The Effect of Intraday Periodicity on Realized Volatility Measures

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Abstract

U-shaped intraday periodicity (IP) is a typical stylized fact characterizing intraday returns on risky assets. In this study we focus on the realized volatility and bipower variation estimators for daily integrated volatility (IV) which are based on intraday returns following a discrete-time model with IP. We demonstrate that neglecting the impact of IP on realized estimators may lead to non-valid statistical inference concerning IV for the commonly available number of intraday returns, moreover, the size of daily jump tests may be distorted. Given the functional form of IP, we derive corrections for the realized measures of IV. We show in a Monte Carlo and an empirical study that the proposed corrections improve commonly point and interval estimators of the IV and tests for jumps.

Keywords: bipower variation, daily integrated volatility, jump detection, realized volatility

1 Introduction

The availability of high-frequency data allows the construction of precise estimators of daily integrated volatility (IV) for risky asset returns. The realized volatility (RV) defined as a sum squared intraday returns is known as a consistent estimator of daily IV in absence of jumps. Other realized measures such as the bipower variation (BV) should be used for IV estimation in presence of jumps during the day. Barndorff-Nielsen and Shephard (2002) derive the asymptotic properties of these measures under quite mild assumptions on the underlying continuous pricing process with jumps. The common practice is to use BV for a day where a statistical test decision is to reject the hull hypothesis of "no jumps" and RV otherwise.

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On the other hand the presence of a persistent intraday pattern (IP) which usually takes a U-form during the trading time of the day is a well documented empirical feature of the intraday volatility (Wood et al. (1985), Harris (1986)). As it appears to be highly correlated with the intraday variation of trading volume, Admati and Pfleiderer (1988) propose to explain the daily U-shape by strategic interaction of traders around market openings and closures. For longer time periods (week, month, etc.) the IP could be explained by the impact of slowly varying macroeconomic fundamentals (Andersen and Bollerslev (1998b), Andersen et al. (2001, 2003)). In this paper we investigate the impact of IP on the finite sample properties of RV and BV estimators of daily IV. To the best of our knowledge, this research agenda has not been investigated yet, although there is a vast amount of literature concerning modeling and estimating IP (Engle et al. (1990), Hamao et al. (1990), Boudt et al. (2011), Engle and Sokalska (2012), Andersen et al. (2012)). Thus our results provide uselful insights in the differences between the asymptotic theory and the practical performance of realized measures based on intraday data. For this purpose we consider the discrete time model without jumps in the spirit of Andersen and Bollerslev (1997), where the variance of intraday returns is written as a product of the deterministic periodic and stochastic volatility components. The IP is assumed to be constant for all days, whereas the stochastic volatility part is changing over time, e.g. from one day to another. We show that for the commonly available number of intraday returns, say M, neglecting the impact of IP would lead to non-valid statistical inference concerning daily IV and may distort the size of commonly used tests for jumps. For a given IP, we compute the first and the second moments of RV and BV, moreover, we derive the impact of IP on the realized tri-power (TP) and quad-power (QP) estimators of daily integrated quarticity (IQ) required for statistical inference about IV. We also establish the asymptotic bivariate distribution of these measures as $M \to \infty$ and suggest a correction for tests on daily jumps based on the distance between RV and BV estimators (Huang and Tauchen (2005), Barndorff-Nielsen and Shephard (2006)).

Our major finding is that for the commonly available number of intraday returns the impact of IP should be explicitly addressed in any statistical inference concerning IV. While the RV estimator of IV is unaffected by IP, BV has a finite sample bias which can only be neglected for a extremely large sample sizes, which are usually not available in applications. Moreover, in the computation of the second moments (IQ) of realized estimators one needs to account for a scaling factor (independently of the sample size), which depends on the functional form of the IP. Also tests for a daily jump component have to be corrected because otherwise they do not keep their nominal level and lead to a detection of spurious jumps. For a given form of IP, we provide corrections and demonstrate the necessity of these modifications accounting for IP in any valid statistical inference concerning IV.

Our results are supported by a Monte Carlo study, where we investigate the impact of IP on various commonly used realized measures and tests for jumps and the "IP corrected" procedures proposed in this paper. In the empirical application we estimate the IP and provide jump

test results for the Dow Jones daily volatility. We show that the corrected tests detect a substantially smaller number of days with jumps than the unadjusted ones. Hence, our results underscore that a common practice to estimate the daily IV (Andersen et al. (2011)) as "start by testing for jumps during the day, then use BV if the Null 'no jumps' is rejected and RV otherwise" could be misleading if a pronounced IP is neglected. Thus, accounting for IP form is crucial for computing IV estimators widely used for portfolio selection, option pricing or Value-at-Risk calculation purposes.
The rest of the paper is organized as follows. In Section 2 we introduce the model for intraday returns and discuss the realized estimators for daily IV. The theoretical results are derived in Section 3 where we establish both finite sample and asymptotic stochastic properties of realized IV estimators for a given IP pattern. Our approach is illustrated in Section 4 by means of a Monte Carlo simulation study and in Section 5 by an empirical application. Section 6 concludes the paper whereas the proofs are put in the Appendix.

2 Modeling and measuring daily volatility based on intraday information

We start from a very general jump-diffusion model for log price increments in order to define the objects of our interest, namely daily integrated volatility and daily integrated quarticity. Next, we present realized estimators of daily volatility and quarticity which are based on M intraday returns. These realized measures are consistent estimators for $M \to \infty$, however, in practice M is finite due to market microstructure noise. Thus, it is of importance to study the finite sample stochastic properties of realized estimators. For this reason we then consider a discrete time model for intraday returns with an explicit specification of intraday periodicity (IP). Our aim is to investigate the impact of IP on realized measures for finite values of M.

2.1 Model for intraday returns and realized measures

A commonly assumed general model presumes that log-prices of risky assets $p(t) = \ln P_t$ follow a continuous time process with a jump component (Andersen et al. (2007)):

$$dp(t) = \mu(t)dt + \sigma(t)d\mathcal{W}_t + \kappa(t)dq(t), \tag{2.1}$$

where $\mathcal{W}_t$ is a Brownian motion. The jump occurrence is governed by a counting process $q(t)$ and the size of jumps is given by $\kappa(t)$. As the drift component $\mu(t)$ is rather small and hardly predictable, we assume zero mean $\mu(t) = 0$ without loss of generality.
We consider a day t as the period of interest with the daily return $r_t = p_t - p_{t-1}$. Our attention

is focused on the daily integrated volatility (IV), which is defined for day t as

$$IV_t = \sigma_t^2 = \int_{t-1}^{t} \sigma^2(u)du.$$

In order to make statistical inference about IV measures we also need the statement concerning the integrated quarticity (IQ) defined as

$$IQ_t = \int_{t-1}^{t} \sigma^4(u)du.$$

The availability of intraday returns allows to construct precise *realized* estimators (Andersen and Bollerslev (1998a)) for the daily IV which are consistent in the general model (2.1). Assume that there are M equally spaced intraday returns for day t denoted by $r_{t,m} = p_{t,m} - p_{t,m-1}$ with $t = 1, ..., T$ and $m = 1, ..., M$. Note that the daily return r_t is obtained as a sum of intraday returns, i.e. $r_t = \sum_{m=1}^{M} r_{t,m}$. The most popular IV estimator is the realized volatility (RV) measure which is given as

$$RV_t = \sum_{m=1}^{M} r_{t,m}^2.$$

Barndorff-Nielsen and Shephard (2002) show the consistency of RV_t for IV_t in model in (2.1) in the case of no jumps, i.e. $RV_t \to IV_t$ as $M \to \infty$. Although RV_t possesses a set of appealing properties, it is not appropriate in the presence a non-zero jump component as $RV_t \to IV_t + \sum_{j=1}^{J_t} \kappa_{t,j}^2$ where $\kappa_{t,j}$ is the size of the jth jumps and J_t is a number of jumps at day t. The bipower variation (BV)

$$BV_t = \frac{\pi}{2} \sum_{m=2}^{M} |r_{t,m}||r_{t,m-1}|.$$

proposed by Barndorff-Nielsen and Shephard (2004) is a jump-robust and consistent estimator of IV even in the presence of jumps, i.e. $BV_t \to IV_t$ as $M \to \infty$. However, RV has a smaller variance than BV if there are no jumps as it is shown in Theorem 3 of Barndorff-Nielsen and Shephard (2006). For this reason the common practice (Huang and Tauchen (2005)) is to start from testing for a jump component during each day t. Then, in case of a significantly large positive distance between RV and BV indicating jumps, one should use BV; otherwise the application of RV is recommended.

Intraday returns are also useful for the purpose of estimating the unknown integrated quarticity (IQ) required for computing variances of RV and BV measures. The realized quarticity (RQ)

is given as (Andersen et al. (2014))

$$RQ_t = \frac{M}{3} \sum_{m=1}^{M} r_{t,m}^4,$$

which is consistent estimator of IQ is case of no jumps, i.e. $RQ_t \to IQ_t$ as $M \to \infty$. However, to address jumps, Barndorff-Nielsen and Shephard (2004) suggest to use the realized tri-power (TP) and quad-power (QP) measures defined by

$$QP_t = M \cdot \frac{\pi^2}{4} \cdot \sum_{m=4}^{M} |r_{t,m-3}||r_{t,m-2}||r_{t,m-1}||r_{t,m}|, \tag{2.2}$$

$$TP_t = M\mu_{\frac{4}{3}}^{-3} \sum_{m=3}^{M} |r_{t,m-2}|^{\frac{4}{3}}|r_{t,m-1}|^{\frac{4}{3}}|r_{t,m}|^{\frac{4}{3}} \tag{2.3}$$

where $\mu_r = 2^{r/2}\Gamma((r+1)/2)/\Gamma(1/2)$ is the rth absolute moment of a standard normal distribution and $\Gamma(\cdot)$ denotes the Gamma function so that $\mu_{4/3} = 0.8309$.
The realized estimators are of immense practical importance for estimation and inference concerning daily volatility. However, although both RV and BV estimators have appealing stochastic properties as $M \to \infty$, their practical implementation is usually based on (say) 10 min intraday returns which makes, for example, $M = 36$ intraday observations for a six hours trading day. Thus, the number of intraday returns M used for construction of realized measures is comparatively small in applications. This happens due to highly persistent empirical (stylized) facts which hinder the use of ultra high frequency data for construction of realized estimators (McAleer and Medeiros (2008)). In particular, such empirical features as market microstructure noise (MMN) should be accounted for. One possibility to overcome this problem is to robustify the realized estimators or the sampling schemes as e.g. in Bandi and Russell (2008) and the following literature. However, the common practice to overcome MMN-related problems for construction of realized estimators remains to use not very frequent sampling, i.e. intraday returns at 5-, 10-, or 15-min frequencies (Andersen et al. (2011)). For these reasons finite M stochastic properties of realized measures are of great practical interest.

2.2 Discrete time model for intraday returns

In order to investigate the impact of IP on realized measures for the the commonly used number M of intraday returns, we now consider a discrete time model where IP is specified explicitly. There is a substantial scope of recent literature concerning discrete-time modeling of intraday returns whereas the IP is assumed to be a multiplicative scaling component (Boudt et al. (2011), Engle and Sokalska (2012), Bekierman and Gribisch (2016)). Following the approach of Andersen and Bollerslev (1997), we assume the stochastic model for intraday return without jumps given as

$$
\begin{aligned}
r_{t,m} &= \sigma_{t,m} \cdot u_{t,m}, \qquad \text{with} \qquad u_{t,m} \overset{\text{iid}}{\sim} \mathcal{N}(0,1), \\
\sigma^2_{t,m} &= \frac{1}{M} \cdot s^2_{t,m} \cdot \gamma^2_{t,m},
\end{aligned}
$$

where $s_{t,m} > 0$ is the deterministic IP volatility component, whereas $\gamma^2_{i,t} > 0$ is the stochastic volatility. We presume that the stochastic volatility component remains constant over day t (Andersen and Bollerslev (1998b), Hecq et al. (2012)), i.e. $\gamma_{t,m} = \sigma_t$ for all $m = 1, ..., M$ but may change from one day to another. In Section 4 we relax this assumption in the Monte Carlo simulation study by assuming that the intraday stochastic volatility follows a GARCH(1,1) diffusion as e.g. in Goncalves and Meddahi (2009).

In line with the current literature (Hecq et al. (2012)), we assume that the IP remains constant at different days, i.e. the time index is skipped with $s_{t,m} = s_m$. Moreover, the periodic component is standardized such it sums up to 1 over the day:

$$
\frac{1}{M} \sum_{m=1}^{M} s^2_m = 1.
$$

Of course, a special case $s_m = 1$ for all $m = 1, ..., M$ corresponds to no IP. Thus, in our model we separate the intraday periodic component s_m and interday stochastic component σ_t by writing

$$
\sigma^2_{t,m} = \frac{1}{M} \cdot s^2_m \cdot \sigma^2_t. \tag{2.4}
$$

Our model without jumps in (2.4) and (2.4) is fairly simple compared to much more advanced approaches (Bekierman and Gribisch (2016)). However, it is still widely used for modeling IP (Boudt et al. (2011), Hecq et al. (2012), Engle and Sokalska (2012)). The measure of interest, the IV for the day t, is given as

$$
IV_t = Var(r_t) = \sum_{m=1}^{M} Var(r_{t,m}) = \frac{1}{M} \sum_{m=1}^{M} s^2_m \sigma^2_t = \sigma^2_t,
$$

whereas the IQ can be written by (Andersen et al. (2014))

$$
IQ_t = E(RQ_t) = \frac{M}{3} \sum_{m=1}^{M} E(r^4_{t,m}) = \frac{\sigma^4_t}{M} \sum_{m=1}^{M} s^4_m. \tag{2.5}
$$

3 The impact of intraday periodicity on RV and BV

Now we provide our statements about the impact of IP on the stochastic properties of realized measures for the model in (2.4) and (2.4). For convenience, we model IP by replacing the normalized function s_m by

$$s_m^2 = \frac{g(\frac{m}{M})}{g_M}, \qquad \text{with} \qquad g_M = \frac{1}{M}\sum_{m=1}^{M} g(\tfrac{m}{M}), \tag{3.1}$$

where $g : [0,1] \mapsto \mathbb{R}$ is a given function. The functional form of $g(\cdot)$ could be very flexible and is subject to very general regularity conditions specified in the following propositions. Empirically, it is usually of U-shape for different classes of risky assets due to strategic interaction of traders during opening and closing trading hours (Admati and Pfleiderer (1988)).

3.1 Finite sample results

For the discrete model of intraday returns (2.4)-(2.4) and given the form of IP $\{g(m/M)\}_{m=1}^{M}$, we derive some stochastic properties of RV and BV estimators of daily IV in the following proposition for finite M.

Proposition 1 *Assume that the IP component is given by (3.1) for some function* $g : [0,1] \mapsto \mathbb{R}$.
(A) The estimator RV_t *for daily* IV *is unbiased so that* $E[RV_t] = IV_t$*. The estimator* BV_t *is biased, that is* $E[BV_t] = \sigma_t^2(1 - R_M) = IV_t(1 - R_M)$*, where the factor* R_M *is given by*

$$R_M = \left(g(\tfrac{1}{M}) + \sum_{m=2}^{M} g(\tfrac{m}{M})^{1/2}[g(\tfrac{m}{M})^{1/2} - g(\tfrac{m-1}{M})^{1/2}] \right) / \sum_{m=1}^{M} g(\tfrac{m}{M}).$$

If $g(\cdot)$ *is continuously differentiable on interval* $[0,1]$*, we have as* $M \to \infty$

$$M \cdot R_M = \left[\frac{1}{2}\frac{\int_0^1 g'(x)dx}{\int_0^1 g(x)dx} + \frac{g(0)}{\int_0^1 g(x)dx} \right] \cdot \big(1 + o(1)\big),$$

so that $\lim_{M\to\infty} R_M = 0$*, i.e.* BV_t *is an asymptotically unbiased estimator of* IV*.*
(B) The (co)variances of RV_t *and* BV_t *are given as*

$$\begin{aligned}
Var(BV_t) &= \frac{\pi^2}{4}\frac{\sigma_t^4}{M^2 g_M^2}\left\{ \left(1 - \frac{4}{\pi^2}\right)\sum_{m=2}^{M} g(\tfrac{m}{M})g(\tfrac{m-1}{M}) + \left(\frac{4}{\pi} - \frac{8}{\pi^2}\right)\sum_{m=3}^{M} g(\tfrac{m}{M})^{1/2} g(\tfrac{m-1}{M}) g(\tfrac{m-2}{M})^{1/2} \right\}, \\
Var(RV_t) &= \frac{2\sigma_t^4}{M^2 g_M^2}\sum_{m=1}^{M} g^2(\tfrac{m}{M}), \\
Cov(RV_t, BV_t) &= \frac{\sigma_t^4}{M^2 g_M^2}\left[\sum_{m=2}^{M} g(\tfrac{m-1}{M})^{1/2} g(\tfrac{m}{M})^{3/2} + \sum_{m=1}^{M-1} g(\tfrac{m+1}{M})^{1/2} g(\tfrac{m}{M})^{3/2} \right]
\end{aligned}$$

Moroever, if $M \to \infty$ we have

$$\begin{aligned} Var(BV_t) &= \frac{\sigma_t^4}{M}\Big(\frac{\pi^2}{4} - 3 + \frac{\pi}{4}\Big)\xi\ (1 + o(1)),, \\ Var(RV_t) &= \frac{2\sigma_t^2}{M}\xi\ (1 + o(1)), \\ Cov(RV_t, BV_t) &= \frac{2\sigma_t^4}{M}\xi\ (1 + o(1)). \end{aligned}$$

where the scaling factor is given by

$$\xi = \frac{\int_0^1 g^2(x)dx}{(\int_0^1 g(x)dx)^2} \tag{3.2}$$

Moreover, we have $\xi \geq 1$ and $\xi = 1$ if and only if $g(\cdot)$ is almost everywhere constant (uniform IP).

Thus, in the case of IP, RV is an unbiased estimator for IV but BV has a finite sample bias which should be corrected for finite M. Since the expectation of BV is given by

$$E[BV_t] = \frac{\sigma_t^2}{M \cdot g_M} \cdot \sum_{m=2}^{M} g(\tfrac{m}{M})^{1/2} g(\tfrac{m-1}{M})^{1/2} = \frac{\sigma_t^2}{M} \cdot \sum_{m=2}^{M} s_m s_{m-1},$$

we suggest the following bias-corrected measure $\widetilde{BV}_t$ for finite M:

$$\widetilde{BV}_t = \frac{\pi}{2} \cdot M \cdot \left(\sum_{m=2}^{M} s_m s_{m-1}\right)^{-1} \cdot \sum_{m=2}^{M} |r_m| \cdot |r_{m-1}| = M \left(\sum_{m=2}^{M} s_m s_{m-1}\right)^{-1} BV_t.$$

Note that for an IP slowly changing over day time with $s_m \approx s_{m-1}$ we get $E[BV_t] = IV_t \cdot (M - 1)/M$, i.e. in this case BV remains an approximately unbiased estimator of IV.
In the following proposition we provide the expectation of realized TP and QP measures for IQ under our model in (2.4) and (2.4) for the given IP $g(\cdot)$

Proposition 2 *Assume that IP is given by (3.1), then the expectations of RQ_t, QP_t and TP_t are given as*

$$\begin{aligned} E[RQ_t] &= \frac{\sigma_t^4}{M \cdot g_M^2} \sum_{m=4}^{M} g(\tfrac{m}{M})^2, \\ E[QP_t] &= \frac{\sigma_t^4}{M \cdot g_M^2} \sum_{m=4}^{M} [g(\tfrac{m-3}{M}) g(\tfrac{m-2}{M}) g(\tfrac{m-1}{M}) g(\tfrac{m}{M})]^{\frac{1}{2}}, \\ E[TP_t] &= \frac{\sigma_t^4}{M \cdot g_M^2} \sum_{m=3}^{M} [g(\tfrac{m-2}{M}) g(\tfrac{m-1}{M}) g(\tfrac{m}{M})]^{\frac{2}{3}}. \end{aligned}$$

Moreover, if $g(\cdot)$ is square integrable we have as $M \to \infty$

$$\lim_{M\to\infty} E[RQ_t] = E[TP_t] = E[QP_t] \;=\; \sigma_t^4 \cdot \xi = IQ_t,$$

where the scaling factor ξ *comprising the impact of IP is given by* (3.2).

Then we suggest the following finite M IP corrections for TP and QP:

$$\begin{aligned}
\widetilde{RQ}_t &= M\Big(\sum_{m=1}^{M} s_m^4\Big)^{-1} RQ_t, \\
\widetilde{QP}_t &= M\Big(\sum_{m=4}^{M} (s_m s_{m-1} s_{m-2} s_{m-3})^{1/2}\Big)^{-1} QP_t, \\
\widetilde{TP}_t &= M\Big(\sum_{m=3}^{M} (s_m s_{m-1} s_{m-2})^{2/3}\Big)^{-1} TP_t.
\end{aligned}$$

3.2 Asymptotic results

The asymptotic distribution for realized measures given the IP is required for statistical inference concerning IV as well as for conducting tests for daily jumps. We provide the corresponding bivariate limit distribution for RV_t and BV_t as $M \to \infty$ for our discrete time model of intraday returns.

Theorem 1 *Consider model (2.4) and (2.4) and assume that the IP component is given by (3.1) with a continuously differentiable function* $g : [0,1] \mapsto \mathbb{R}$. *Then, as* $M \to \infty$,

$$M^{1/2} \cdot IQ_t^{-1/2} \cdot \begin{pmatrix} RV_t - \sigma_t^2 \\ BV_t - \sigma_t^2 \end{pmatrix} \xrightarrow{L} \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} v_{rr} & v_{rq} \\ v_{qr} & v_{qq} \end{bmatrix} \right).$$

with the constants $v_{rr} = 2$, $v_{qr} = v_{rq} = 2$ *and* $v_{qq} = \pi^2/4 + \pi - 3$. *The quarticity IQ is given by* $IQ_t = \xi \cdot \sigma_t^4$ *and can be consistently estimated by either* TP_t *or* QP_t.

Thus, a pronounced IP with $\xi \geq 1$ causes more variability of IV estimators compared to the case of no IP where $\xi = 1$. The asymptotic $(1-\alpha)$-confidence interval for daily IV based on RV and RQ measures is given in case of no jumps as

$$CI_t(1-\alpha) = [RV_t + z_{\alpha/2} \cdot v_{rr}^{1/2} \cdot RQ_t^{1/2}/M^{1/2}, \quad RV_t - z_{\alpha/2} \cdot v_{rr}^{1/2} \cdot RQ_t^{1/2}/M^{1/2}],$$

where $z_{\alpha/2}$ is the $\alpha/2$-quantile of the standard normal distribution. The confidence intervals based on BV, TP and QP are constructed in the same way.
The results in Proportion 2 and Theorem 1 are useful for making statistical inference concerning the IV. If M is sufficiently large one could robustly estimate IQ by TP or QP as in (2.3) or (2.2) without estimating ξ separately. However, as we show later in the Monte Carlo simulation, the approximation $TP_t \approx IQ_t$ as in Proposition 2 is only precise enough for a fairly large M. Based on our Monte Carlo results, we strictly recommend to use the IP-corrected estimators $\widetilde{QP}_t$ or $\widetilde{TP}_t$ for the construction of confidence intervals.

3.3 Test for jumps

In order to decide, whether RV or BV estimator should be used for measuring IV at day t, one needs to make a test for jumps during this day. Here we consider tests for jumps during this day (Barndorff-Nielsen and Shephard (2006), Huang and Tauchen (2005))[1], which are based on a "standardized" difference $RV_t - BV_t$. Several tests where a jump is detected if the distance between RV and BV is statistically significant are investigated by Huang and Tauchen (2005). For the sake of brevity we concentrate here on the test which uses the realized quad-power variation (QP), whereas tests using TP are constructed in a similar way.
A popular test is based on the statistic

$$T_t = \frac{(RV_t - BV_t)/RV_t}{\left(\frac{\pi^2/4+\pi-5}{M} \cdot \max\left\{1, \frac{QP_t}{BV_t^2}\right\}\right)^{1/2}}, \tag{3.3}$$

which we correct in order to address the impact of IP, that is

$$\widetilde{T}_t = \frac{(RV_t - \widetilde{BV}_t)/RV_t}{\left(\frac{\pi^2/4+\pi-5}{M} \cdot \xi \cdot \max\left\{1, \frac{\widetilde{QP}_t}{\widetilde{BV}_t^2}\right\}\right)^{1/2}}, \tag{3.4}$$

We have (even in case of jumps) $BV_t^2 \to \sigma_t^4$, $\widetilde{BV}_t^2 \to \sigma_t^4$, $QP_t \to \xi\sigma_t^4$ and $\widetilde{QP}_t \to \sigma_t^4$ as $M \to \infty$ and therefore a straightforward application of Theorem 1 and the Delta method shows that under the null hypothesis of 'no jumps' the statistics T_t and $\widetilde{T}_t$ have an asymptotic standard normal distribution $\mathcal{N}(0,1)$ as $M \to \infty$ due to $\lim_{M\to\infty} \max\{QP_t/BV_t^2\} = \max\{1,\xi\} = 1$, by Theorem 1.
Note that there is no difference between the corrected and original test if M is very large. However, we show in Section 4 by means of a Monte Carlo study that for realistic sample sizes the differences are substantial. Even in the case of a quite large $M = 1152$ we observe test size distortions, see Figure 5. Thus, in order to correct the tests for the impact of IP one has to estimate the finite sample correction factors which are based on estimators of $g(\cdot)$.

4 Simulation study

We illustrate our theoretical findings by means of a Monte Carlo simulation study. First we introduce the functional form of the IP and discuss the choice of the parameters within the model in (2.4)-(2.4) whereas the volatility component σ_t^2 assumed to be constant during the day. The generated intraday returns are used for the construction of the realized measures. We study the impact of IP on the obtained estimators in Section 4.2. Then we analyse the performance of jump tests given the IP form for both constant as well as stochastic volatility

[1]But not tests for jumps in a selected intraperiod return, as in Lee and Mykland (2008), Boudt et al. (2011).

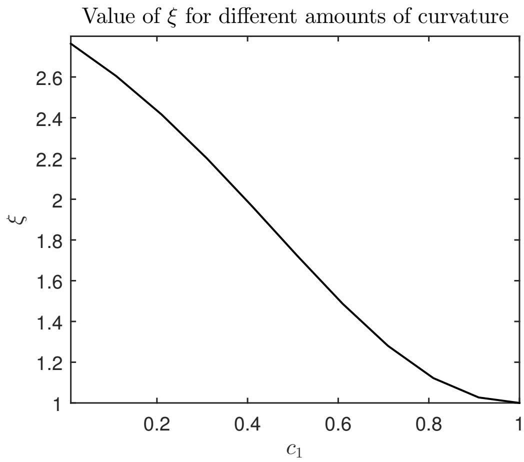

Figure 1: *Values of the scaling factor ξ defined in* (3.2) *for different amounts of curvature.*

whereas the latter model is presented in Section 4.3.

4.1 Design of the study

We consider three different settings with $M = 24$, $M = 48$ and $M = 1152$, which corresponds to sampling frequencies of 20 min, 10 min and 25 sec for an 8-hours trading day, respectively. Note that in the case $M = 1152$ there exists an adverse impact of MMN, and we only consider this case to illustrate the large sample properties of the estimators of IV in presence of IP. We generate M intraday day returns for each of $T = 10^4$ days with $r_{m,t} \sim \mathcal{N}(0, \sigma_t^2 s_m^2/M)$. The value σ_t^2 is fixed to $\sigma^2 = 1$ which also implies that $IV = \sigma^2 = 1$ as well as $\sigma^4 = 1$. In line with empirical evidence, we assume a quadratic convex U-shape for the intraday profile with

$$g(m/M) = c_1 + c_2(m - M/2)^2, \qquad c_1, c_2 > 0.$$

The standardized IP values are obtained by $s_m^2 = g(m/M)/g_M$. Taking into account the constraint $\sum_{m=1}^{M} s_m^2 = M$, the parameter c_2 determines the curvature and can be exactly computed given a fixed value of c_1, i.e.

$$c_2 = \frac{(1 - c_1) \cdot 12}{M^2 + 2}.$$

We consider $c_1 \in \{0.01, 0.11, \ldots, 0.91, 1\}$ whereas $c_1 = 1$ corresponds to no IP (no curvature) and for $c_1 \to 0^+$ the curvature is most pronounced. The values of the scaling factor ξ defined in (3.2) as a function of c_1 are plotted in Figure 1 where we observe that ξ is substantially larger than one even for moderate curvature with values of c_1 close to one.
As in practice the IP is unknown, we estimate it in our study as well. Parametric and non-parametric methods for estimating the IP have been proposed by Andersen and Bollerslev (1997) and Taylor and Xu (1997), respectively. In order to construct estimators for $\hat{g}(\cdot)$ and $\hat{s}^2(\cdot)$, we apply the robust non-parameter weighted standard deviation (WSD) estimator of

Boudt et al. (2011). By contrasting the finite sample variance of RV obtained from Proposition 1 as $(2/M^2)\cdot\sigma^4\cdot\sum s_m^4$ and its asymptotic counterpart $(2/M)\cdot\sigma^4\cdot\xi$ from Theorem 1, we suggest to use the finite sample estimator

$$\hat{\xi} = \frac{1}{M}\cdot\sum_{m=1}^{M}\hat{s}_m^4.$$

4.2 The impact of IP on realized measures

Here we investigate the IP impact on daily realized measures such as BV and RV for IV and TP and QP for IQ. After generating intraday returns, we compute RV_t, BV_t, TP_t QP_t for each day $t = 1, ..., T$. Next, we apply the usual finite sample corrections of these multipower measures required as the number of sum elements is smaller than M, e.g. $M/(M-1)$ for BV, $(M-3)/M$ for QP etc. We denote these measures as 'uncorrected' in order to distinguish them from IP-corrected measures denoted by $\widetilde{BV}_t$, etc. Then we average realized measures over $T = 10^4$ days by computing $\overline{RV} = (1/T)\cdot\sum_{t=1}^{T} RV_t$ etc. They are shown for different M and c_1 values in Figure 3 for RV and BV, and in Figure 4 for TP and QP. The uncorrected measures are shown in the left panels, whereas IP-corrected on the right ones.

Consistent with our theoretical considerations, the mean of RV is not affected by IP. The bias in BV due to IP is strongly pronounced for $M = 24$ and $M = 48$ for large and medium curvature but almost disappears for large $M = 1152$. The bias-corrected mean of $\widetilde{BV}$ is close to the true IV for all considered values of M, so the suggested correction appears to be useful even for the relatively large sample size $M = 1152$.

The means of both uncorrected TP and QP as well as of corrected $\widetilde{TP}$ and $\widetilde{QP}$ reported in Figure 4 show quite distinct behavior for different values of M. We observe in the left panel in Figure 4 that for $M = 1152$ the means of the uncorrected estimates are rather close to the scaling factor ξ as it is shown in Proposition 2 (note that $\sigma^4 = 1$). However, for small values such as $M = 24$ and $M = 48$ the behaviour of these measures is completely different. In particular, the mean of the uncorrected TP is even smaller than 1 in the case of large curvatures and $M = 24$. In the right panel of Figure 4 we observe that the proposed IP-corrections work well for all considered values of M. Summarizing, the IP has a substantial influence on BV as well as on TP and QP measures. The proposed IP-corrections are necessary in order to obtain valid statistical inference.

4.3 The impact of IP on jump tests

Next, we investigate the impact of IP on the size and power of the jump tests for both uncorrected (3.3) and corrected (3.4) test statistics. Given the nominal size 5%, we compute the actual size of the tests for different values of M and curvature ξ which is determined by c_1.

In order to generate intraday return we use both, a constant intraday volatility model as described earlier as well as a stochastic intraday volatility model. The latter is defined by assuming that $r_{m,t} \sim \mathcal{N}(0, s_m^2 \cdot \gamma_{t,m}^2/M)$. The stochastic volatility is governed by the process

$$\Delta\gamma_{t,m}^2 = 0.035(0.636 - \gamma_{t,m}^2)\,\Delta t + 0.144\gamma_{t,m}^2 u_{t,m}\sqrt{\Delta t},$$

with $\Delta t = 1/M$ and $u_{t,m} \sim \mathcal{N}(0,1)$. The initial value is set to $\gamma_{1,1}^2 = 1$. The model is a discretized version of the GARCH (1,1) diffusion studied by Andersen and Bollerslev (1998a) and Goncalves and Meddahi (2009). Note that the process for $\Delta\gamma_{t,m}^2$ is mean-reverting which ensures boundedness.

The test size for different curvatures measured by c_1 is shown in Figure 5 for both constant and stochastic intraday volatility models. The results are very similar for both volatility specifications. In case of no curvature, the actual test size is equal to the nominal 5%. For sample sizes $M = 24$ and $M = 48$, the uncorrected tests exceed their nominal level substantially for $c_1 = 0.01$ (large curvature) and approaches the nominal level (5%) slowly as the curvature decreases. Test size distortion due to IP is still present for $M = 1152$ for pronounced IPs. On the other hand the corrected tests proposed in this paper keep their nominal level for all c_1 and M values. These observations support the necessity to use the IP-corrected test statistic given in (3.4).

The test of Kolmogorov-Smirnov is applied in order to check whether the empirical distribution of test statistics T_t and $\widetilde{T}_t$ for $M = 1152$ and $M = 11520$ deviates from the standard normal distribution $\mathcal{N}(0,1)$ under the null hypothesis 'no jumps'. We consider different curvatures with the largest $\xi = 2.7635$ to no curvature case $\xi = 1$ for both constant and stochastic volatility. The corresponding p-values are provided in Table 1. In the case $M = 11520$ which is completely unrealistic from the practical point of view the p-values are large for all scaling factors ξ so that the null hypothesis of a standard normal distribution cannot be rejected. However, for $M = 1152$, the p-values are rather close to zero even for quite moderate curvature in the case of the uncorrected test statistic. We observe the p-values for the corrected test statistic are larger compared to the uncorrected one, however, they are still quite close to zero for large curvature in the casef $M = 1152$,. These findings point on the necessity of IP correction for the jump test as well as that even $M = 1152$ is not enough to guarantee the correct test size for large intraday curvatures.

For the power analysis, we model one additive jump per day of a fixed size κ which appears at a random time point during the daily trading. We consider jumps of small ($\kappa = 0.1$ $\kappa = 0.2$), medium ($\kappa = 0.5$) and large ($\kappa = 1$) size. We analyse three different amounts of curvature ξ with the nominal size set to be 5%. The results of the power analysis for different jump sizes and confidence levels are collected in Table 2 for constant intraday volatility (upper part) and for stochastic intraday volatility (lower part).

For the case of constant volatility, as expected, the power of the corrected and uncorrected test are identical in case of no curvature $\xi = 1$. Generally, the power of both tests is not very high for small M and jumps of small size κ. For medium and large jumps and small M, the uncorrected test has higher power, but this seemingly superiority can be explained

constant intraday volatility				stochastic intraday volatility			
$M = 1152$		T_t	$\widetilde{T}_t$	$M = 1152$		T_t	$\widetilde{T}_t$
$c_1 = 0.01$	ξ =2.7635	0.0000	0.0186	$c_1 = 0.01$	ξ =2.7635	0.0000	0.0008
$c_1 = 0.11$	ξ =2.6047	0.0000	0.0261	$c_1 = 0.11$	ξ =2.6047	0.0000	0.0009
$c_1 = 0.21$	ξ =2.4163	0.0000	0.0334	$c_1 = 0.21$	ξ =2.4163	0.0000	0.0010
$c_1 = 0.31$	ξ =2.2010	0.0000	0.0435	$c_1 = 0.31$	ξ =2.2010	0.0000	0.0007
$c_1 = 0.41$	ξ =1.9655	0.0000	0.0416	$c_1 = 0.41$	ξ =1.9655	0.0000	0.0015
$c_1 = 0.51$	ξ =1.7217	0.0001	0.0526	$c_1 = 0.51$	ξ =1.7217	0.0000	0.0014
$c_1 = 0.61$	ξ =1.4865	0.0016	0.0920	$c_1 = 0.61$	ξ =1.4865	0.0000	0.0052
$c_1 = 0.71$	ξ =1.2800	0.0213	0.1228	$c_1 = 0.71$	ξ =1.2800	0.0000	0.0098
$c_1 = 0.81$	ξ =1.1219	0.1596	0.0639	$c_1 = 0.81$	ξ =1.1219	0.0012	0.0255
$c_1 = 0.91$	ξ =1.0270	0.4789	0.0545	$c_1 = 0.91$	ξ =1.0270	0.0349	0.1886
$c_1 = 1$	ξ =1.0000	0.0562	0.0562	$c_1 = 1$	ξ =1.0000	0.4190	0.4190
$M = 11520$		T_t	$\widetilde{T}_t$	$M = 11520$		T_t	$\widetilde{T}_t$
$c_1 = 0.01$	ξ =2.7635	0.9305	0.9921	$c_1 = 0.01$	ξ =2.7635	0.5691	0.2372
$c_1 = 0.11$	ξ =2.6047	0.8893	0.9953	$c_1 = 0.11$	ξ =2.6047	0.6927	0.3238
$c_1 = 0.21$	ξ =2.4163	0.9101	0.9998	$c_1 = 0.21$	ξ =2.4163	0.8463	0.5438
$c_1 = 0.31$	ξ =2.2010	0.8684	0.9993	$c_1 = 0.31$	ξ =2.2010	0.8714	0.6839
$c_1 = 0.41$	ξ =1.9655	0.8867	0.9980	$c_1 = 0.41$	ξ =1.9655	0.6717	0.6497
$c_1 = 0.51$	ξ =1.7217	0.7480	0.9707	$c_1 = 0.51$	ξ =1.7217	0.7383	0.5028
$c_1 = 0.61$	ξ =1.4865	0.9112	0.9614	$c_1 = 0.61$	ξ =1.4865	0.6774	0.5359
$c_1 = 0.71$	ξ =1.2800	0.6836	0.9291	$c_1 = 0.71$	ξ =1.2800	0.6472	0.4395
$c_1 = 0.81$	ξ =1.1219	0.9831	0.9928	$c_1 = 0.81$	ξ =1.1219	0.8830	0.8950
$c_1 = 0.91$	ξ =1.0270	0.9991	0.9950	$c_1 = 0.91$	ξ =1.0270	0.9259	0.9440
$c_1 = 1$	ξ =1.0000	0.7738	0.7738	$c_1 = 1$	ξ =1.0000	0.8701	0.8701

Table 1: *Kolmogorov-Smirnov p-values for both uncorrected and corrected jump test statistics* T *and* $\widetilde{T}$ *which are standard normally distributed under* H_0.

by the fact the uncorrected tests exceed the nominal level substantially. Even for $M = 1152$ the power is low for both tests in the case of small jumps but approaches 1 as the jump size increases. In the stochastic volatility the p-values are close but slightly higher to those in the constant volatility case confirming that more intraday variability causes more frequent rejects of the null hypothesis (Andersen et al. (2012)). This is in line with the findings of Huang and Tauchen (2005), whose simulations show that time varying intraday volatility following a simple stochastic volatility model does not have an adverse effect on the performance of the jump tests.

constant intraday volatility σ_t^2									
uncorrected test, high curvature ($\xi = 2.7635$)					corrected test, high curvature ($\xi = 2.7635$)				
M / κ	0.1	0.2	0.5	1	M / κ	0.1	0.2	0.5	1
24	0.2245	0.2355	0.4200	0.8059	24	0.0524	0.0536	0.1028	0.4328
48	0.1805	0.2012	0.4492	0.8984	48	0.0586	0.0622	0.1895	0.7584
1152	0.0988	0.2540	0.9955	1.0000	1152	0.0722	0.2099	0.9948	1.0000
uncorrected test, medium curvature ($\xi = 1.9655$)					corrected test, medium curvature ($\xi = 1.9655$)				
M / κ	0.1	0.2	0.5	1	M / κ	0.1	0.2	0.5	1
24	0.1719	0.1786	0.3412	0.7860	24	0.0525	0.0536	0.1116	0.4999
48	0.1494	0.1674	0.4334	0.9124	48	0.0575	0.0619	0.2130	0.8291
1152	0.1019	0.2950	0.9983	1.0000	1152	0.0809	0.2554	0.9982	1.0000
uncorrected test, no curvature ($\xi = 1$)					corrected test, no curvature ($\xi = 1$)				
M / κ	0.1	0.2	0.5	1	M / κ	0.1	0.2	0.5	1
24	0.0517	0.0590	0.1899	0.6769	24	0.0517	0.0590	0.1899	0.6769
48	0.0526	0.0727	0.3264	0.9284	48	0.0526	0.0727	0.3264	0.9284
1152	0.0972	0.3930	1.0000	1.0000	1152	0.0972	0.3930	1.0000	1.0000

stochastic intraday volatility $\sigma_{t,m}^2$									
uncorrected test, high curvature ($\xi = 2.7635$)					corrected test, high curvature ($\xi = 2.7635$)				
M / κ	0.1	0.2	0.5	1	M / κ	0.1	0.2	0.5	1
24	0.2273	0.2450	0.4223	0.8124	24	0.0512	0.0587	0.1069	0.4288
48	0.1719	0.2032	0.4513	0.9001	48	0.0545	0.0641	0.1914	0.7639
1152	0.1047	0.2608	0.9961	1.0000	1152	0.0779	0.2184	0.9960	1.0000
uncorrected test, medium curvature ($\xi = 1.9655$)					corrected test, medium curvature ($\xi = 1.9655$)				
M / κ	0.1	0.2	0.5	1	M / κ	0.1	0.2	0.5	1
24	0.1731	0.1845	0.3420	0.7916	24	0.0552	0.0622	0.1944	0.6889
48	0.1409	0.1697	0.4337	0.9143	48	0.0511	0.0704	0.3302	0.9282
1152	0.1056	0.3014	0.9989	1.0000	1152	0.0918	0.4021	1.0000	1.0000
uncorrected test, no curvature ($\xi = 1$)					corrected test, no curvature ($\xi = 1$)				
M / κ	0.1	0.2	0.5	1	M / κ	0.1	0.2	0.5	1
24	0.0552	0.0622	0.1944	0.6889	24	0.0517	0.0590	0.1899	0.6769
48	0.0511	0.0704	0.3302	0.9282	48	0.0526	0.0727	0.3264	0.9284
1152	0.0918	0.4021	1.0000	1.0000	1152	0.0972	0.3930	1.0000	1.0000

Table 2: *Power of the jump test for different M and curvature for constant and stochastic intraday volatility. The test size is set to be $\alpha = 0.05$.*

5 Empirical study

5.1 Design of the study

In the empirical study, we consider intraday returns of the Dow Jones Industrial Average Index with the aim to make statements concerning daily IV. Our data set consists of intraday observations for the period from January 1996 to December 2010. In our application we consider 15 and 10 min intraday returns because of two reasons. First, it is done due to reported difficulties in estimating integrated quarticity from higher frequency data (Andersen et al. (2014), Bandi and Russell (2008)). Secondly, the conditional normality is more appropriate for these frequencies of intraday data than for higher ones where the use of t-distribution could be more suitable (Bekierman and Gribisch (2016)). As IP estimation requires a consistent number of observations per day, we skip days with missing observations. The final sample consists of 3329 days with $M = 25$ (15-min) or $M = 38$ (10-min) observations per day. Due to a possible bias caused by overnight effects, we exclude the first observation of the day.

The study is organized as follows: first, we estimate the intraday IP s_m, curvature measure ξ and the correction factors derived in Section 3. Then we present descriptive statistics for the realized measures RV, BV , TP and QP. Finally, we conduct both the uncorrected and IP-corrected jump test and analyze differences in their performance.

5.2 Estimation of intraday pattern and descriptive statistics

As in the simulation study, the IP is estimated from the data via the non-parametric WSD estimator proposed by Boudt et al. (2011). We choose this estimator as it is robust to jumps and does not require a-priori specification of the IP's functional form. During the considered period of time the IP form remains almost unchanged whereas for monitoring changes in IP shapes one can apply methods from functional data analysis as in Kokoszka and Reimherr (2013) and Gabrys et al. (2013).

The estimated IP components for 15 and 10 minute returns are illustrated in Figure 2. The components are normalized such that they sum up to M. For both sampling frequencies, the pattern has a convex U-shape as one could expected from both theory (Admati and Pfleiderer (1988)) and previous empirical work (Andersen and Bollerslev (1997)). It is high during morning and evening hours and low during the lunch break. In numerical terms, volatility is about twice as high during the peak in the morning compared to the middle of the day. Both patterns look very similar, although in case of the 10 minute returns there are more pronounced small activity spikes which are smoothed out if sampling frequency is lower.

Given the estimated IP, we calculate correction factors for realized BV, QP and TP as well as the estimate for ξ, all reported in Table 3. The estimated scaling factor $\hat{\xi}$ is quite close to 1 and numerically almost identical for both sampling frequencies. Note, however, that the factor ξ corresponds to the asymptotic case $M \to \infty$ whereas we have finite M here. Thus,

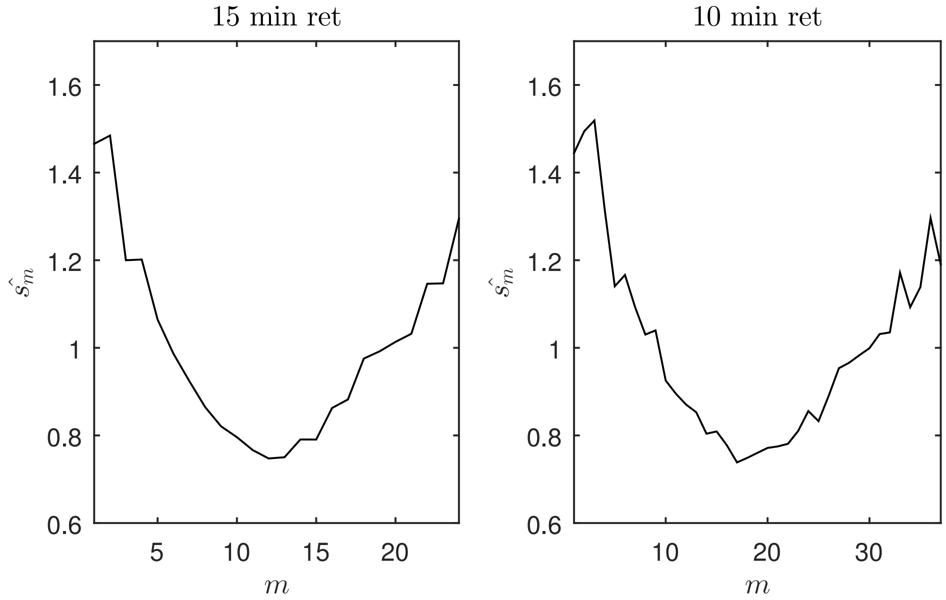

Figure 2: *Intraday patterns of Dow Jones, full sample WSD estimation.*

the estimated finite sample corrections for BV, TP, and QP in Table 3 are of more practical importance. The correction factor for BV defined by the ratio $BV/\widetilde{BV}$ is small for both sampling frequencies, indicating that BV is biased downwards 6% resp. 3% for 15 resp. 10 min returns. The correction factors for TP and QP are much larger, e.g. $TP/\widetilde{TP}$=1.2283 in case of 15 min returns so that there is a downward bias of almost 20% in QP. Thus, even for IP with low curvature there is a substantial downward bias in the estimators of IQ. For 10 min returns, the bias reduces to $QP/\widetilde{QP} = 1.1225$ which is consistent with the simulation results where we find that the finite sample IP-bias reduces with increase of M. The correction factors for TP are lower in magnitude than those for QP but still far away from negligible.

Correction factor	15-min ($M = 24$)	10-min ($M = 37$)
$BV/\widetilde{BV}$	1.0619	1.0375
$TP/\widetilde{TP}$	1.1325	1.0610
$QP/\widetilde{QP}$	1.2283	1.1225
$\hat{\xi}$	1.0445	1.0452

Table 3: *Empirically estimated IP correction factors, full sample.*

Next, we provide descriptive statistics of realized estimators requires for jump tests as described in Section 3.3. In Table 4 we report the full sample averages of both uncorrected and IP-corrected measures. The average RV is stable across sampling frequencies, whereas the average BV is biased downwards compared to RV even after the IP correction. The average corrected TP is almost the same for the both 15 and 10 min sampling frequencies, so the finite M IP correction seems to function properly here. Alternatively, QP shows more variability although the corrected values are close to those of TP. Another important quantity is the relative jump component $(RV - BV)/RV$ which, as expected, reduces substantially after correcting BV for IP bias both for 15 and 10 min frequencies. Thus, it is not a surprise that the average uncorrected test statistics T_{QP} is much larger than of the corrected $\widetilde{T}_{QP}$.

Uncorrected estimators			Corrected estimators		
average	15-min	10-min	average	15-min	10-min
$RV \times 10^{-3}$	0.0944	0.0971	$RV \times 10^{-3}$	0.0944	0.0971
$BV \times 10^{-3}$	0.0840	0.0899	$\widetilde{BV} \times 10^{-3}$	0.0892	0.0932
$TP \times 10^{-6}$	0.0433	0.0465	$\widetilde{TP} \times 10^{-6}$	0.0490	0.0493
$QP \times 10^{-6}$	0.0316	0.0442	$\widetilde{QP} \times 10^{-6}$	0.0388	0.0497
$(RV - BV)/RV$	0.1100	0.0748	$(RV\text{-}\widetilde{BV})/RV$	0.0549	0.0401
T_{QP}	0.6727	0.6424	$\widetilde{T}_{QP}$	0.3260	0.3754

Table 4: *Means of relevant quantities for both uncorrected and corrected estimators.*

5.3 Performance of jump tests

Finally, we investigate the percentage of days identified as jump days both by the corrected and uncorrected tests given in (3.3) and (3.4), respectively. We consider tests exploiting both QP and TP measures of IV. The results are reported in Table 5 for different α-significance levels.

Uncorrected test						Corrected test					
based on QP						based on $\widetilde{QP}$					
α, %	10	5	1	0.5	0.1	α, %	10	5	1	0.5	0.1
15-min	27.87	18.17	7.50	4.77	1.65	15-min	19.07	12.31	4.62	2.94	0.90
10-min	26.55	18.35	6.84	4.71	1.68	10-min	20.54	13.24	4.71	3.24	1.02
based on TP						based on $\widetilde{TP}$					
α, %	10	5	1	0.5	0.1	α, %	10	5	1	0.5	0.1
15-min	27.40	17.72	7.21	4.48	1.53	15-min	18.89	11.93	4.42	2.82	0.87
10-min	26.31	17.72	6.49	4.39	1.50	10-min	20.19	12.32	4.57	3.15	1.02

Table 5: *Percentage of days with jumps detected for different significance levels* α.

First notice that in all cases the percentage of detected jumps is higher than it is expected under H_0. There are only minor differences between the reject percentages for 15 and 10 min frequencies, moreover, the results are quite similar for QP and TP measures. Compared to the uncorrected test, the IP-corrected test detects a smaller percentage of jumps for all significance levels. This is consistent with the findings in Table 4 that the relative jump component $(RV - BV)/RV$ is smaller in case of the corrected test. Moreover, it is in line with our results in the simulation study in Section 4.3, where we show that ignoring IP correction for tests could lead to detection of spurious jumps. Another difference between uncorrected and IP-corrected test results is that for the former more jumps are detected at 15 min frequency whereas for the latter it is vice versa. As expected, there is not a single day where *only* the corrected test detects a jump. This can be interpreted as the evidence that additional jumps detected by the uncorrected test may indeed be spurious.

6 Summary

Based on intraday high frequency observations, realized volatility (RV) and bipower variation (BV) are used as estimators of daily integrated volatility whereas tri-power (TP) and quad-power (QP) variations serve as measures for integrated quarticity. In this paper we investigate the impact of intraday periodicity (IP) on the finite sample properties of these realized measures. For our analysis we assume the discrete time model for intraday returns on risky assets and postulate a multiplicative deterministic IP component which is usually of U-shape empirically. Although asymptotically the impact of IP is negligible, it is shown in this paper that finite sample corrections of BV as well as of TP and QP measures are necessary to obtain valid statistical inference. Moreover, we also demonstrate that tests for a jump component need a finite sample IP-correction, because otherwise the commonly used tests exceed their nominal level substantially. Our results are illustrated by means of a Monte Carlo simulation study for both constant and stochastic intraday volatility models. Finally, we estimate IP correction factors for the intraday returns of the Dow Jones Industrial Average Index.

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Appendix: proofs

Proof of Proposition 1. Note that the statement for RV_t is obvious and consider only $BV_t = \frac{\pi}{2}\sum_{m=2}^{M}|r_{t,m}||r_{t,m-1}|$. We use the fact that $E[|X|] = \sqrt{2/\pi}\,\sigma$ if X is a centered normal distributed random variable with variance σ^2, which implies

$$E[BV_t] = \frac{\sigma_t^2}{M}\sum_{m=2}^{M} s_m s_{m-1} = \sigma_t^2 \frac{\sum_{m=2}^{M}[g(\frac{m}{M})g((m-1)/M)]^{1/2}}{\sum_{m=1}^{M} g(\frac{m}{M})} = \sigma_t^2(1-R_M),$$

where the term R_M is given by

$$R_M = \frac{g(\frac{1}{M}) + \sum_{m=2}^{M} g(\frac{m}{M})^{1/2}(g(\frac{m}{M})^{1/2} - g(\frac{m-1}{M})^{1/2})}{\sum_{m=1}^{M} g(\frac{m}{M})}.$$

If g is continuously differentiable we obtain by the mean value theorem

$$R_M = \frac{\frac{1}{M}\sum_{m=2}^{M} g(\frac{m}{M})^{1/2}(g^{1/2})'(\xi_m) + g(1/M)}{\sum_{m=2}^{M} g(\frac{m}{M})} ,$$

where $\xi_m \in \big((m-1)/M, m/M\big)$ $(m = 2, \ldots, M)$. Consequently, using the fact that $(g^{1/2})'(x) = \frac{1}{2}\frac{g'(x)}{g^{1/2}(x)}$ and an approximation of the sums by a Riemann integral it follows that

$$M \cdot R_M = \Big(\frac{1}{2}\frac{\int_0^1 g'(x)dx}{\int_0^1 g(x)dx} + \frac{g(0)}{\int_0^1 g(x)dx}\Big) \cdot (1 + o(1)) ,$$

which proves the statement (A) of Proposition 1.

For the statement (B) in order to compute$V(BV_t)$ we first look at $E(BV_t)^2$, setting $r_{t,m} = r_m$ for the sake of a simple notation. We have

$$E(BV_t)^2 = \Big[\frac{\sigma_t^2}{M}\sum_{m=2}^{M} s_m s_{m-1}\Big]^2 = \frac{\sigma^4}{M^2}\Big[\sum_{m=2}^{M} s_m s_{m-1}\Big]^2 = \frac{\pi^2}{4}E\Big[\sum_{m_1=2}^{M}\sum_{m_2=2}^{M} |r_{m_1}||r_{m_1-1}|]|r_{m_2}||r_{m_2-1}|\Big].$$

As the random variables r_m are independent, the double sum contains three types of non-vanishing expectations

$$
\begin{aligned}
E(|r_{m_1}^2||r_{m_1-1}^2|) &= \frac{\sigma^4}{M^2} s_m^2 s_{m-1}^2 \\
E(|r_{m_1}||r_{m_1-1}^2||r_{m_1-2}|) &= \sqrt{\frac{2\sigma^2}{\pi M} s_{m_1}^2} \frac{\sigma^2}{M} s_{m_1-1}^2 \sqrt{\frac{2\sigma^2}{\pi M 2} s_{m_1-2}^2} = \frac{2\sigma^4}{\pi M^2} s_{m_1} s_{m_1-1}^2 s_{m_1-2} \\
E(|r_{m_1}||r_{m_1-1}||r_{m_2}||r_{m_2-1}|) &= \frac{4\sigma^4}{\pi^2 M^2} s_{m_1} s_{m_1-1} s_{m_2} s_{m_2-1}.
\end{aligned}
$$

This yields

$$
\begin{aligned}
V(BV_t) = \frac{\pi^2}{4} \frac{\sigma^4}{M^2} \Big\{ \sum_{m_1=2}^{M} s_m^2 s_{m-1}^2 + \frac{4}{\pi} \sum_{m_1=3}^{M} s_{m_1} s_{m_1-1}^2 s_{m_1-2} \\
+ \frac{4}{\pi^2} \underbrace{\sum_{m_1=2}^{M} \sum_{m_2=2}^{M}}_{|m_1-m_2|>1} s_{m_1} s_{m_1-1} s_{m_2} s_{m_2-1} \Big\} - \frac{\sigma^4}{M^2} \Big[\sum_{m=2}^{M} s_m s_{m-1} \Big]^2.
\end{aligned}
$$

Observing the definition of s_m^2 in (3.1), we get

$$
\begin{aligned}
V(BV_t) = \frac{\pi^2}{4} \frac{\sigma^4}{M^2 g_M^2} \Big\{ (1 - \frac{4}{\pi^2}) \sum_{m_1=2}^{M} g(\tfrac{m_1}{M}) g(\tfrac{m_1-1}{M}) \\
+ \frac{4}{\pi} \left(1 - \frac{2}{\pi}\right) \sum_{m_1=3}^{M} \big\{ g(\tfrac{m_1}{M}) \big\}^{1/2} g(\tfrac{m_1-1}{M}) \big\{ g(\tfrac{m_1-2}{M}) \big\}^{1/2} \Big\},
\end{aligned}
$$

which proves the first assertion regarding the variance of BV_t. The approximation finally follows by interpreting the sum as approximations of a Riemann integral. The expressions $V(RV_t)$ and the covariance $Cov(RV_t, BV_t)$ are obtained by similar arguments, and the statement (B) of Proposition 1 follows.

Proof of Theorem 1. It follows from Proposition 2 that

$$
\begin{aligned}
V(BV_t) &= \frac{\sigma^4}{M} (\frac{\pi^2}{4} + \pi - 3) \xi \cdot (1 + o(1)), \\
V(RV_t) &= 2 \frac{\sigma^4}{M} \xi \cdot (1 + o(1)), \\
Cov(RV_t, BV) &= 2 \frac{\sigma^4}{M} \xi \cdot (1 + o(1)).
\end{aligned}
$$

where

$$
\xi = \int_0^1 g^2(x) dx \Big/ \Big(\int_0^1 g(x) dx \Big)^2 \geq 1.
$$

by the Cauchy Schwarz inequality. Therefore Theorem 1 follows by a straightforward application of a CLT for triangular arrays of m-dependent random variables [see Romano and Wolf (2000)] and the Cramer Wold device.

Proof of Proposition 2. Recall the definition of QP_t in (2.2) and (3.1), then

$$E[QP_t] = M\frac{\pi^2}{4}\left\{\sum_{m=4}^{M} E\left[|r_{m-3}|\right]E\left[|r_{m-2}|\right]E\left[|r_{m-1}|\right]E\left[|r_m|\right]\right\} = \frac{\sigma^4}{M}\sum_{m=4}^{M} s_{m-4}s_{m-2}s_{m-1}s_m$$

$$= \frac{\sigma_t^4}{M}(g_M)^{-2}\sum_{m=4}^{M}\left[g(\tfrac{m-3}{M})g(\tfrac{m-2}{M})g(\tfrac{m-1}{M})g(\tfrac{m}{M})\right]^{1/2} = \frac{\sigma_t^4}{M}\xi\ (1+o(1)),$$

where the last line follows again by a Riemann integral. Next consider TP_t defined in (2.3) and observe that $E[|Z|^{4/3}] = \mu_{\frac{4}{3}}\sigma^4 = \frac{\sigma^4}{\sqrt{\pi}}2^{2/3}\Gamma(\frac{7}{6})$ if Z is a centered normal distributed random variable with variance σ^2. This yields observing (3.1)

$$E[TP_t] = M\mu_{\frac{4}{3}}^{-3}\sum_{m=3}^{M}\left\{E\left[|r_{m-2}|^{\frac{4}{3}}\right]E\left[|r_{m-1}|^{\frac{4}{3}}\right]E\left[|r_m|^{\frac{4}{3}}\right]\right\} = \frac{\sigma^4}{M}\sum_{m=3}^{M} s_{m-2}^{\frac{4}{3}}s_{m-1}^{\frac{4}{3}}s_m^{\frac{4}{3}}$$

$$= \frac{\sigma_t^4}{M}(g_M)^{-2}\sum_{m=3}^{M}\left[g(\tfrac{m-2}{M})g(\tfrac{m-1}{M})g(\tfrac{m}{M})\right]^{2/3} = \frac{\sigma_t^4}{M}\xi\ (1+o(1)).$$

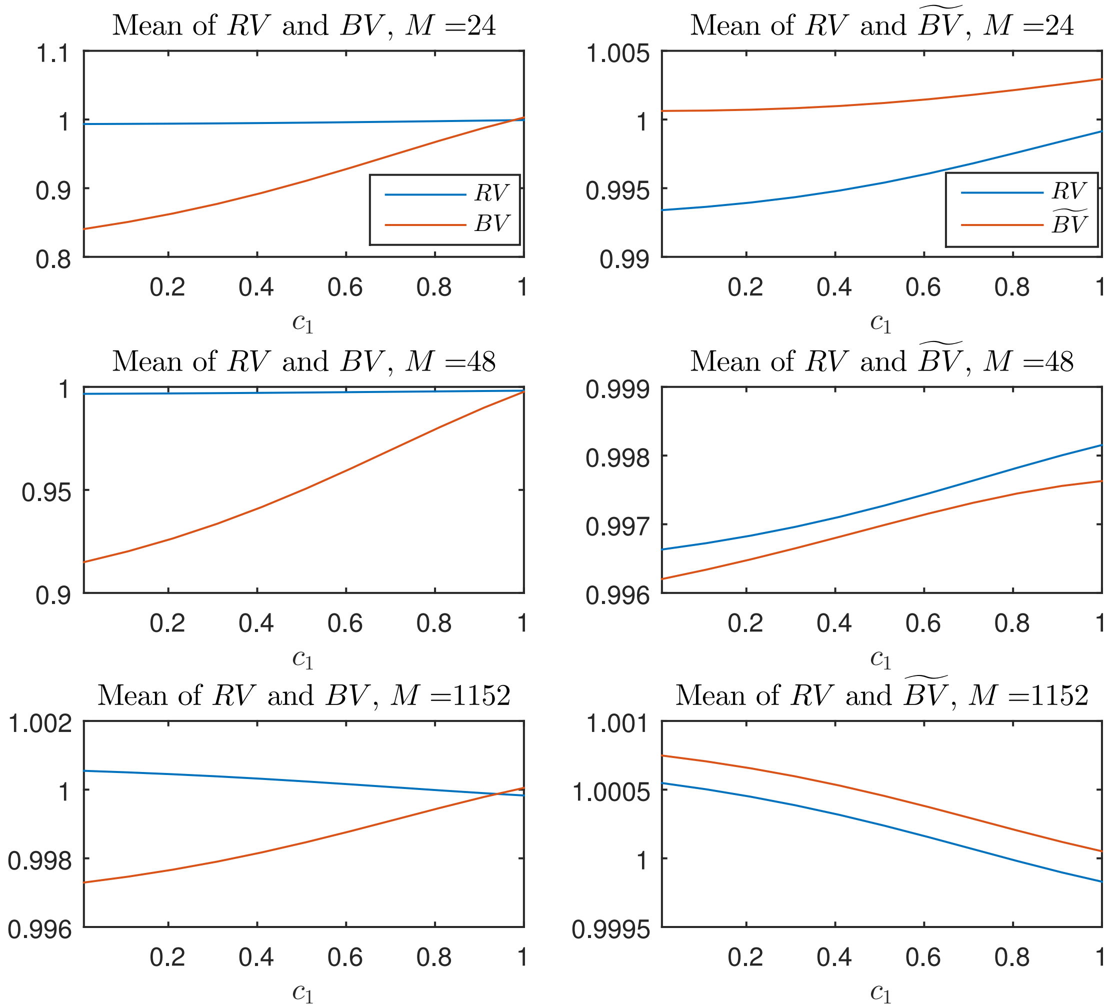

Figure 3: *Bias in RV and BV measure, uncorrected (left) and corrected (right) as a function of curvature ($c_1 = 1$ means no curvature) for different numbers of intraday returns M.*

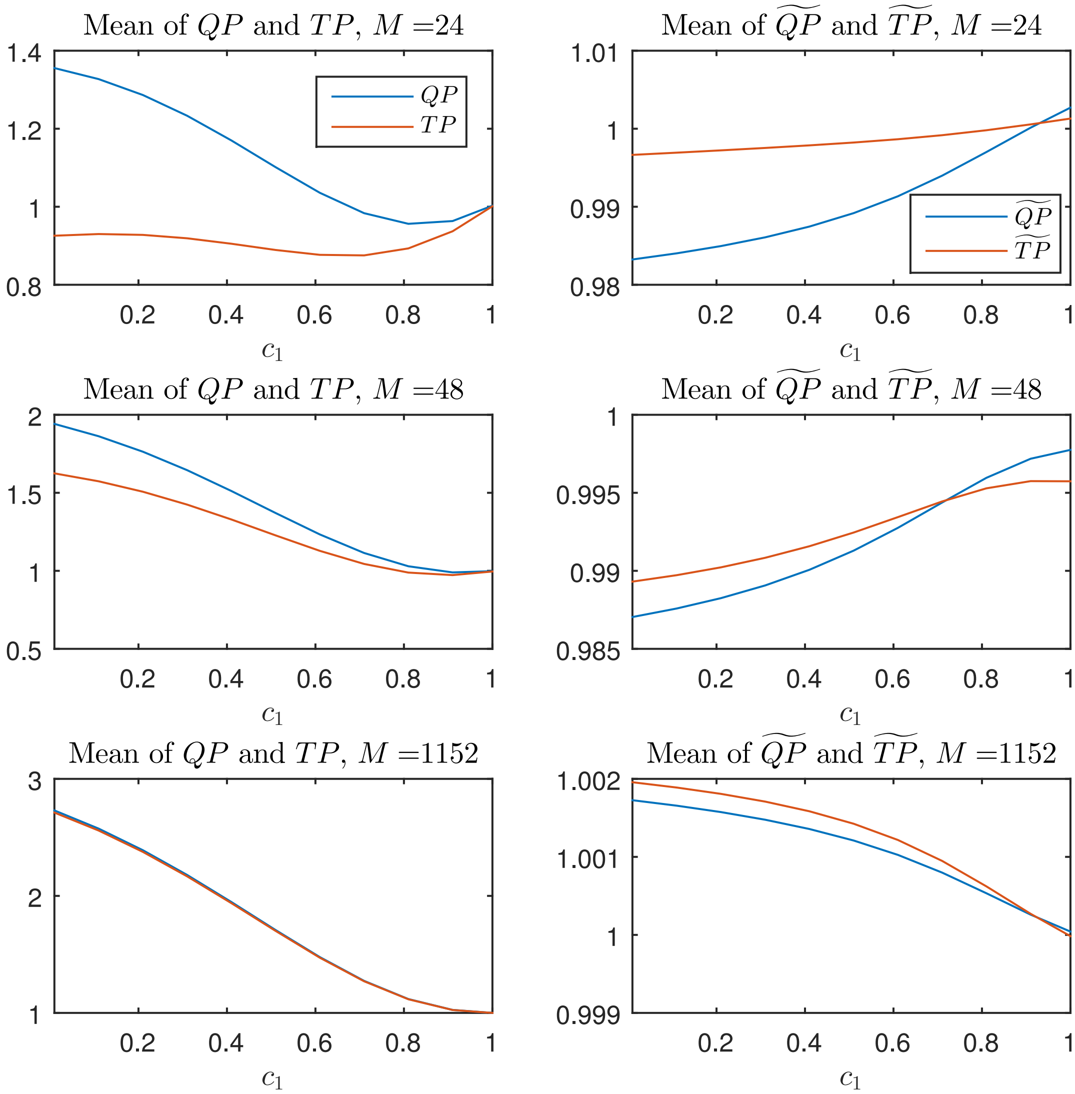

Figure 4: *QP and TP, uncorrected (left) and corrected (right) are a function of curvature ($c_1 = 1$ means no curvature) for different numbers of intraday returns M. Note that for no IP ($c_1 = 1$) we have $QP = TP = 1$.*

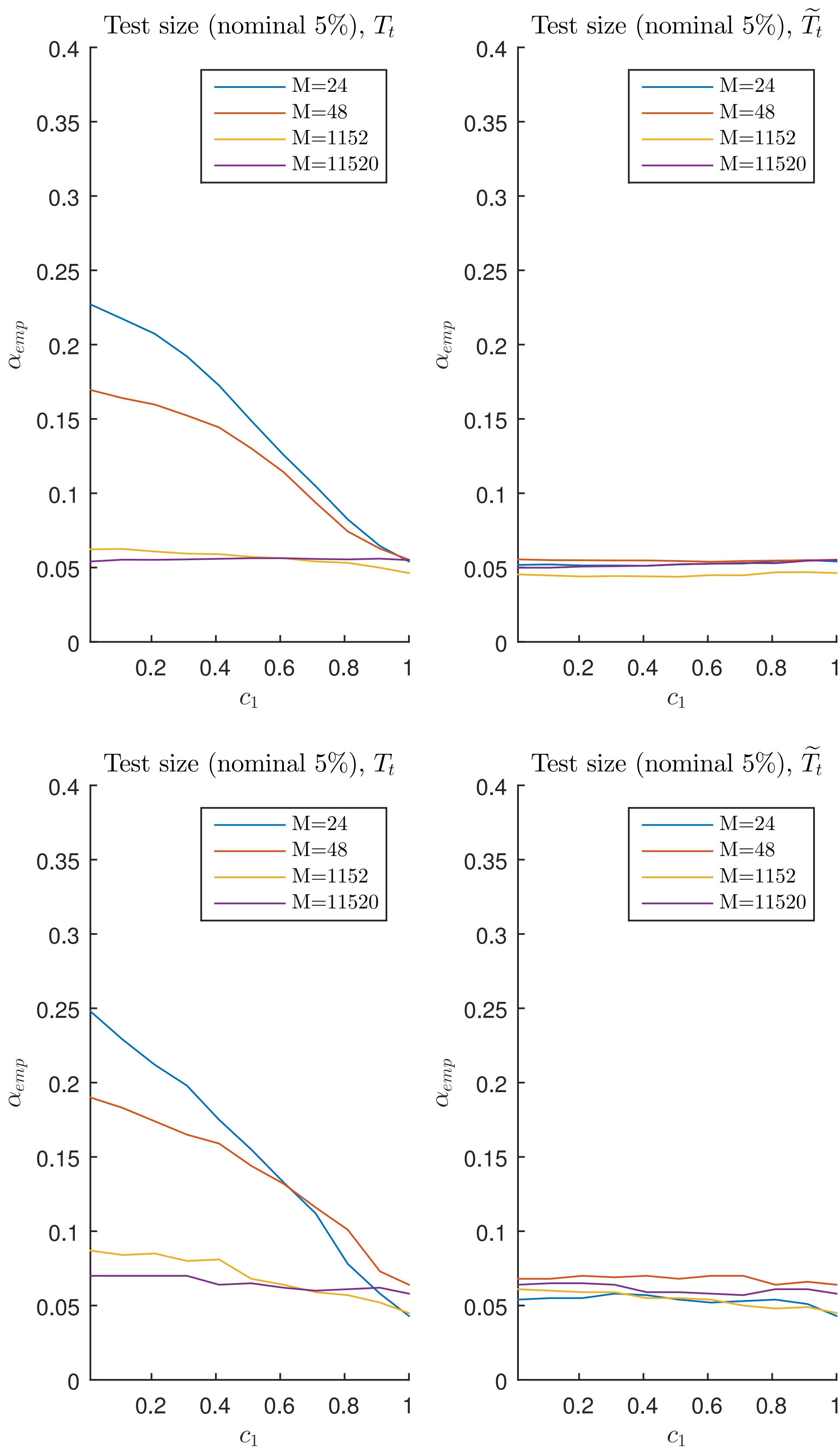

Figure 5: *Empirical test size for different M for constant (upper plots) and stochastic (lower plots) volatility for nominal size 5%.*