}{2}\partial_2^2(F_j(F_j^{-1}(p|x)|x))f_j(x) \right), \end{aligned}$$

and the asymptotic variance is given by

$$\begin{aligned}
V^{NW} &= 2p(1-p)\Big\{p(1-p)\int (K_r * K_r)^2(u)du \\
&\times \int_0^1 \Big(\big(\partial_1(F_1^{-1}(p|x))\big)^2 \frac{f_1(x)f_2^2(x)}{a_1} + \big(\partial_1(F_2^{-1}(p|x))\big)^2 \frac{f_1^2(x)f_2(x)}{a_2}\Big)^2 dx \\
&+2c\int_0^1 \Big(\big(\partial_1(F_1^{-1}(p|x))\big)^2 \frac{f_1(x)f_2^2(x)}{a_1} + \big(\partial_1(F_2^{-1}(p|x))\big)^2 \frac{f_1^2(x)f_2(x)}{a_2}\Big) \\
&\times (f_2(x)C_1(x) - f_1(x)C_2(x))^2 dx\Big\}.
\end{aligned}$$

Similarly, if the local linear estimate is used as initial estimate for the conditional distribution function, it follows that

$$n\sqrt{h_r}\left(T_n^{LL} - h_r^4\, B_1^{LL} - \frac{1}{nh_r}B_2^{LL}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, V^{LL}), \tag{3.3}$$

where the terms B_1^{LL} and B_2^{LL} are defined as

$$\begin{aligned}
B_1^{LL} &= \mu_2^6(K_r)\int_0^1 f_1^4(x)f_2^4(x)(\overline{C}_1(x) - \overline{C}_2(x))^2 dx, \\
B_2^{LL} &= p(1-p)\mu_0^{(2)}(K_r)\mu_2^4(K_r) \\
&\times \int_0^1 \Big(\big(\partial_1(F_1^{-1}(p|x))\big)^2 \frac{f_1^3(x)f_2^4(x)}{a_1} + \big(\partial_1(F_2^{-1}(p|x))\big)^2 \frac{f_1^4(x)f_2^3(x)}{a_2}\Big) dx, \\
\overline{C}_j(x) &= \frac{1}{2}\partial_1(F_j^{-1}(p|x))\partial_2^2(F_j(F_j^{-1}(p|x)|x)),
\end{aligned}$$

and the asymptotic variance has the form

$$\begin{aligned}
V^{LL} &= 2p(1-p)\mu_2^8(K_r)\Big\{p(1-p)\int_0^1 (K_r * K_r)^2(u)du \\
&\times \int_0^1 \Big(\big(\partial_1(F_1^{-1}(p|x))\big)^2 \frac{f_1^3(x)f_2^4(x)}{a_1} + \big(\partial_1(F_2^{-1}(p|x))\big)^2 \frac{f_1^4(x)f_2^3(x)}{a_2}\Big)^2 dx \\
&+2c\mu_2^2(K_r)\int_0^1 \Big(\big(\partial_1(F_1^{-1}(p|x))\big)^2 \frac{f_2(x)}{a_1} + \big(\partial_1(F_2^{-1}(p|x))\big)^2 \frac{f_1}{a_2}\Big) f_1^7(x)f_2^7(x) \\
&\times (\overline{C}_1(x) - \overline{C}_2(x))^2 dx\Big\}.
\end{aligned}$$

The proof of Theorem 3.1 is complicated and therefore deferred to the Appendix. In the following we discuss the asymptotic properties of the statistic T_n under local and fixed alternatives. In the case of local alternatives of the form

$$F_1^{-1}(p|x) = F_2^{-1}(p|x) + \frac{g(x,p)}{\sqrt{n}\, h_r^{1/4}}$$

it follows by a careful inspection of the proof of Theorem 3.1 that

$$n\sqrt{h_r}\left(T_n^{NW} - h_r^4\, B_1^{NW} - \frac{1}{nh_r} B_2^{NW}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(\gamma_{NW}^2(p), V^{NW})$$
$$n\sqrt{h_r}\left(T_n^{LL} - h_r^4\, B_1^{LL} - \frac{1}{nh_r} B_2^{LL}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(\gamma_{LL}^2(p), V^{LL})$$

where $B_1^{NW}, B_2^{NW}, V^{NW}, B_1^{LL}, B_2^{LL}, V^{LL}$ are defined in Theorem 3.1 and the quantities $\gamma_{NW}^2(p)$ and $\gamma_{LL}^2(p)$ are given by

$$\gamma_{NW}^2(p) = \int_0^1 f_1^2(x) f_2^2(x) g^2(x,p) dx, \quad \gamma_{LL}^2(p) = \mu_2^4(K_r)\int_0^1 f_1^4(x) f_2^4(x) g^2(x,p) dx,$$

respectively. The following result considers the asymptotic properties under a fixed alternative. In this case the statistics T_n^{NW} and T_n^{LL} are also asymptotically normal distributed, where they have to be centered by M_{NW}^2 respectively M_{LL}^2 and the variance is of order n^{-1}.

Theorem 3.2. *If $J = 2$, the assumptions stated in Section 2 and assumptions (A) - (F) are satisfied, we have under a fixed alternative*

$$\sqrt{n}\left(T_n^{NW} + h_r^2(\tilde{B}_1^{NW} - \tilde{B}_2^{NW}) - M_{NW}^2\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tilde{V}^{NW}), \tag{3.4}$$

where M_{NW}^2 is defined in (2.16), the terms $\tilde{B}_1^{NW}$ and $\tilde{B}_2^{NW}$ are given by

$$\begin{aligned}\tilde{B}_1^{NW} &= 2\int_0^1 (f_1(x)C_2(x) - f_2(x)C_1(x))\,(F_2^{-1}(p|x) - F_1^{-1}(p|x)) f_1(x) f_2(x) dx,\\ \tilde{B}_2^{NW} &= \mu_2(K_r)\int_0^1 \left(f_2^2(x)f_1(x)f_1''(x) + f_1^2(x)f_2(x)f_2''(x)\right)\left(F_2^{-1}(p|x) - F_1^{-1}(p|x)\right)^2 dx\,,\end{aligned}$$

and the quantities C_1 and C_2 are defined in Theorem 3.1. The asymptotic variance is given by

$$\begin{aligned}\tilde{V}^{NW} &= 4p(1-p)\sum_{j=1}^2 \frac{1}{a_j} E\left[\left(\frac{\partial_1(F_j^{-1}(p|X_{1j}))}{f_j(X_{1j})}\right)^2 f_1^4(X_{1j}) f_2^4(X_{1j})(F_2^{-1}(p|X_{1j}) - F_1^{-1}(p|X_{1j}))^2\right]\\ &\quad +4\sum_{j=1}^2 \frac{1}{a_j} Var\left(\frac{f_1^2(X_{1j}) f_2^2(X_{1j})(F_2^{-1}(p|X_{1j}) - F_1^{-1}(p|X_{1j}))^2}{f_j(X_{1j})}\right)\end{aligned}$$

Similarly, if the local linear estimate is used as an initial estimate for the conditional distribution function it follows

$$\sqrt{n}\left(T_n^{LL} - h_r^2\,(\tilde{B}_1^{LL} - \tilde{B}_2^{LL}) - M_{LL}^2\right) \longrightarrow \mathcal{N}(0, \tilde{V}^{LL})\,, \tag{3.5}$$

where the terms $\tilde{B}_1^{LL}$ and $\tilde{B}_2^{LL}$ are given by

$$\tilde{B}_1^{LL} = 2\mu_2^5(K_r)\int_0^1 \left(\bar{C}_2(x) - \bar{C}_1(x)\right)(F_2^{-1}(p|x) - F_1^{-1}(p|x)) f_1^4(x) f_2^4(x) dx,$$

$$\tilde{B}_2^{LL} = \mu_2^3(K_r)\int_0^1 f_1^2(x)f_2^2(x)\left\{f_2^2(x)\left(\frac{1}{2}\mu_4(K_r)f_1^{''}(x)f_1(x)-\mu_2^2(K_r)\left(f_1^{'}\right)^2(x)\right)\right.$$
$$\left.+ f_1^2(x)\left(\frac{1}{2}\mu_4(K_r)f_2^{''}(x)f_2(x)-\mu_2^2(K_r)\left(f_2^{'}\right)^2(x)\right)\right\}\left(F_2^{-1}(p|x)-F_1^{-1}(p|x)\right)^2 dx$$

and the asymptotic variance is given by

$$\tilde{V}^{LL} = 4\mu_2^8(K_r)p(1-p)\sum_{j=1}^2\frac{1}{a_j}E\left[\left(\frac{\partial_1 F_j^{-1}(p|X_{1j})}{f_j(X_{1j})}\right)^2 f_1^8(X_{1j})f_2^8(X_{1j})\left(F_1^{-1}(p|X_{1j})-F_2^{-1}(p|X_{1j})\right)^2\right]$$
$$+16\mu_2^8(K_r)\sum_{j=1}^2\frac{1}{a_j}Var\left(\frac{1}{f_j(X_{1j})}f_1^4(X_{1j})f_2^4(X_{1j})\left(F_1^{-1}(p|X_{1j})-F_2^{-1}(p|X_{1j})\right)^2\right).$$

Remark 3.3. The bias and variance terms in Theorem 3.1 and 3.2 are rather complicated and sdepend on several features of the data generating process. Under additional assumptions these expressions simplify. For example, if the densities of the predictors and the *whole* conditional distributions are identical, i.e. $f_1(x) = f_2(x)$ and $F_2(y|x) = F_1(y|x)$, then it is easy to see that

$$B_1^{NW} = B_1^{LL} = 0$$

in Theorem 3.1 (note that the hypothesis H_0 does not imply equality for the densities of the explanatory variable or the distributions $F_i(y|x)$). Similarly, if $h_r = o(n^{-1/5})$ we have $c = 0$ and the representations of the variances in V^{NW} and V^{LL} in Theorem 3.1 are substantially simpler.

In the remaining part of this section we state the corresponding result in the case of $J \geq 2$ samples. The basic structure of the results is the same, but the corresponding variance terms are substantially more complicated.

Theorem 3.4. *Let the assumptions of Section 2 and assumptions (A) - (F) be satisfied.*

(a) Under the null hypothesis H_0 in (2.1) the weak convergence (3.2) and (3.3) hold, where the terms $B_1^{NW}, B_2^{NW}, V^{NW}$ and $B_1^{LL}, B_2^{LL}, V^{LL}$ are given by

$$B_1^{NW} = \sum_{j=1}^J\sum_{i=1}^{j-1}\int_0^1 (f_j(x)C_i(x)-f_i(x)C_j(x))^2dx,$$
$$B_2^{NW} = p(1-p)\mu_0^{(2)}(K_r)\sum_{j=1}^J\sum_{i=1}^{j-1}\int_0^1\left(\left(\partial_1(F_j^{-1}(p|x))\right)^2\frac{f_i^2(x)f_j(x)}{a_j}\right.$$
$$\left.+\left(\partial_1(F_i^{-1}(p|x))\right)^2\frac{f_i(x)f_j^2(x)}{a_i}\right)dx,$$

$$
\begin{aligned}
V^{NW} &= 2p^2(1-p)^2 \int (K_r * K_r)^2(u)du \left\{ \sum_{j=1}^{J} \sum_{i=1}^{j-1} E \left[\frac{1}{f_i(X_{1i})} \right. \right. \\
&\quad \times \left. \left(\frac{f_i(X_{1i}) f_j^2(X_{1i})}{a_i} \left(\partial_1(F_i^{-1}(p|X_{1i}))\right)^2 + \frac{f_i^2(X_{1i}) f_j(X_{1i})}{a_j} \left(\partial_1(F_j^{-1}(p|X_{1i}))\right)^2 \right)^2 \right] \\
&\quad \left. + 2 \sum_{j=1}^{J} \sum_{i=1}^{j-1} \sum_{k=1}^{i-1} \sum_{l\in\{i,j,k\}} \frac{1}{a_l^2} E \left[\frac{\left(\partial_1(F_l^{-1}(p|X_{1l}))\right)^4}{f_l(X_{1l})} f_i^2(X_{1l}) f_j^2(X_{1l}) f_k^2(X_{1l}) \right] \right\} \\
&\quad +4cp(1-p) \sum_{j=1}^{J} \sum_{i=1}^{j-1} \sum_{l\in\{i,j\}} \frac{1}{a_l} \\
&\quad \times E \left[\left(\frac{\partial_1(F_l^{-1}(p|X_{1l}))}{f_l(X_{1l})} \right)^2 f_i^2(X_{1l}) f_j^2(X_{1l}) \left(f_j(X_{1l}) C_i(X_{1l}) - f_i(X_{1l}) C_j(X_{1l}) \right)^2 \right] \\
&\quad +8cp(1-p) \sum_{j=1}^{J} \sum_{i=1}^{j-1} \sum_{k=1}^{i-1} \sum_{l\in\{i,j,k\}} \frac{1}{a_l} E \left[\frac{\left(\partial_1(F_l^{-1}(p|X_{1l}))\right)^2}{f_l(X_{1l})} f_i(X_{1l}) f_j(X_{1l}) f_k(X_{1l}) \right. \\
&\quad \times \left. \prod_{m\in\{i,j,k\}\setminus\{l\}} \left(f_m(X_{1l}) C_l(X_{1l}) - f_l(X_{1l}) C_m(X_{1l}) \right) \right]
\end{aligned}
$$

and

$$
\begin{aligned}
B_1^{LL} &= \mu_2^6(K_r) \sum_{j=1}^{J} \sum_{i=1}^{j-1} \int_0^1 f_i^4(x) f_j^4(x) \left(\overline{C}_i(x) - \overline{C}_j(x) \right)^2 dx, \\
B_2^{LL} &= p(1-p) \mu_0^{(2)}(K_r) \mu_2^4(K_r) \\
&\quad \times \sum_{j=1}^{J} \sum_{i=1}^{j-1} \int_0^1 \left(\left(\partial_1(F_i^{-1}(p|x))\right)^2 \frac{f_i^3(x) f_j^4(x)}{a_i} + \left(\partial_1(F_j^{-1}(p|x))\right)^2 \frac{f_i^4(x) f_j^3(x)}{a_j} \right) dx \\
V^{LL} &= p^2(1-p)^2 \mu_2^8(K_r) \int_0^1 (K_r * K_r)^2(u)du \left\{ \sum_{j=1}^{J} \sum_{i=1}^{j-1} E \left[\frac{1}{f_i(X_{1i})} \right. \right. \\
&\quad \times \left. \left(\frac{f_i^3(X_{1i}) f_j^4(X_{1i})}{a_i} \left(\partial_1(F_i^{-1}(p|X_{1i}))\right)^2 + \frac{f_i^4(X_{1i}) f_j^3(X_{1i})}{a_j} \left(\partial_1(F_j^{-1}(p|X_{1i}))\right)^2 \right)^2 \right] \\
&\quad \left. + 2 \sum_{j=1}^{J} \sum_{i=1}^{j-1} \sum_{k=1}^{i-1} \sum_{l\in\{i,j,k\}} \frac{1}{a_l^2} E \left[\left(\partial_1(F_l^{-1}(p|X_{1l}))\right)^4 f_i^4(X_{1l}) f_j^4(X_{1l}) f_k^4(X_{1l}) f_l(X_{1l}) \right] \right\} \\
&\quad +4cp(1-p) \mu_2^{10}(K_r) \left\{ \sum_{j=1}^{J} \sum_{i=1}^{j-1} \sum_{l\in\{i,j\}} \frac{1}{a_l} \right.
\end{aligned}
$$

$$E\left[\left(\frac{\partial_1(F_l^{-1}(p|X_{1l}))}{f_l(X_{1l})}\right)^2 f_i^8(X_{1l})f_j^8(X_{1l})\left(\overline{C}_i(X_{1l})-\overline{C}_j(X_{1l})\right)^2\right]$$
$$+2\sum_{j=1}^{J}\sum_{i=1}^{j-1}\sum_{k=1}^{i-1}\sum_{l\in\{i,j,k\}} E\left[\frac{\left(\partial_1(F_l^{-1}(p|X_{1l}))\right)^2}{a_l} f_i^4(X_{1l})f_j^4(X_{1l})f_k^4(X_{1l})f_l^2(X_{1l})\right.$$
$$\left.\left.\times\prod_{m\in\{i,j,k\}\setminus\{l\}}(\overline{C}_l(X_{1l})-\overline{C}_m(X_{1l}))\right]\right\}.$$

(b) *Under a fixed alternative the weak convergence (3.4) and (3.5) hold, where the terms* $\tilde{B}_1^{NW},\tilde{B}_2^{NW},\tilde{V}^{NW}$ *and* $\tilde{B}_1^{LL},\tilde{B}_2^{LL},\tilde{V}^{LL}$ *are defined by*

$$\tilde{B}_1^{NW} = 2\sum_{j=1}^{J}\sum_{i=1}^{j-1}\int_0^1 (f_j(x)C_i(x)-f_i(x)C_j(x))\,(F_j^{-1}(p|x)-F_i^{-1}(p|x))f_i(x)f_j(x)dx,$$

$$\tilde{B}_2^{NW} = \mu_2(K_r)\sum_{j=1}^{J}\sum_{i=1}^{j-1}\int_0^1 \left(f_j^2(x)f_i(x)f_i^{''}(x)+f_i^2(x)f_j(x)f_j^{''}(x)\right)\left(F_j^{-1}(p|x)-F_i^{-1}(p|x)\right)^2 dx,$$

$$\tilde{V}^{NW} = 4p(1-p)\sum_{j=1}^{J}\sum_{i=1}^{j-1}\sum_{l\in\{i,j\}}\frac{1}{a_l}$$
$$\times E\left[\left(\frac{\partial_1(F_l^{-1}(p|X_{1l}))}{f_l(X_{1l})}\right)^2 f_i^4(X_{1l})f_j^4(X_{1l})(F_i^{-1}(p|X_{1l})-F_j^{-1}(p|X_{1l}))^2\right]$$
$$+8p(1-p)\sum_{j=1}^{J}\sum_{i=1}^{j-1}\sum_{k=1}^{i-1}\sum_{l\in\{i,j,k\}} E\left[\frac{1}{a_l}\left(\partial_1(F_l^{-1}(p|X_{1l}))\right)^2\right.$$
$$\left.\times\, f_i^2(X_{1l})f_j^2(X_{1l})f_k^2(X_{1l})\prod_{m\in\{i,j,k\}\setminus\{l\}}(F_l^{-1}(p|X_{1l})-F_m^{-1}(p|X_{1l}))\right]$$
$$+4\sum_{j=1}^{J}\sum_{i=1}^{j-1}\left\{\sum_{l\in\{i,j\}}\frac{1}{a_l}Var\left(\frac{f_i^2(X_{1l})f_j^2(X_{1l})(F_i^{-1}(p|X_{1l})-F_j^{-1}(p|X_{1l}))^2}{f_l(X_{1l})}\right)\right.$$
$$\left.+\,8\sum_{k=1}^{i-1}\sum_{l\in\{i,j,k\}}\frac{1}{a_l}Var\left(f_i(X_{1l})f_j(X_{1l})f_k(X_{1l})\prod_{m\in\{i,j,k\}\setminus\{l\}}(F_l^{-1}(p|X_{1l})-F_m^{-1}(p|X_{1l}))\right)\right\}$$

and

$$\tilde{B}_1^{LL} = 2\mu_2^5(K_r)\sum_{j=1}^{J}\sum_{i=1}^{j-1}\int_0^1 \left(\bar{C}_j(x)-\bar{C}_i(x)\right)(F_j^{-1}(p|x)-F_i^{-1}(p|x))f_j^4(x)f_i^4(x)dx,$$

$$\tilde{B}_2^{LL} = \mu_2^3(K_r)\sum_{j=1}^{J}\sum_{i=1}^{j-1}\int_0^1 f_i^2(x)f_j^2(x)\left\{f_j^2(x)\left(\frac{1}{2}\mu_4(K_r)f_i^{''}(x)f_i(x)-\mu_2^2(K_r)\left(f_i^{'}\right)^2(x)\right)\right.$$

$$+ f_i^2(x)\left(\frac{1}{2}\mu_4(K_r)f_j^{''}(x)f_j(x) - \mu_2^2(K_r)\left(f_j^{'}\right)^2(x)\right)\Bigg\}\left(F_j^{-1}(p|x) - F_i^{-1}(p|x)\right)^2 dx,$$

$$\begin{aligned}
\tilde{V}^{LL} &= 4\mu_2^8(K_r)p(1-p)\Bigg\{\sum_{j=1}^{J}\sum_{i=1}^{j-1}\sum_{l\in\{i,j\}}\frac{1}{a_l} \\
&\times E\left[\left(\frac{\partial_1 F_l^{-1}(p|X_{1l})}{f_l(X_{1l})}\right)^2 f_i^8(X_{1l})f_j^8(X_{1l})\left(F_i^{-1}(p|X_{1l}) - F_j^{-1}(p|X_{1l})\right)^2\right] \\
&+ 2\sum_{j=1}^{J}\sum_{i=1}^{j-1}\sum_{k=1}^{i-1}\sum_{l\in\{i,j,k\}} E\left[\frac{\left(\partial_1(F_l^{-1}(p|X_{1l}))\right)^2}{a_l} f_l^2(X_{1l})f_i^4(X_{1l})f_j^4(X_{1l})f_k^4(X_{1l})\right. \\
&\times \left.\left.\prod_{m\in\{i,j,k\}\setminus\{l\}}(F_l^{-1}(p|X_{1l}) - F_m^{-1}(p|X_{1l}))\right]\right\} \\
&+16\mu_2^8(K_r)\sum_{j=1}^{J}\sum_{i=1}^{j-1}\sum_{l\in\{i,j\}}\frac{1}{a_l}Var\left(\frac{1}{f_l(X_{1l})}f_i^4(X_{1l})f_j^4(X_{1l})\left(F_i^{-1}(p|X_{1l}) - F_j^{-1}(p|X_{1l})\right)^2\right) \\
&+32\mu_2^8(K_r)\sum_{j=1}^{J}\sum_{i=1}^{j-1}\sum_{k=1}^{i-1}\Bigg\{\sum_{l\in\{i,j,k\}} E\left[\frac{1}{a_l}f_l^2(X_{1l})f_i^4(X_{1l})f_j^4(X_{1l})f_k^4(X_{1l})\right. \\
&\times \left.\prod_{m\in\{i,j,k\}\setminus\{l\}}(F_l^{-1}(p|X_{1l}) - F_m^{-1}(p|X_{1l}))^2\right] \\
&-\frac{1}{a_l}\prod_{m\in\{i,j,k\}\setminus\{l\}} E\left[f_m^4(X_{1l})f_l^3(X_{1l})(F_l^{-1}(p|X_{1l}) - F_m^{-1}(p|X_{1l}))^2\right]\Bigg\}.
\end{aligned}$$

Remark 3.5. The results of Theorem 3.1, 3.2 and 3.4 can be used to obtain an asymptotic level α test by rejecting the null hypothesis for large values of T_n^{NW} or T_n^{LL}. A bootstrap version of this test will be investigated by means of a simulation study.
Note that the asymptotic properties of the test statistics under fixed alternatives can be used to study the power function of the resulting test. For example, in the case of $J = 2$ samples, the power of the test, which rejects the null hypothesis for large values of the statistic T_n^{LL}, is approximately given by

$$P\left(H_0 \text{ rejected} \mid H_1 \text{ is true}\right) \approx 1 - \Phi\left(-\frac{\sqrt{n}(M_{LL}^2 + h_r^2(\tilde{B}_1^{LL} - \tilde{B}_2^{LL}))}{\sqrt{\tilde{V}^{LL}}}\right), \quad (3.6)$$

where Φ denotes the distribution function of the standard normal distribution and the quantities $M_{LL}^2, \tilde{B}_1^{LL}, \tilde{B}_2^{LL}$ and $\tilde{V}^{LL}$ are defined in Theorem 3.2 We will use this approximation to explain some of the finite sample properties of the proposed test in the following section.

Remark 3.6. Note that the results of this section can easily be generalized to a multivariate predictor, by simply using a multivariate Nadaraya-Watson or local linear estimate of the conditional distribution function in the initial step [see e.g. Härdle, Müller, Sperlich and Werwatz (2004)] and calculating the L^2-distance over the cube $[0,1]^d$ (where d is the dimension of the predictors and $[0,1]^d$ the support of the corresponding density). The details are omitted for the sake of brevity. However, it should be mentioned here that some care is necessary if the test based on T_n is applied in the case of a multivariate predictor, because of the curse of dimensionality. If $d \geq 3$ it is usually difficult to estimate the conditional quantile curve with sufficient precision, and as a consequence a test for the equality of the conditional quantile curves will not be very accurate.

4 Finite sample properties

In order to investigate the performance of the proposed test for finite samples, we have performed a small simulation study. It is well known that the approximation of the nominal level of tests based on the L^2-distance between two nonparametric estimates is usually rather poor [see e.g. Fan and Linton (2003)]. For this reason we propose to use a smoothed residual bootstrap to obtain critical values. To be precise, let

$$\widehat{U}_{ij} = Y_{ij} - \widehat{F}_j^{-1}(p|X_{ij}) \quad (i = 1, \ldots, n_j;\ j = 1, \ldots, J) \tag{4.1}$$

be the estimated quantile-residuals, where $\widehat{F}_j^{-1}(p|\cdot)$ is the estimator of the $p-$th quantile-function, calculated from the $j-$th sample. We now randomly draw with replacement from the estimated residuals in each sample (name the resulting random variables U_{ij}^*) and add independent normally distributed random variables τ_{ij}, with expectation $\mu_p(\delta)$ and variance δ^2, where $\mu_p(\delta)$ is chosen to guarantee that τ_{ij} has $p-$quantile 0. The obtained quantities $U_{ij}^B = \hat{U}_{ij}^* + \tau_{ij}$ are the required bootstrap residuals. The bootstrap data (X_{ij}^B, Y_{ij}^B) are now defined as

$$\begin{aligned} X_{ij}^B &= X_{ij}, \\ Y_{ij}^B &= \widehat{F}^{-1}(p|X_{ij}) + U_{ij}^B, \end{aligned}$$

where $\widehat{F}^{-1}(p|\cdot)$ is an estimator of the conditional quantile-function calculated from the pooled data. From the bootstrap sample we calculate the bootstrap statistic T_n^B, and the $\alpha-$quantile of the test statistic T_n is estimated on the basis of R bootstrap replications. More precisely, if t^* denotes the $(1-\alpha)$-quantile of the bootstrap sample $T_n^{B(1)}, \ldots, T_n^{B(R)}$, the null hypothesis is rejected if

$$T_n > t^*. \tag{4.2}$$

Remark 4.1. Note that the wild bootsrap proposed by Sun (2006) will not work for the test statistic proposed in this paper. The reason is that the wild bootsrap is constructed to obtain bootstrap residuals with $p-$quantile zero where the second and third moments are as close as possible to the corresponding moments of the true residuals. However, the asymptotical variance of our estimator contains the term $\partial_1 F_j^{-1}(p|x) = 1/f_{j,Y}(F_j^{-1}(p|x)|x)$ and the wild bootsrap residuals do not reproduce this quantity correctly.
In contrast to the wild bootsrap of Sun (2006), the proposed bootsrap procedure aims at producing residuals with density close to the density of the true residuals $Y_i - F_j^{-1}(p|X_i)$. In order to heuristically understand why this actually works, observe that the density of the bootstrap residuals conditional on the data is of the form:

$$f_j^B(y) = \int \phi_{\mu_p(\delta),\delta^2}(u-y)d\hat{F}_j^U(u) = \frac{1}{n\delta}\sum_i \phi_{0,1}\left(\frac{\widehat{U}_{ij} - \mu_p(\delta) - y}{\delta}\right),$$

where $\hat{F}_j^U$ denotes the empirical distribution function of $\widehat{U}_{1j}, \cdots, \widehat{U}_{n_j j}$ and $\phi_{\mu_p(\delta),\delta^2}$ the density of a $\mathcal{N}(\mu_p(\delta), \delta^2)$ random variable. In the case $p = 50\%$, which corresponds to $\mu_p(\delta) = 0$, this is the Kernel density estimator of Parzen (1962) with a Gaussian Kernel and bandwidth δ (for other values of p, this will hold asymptotically since $\mu_p(\delta)$ tends to zero as $\delta \to 0$). Hence the density of the bootstrap residuals conditional on the data is close to the true density of the residuals. This argument demonstrates that the smoothing parameter δ corresponds to a bandwidth in density estimation and should be chosen accordingly. In particular, this motivates the choice (4.7).

Remark 4.2. The bootstrap proposed here only works for i.i.d. errors. However, by replacing the estimator $\hat{F}_j^U$ in Remark 4.1 with a conditional version (which would correspond to locally drawing residuals with replacement) it can easily be extended to the general case.

In the simulation study we compared $J = 2$ quantile curves. The nonparametric estimates $\widehat{F}_j^{-1}(p|\cdot)$ were calculated using local-linear weights, the Epanechnikov kernel

$$K_r(x) = K_d(x) = \frac{3}{4}(1-x^2)I_{[-1,1]}(x) \tag{4.3}$$

and the bandwidths

$$h_{r,j} = \left(\frac{p(1-p)}{\phi(\Phi^{-1}(p))^2}\right)^{1/5}\left(\frac{\sigma^2}{n_j}\right)^{3/10}, \tag{4.4}$$

where σ^2 denotes the variance of the residuals of the data-generating process, and ϕ, Φ denote the density and distribution function from the standard normal distribution, respectively. This choice of bandwidths is motivated by Dette and Neumeyer (2001) and Yu and Jones (1998). The estimate $\widehat{F}^{-1}(p|\cdot)$ from the pooled sample was calculated using the bandwidth

$$h_r = \left(\frac{p(1-p)}{\phi(\Phi^{-1}(p))^2}\right)^{1/5}\left(\frac{\sigma^2}{n_1+n_2}\right)^{3/10} \tag{4.5}$$

Table 1: *Rejection probabilities of the test (4.2) of equal* 50% *quantile curves under the null hypothesis for models (4.8) and (4.9) with normally distributed errors. The numbers in brackets denote the corresponding rejection probabilities of the test of Dette and Neumeyer (2001).*

$p = 0.5$					
model $g_1(t) = g_2(t) = t^2$					
(n_1, n_2)	(25,25)	(25,50)	(50,50)	(50,100)	(100,100)
$\alpha = 5\%$	5.70% (5.5%)	5.94% (5.61%)	5.74% (4.12%)	5.42% (5.42%)	4.94% (4.62%)
$\alpha = 10\%$	10.64% (9.51%)	11.00% (10.62%)	10.56% (7.85%)	9.92% (10.40%)	9.94% (10.05%)
$\alpha = 20\%$	21.00% (18.81%)	20.20% (17.78%)	20.34% (15.71%)	18.88% (20.01%)	19.14% (19.62%)
model $g_1(t) = g_2(t) = \cos(\pi t)$					
$\alpha = 5\%$	5.84% (3.82%)	5.42% (4.57%)	5.90% (4.85%)	5.52% (4.62%)	5.30% (5.11%)
$\alpha = 10\%$	10.44% (8.15%)	10.10% (8.61%)	10.34% (8.93%)	10.42% (8.90%)	10.40% (10.09%)
$\alpha = 20\%$	20.74% (14.6%)	19.82% (16.20%)	19.76% (17.74%)	19.28% (18.71%)	21.30% (20.11%)

Table 2: *Rejection probabilities of the test (4.2) of equal* 25% *quantile curves under the null hypothesis for models (4.8) and (4.9) with normally distributed errors.*

$p = 0.25$					
model $g_1(t) = g_2(t) = t^2$					
(n_1, n_2)	(25,25)	(25,50)	(50,50)	(50,100)	(100,100)
$\alpha = 5\%$	5.64%	5.40%	5.88%	5.82%	5.40%
$\alpha = 10\%$	10.78%	10.68%	11.30%	10.74%	10.68%
$\alpha = 20\%$	20.84%	20.76%	21.12%	21.42%	20.76%
model $g_1(t) = g_2(t) = \cos(\pi t)$					
$\alpha = 5\%$	6.20%	5.60%	5.54%	6.02%	6.90%
$\alpha = 10\%$	11.14%	10.52%	10.50%	10.60%	11.35%
$\alpha = 20\%$	21.42%	20.98%	20.92%	19.92%	20.50%

and we used $h_{r,1}$ and $h_{r,2}$ also for the calculation of the test statistic in the bootstrap procedure. The bandwidth h_d was always chosen as $h_d = h_r^{1.3}$ as in Dette and Volgushev (2008).

The choice of the function G is not very critical (c.f. Dette, Volgushev 2008), and we set G equal to the distribution function of a normally distributed random variable with mean μ_G and varinace σ_G^2 where μ_G was chosen as the sample mean of $Y_{1j}, ..., Y_{n_j j}$ for the calculation of $\hat{F}_j^{-1}$, as the sample mean of the pooled $Y-$data for $\hat{F}^{-1}$ and as the sample mean of $Y_{1j}^B, ..., Y_{n_j j}^B$ for the quantile estimators in the bootstrap. The same applies for σ_G^2 which was taken to equal the corresponding sample variances.

The data were generated by

$$Y_{ij} = g_j(X_{ij}) + U_{ij} \quad (i = 1, \ldots, n_j;\ j = 1, 2), \tag{4.6}$$

where the random variables X_{ij} were uniformly distributed on the interval $[0, 1]$ and U_{ij} were normally distributed with mean 0 and variance σ^2. For the smoothing of the bootstrap residuals we used different δ's for each group if the sample sizes were different, i.e.

$$\delta_j = \left(\frac{\sqrt{2}}{\sigma^3} \right)^{-1/5} n_j^{-1/4}. \tag{4.7}$$

Following Dette and Neumeyer (2001) we considered two cases for the simulation of the nominal level, that is

$$g_1(t) = g_2(t) = t^2, \tag{4.8}$$

$$g_1(t) = g_2(t) = \cos(\pi t), \tag{4.9}$$

and the variance was chosen as $\sigma^2 = 1$. We resampled $R = 99$ times and rejection probabilities were calculated by 2000 simulation runs. The simulated rejection probabilities for testing the equality of the 50% and 25% quantile curves are displayed in Table 1 and 2, respectively. We observe a rather precise approximation of the nominal level in all cases. For the sake of comparison, Table 1 contains also the simulated level of the wild bootstrap test proposed by Dette and Neumeyer (2001), which is based on an L^2-distance of the estimates for the mean regression curves from both samples and therefore most similar to the approach presented in this paper. The results are fairly comparable, where we observe a slightly better approximation of the 20% level by the procedure (4.2).

It might also be of interest to study the robustness properties of both tests. For this purpose we have simulated data according model (4.6) where 80% of the random errors U_{ij} are standard normally distributed and the remaining 20% are Cauchy distributed. The corresponding results are display in Table 3, and we observe that the nominal level of the test (4.2) is slightly underestimated in the presence of errors with an infinite variance. On the other hand, the test of Dette and Neumeyer (2001) yields a more substantial discrepancy between the nominal and the actual level in the presence of Cauchy distributed errors. This effect is clearly visible in the case (4.8) and also in the model (4.9) if $\alpha = 20\%$.

Table 3: *Rejection probabilities of the test (4.2) of equal* 50% *quantile curves under null hypothesis for models (4.8) and (4.9) with* 80% *normally and* 20% *Cauchy distributed errors. The numbers in brackets denote the corresponding rejection probabilities of the test of Dette and Neumeyer (2001).*

$p = 0.5$					
model $g_1(t) = g_2(t) = t^2$					
(n_1, n_2)	(25,25)	(25,50)	(50,50)	(50,100)	(100,100)
$\alpha = 5\%$	4.1% (2.7%)	5.0% (3.0%)	4.6% (3.3%)	4.2% (3.5%)	5.6% (4.0%)
$\alpha = 10\%$	7.8% (6.9%)	9.1% (5.1%)	9.1% (5.4%)	9.3% (5.3%)	10.0% (7.6%)
$\alpha = 20\%$	17.0% (13.3%)	17.9% (11.9%)	18.6% (11.2%)	19.8% (9.9%)	20.4% (11.3%)
model $g_1(t) = g_2(t) = \cos(\pi t)$					
$\alpha = 5\%$	4.5% (3.2%)	4.5% (4.8%)	4.4% (5.7%)	6.2% (4.1%)	4.3% (5.5%)
$\alpha = 10\%$	8.9% (6.1%)	8.6% (7.9%)	9.2% (7.9%)	12.1% (5.9%)	10.9% (9.9%)
$\alpha = 20\%$	18.8% (12.7%)	17.8% (13.5%)	18.0% (12.9%)	20.7% (10.0%)	18.8% (13.8%)

Table 4: *Rejection probabilities of the test (4.2) of equal* 50% *quantile curves under various alternatives with normal errors. The numbers in brackets denote the corresponding rejection probabilities of the test of Dette and Neumeyer (2001).*

$p = 0.5$						
	$n_1 = n_2 = 25$			$n_1 = n_2 = 50$		
model	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 20\%$	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 20\%$
(a)	53.0% (60.7%)	66.6% (73.3%)	79.0% (85.3%)	84.0% (97.4%)	92.2% (98.9%)	96.6% (99.4%)
(b)	45.8% (87.9%)	60.4% (93.2%)	73.8% (97.1%)	82.4% (100%)	91.2% (100%)	95.8% (100%)
(c)	55.8% (61.9%)	67.0% (72.4%)	79.0% (82.1%)	82.2% (91.6%)	88.6% (96.1%)	95.6% (98.3%)
(d)	97.8% (98.7%)	98.8% (99.5%)	99.4% (100%)	100% (100%)	100% (100%)	100% (100%)
(e)	52% (20.6%)	64.8% (26.9%)	77.2% (35.8%)	87.6% (58.0%)	91.6% (65.5%)	95.6% (74.3%)
(f)	94.8% (77.0%)	98.2% (82.9%)	99.6% (88.0%)	100% (99.7%)	100% (99.9%)	100% (99.9%)

Table 5: *Rejection probabilities of the test (4.2) of equal* 25% *quantile curves under various alternatives with normal errors.*

$p = 0.25$						
	$n_1 = n_2 = 25$			$n_1 = n_2 = 50$		
model	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 20\%$	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 20\%$
(a)	44.0%	60.0%	74.4%	83.6%	91.2%	97.2%
(b)	44.0%	56.6%	69.6%	79.4%	87.8%	94.6%
(c)	55.4%	64.0%	77.0%	80.8%	86.6%	92.6%
(d)	93.0%	96.4%	99.0%	99.8%	99.8%	100%
(e)	48.2%	61.2%	74.2%	78.4%	85.8%	93.0%
(f)	94.4%	97.2%	99.0%	100%	100%	100%

For the investigation of the power properties of the new test we considered the following models

$$
\begin{array}{ll}
(a) & g_1(t) = -g_2(t) = 0.5\cos(2\pi t) \\
(b) & g_1(t) = -g_2(t) = 0.5\sin(2\pi t) \\
(c) & g_1(t) = g_2(t) - t = \cos(\pi t) \\
(d) & g_1(t) = g_2(t) - 1 = \cos(\pi t) \\
(e) & g_1(t) = g_2(t) - t = \cos(2\pi t) \\
(f) & g_1(t) = g_2(t) - 1 = \cos(2\pi t)
\end{array}
$$

which have also been investigated by Dette and Neumeyer (2001). The variance of the random errors U_{ij} was chosen as $\sigma^2 = 0.5$. Following these authors, the simulated rejection probabilities from 1000 simulation runs are displayed in Table 4 and 5 corresponding to the estimation of the 50% and 25% quantile curve, respectively. The results indicate that the alternatives are detected with reasonable probabilities in all cases under consideration. A comparison of the rejection probabilities in Table 4 and 5 shows that a difference between the 25% quantile curves is detected with slightly lower probability as a difference between the 50% quantile curves. A heuristic argument for this observation is that usually the 25% quantile curve is harder to estimate. However, a more rigorous explanation of this phenomenon is possible on the basis of Theorem 3.2, which gives the asymptotic distribution of the test statistic under fixed alternatives. Note that Remark 3.5 shows that of the power of the test is determined (in first order) by $\frac{M_{LL}^2}{\sqrt{\tilde{V}^{LL}}}$, in particular the power is an increasing function of this quantity. Consequently, the ratio $\frac{M_{LL}^2}{\sqrt{\tilde{V}^{LL}}}$ could be used to get some idea about the power properties of the new test.
In the scenario considered in our simulation study we have uniformly distributed predictors and normally distributed errors with mean 0 and variance σ^2, which yields

$$
\frac{M_{LL}^2}{\sqrt{\tilde{V}^{LL}}} = \begin{cases} \frac{1}{8}\left(\frac{1}{2}\frac{p(1-p)}{\phi_\sigma(\Phi_\sigma^{-1}(p))^2} + \frac{1}{2}\right)^{-1/2} & \text{, in models } (a) \text{ and } (b) \\ \frac{1}{12}\left(\frac{1}{3}\frac{p(1-p)}{\phi_\sigma(\Phi_\sigma^{-1}(p))^2} + \frac{16}{45}\right)^{-1/2} & \text{, in models } (c) \text{ and } (e) \\ \frac{1}{4}\left(\frac{p(1-p)}{\phi_\sigma(\Phi_\sigma^{-1}(p))^2}\right)^{-1/2} & \text{, in models } (d) \text{ and } (f), \end{cases}
$$

where ϕ_σ and Φ_σ^{-1} denote the density and the quantile function of a centered normally distributed random variable with variance σ^2, respectively. These quantities are maximal for $p = 0.5$ and are given in Table 6 for $\sigma^2 = 0.5$. The empirically observed differences in the the power for the 25%- and 50% quantile curves [see Table 4 and 5] can be qualitatively explained by the differences between the values in Table 6.

For a sake of comparison, Table 4 also contains the rejection probabilities of the test of Dette and Neumeyer (2001) for the six scenarios. We observe a comparable behaviour for the alternatives (a), (c) and (d) [with slight advantages of the test proposed by Dette and Neumeyer (2001)]. In the scenario (b) the test of Dette and Neumeyer (2001) yields much larger rejection probabilities than the test (4.2), while in the remaining cases (e) and (f) the test based on quantile function is more powerful. The improvements are substantial for the alternative (e).

Table 6: $M^2_{LL}/\sqrt{\tilde{V}^{LL}}$ *for the different models considered in the simulation study.*

	Models		
	(a) and (b)	(c) and (e)	(d) and (f)
$p = 0.25$	0.128	0.102	0.260
$p = 0.5$	0.133	0.106	0.282

Table 7: *Rejection probabilities of the test (4.2) of equal* 50% *quantile curves under various alternatives with* 80% *normally and* 20% *Cauchy distributed errors. The numbers in brackets denote the corresponding rejection probabilities of the test of Dette and Neumeyer (2001).*

$p = 0.5$						
	$n_1 = n_2 = 25$			$n_1 = n_2 = 50$		
model	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 20\%$	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 20\%$
(a)	30.2% (18.5%)	45.3% (25.5%)	64.1% (34.1%)	71.2% (30.7%)	81.9% (37.3%)	90.1% (44.5%)
(b)	23.9% (58.4%)	41.1% (68.3%)	60.1% (75.8%)	67.3% (77.4%)	79.7% (79.7%)	89.8% (83.2%)
(c)	30.2% (33.1%)	46.8% (39.1%)	62.6% (46.3%)	63.1% (36.0%)	74.1% (42.9%)	84.5% (49.7%)
(d)	71.0% (58.7%)	83.3% (63.9%)	91.8% (69.5%)	94.1% (62.7%)	96.3% (65.8%)	96.7% (70.1%)
(e)	31.5% (15.2%)	44.2% (21.0%)	60.4% (27.4%)	65.1% (22.4%)	75.0% (27.2%)	83.9% (34.8%)
(f)	67.5% (47.0%)	81.4% (53.1%)	91.5% (59.7%)	92.0% (58.1%)	94.6% (61.4%)	95.5% (63.9%)

We conclude the study of the finite sample properties with a brief investigation of the impact of outliers on the power of the new test and the test of Dette and Neumeyer (2001). For this purpose we considered the same models and parameters as in the previous paragraph, but replaced 20% of the errors by Cauchy-distributed random variables multiplied by σ. The results are displayed in Table 7 for the new test (4.2) and the test of Dette and Neumeyer (2001). Compared to the case of 100% normally distributed errors, we observe a loss in power for both tests. For the new test the rejection probabilities are in average about 26% smaller for the sample sizes $n_1 = n_2 = 25$ and about 10% smaller for sample sizes $n_1 = n_2 = 50$. For the test of Dette and Neumeyer (2001), the average loss of power is more substantial and given by 36% and 44% for the cases $n_1 = n_2 = 25$ and $n_1 = n_2 = 50$, respectively. As a consequence, the test (4.2) nearly always yields larger rejection probabilities in the case of 20% Cauchy distributed errors.

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5 Appendix: Proofs

To keep the notation simple we concentrate on the case of $J = 2$ samples, $N_j = n_j$ and the Nadaraya-Watson estimate. The corresponding statements for the local linear estimate and more than 2 samples follow by exactly the same arguments with an additional amount of notation.

5.1 Proof of Theorem 3.1.

We use the notation $H_j^{-1}(p|x) = G(F_j^{-1}(p|x))$, $\tilde{G}(x) = (G^{-1})'(H_1^{-1}(p|x))$ and obtain by a Talor-expansion under the null hypothesis H_0 (note that the distribution function G is strictly monotone)

$$
\begin{aligned}
(5.1)\ T_n &= \int (\widehat{F}_1^{-1}(p|t) - F_1^{-1}(p|t) + F_2^{-1}(p|t) - \widehat{F}_2^{-1}(p|t))^2 \widehat{w}_{12}(t)dt \\
&= \int \tilde{G}^2(t) \left(\widehat{H}_1^{-1}(p|t) - H_1^{-1}(p|t) - (\widehat{H}_2^{-1}(p|t) - H_2^{-1}(p|t))\right)^2 \widehat{w}_{12}(t)dt \\
&\quad +2 \int \tilde{G}(t)\widehat{w}_{12}(t)(\widehat{H}_1^{-1}(p|t) - \widehat{H}_2^{-1}(p|t)) \\
&\quad \times \left[(G^{-1})''(\xi_1)(\widehat{H}_1^{-1}(p|t) - H_1^{-1}(p|t))^2 - (G^{-1})''(\xi_2)(\widehat{H}_2^{-1}(p|t) - H_2^{-1}(p|t))^2\right] dt \\
&\quad + \int \widehat{w}_{12}(t)\{[(G^{-1})''(\xi_1)]^2(\widehat{H}_1^{-1}(p|t) - H_1^{-1}(p|t))^4 \\
&\quad -2(G^{-1})''(\xi_1)(G^{-1})''(\xi_2)(\widehat{H}_1^{-1}(p|t) - H_1^{-1}(p|t))^2(\widehat{H}_2^{-1}(p|t) - H_2^{-1}(p|t))^2 \\
&\quad +[(G^{-1})''(\xi_2)]^2(\widehat{H}_2^{-1}(p|t) - H_2^{-1}(p|t))^4\}dt,
\end{aligned}
$$

where the random variables ξ_1 and ξ_2 satisfy $|\xi_j - H_j^{-1}(p|t)| \leq |\widehat{H}_j^{-1}(p|t) - H_j^{-1}(p|t)|$. Under the assumptions of Theorem 3.1 it follows from Dette and Volgushev (2008) that

$$
\widehat{H}_j^{-1}(p|t) - H_j^{-1}(p|t) = O_p(h_r^2) + O_p\left(\frac{1}{\sqrt{nh_r}}\right),
$$

and as a consequence the last two integrals in (5.1) are of order $o_p((n\sqrt{h_r})^{-1})$. Therefore it remains to consider the first integral, which will be denoted by $T_n^{(1)}$ throughout this section. From the definition of $\widehat{H}_j^{-1}(p|x)$ in (2.7) we obtain by a further Taylor expansion

$$
\widehat{H}_j^{-1}(p|x) - H_j^{-1}(p|x) = \Delta_j^{(1)}(p|x) + \Delta_j^{(2)}(p|x) + \Delta_j^{(3)}(p|x) + \Delta_j^{(4)}(p|x),
$$

where

$$\Delta_j^{(1)}(p|x) := -\frac{1}{n_j h_d}\sum_{i=1}^{n_j} K_d\left(\frac{F_j(g_{ij}|x)-p}{h_d}\right)\left(\widehat{F}_j(g_{ij}|x)-F_j(g_{ij}|x)\right),$$

$$\Delta_j^{(2)}(p|x) := -\frac{1}{2n_j h_d^2}\sum_{i=1}^{n_j} K_d^{'}\left(\frac{F_j(g_{ij}|x)-p}{h_d}\right)\left(\widehat{F}_j(g_{ij}|x)-F_j(g_{ij}|x)\right)^2,$$

$$\Delta_j^{(3)}(p|x) := -\frac{1}{6n_j h_d^3}\sum_{i=1}^{n_j} K_d^{''}\left(\frac{\xi_{ij}-p}{h_d}\right)\left(\widehat{F}_j(g_{ij}|x)-F_j(g_{ij}|x)\right)^3,$$

$$\Delta_j^{(4)}(p|x) := \frac{1}{n_j h_d}\int_{-\infty}^{p}\sum_{i=1}^{n_j} K_d\left(\frac{F_j(g_{ij}|x)-u}{h_d}\right)du - H_j^{-1}(p|x),$$

we used the notation $g_{ij} := G^{-1}\left(\frac{i}{n_j}\right)$, and the random variables ξ_{ij} satisfy $|\xi_{ij}-F_j(g_{ij}|x)| \leq |\widehat{F}_j(g_{ij}|x)-F_j(g_{ij}|x)|$. Using similar arguments as in Dette and Volgushev (2008) it follows that

$$\Delta_j^{(2)}(p|x) = o_p\left(\frac{1}{nh_r h_d}\right)+o_p\left(\frac{h_r^4}{h_d}\right),$$

$$\Delta_j^{(3)}(p|x) = O_p\left(\frac{h_r^6}{h_d^{5/2}}\right),$$

$$\Delta_j^{(4)}(p|x) = \frac{1}{2}\mu_2(K_d)h_d^2\partial_1^2(H_j^{-1}(p|x))+O\left(\frac{1}{nh_d}\right).$$

An application of the Cauchy-Schwarz-inequality yields for the quantities

$$T_{n,kl} := \int_0^1 \tilde{G}^2(x)\left(\Delta_2^{(k)}(p|x)-\Delta_1^{(k)}(p|x)\right)\left(\Delta_2^{(l)}(p|x)-\Delta_1^{(l)}(p|x)\right)\widehat{w}_{12}(x)dx = o_p\left(\frac{1}{n\sqrt{h_r}}\right)$$

for all $(k,l)\neq(1,1)$. This implies

$$\begin{aligned} (5.2)\quad T_n^{(1)} &= T_{n,11}+o_p\left(\frac{1}{n\sqrt{h_r}}\right) \\ &= \int_0^1 \tilde{G}(x)^2\left(f_1(x)\widehat{f}_2(x)\Delta_2^{(1)}(p|x)-f_2(x)\widehat{f}_1(x)\Delta_1^{(1)}(p|x)\right)^2dx+o_p\left(\frac{1}{n\sqrt{h_r}}\right), \end{aligned}$$

where we have used the definition of $\widehat{w}_{12}(x) = (\widehat{f}_1(x)\widehat{f}_2(x))^2$. Now we define the independent identically distributed random variables

$$Z_{kj}(x) := \frac{-1}{n_j^2 h_d h_r}\sum_{l=1}^{n_j} K_d\left(\frac{F_j(g_{lj}|x)-p}{h_d}\right)K_r\left(\frac{X_{kj}-x}{h_r}\right)(I\{Y_{kj}\leq g_{lj}\}-F_j(g_{lj}|x)).$$

Remembering the definition of the Nadaray-Watson-weights, we get

$$\widehat{f}_j(x)\Delta_j^{(1)}(p|x) = \sum_{k=1}^{n_j} Z_{kj}(x).$$

Further we define

$$\tilde{Z}_k(x) := \begin{cases} f_1(x)(Z_{k2}(x) - E\left[Z_{k2}(x)\right]) & \text{for } 1 \le k \le n_2 \\ -f_2(x)(Z_{(k-n_2)1}(x) - E\left[Z_{(k-n_2)1}(x)\right]) & \text{for } n_2+1 \le k \le n \end{cases} \tag{5.3}$$

which are centered independent random variables. Using this notation we obtain from (5.2) the following representation for the statistic $T_n^{(1)}$

$$T_n^{(1)} = \int_0^1 [A_1(x) + 2A_2(x) + A_3(x)]dx + o_p\left(\frac{1}{n\sqrt{h_r}}\right), \tag{5.4}$$

where

$$\begin{aligned} A_1(x) &= \tilde{G}^2(x)\left(\sum_{k=1}^n \tilde{Z}_k(x)\right)^2 \\ A_2(x) &= \tilde{G}^2(x)\left(\sum_{k=1}^n \tilde{Z}_k(x)\right)\left(f_1(x)E\left[\widehat{f}_2(x)\Delta_2^{(1)}(p|x)\right] - f_2(x)E\left[\widehat{f}_1(x)\Delta_1^{(1)}(p|x)\right]\right) \\ A_3(x) &= \tilde{G}^2(x)\left(f_1(x)E\left[\widehat{f}_2(x)\Delta_2^{(1)}(p|x)\right] - f_2(x)E\left[\widehat{f}_1(x)\Delta_1^{(1)}(p|x)\right]\right)^2. \end{aligned}$$

Obviously we have $E\left[A_2(x)\right] = 0$ and straightforward but tedious calculations [see Wagener (2008)] yields

$$\begin{aligned} E\left[A_1(x)\right] &= p(1-p)\mu_0^{(2)}(K_r)\Big\{\left(\partial_1(F_2^{-1}(p|x))\right)^2\frac{f_1^2(x)f_2(x)}{n_2h_r} \\ &\quad + \left(\partial_1(F_1^{-1}(p|x))\right)^2\frac{f_2^2(x)f_1(x)}{n_1h_r}\Big\} + o\left(\frac{1}{n\sqrt{h_r}}\right) \end{aligned} \tag{5.5}$$

and

$$A_3(x) = h_r^4\left(f_2(x)C_1(x) - f_1(x)C_2(x)\right)^2 + O(h_r^5). \tag{5.6}$$

Similarly, it follows by Markov's-inequality

$$A_1(x) = \tilde{G}^2(x)\sum_{k=1}^n\sum_{l\neq k}\tilde{Z}_k(x)\tilde{Z}_l(x) + E\left[A_1(x)\right] + o_p\left(\frac{1}{n\sqrt{h_r}}\right) \tag{5.7}$$

and we denote the first sum in (5.7) by $\tilde{A}_1(x)$. For the variances of $\tilde{A}_1(x)$, $A_2(x)$ and the covariance we obtain

$$\begin{aligned} Var\left(\int_0^1 \tilde{A}_1(x)dx\right) &= \frac{2p^2(1-p)^2}{h_r}\int (K_r * K_r)^2(u)du \\ &\quad \times \int_0^1\left(\left(\partial_1(F_1^{-1}(p|x))\right)^2\frac{f_1(x)f_2^2(x)}{n_1} + \left(\partial_1(F_2^{-1}(p|x))\right)^2\frac{f_1^2(x)f_2(x)}{n_2}\right)^2 dx \end{aligned}$$

$$+o\left(\frac{1}{n^2h_r}\right),$$

$$Var\left(\int_0^1 A_2(x)dx\right) = p(1-p)h_r^4\int_0^1\left(\left(\partial_1(F_1^{-1}(p|x))\right)^2\frac{f_1(x)f_2^2(x)}{n_1}+\left(\partial_1(F_2^{-1}(p|x))\right)^2\frac{f_1^2(x)f_2(x)}{n_2}\right)$$
$$\times(f_2(x)C_1(x)-f_1(x)C_2(x))^2dx+o\left(\frac{h_r^4}{n}\right)$$

and

$$Cov\left(\int_0^1\tilde{A}_1(x)dx,\int_0^1 A_2(x)dx\right)=o\left(\frac{1}{n^2h_r}\right)$$

using Fubini's Theorem. Finally we define random variables to apply the central limit theorem of de Jong (1996) as follows:

$$W_I := \begin{cases} n\sqrt{h_r}2\int_0^1\tilde{G}^2(x)\tilde{Z}_k(x)\tilde{Z}_l(x)dx & \text{if } I=\{k,l\} \\ n\sqrt{h_r}2\int_0^1\tilde{G}^2(x)\tilde{Z}_k(x)\left(f_1(x)E\left[\widehat{f}_2(x)\Delta_2^{(1)}(x)\right]-f_2(x)E\left[\widehat{f}_1(x)\Delta_1^{(1)}(x)\right]\right)dx & \text{if } I=\{k\} \\ 0 & \text{in all other cases.}\end{cases}$$

Obviously these random variables are measurable with respect to the sigma field $\mathcal{F}_I := \sigma\{U_i|i\in I\}$ where $U_k=(X_{kj_k},Y_{kj_k})$, $j_k=1$ if $k>n_2$ and $j_k=2$ otherwise. Moreover these random variables fulfil

$$E\left[W_{I_1}|\mathcal{F}_{I_2}\right]=0,\ \text{if } I_1\not\subset I_2.$$

This means that condition (a) and (b) on the top of page 106 in de Jong (1996) are satisfied. Therefore it remains to check the two other conditions of Theorem 1 on page 107 in this reference, which can be done by a straightforward but tedious calculation. Consequently, Theorem 1 of de Jong (1996) is applicable in the present context and it follows

$$W(n):=n\sqrt{h_r}\left(\int_0^1\tilde{A}_1(x)dx+2\int_0^1 A_2(x)dx\right)=\sum_{|I|\le 2}W_I\xrightarrow{\mathcal{D}}\mathcal{N}(0,V^{NW}).$$

Observing the representation for the expectations of the random variables $A_1(x)$ and $A_3(x)$ in (5.5) and (5.6), respectively and the representation

$$T_n=\int_0^1 A_1(x)dx+2\int_0^1 A_2(x)dx+\int_0^1 A_3(x)dx+o_p\left(\frac{1}{n\sqrt{h_r}}\right)$$

the assertion of the Theorem 3.1 for the Nadaraya-Watson weights follows. □

5.2 Proof of Theorem 3.2.

The proof of this theorem uses the same techniques as the one above and, for the sake of brevity, only the main steps are presented. Under a fixed alternative the teststatics splits into

$$T_n=T_n^{(1)}+2T_n^{(2)}+T_n^{(3)}+o_p\left(\frac{1}{\sqrt{n}}\right),$$

where $T_n^{(1)}$ is defined in the proof of Theorem 3.1 and the other quantities are given by

$$\begin{aligned} T_n^{(2)} &= \int_0^1 \tilde{G}(x)\left(\widehat{H}_1^{-1}(p|x) - H_1^{-1}(p|x) - (\widehat{H}_2^{-1}(p|x) - H_2^{-1}(p|x))\right)\left(F_2^{-1}(p|x) - F_1^{-1}(p|x)\right)\widehat{w}_{12}(x)dx, \\ T_n^{(3)} &= \int_0^1 \left(F_2^{-1}(p|x) - F_1^{-1}(p|x)\right)^2 \widehat{w}_{12}(x)dx. \end{aligned}$$

An inspection of the proof of Theorem 3.1 shows that under a fixed alternative the random variable $T_n^{(1)}$ is of order $o_p\left(\frac{1}{\sqrt{n}}\right)$ (one has to investigate $T_{n,k4}$ more carefully). Straightforward calculations yield

$$\begin{aligned} T_n^{(2)} &= \int_0^1 \tilde{G}(x)\left(f_1(x)\sum_{k=1}^{n_2} Z_{k2}(x) - f_2(x)\sum_{k=1}^{n_1} Z_{k1}(x)\right)\left(F_2^{-1}(p|x) - F_1^{-1}(p|x)\right) f_1(x)f_2(x)dx \\ &\quad + o_p\left(\frac{1}{\sqrt{n}}\right), \\ E\left[T_n^{(2)}\right] &= -h_r^2\int_0^1 \left(f_1(x)C_2(x) - f_2(x)C_1(x)\right)\left(F_2^{-1}(p|x) - F_1^{-1}(p|x)\right) f_1(x)f_2(x)dx + o\left(\frac{1}{\sqrt{n}}\right), \\ Var\left(T_n^{(2)}\right) &= p(1-p)\int_0^1 \left[\frac{f_1^2(x)f_2(x)}{n_2}\left(\partial_1(F_2^{-1}(p|x))\right)^2 + \frac{f_1(x)f_2^2(x)}{n_1}\left(\partial_1(F_1^{-1}(p|x))\right)^2\right] \\ &\quad \times (F_2^{-1}(p|x) - F_1^{-1}(p|x))^2 f_1^2(x)f_2^2(x)dx + o\left(\frac{1}{n}\right). \end{aligned}$$

The statistic $T_n^{(2)}$ is a sum of independent random variables. Because of the randomness of $T_n^{(3)}$, caused by the random weights $\widehat{w}_{12}(x)$, which are needed to handle the random denominator problem, we can not apply the central limit theorem of Lindeberg to $T_n^{(3)}$. Some calculations give

$$\begin{aligned} E\left[T_n^{(3)}\right] &= \int_0^1 \left\{f_1^2(x)f_2^2(x) + h_r^2\mu_2(K_r)\left(f_2^2(x)f_1(x)f_1''(x) + f_1^2(x)f_2(x)f_2''(x)\right)\right\} \\ &\quad \times (F_2^{-1}(p|x) - F_1^{-1}(p|x))^2 dx + O\left(\frac{1}{n}\right) \end{aligned}$$

and

$$\begin{aligned} Var\left(T_n^{(3)}\right) &= 4\left\{\frac{1}{n_2}\int_0^1 \left(F_2^{-1}(p|x) - F_1^{-1}(p|x)\right)^4 f_1^4(x)f_2^3(x)dx \right. \\ &\quad \left. + \frac{1}{n_1}\int_0^1 \left(F_2^{-1}(p|x) - F_1^{-1}(p|x)\right)^4 f_1^3(x)f_2^4(x)dx\right\} \\ &\quad -4\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\left(\int_0^1 \left(F_2^{-1}(p|x) - F_1^{-1}(p|x)\right)^2 f_1^2(x)f_2^2(x)dx\right)^2 + o\left(\frac{1}{n}\right). \end{aligned}$$

and so the variances of $T_n^{(2)}$ and $T_n^{(3)}$ are of the same order. The covariance of $T_n^{(2)}$ and $T_n^{(3)}$ is of order $o\left(n^{-1}\right)$. We are able to rewrite the teststatistic in the following way.

$$\sqrt{n}\left(T_n - E\left[T_n\right]\right) = \sqrt{n}\left(2T_n^{(2)} + T_n^{(3)} - 2E\left[T_n^{(2)}\right] - E\left[T_n^{(3)}\right]\right) + o_p(1) = \sum_{I\subset\{1,\dots,n\},|I|\le 4} \tilde{W}_I + o_p(1)$$

where the random variables $\tilde{W}_I$ again fulfil the conditions on page 106 and 107 of de Jong (1996) (due to lack of space, we do not give the exact definitions of $\tilde{W}_I$). So we can apply Theorem 1 of de Jong (1996) and the assertion of Theorem 3.2 follows.

Remark 3.8 Sun (2006) also considered samples that include discrete variables. We can easily generalize our test statistic in the following way for samples of this type. To be precise, let $X_{ij} = (X_{1,ij}, X_{2,ij})$, where $X_{1,ij}$ are discrete variables taking values in a given set, say χ, and $X_{2,ij}$ are continuous variables satisfying the assumptions made throughout this paper. We define the weights for the initial estimate of the conditional distribution function as follows:

$$\overline{w}_{ij}((x_1, x_2)) = \tilde{w}_{ij}(x_2)\left(I\{X_{1,ij} = x_1\} + \lambda I\{X_{1,ij} \neq x_1\}\right),$$

where $\tilde{w}_{ij}$ are Nadaraya-Watson weights or local linear weights respectively and $\lambda \geq 0$ is an additional bandwidth satisfying

$$n\sqrt{h_r}h_r^4\lambda = o(1), \quad \frac{\lambda}{\sqrt{h_r}} = o(1).$$

In this case we define

$$T_n = \sum_{x_1 \in \chi} \int_0^1 \sum_{j=1}^{J} \sum_{i=1}^{j-1} (\hat{F}_j^{-1}(p|(x_1, x_2)) - \hat{F}_i^{-1}(p|(x_1, x_2)))^2 \hat{w}_{ij}(x_2) dx_2.$$

Analogous results to Theorem 3.1, Theorem 3.2 and Theorem 3.4 hold in this case. For example, in the local linear case we have

$$n\sqrt{h_r}\,(T_n - \overline{B}_1 - \overline{B}_2) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \overline{V}),$$

where the asymptotic and bias variance are given by

$$\begin{aligned}
\overline{B}_1 &= \mu_2^6(K_r) \sum_{x_1 \in \chi} \int_0^1 f_1^4(x_1, x_2) f_2^4(x_1, x_2)(\overline{C}_1(x_1, x_2) - \overline{C}_2(x_1, x_2))^2 dx_2,\\
\overline{B}_2 &= p(1-p)\mu_0^{(2)}(K_r)\mu_2^4(K_r) \sum_{x_1 \in \chi} \int_0^1 (\partial_1(F_1^{-1}(p|x_1, x_2)))^2 \frac{f_1^3(x_1, x_2) f_2^4(x_1, x_2)}{a_1}\\
&\quad + (\partial_1(F_2^{-1}(p|x_1, x_2)))^2 \frac{f_1^4(x_1, x_2) f_2^3((x_1, x_2))}{a_2} dx_2,\\
\overline{C}_j(x_1, x_2) &= \frac{1}{2}\partial_1(F_j^{-1}(p|x_1, x_2))\partial_{x_2}^2(F_j(F_j^{-1}(p|x_1, x_2)|x_1, x_2)),
\end{aligned}$$

$$\begin{aligned}
\overline{V} &= 2p(1-p)\mu_2^8(K_r) \sum_{x_1 \in \chi} \left\{ p(1-p) \int_0^1 (K_r * K_r)^2(u) du \right.\\
&\quad \times \int_0^1 \left(\left(\partial_1(F_1^{-1}(p|x_1, x_2))\right)^2 \frac{f_1^3(x_1, x_2) f_2^4(x_1, x_2)}{a_1} + \left(\partial_1(F_2^{-1}(p|x_1, x_2))\right)^2 \frac{f_1^4(x_1, x_2) f_2^3(x_1, x_2)}{a_2} \right)^2 dx_2
\end{aligned}$$

$$+2c\mu_2^2(K_r)\int_0^1\left(\left(\partial_1(F_1^{-1}(p|x_1,x_2))\right)^2\frac{f_2(x_1,x_2)}{a_1}+\left(\partial_1(F_2^{-1}(p|x_1,x_2))\right)^2\frac{f_1(x_1,x_2)}{a_2}\right)$$
$$\times f_1^7(x_1,x_2)f_2^7(x_1,x_2)(\overline{C}_1(x)-\overline{C}_2(x))^2dx\Big\},$$

respectively, f_j ($j=1,2$) denotes the joint density of $(X_{1,ij},X_{2,ij})$.

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