Large sample distribution for fully functional periodicity tests

Siegfried Hörmann

Institute for Statistics
Graz University of Technology

Based on joint work with
Piotr Kokoszka (Colorado State) and Gilles Nisol (ULB)
Clément Cerovecki (ULB) and Vaidotas Characiejus (ULB)

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Pollution data

\[ N = 167 \] observations of PM10 measured in Graz (Austria) from 10/1/15 to 03/15/16.

Is there a significant difference in the daily average w.r.t. day of week?
Pollution data

We use ANOVA:

$$H_0 : \mu_{MO} = \cdots = \mu_{SU} \quad \text{v.s.} \quad H_A : H_0 \text{ doesn't hold.}$$

We obtain \( p \)-value of 0.72 \( \implies \) do not reject \( H_0 \).
Pollution data

Same time period with NO (nitrogen monoxide).

We obtain \( p \)-value of 0.25 \( \implies \) do not reject \( \mathcal{H}_0 \).
Dependence

Autocorrelation functions of daily averages of PM10 and NO.

Series PM10

Series NO
PM10–data in Graz are available in 30 minutes resolution.

⇒ Functional time series

Goal: check whether there is underlying periodic signal
Weekday averages

Intraday mean curves from half-hourly observations.
Three models for periodic functional data

\[
Y_t(u) = \mu(u) + [\alpha \cos(t\theta) + \beta \sin(t\theta)] \times w_0(u) + Z_t(u), \quad (1)
\]

\[
Y_t(u) = \mu(u) + s_t \times w_0(u) + Z_t(u), \quad (2)
\]

\[
Y_t(u) = \mu(u) + w_t(u) + Z_t(u), \quad (3)
\]

with

\[
s_t = s_{t+d} \quad \text{and} \quad w_t(u) = w_{t+d}(u)
\]

and

\[
\sum_{t=1}^{d} s_t = 0 \quad \text{and} \quad \sum_{t=1}^{d} w_t(u) = 0.
\]
Part 1: $\theta$ is known.

Part 2: $\theta$ is unknown.

Noise: We start with

$$Z_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \Gamma).$$

Later we generalize to very general stationary processes.
Projections

Choose a set of functions \( v_1(u), \ldots, v_p(u) \) and compute

\[
Y_{tk} = \int Y_t(u) v_k(u) du,
\]

which gives rise to the vector models

\[
Y_t = \mu + [\alpha \cos(t\theta) + \beta \sin(t\theta)] \times w_0 + Z_t;
\]

\[
Y_t = \mu + s_t \times w_0 + Z_t;
\]

\[
Y_t = \mu + w_t + Z_t.
\]
LR tests for projections

Let us assume that \( d \) is odd and set \( q = (d - 1)/2 \), \( \vartheta_k = \frac{2\pi k}{d} \).

**Theorem**

Assume \( \Sigma = \text{Var}(Z_1) > 0 \) known. LR-test statistics are:

\[
T_{M1} := \| A(\vartheta_1) \Sigma^{-1} A'(\vartheta_1) \|
\]

\[
T_{M2} := \| A(\vartheta_1, \ldots, \vartheta_q) \Sigma^{-1} A'(\vartheta_1, \ldots, \vartheta_q) \|
\]

\[
T_{M3} := \| A(\vartheta_1, \ldots, \vartheta_q) \Sigma^{-1} A'(\vartheta_1, \ldots, \vartheta_q) \|_{\text{tr}}
\]

Here \( A(\theta_{i_1}, \ldots, \theta_{i_k}) = [R(\theta_{i_1}), \ldots, R(\theta_{i_k}), C(\theta_{i_1}), \ldots, C(\theta_{i_k})]' \)

where

\[
R(\theta) + iC(\theta) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} Y_k e^{-ik\theta}.
\]
Model 3: relation to MANOVA

Note that

$$T^{M3} = \left\| A(\varphi_1, \ldots, \varphi_q) \Sigma^{-1} A'(\varphi_1, \ldots, \varphi_q) \right\|_{tr} \sim T^{MAV}$$

where

$$T^{MAV} = \sum_{k=1}^{d} \frac{N}{d} (\bar{Y}_k - \bar{Y})' \Sigma^{-1} (\bar{Y}_k - \bar{Y}),$$

where $\bar{Y}_k = \frac{1}{n} \sum_{t=1}^{n} Y_{(t-1)d+k}, 1 \leq k \leq d$. (Balanced design).
Remarks

- **Approach is easy** and purely multivariate.

- In practice we **replace** \( \Sigma \) by estimator.

  NOTE: If we plug in \( \hat{\Sigma}_A \) this test is asymptotically equivalent to Wilks’ Lambda.

- For all three tests explicit null-distributions are available.

- **The downside of this method**: we need to have a good basis which picks up the signal.
The standard multivariate test of MacNeill for Model 1 is

\[ S^{M_1} := \| \tilde{D}_N(\vartheta_1) \|^2 = \| A(\vartheta_1) \Sigma^{-1} A'(\vartheta_1) \|_{tr}. \]

(Here \( \tilde{D}_N \) is the DFT of standardized data.)
The tests are computed from the standardized data

\[ \Sigma^{-1/2} Y_1, \ldots, \Sigma^{-1/2} Y_N. \]

\(\implies\) asymptotically pivotal tests;

\(\implies\) component processes with larger variation are not concealing potential periodic patterns in components with little variance.

While this is clearly a very desirable property for multivariate data samples, one may favor a different perspective for functional data.
Fully functional approach

Let \( A(\theta_{i_1}, \ldots, \theta_{i_k}) \) be a \( 2k \)-vector of functions computed from functional DFT

\[
\frac{1}{\sqrt{N}} \sum_{t=1}^{N} Y_t(u) e^{-it\theta}.
\]

Fully functional test for Model (3) \( \iff \) ANOVA model:

\[
T^{F_3} := \| A(\varphi_1, \ldots, \varphi_q) A'(\varphi_1, \ldots, \varphi_q) \|_{tr}.
\]

Again

\[
T^{F_3} \sim T^{FAV} = \sum_{k=1}^{d} \frac{N}{d} \| \overline{Y}_k - \overline{Y} \|^2.
\]

Asymptotic distribution of \( T^{F_3} \) (functional of DFT's) can be derived and then carried over to \( T^{FAV} \).
Theorem

Under $L^2$-m-approximability the functional ANOVA test statistic satisfies

$$
\sum_{k=1}^{d} \frac{N}{d} \| \bar{Y}_k - \bar{Y} \|^2 \xrightarrow{d} 2 \sum_{k=1}^{(d-1)/2} \Xi_k,
$$

where $\Xi_k \sim \text{HExp}(\lambda_1(\vartheta_k), \lambda_2(\vartheta_k), \ldots)$, and $\lambda_\ell(\vartheta_k)$ are the eigenvalues of $F_{\vartheta_k}$.

Remarks:

1. non-pivotal test;
2. consistent estimation of spectral density operator and its eigenvalues is possible;
3. good empirical approximation.
Empirical levels

- Synthetic but realistic data: PM-residuals resampled.
- Used to generate a functional MA(5) process.
- 1,000 repetitions.

<table>
<thead>
<tr>
<th></th>
<th>$T_{M1}$</th>
<th>$T_{M2}$</th>
<th>$T_{M3}$</th>
<th>$T_{F3}$</th>
<th>$T_{M1}$</th>
<th>$T_{M2}$</th>
<th>$T_{M3}$</th>
<th>$T_{F3}$</th>
</tr>
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<tr>
<td>FF</td>
<td>5.0</td>
<td>4.7</td>
<td></td>
<td></td>
<td>5.0</td>
<td>4.7</td>
<td></td>
<td>8.3</td>
</tr>
<tr>
<td>p = 1</td>
<td>5.0</td>
<td>3.9</td>
<td>3.9</td>
<td>5.0</td>
<td>9.6</td>
<td>8.7</td>
<td>8.7</td>
<td>9.9</td>
</tr>
<tr>
<td></td>
<td>5.9</td>
<td>4.4</td>
<td>4.4</td>
<td>5.9</td>
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<td>9.9</td>
<td>9.9</td>
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<td>3.9</td>
<td>6.4</td>
<td>11.2</td>
<td>8.7</td>
<td>8.0</td>
<td>9.1</td>
</tr>
<tr>
<td></td>
<td>6.0</td>
<td>5.4</td>
<td>4.1</td>
<td>6.0</td>
<td>10.6</td>
<td>9.8</td>
<td>9.1</td>
<td></td>
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<tr>
<td>p = 3</td>
<td>6.8</td>
<td>3.8</td>
<td>3.9</td>
<td>6.8</td>
<td>12.2</td>
<td>7.9</td>
<td>6.9</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5.8</td>
<td>5.5</td>
<td>4.2</td>
<td>5.8</td>
<td>10.6</td>
<td>9.6</td>
<td>7.9</td>
<td></td>
</tr>
<tr>
<td>p = 5</td>
<td>7.4</td>
<td>6.4</td>
<td>6.2</td>
<td>7.4</td>
<td>15.5</td>
<td>11.6</td>
<td>11.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6.7</td>
<td>6.4</td>
<td>5.7</td>
<td>6.7</td>
<td>12.5</td>
<td>12.2</td>
<td>10.7</td>
<td></td>
</tr>
</tbody>
</table>

**Table 1:** Empirical size (in %) at the nominal level $\alpha$ of 5% and 10% for dependent time series with sample sizes $N = 210$ (top rows) and $N = 420$ (bottom rows).
To study a realistic alternative we add back the weekday means to previous time series.

The signal-to-noise ratio in this example is approx $1/30$.

<table>
<thead>
<tr>
<th></th>
<th>$T^{M_1}$</th>
<th>$T^{M_2}$</th>
<th>$T^{M_3}$</th>
<th>$T^{F_3}$</th>
<th></th>
<th>$T^{M_1}$</th>
<th>$T^{M_2}$</th>
<th>$T^{M_3}$</th>
<th>$T^{F_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>FF</td>
<td></td>
<td></td>
<td></td>
<td>72.9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>99.9</td>
</tr>
<tr>
<td>$p = 1$</td>
<td>14.3</td>
<td>26.5</td>
<td>26.5</td>
<td></td>
<td>$p = 1$</td>
<td>21.7</td>
<td>56.6</td>
<td>56.6</td>
<td></td>
</tr>
<tr>
<td>$p = 2$</td>
<td>50.2</td>
<td>89.4</td>
<td>89.9</td>
<td></td>
<td>$p = 2$</td>
<td>76.9</td>
<td>99.7</td>
<td>99.8</td>
<td></td>
</tr>
<tr>
<td>$p = 3$</td>
<td>73.4</td>
<td>96.4</td>
<td>98.1</td>
<td></td>
<td>$p = 3$</td>
<td>92.2</td>
<td>100</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>$p = 5$</td>
<td>99.22</td>
<td>100</td>
<td>100</td>
<td></td>
<td>$p = 5$</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Empirical rates (in %) when testing at nominal level $\alpha$ of 5%. Results are based on 1,000 Monte Carlo simulation runs.
### Real data tests for NO data

<table>
<thead>
<tr>
<th></th>
<th>$T^M_1$</th>
<th>$T^M_2$</th>
<th>$T^M_3$</th>
<th>$T^F_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>FF (100%)</td>
<td></td>
<td></td>
<td></td>
<td>0.006</td>
</tr>
<tr>
<td>$v(u) \equiv 1$</td>
<td>0.305</td>
<td>0.099</td>
<td>0.099</td>
<td></td>
</tr>
<tr>
<td>$p = 1$ (68%)</td>
<td>0.247</td>
<td>0.076</td>
<td>0.076</td>
<td></td>
</tr>
<tr>
<td>$p = 2$ (81%)</td>
<td>0.495</td>
<td>0.204</td>
<td>0.172</td>
<td></td>
</tr>
<tr>
<td>$p = 3$ (87%)</td>
<td>$&lt; 10^{-5}$</td>
<td>$&lt; 10^{-5}$</td>
<td>$&lt; 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>$p = 5$ (97%)</td>
<td>$&lt; 10^{-5}$</td>
<td>$&lt; 10^{-5}$</td>
<td>$&lt; 10^{-5}$</td>
<td></td>
</tr>
</tbody>
</table>

**Table 3:** The $p$-values for NO data. In parentheses, the percentage of variance explained by the first $p$ principal components on which the curves are projected.
Uniform Test

Assume now that we would like to test for a periodic pattern when the frequency $\theta$ is unknown.

We consider a test based on

$$M_N = \max_{j=1,\ldots,q} \| D_N(\theta_j) \|^2 = \max_{j=1,\ldots,q} \| A(\theta_j)A'(\theta_j) \|_\text{tr}.$$
Scalar setting

In the scalar setting, we know that

\[ M_N - \log q \xrightarrow{d} \mathcal{G}, \quad q \sim N/2 \quad (4) \]

where \( \mathcal{G} \) is the standard Gumbel distribution.

**Simple explanation:** If data are iid Gaussian, then

\[ |D_N(\theta_j)|^2 \overset{iid}{\sim} \text{Exp}(1). \]

**Without Gaussianity:** we typically only have that

\[ (|D_N(\omega_1)|^2, \ldots, |D_N(\omega_q)|^2) \xrightarrow{d} (E_1, \ldots, E_q), \quad E_j \overset{iid}{\sim} \text{Exp}(1) \]

if \( q \) is fixed. Nevertheless, Davis and Mikosch (1999) obtained (4) for i.i.d. sequences and linear processes, and later to a more general notion of dependence in Lin and Liu (2009).
Main results

For \( p \geq 1 \), we define \( M^p_N = \max_{j=1, \ldots, q} \| D^p_N(\omega_j) \|^2 \), where

\[
D^p_N(\omega) \overset{\text{K.L.}}{=} N^{-1/2} \sum_{t=1}^{N} \sum_{k=1}^{p} \langle X_t, v_k \rangle v_k e^{-i t \omega}
\]

and the \( v_k \)'s are the eigenfunctions of the operator \( C_0 \), associated to eigenvalues \( \lambda_1 > \cdots > \lambda_p \).

**Theorem 1.** \( \frac{M^p_N - b^p_N}{\lambda_1} \xrightarrow{d} G \), for a fixed \( p \geq 1 \)

**Theorem 2.** \( \frac{M^{pN}_N - b^{pN}_N}{\lambda_1} \xrightarrow{d} G \), for some \( p = p_N \nearrow +\infty \)

**Theorem 3.** \( \frac{M_N - b_N}{\lambda_1} \xrightarrow{d} G \)

where

\[
b^p_N = \lambda_1 \log q - \lambda_1 \sum_{j=2}^{p} \log(1 - \lambda_j / \lambda_1)
\]

\[
b_N = \lim_{p \to +\infty} b^p_N \geq \lambda_1 \log q + E \| X_0 \|^2
\]
Idea of proof (with $\lambda_1 = 1$)

Write

$$M_N - b_N = \underbrace{M_N - M^p_N}_{A_1} + \underbrace{M^p_N - b^p_N}_{A_2} + \underbrace{b^p_N - b_N}_{A_3}.$$

- Obviously $A_3 \to 0$ when $p \to \infty$. ✓
- We can show that $A_1 \to 0$ if $p \to \infty$ fast enough. ✓
- The challenge is $A_2$! We will need that $p$ grows slowly enough and hope that the intersection of sequences $p = p_N$ in $A_2$ and $A_1$ is not empty.
Idea of proof

Let $\tilde{M}_N^p$ be computed as $M_N^p$, with a Gaussians sequence $Y_1, \ldots, Y_N$ such that $\text{Var}(Y_1) = \text{Var}(X_1)$, and we define

$$\rho_{N,p} = \sup_{x \in \mathbb{R}} |P(M_N^p \leq x) - P(\tilde{M}_N^p \leq x)|$$

Then

$$\left| P(M_N^p - b_N^p \leq x) - e^{-e^{-x}} \right| \leq \rho_{N,p} + \left| P(\tilde{M}_N^p - b_N^p \leq x) - e^{-e^{-x}} \right|. \quad \checkmark$$
Gaussian approximation

We have that

$$\{ M^p_N \leq x \} = \{ S^X_N \in A \},$$

where

$$X_t = \begin{pmatrix} \langle X_t, v_1 \rangle \cos(\omega_1) \\ \langle X_t, v_1 \rangle \sin(\omega_1) \\ \vdots \\ \langle X_t, v_p \rangle \sin(\omega_q) \end{pmatrix} \in \mathbb{R}^{2pq} \quad (2pq \approx Np)$$

and $A$ is $2p$–sparsely convex, i.e. finite intersections of sets whose indicator function depends on at most $2p$ components.
Gaussian approximation

We use a result of Chernozhukov, Chetverikov, and Kato (2017)

\[ \sup_{A \in A^{sp}} |P(S_N^X \in A) - P(S_N^Y \in A)|, \]

where \( A^{sp} \) is the class of s–sparsely convex subsets of \( \mathbb{R}^d \).

We had to work a bit around since in this bound there was one constant left, which depends (translated into our setup) in an unspecified way on \( \lambda_p \). Doesn’t matter if \( p \) is fixed!

In the end we deduce that

\[ \rho_{N,p} \lesssim \frac{p^3 \log(N)}{\lambda_p^{1/2} N^{1/6}}. \]
In particular Th. 3 holds if:

- $\lambda_k \sim \rho^k$, $\rho < 1$ and $E\|X_1\|^s < \infty$, with $s > 10$. ($p_N \sim \log(n)$)

- $\lambda_k \sim \frac{1}{k^\nu}$, $\nu > 1$ and $E\|X_1\|^s < \infty$, with $s$ very big. ($p_N \sim N^\gamma$)
All the results are still valid when $b_N$ is replaced by $\hat{b}_n$.

We work under $E\|X_1\|^4 < \infty$, but when $p$ is fixed, we can use a truncation argument to relax it to $E\|X_1\|^{2+\epsilon} < \infty$. Hence, we have extended the result of Davis and Mikosch (1999).

If $p$ is fixed, we have considered the dependent case, i.e. of multivariate linear processes $X_t = \sum_{\ell=1}^{\infty} \Psi_{\ell}(Z_{t-\ell})$, where the $\Psi_{\ell}$'s are matrices and $(Z_t)_{t \in \mathbb{Z}}$ is an i.i.d. sequence in $H$. In that case, we rather work with

$$L_N = \max_{j=1,...,q} \|\hat{F}_{\omega_j}^{\frac{1}{2}} D_N(\omega_j)\|^2$$
Uniform test

Consider a functional time series

$$X_t = s_t + \varepsilon_t$$

where $t \mapsto s_t$ is **periodic** of **unknown** frequency $\omega$ and $(\varepsilon_t)_{t \in \mathbb{Z}}$ is some iid noise in $H$. We wish to test whether

$$\begin{cases} \mathcal{H}_0 : \|s_t\| = 0 & \forall t, \\ \mathcal{H}_1 : \exists t \text{ such that } \|s_t\| > 0 \end{cases}$$

From the previous result, we deduce that under $\mathcal{H}_0$,

$$T = \frac{M_N - b_N}{\lambda_1} \xrightarrow{d} G.$$ 

On the other hand, we have that $T \xrightarrow{P} +\infty$ under $\mathcal{H}_1$. 
Simulation

We consider 3 different alternative hypothesis:

- **low frequency**, \( s_t^{(1)} = 0.5 \cos(t \cdot \frac{2\pi}{40}) \)
- **high frequency**, \( s_t^{(2)} = 0.5 \cos(t \cdot \frac{2\pi}{7}) \)
- **mixture**, \( s_t^{(3)} = 0.3 \cos(t \cdot \frac{2\pi}{40}) + 0.3 \cos(t \cdot \frac{2\pi}{20}) + 0.3 \cos(t \cdot \frac{2\pi}{7}) \)
Simulations

Rejection rates under the *null* and 3 different *alternatives* at level $\alpha = 0.05$
with increasing sample size, computed with 1000 replications.

<table>
<thead>
<tr>
<th></th>
<th>$N = 100$</th>
<th>$N = 200$</th>
<th>$N = 500$</th>
<th>$N = 1000$</th>
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</thead>
<tbody>
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<td>$X^{(0)}$</td>
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<td>0.05</td>
<td>0.06</td>
<td>0.06</td>
</tr>
<tr>
<td>$X^{(1)}$</td>
<td>0.18</td>
<td>0.87</td>
<td>0.97</td>
<td>1.00</td>
</tr>
<tr>
<td>$X^{(2)}$</td>
<td>0.31</td>
<td>0.49</td>
<td>0.97</td>
<td>1.00</td>
</tr>
<tr>
<td>$X^{(3)}$</td>
<td>0.14</td>
<td>0.39</td>
<td>0.87</td>
<td>1.00</td>
</tr>
</tbody>
</table>

\[
X_t^{(i)} = s_t^{(i)} + \varepsilon_t, \quad t = 1, \ldots, n
\]

with $s_t^{(0)} := 0$, and the curves $\varepsilon_t$ are bootstrapped from the PM$_{10}$ data set.
Simulations

We set $\omega^* = \arg\max_{j=1,\ldots,q} \|D_N(\omega_j)\|^2$
We have proposed several tests for periodicity of functional data.
We distinguish between projection based approach and fully functional approach.
Fully functional approach theoretically and practically appealing.
Projection based approach very powerful, but needs tuning.
We developed our tests from LR approach which improves also multivariate theory.
Extension to dependent data.
Extension to unknown length of period.