Large sample distribution for fully functional periodicity tests

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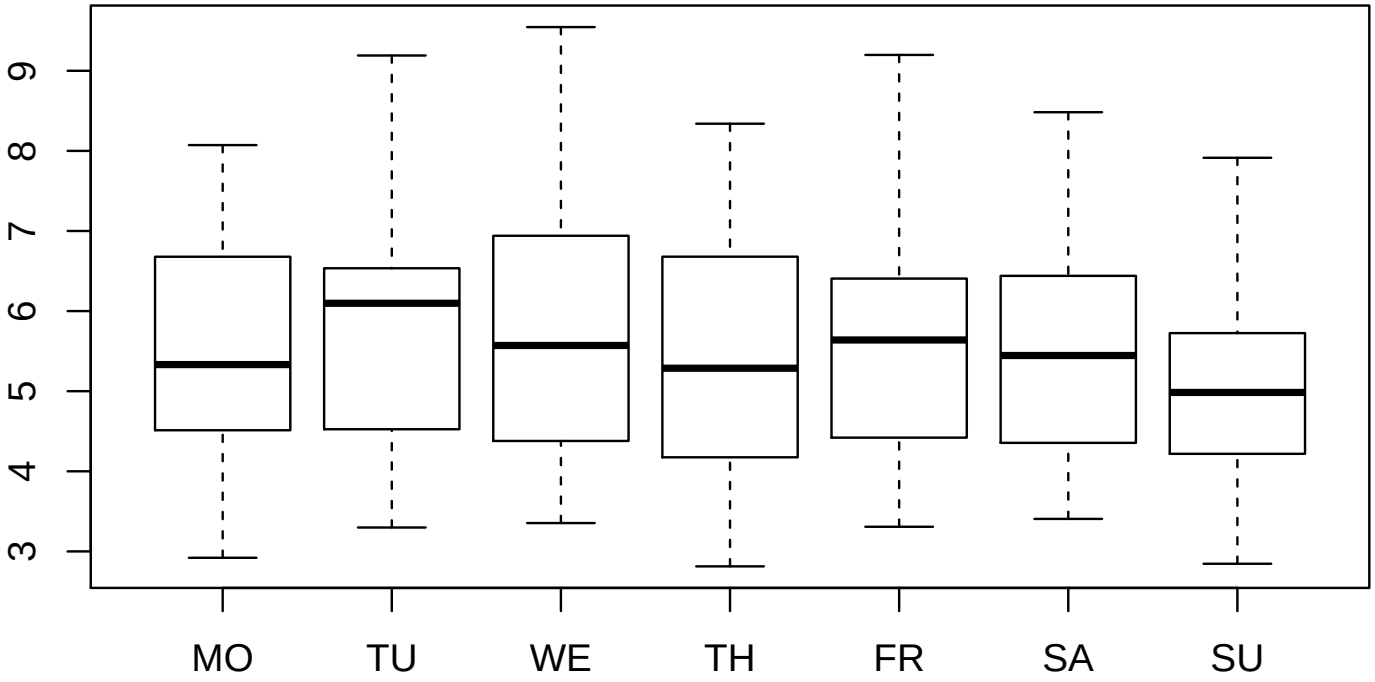
Pollution data

$N = 167$ observations of `PM10` measured in Graz (Austria) from 10/1/15 to 03/15/16.



Is there a significant difference in the daily average w.r.t. day of week?

Pollution data

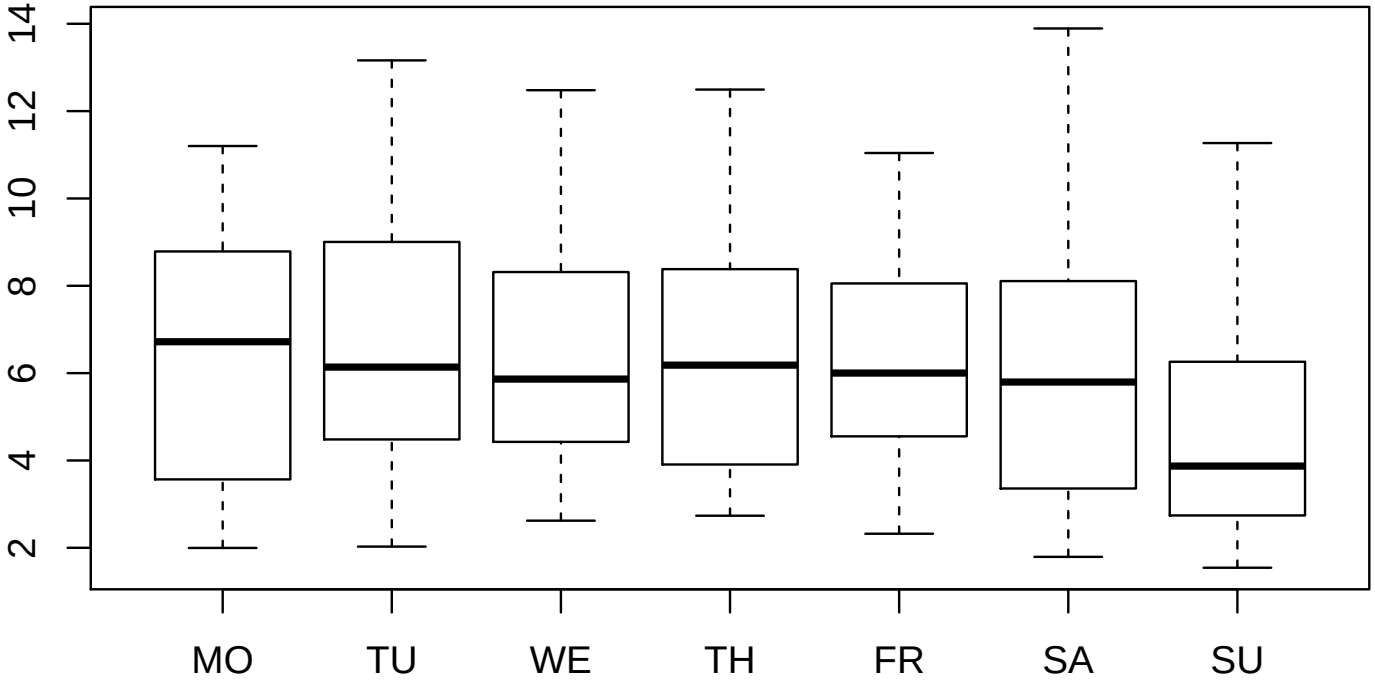
We use ANOVA:

$$\mathcal{H}_0 \colon \mu_{\mathrm{MO}} = \cdots = \mu_{\mathrm{SU}} \quad \text{v.s.} \quad \mathcal{H}_A \colon \mathcal{H}_0 \text{ doesn't hold.}$$

We obtain p-value of 0.72 $\Longrightarrow$ do not reject $\mathcal{H}_0$.

Pollution data

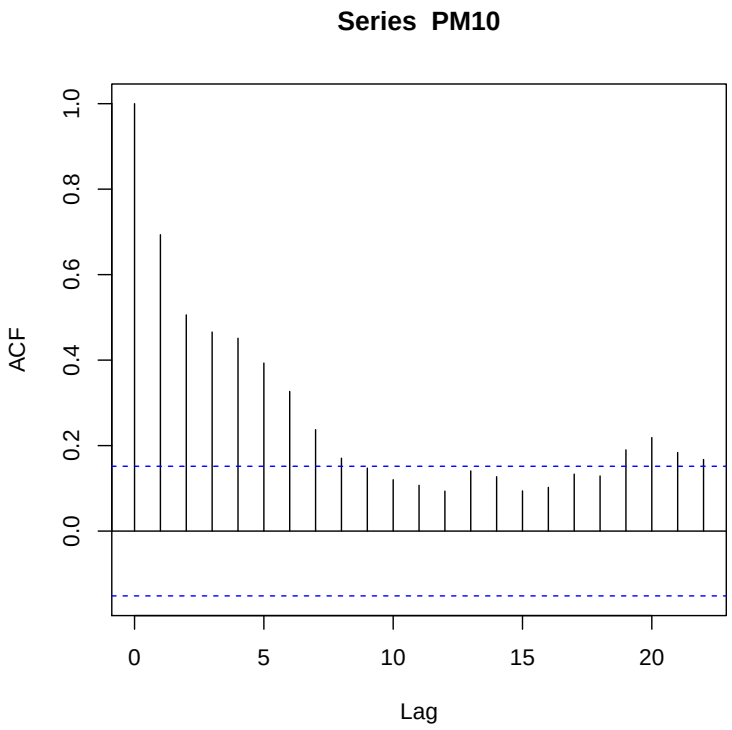
Same time period with `NO` (nitrogen monoxide).

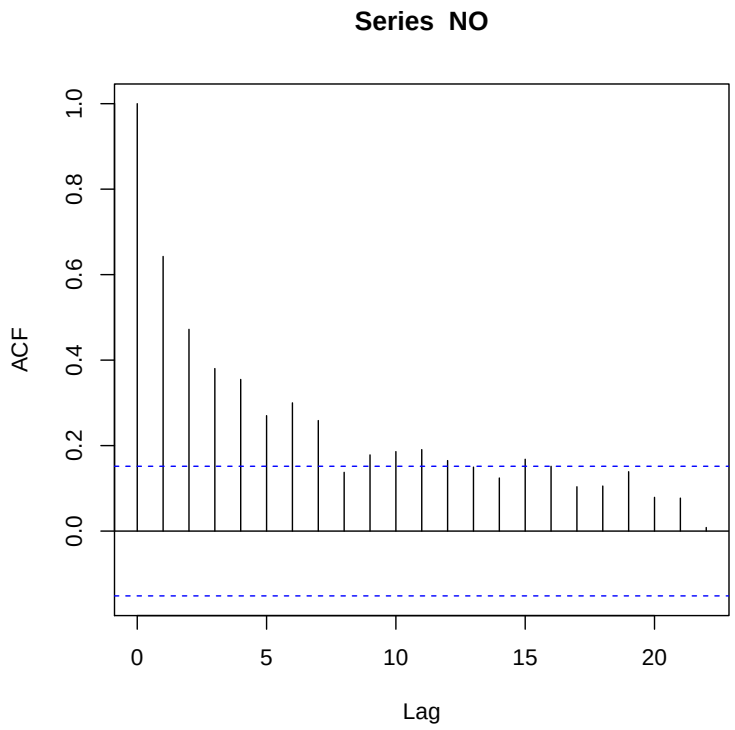


We obtain p-value of 0.25 $\Longrightarrow$ do not reject $\mathcal{H}_0$.

Dependence

Autocorrelation functions of daily averages of `PM10` and `NO`.



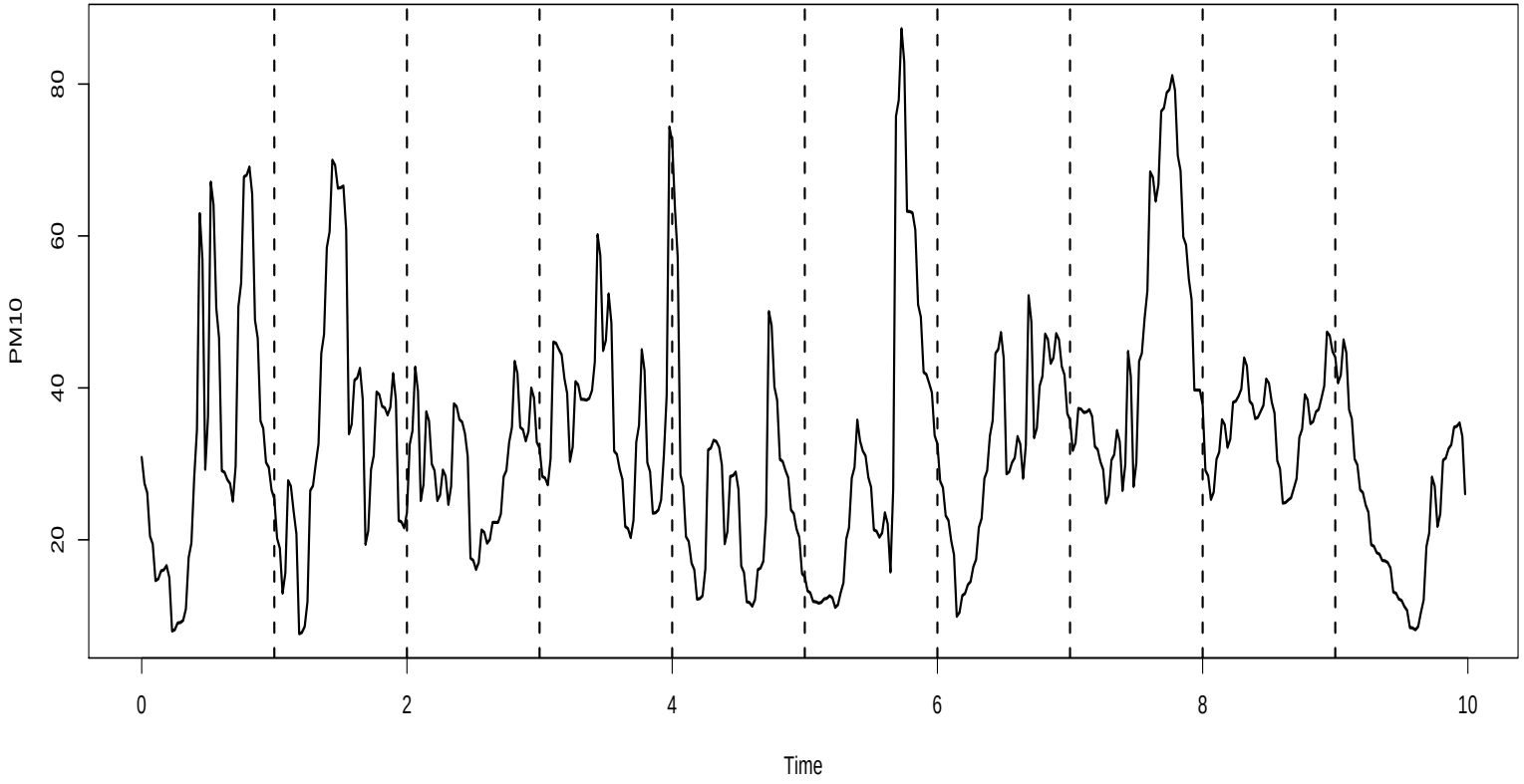


Functional data

PM10–data in Graz are available in 30 minutes resolution.

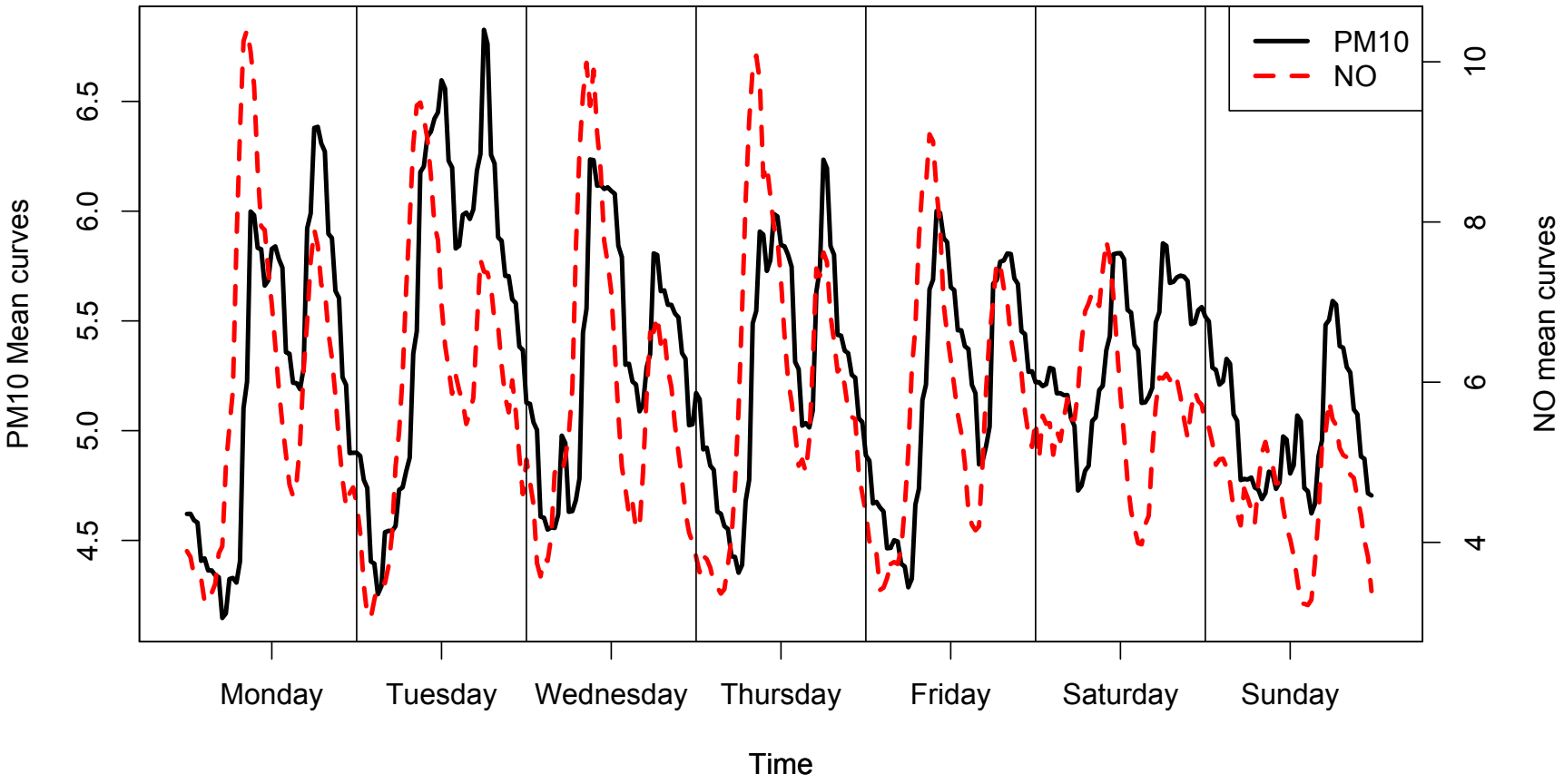
$\Longrightarrow$ Functional time series

Goal: check whether there is underlying periodic signal



Weekday averages

Intraday mean curves from half-hourly observations.



Three models for periodic functional data

$$Y_t(u) = \mu(u) + [\alpha \cos(t\theta) + \beta \sin(t\theta)] \times w_0(u) \quad + Z_t(u), \quad (1)$$

$$Y_t(u) = \mu(u) + s_t \times w_0(u) \quad + Z_t(u), \quad (2)$$

$$Y_t(u) = \mu(u) + w_t(u) \quad + Z_t(u), \quad (3)$$

with

$$s_t = s_{t+d} \quad \text{and} \quad w_t(u) = w_{t+d}(u)$$

and

$$\sum_{t=1}^{d} s_t = 0 \quad \text{and} \quad \sum_{t=1}^{d} w_t(u) = 0.$$

Settings

Part 1: θ is known.

Part 2: θ is unknown.

Noise: We start with

$$Z_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \Gamma).$$

Later we generalize to very general stationary processes.

Projections

Choose a set of functions $v_1(u), \ldots, v_p(u)$ and compute

$$Y_{tk} = \int Y_t(u) v_k(u) du,$$

which gives rise to the vector models

$$\boldsymbol{Y}_t = \boldsymbol{\mu} + [\alpha \cos(t\theta) + \beta \sin(t\theta)] \times \boldsymbol{w}_0 + \boldsymbol{Z}_t;$$

$$\boldsymbol{Y}_t = \boldsymbol{\mu} + s_t \times \boldsymbol{w}_0 + \boldsymbol{Z}_t;$$

$$\boldsymbol{Y}_t = \boldsymbol{\mu} + \boldsymbol{w}_t + \boldsymbol{Z}_t.$$

LR tests for projections

Let us assume that d is odd and set $q = (d-1)/2$, $\vartheta_k = \frac{2\pi k}{d}$.

Theorem

Assume $\boldsymbol{\Sigma} = \mathrm{Var}(\boldsymbol{Z}_1) > 0$ *known. LR-test statistics are:*

$$T^{\mathtt{M}_1} := \left\| \boldsymbol{A}(\vartheta_1)\boldsymbol{\Sigma}^{-1}\boldsymbol{A}'(\vartheta_1) \right\|$$

$$T^{\mathtt{M}_2} := \left\| \boldsymbol{A}(\vartheta_1,\ldots,\vartheta_q)\boldsymbol{\Sigma}^{-1}\boldsymbol{A}'(\vartheta_1,\ldots,\vartheta_q) \right\|$$

$$T^{\mathtt{M}_3} := \left\| \boldsymbol{A}(\vartheta_1,\ldots,\vartheta_q)\boldsymbol{\Sigma}^{-1}\boldsymbol{A}'(\vartheta_1,\ldots,\vartheta_q) \right\|_{\mathbf{tr}}$$

Here $\boldsymbol{A}(\theta_{i_1},\ldots,\theta_{i_k}) = \left[\boldsymbol{R}(\theta_{i_1}),\ldots,\boldsymbol{R}(\theta_{i_k}),\boldsymbol{C}(\theta_{i_1}),\ldots,\boldsymbol{C}(\theta_{i_k})\right]'$ where

$$\boldsymbol{R}(\theta) + \mathrm{i}\boldsymbol{C}(\theta) = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}\boldsymbol{Y}_k e^{-\mathrm{i}k\theta}.$$

Model 3: relation to MANOVA

Note that

$$T^{\mathtt{M}_3} = \left\| \boldsymbol{A}(\vartheta_1, \ldots, \vartheta_q) \boldsymbol{\Sigma}^{-1} \boldsymbol{A}'(\vartheta_1, \ldots, \vartheta_q) \right\|_{\mathbf{tr}} \sim T^{\mathtt{MAV}}$$

where

$$T^{\mathtt{MAV}} = \sum_{k=1}^{d} \frac{N}{d} (\overline{\boldsymbol{Y}}_k - \overline{\boldsymbol{Y}})' \boldsymbol{\Sigma}^{-1} (\overline{\boldsymbol{Y}}_k - \overline{\boldsymbol{Y}}),$$

where $\overline{\boldsymbol{Y}}_k = \frac{1}{n} \sum_{t=1}^{n} \boldsymbol{Y}_{(t-1)d+k}, 1 \leq k \leq d$. (Balanced design).

Remarks

- Approach is easy and purely multivariate.
- In practice we replace $\mathbf{\Sigma}$ by estimator.

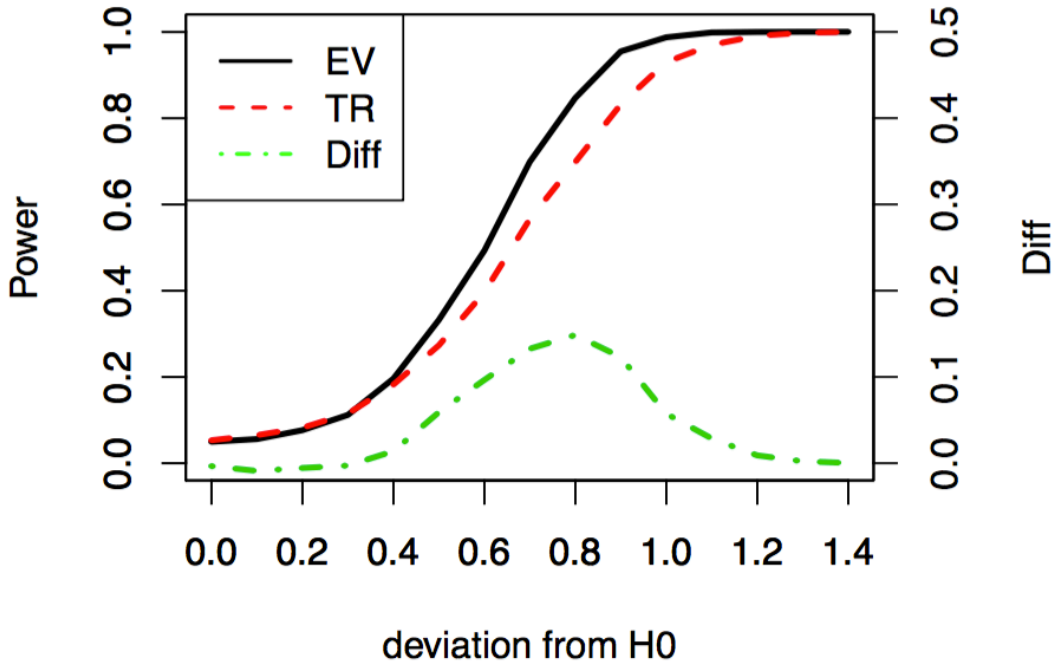
  NOTE: If we plug in $\widehat{\mathbf{\Sigma}}_A$ this test is asymptotically equivalent to Wilks' Lambda.
- For all three tests explicit null-distributions are available.
- The downside of this method: we need to have a good basis which picks up the signal.

Remarks

The standard multivariate test of MacNeill for Model 1 is

$$S^{\mathtt{M}_1} := \|\widetilde{D}_N(\vartheta_1)\|^2 = \left\|\boldsymbol{A}(\vartheta_1)\boldsymbol{\Sigma}^{-1}\boldsymbol{A}'(\vartheta_1)\right\|_{\mathbf{tr}}.$$

(Here $\widetilde{D}_N$ is the DFT of standardized data.)



Rescaling

The tests are computed from the standardized data

$$\boldsymbol{\Sigma}^{-1/2}\boldsymbol{Y}_1, \ldots, \boldsymbol{\Sigma}^{-1/2}\boldsymbol{Y}_N.$$

$\Longrightarrow$ asymptotically pivotal tests;

$\Longrightarrow$ component processes with larger variation are not concealing potential periodic patterns in components with little variance.

While this is clearly a very desirable property for multivariate data samples, one may favor a different perspective for functional data.

Fully functional approach

Let $A(\theta_{i_1}, \ldots, \theta_{i_k})$ be a $2k$-vector of functions computed from functional DFT

$$\frac{1}{\sqrt{N}} \sum_{t=1}^{N} Y_t(u) e^{-\mathrm{i}t\theta}.$$

Fully functional test for Model (3) $\Longleftrightarrow$ ANOVA model:

$$T^{\mathrm{F}_3} := \|A(\vartheta_1, \ldots, \vartheta_q) A'(\vartheta_1, \ldots, \vartheta_q)\|_{\mathrm{tr}}.$$

Again

$$T^{\mathrm{F}_3} \sim T^{\mathtt{FAV}} = \sum_{k=1}^{d} \frac{N}{d} \|\overline{Y}_k - \overline{Y}\|^2.$$

Asymptotic distribution of T^{F_3} (functional of DFT's) can be derived and then carried over to $T^{\mathtt{FAV}}$.

Theorem

Under L^2-m-approximability the functional ANOVA test statistic satisfies

$$\sum_{k=1}^{d} \frac{N}{d} \|\overline{Y}_k - \overline{Y}\|^2 \xrightarrow{d} 2 \sum_{k=1}^{(d-1)/2} \Xi_k,$$

where $\Xi_k \overset{ind.}{\sim} \mathrm{HExp}(\lambda_1(\vartheta_k), \lambda_2(\vartheta_k), \ldots)$, *and* $\lambda_\ell(\vartheta_k)$ *are the eigenvalues of* $\mathcal{F}_{\vartheta_k}$.

Remarks:

1. non-pivotal test;
2. consistent estimation of spectral density operator and its eigenvalues is possible;
3. good empirical approximation.

Empirical levels

- Synthetic but realistic data: PM-residuals resampled.
- Used to generate a functional MA(5) process.
- 1.000 repetitions.

	$\alpha = 5\%$				$\alpha = 10\%$			
	$T^{\mathtt{M}1}$	$T^{\mathtt{M}2}$	$T^{\mathtt{M}3}$	$T^{\mathtt{F}3}$	$T^{\mathtt{M}1}$	$T^{\mathtt{M}2}$	$T^{\mathtt{M}3}$	$T^{\mathtt{F}3}$
FF				5.0				8.3
				4.7				9.9
$p = 1$	5.0	3.9	3.9		9.6	8.7	8.7	
	5.9	4.4	4.4		9.8	9.9	9.9	
$p = 2$	6.4	4.3	3.9		11.2	8.7	8.0	
	6.0	5.4	4.1		10.6	9.8	9.1	
$p = 3$	6.8	3.8	3.9		12.2	7.9	6.9	
	5.8	5.5	4.2		10.6	9.6	7.9	
$p = 5$	7.4	6.4	6.2		15.5	11.6	11.4	
	6.7	6.4	5.7		12.5	12.2	10.7	

Table 1: Empirical size (in %) at the nominal level α of 5% and 10% for dependent time series with sample sizes $N = 210$ (top rows) and $N = 420$ (bottom rows).

Power

To study a realistic alternative we add back the weekday means to previous time series.

The signal-to-noise ratio in this example is approx 1/30.

	$N = 210$				$N = 420$			
	$T^{\mathtt{M}_1}$	$T^{\mathtt{M}_2}$	$T^{\mathtt{M}_3}$	$T^{\mathtt{F}_3}$	$T^{\mathtt{M}_1}$	$T^{\mathtt{M}_2}$	$T^{\mathtt{M}_3}$	$T^{\mathtt{F}_3}$
FF				72.9				99.9
$p = 1$	14.3	26.5	26.5		21.7	56.6	56.6	
$p = 2$	50.2	89.4	89.9		76.9	99.7	99.8	
$p = 3$	73.4	96.4	98.1		92.2	100	100	
$p = 5$	99.22	100	100		100	100	100	

Table 2: Empirical rates (in %) when testing at nominal level α of 5%. Results are based on 1,000 Monte Carlo simulation runs.

Real data tests for NO data

	$T^{\mathtt{M}_1}$	$T^{\mathtt{M}_2}$	$T^{\mathtt{M}_3}$	$T^{\mathtt{F}_3}$
FF (100%)				0.006
$v(u) \equiv 1$	0.305	0.099	0.099	
$p = 1$ (68%)	0.247	0.076	0.076	
$p = 2$ (81%)	0.495	0.204	0.172	
$p = 3$ (87%)	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$	
$p = 5$ (97%)	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$	

Table 3: The p-values for NO data. In parentheses, the percentage of variance explained by the first p principal components on which the curves are projected.

Uniform Test

Assume now that we would like to test for a periodic pattern when the frequency θ is unknown.

We consider a test based on

$$M_N = \max_{j=1,\ldots,q} \|D_N(\theta_j)\|^2 = \max_{j=1,\ldots,q} \|A(\theta_j)A'(\theta_j)\|_{\mathrm{tr}}.$$

Scalar setting

In the scalar setting, we know that

$$M_N - \log q \xrightarrow{d} \mathcal{G}, \quad q \sim N/2 \tag{4}$$

where $\mathcal{G}$ is the standard *Gumbel* distribution.

Simple explanation: If data are iid Gaussian, then

$$|D_N(\theta_j)|^2 \overset{iid}{\sim} \text{Exp}(1).$$

Without Gaussianity: we typically only have that

$$\left(|D_N(\omega_1)|^2, \ldots, |D_N(\omega_q)|^2\right) \xrightarrow{d} (E_1, \ldots, E_q), \quad E_j \overset{iid}{\sim} \mathrm{Exp}(1)$$

if q is fixed. Nevertheless, Davis and Mikosch (1999) obtained (4) for i.i.d. sequences and linear processes, and later to a more general notion of dependence in Lin and Liu (2009).

Main results

For $p \geq 1$, we define $M_N^p = \max_{j=1,\dots,q} \|D_N^p(\omega_j)\|^2$, where

$$D_N^p(\omega) \overset{\text{K.L.}}{=} N^{-1/2} \sum_{t=1}^{N} \sum_{k=1}^{p} \langle X_t, v_k \rangle v_k e^{-\mathbb{i} t\omega}$$

and the v_k's are the **eigenfunctions** of the operator C_0, associated to eigenvalues $\lambda_1 > \cdots > \lambda_p$.

Theorem 1. $\frac{M_N^p - b_N^p}{\lambda_1} \xrightarrow{d} \mathcal{G}$, for a fixed $p \geq 1$

Theorem 2. $\frac{M_N^{p_N} - b_N^{p_N}}{\lambda_1} \xrightarrow{d} \mathcal{G}$, for some $p = p_N \nearrow +\infty$

Theorem 3. $\frac{M_N - b_N}{\lambda_1} \xrightarrow{d} \mathcal{G}$

$$\text{where} \quad \begin{array}{rcl} b_N^p & = & \lambda_1 \log q - \lambda_1 \sum_{j=2}^{p} \log(1 - \lambda_j/\lambda_1) \\ b_N & = & \lim\limits_{p \to +\infty} b_N^p \quad \geq \lambda_1 \log q + E\|X_0\|^2 \end{array}$$

Idea of proof (with $\lambda_1 = 1$)

Write

$$M_N - b_N = \underbrace{M_N - M_N^p}_{A_1} + \underbrace{M_N^p - b_N^p}_{A_2} + \underbrace{b_N^p - b_N}_{A_3}.$$

- Obviously $A_3 \to 0$ when $p \to \infty$. ✓
- We can show that $A_1 \to 0$ if $p \to \infty$ fast enough. ✓
- The challenge is A_2! We will need that p grows slowly enough and hope that the intersection of sequences $p = p_N$ in A_2 and A_1 is not empty.

Idea of proof

Let $\widetilde{M}_N^p$ be computed as M_N^p, with a Gaussians sequence $Y_1, \ldots, Y_N$ such that $\mathrm{Var}(Y_1) = \mathrm{Var}(X_1)$, and we define

$$\rho_{N,p} = \sup_{x \in \mathbb{R}} |P(M_N^p \leq x) - P(\widetilde{M}_N^p \leq x)|$$

Then

$$\left| P\left(M_N^p - b_N^p \leq x\right) - e^{-e^{-x}} \right| \leq \rho_{N,p} + \underbrace{\left| P(\widetilde{M}_N^p - b_N^p \leq x) - e^{-e^{-x}} \right|}_{\checkmark}.$$

Gaussian approximation

We have that

$$\{M_N^p \leq x\} = \{S_N^{\mathbf{X}} \in A\},$$

where

$$\mathbf{X}_t = \begin{pmatrix} \langle X_t, v_1 \rangle \cos(\omega_1) \\ \langle X_t, v_1 \rangle \sin(\omega_1) \\ \vdots \\ \langle X_t, v_p \rangle \sin(\omega_q) \end{pmatrix} \in \mathbb{R}^{2pq} \quad (2pq \approx Np)$$

and A is $2p$–sparsely convex, i.e. finite intersections of sets whose indicator function depends on at most $2p$ components.

Gaussian approximation

We use a result of Chernozhukov, Chetverikov, and Kato (2017)

$$\sup_{A \in \mathcal{A}^{\mathrm{sp}}} |P(S_N^{\mathbf{X}} \in A) - P(S_N^{\mathbf{Y}} \in A)|,$$

where $\mathcal{A}^{\mathrm{sp}}$ is the class of s–sparsely convex subsets of $\mathbb{R}^d$.

We had to work a bit around since in this bound there was one constant left, which depends (translated into our setup) in an unspecified way on λ_p. Doesn't matter if p is fixed!

In the end we deduce that

$$\rho_{N,p} \lesssim \frac{p^3 \log(N)}{\lambda_p^{1/2} N^{1/6}}.$$

Conclusion of the proof

In particular Th. 3 holds if:

- $\lambda_k \sim \rho^k$, $\rho < 1$ and $E\|X_1\|^s < \infty$, with $s > 10$.
  $(p_N \sim \log(n))$
- $\lambda_k \sim \frac{1}{k^\nu}$, $\nu > 1$ and $E\|X_1\|^s < \infty$, with s *very big*.
  $(p_N \sim N^\gamma)$

Comments

- All the results are still valid when b_N is replaced by $\widehat{b}_n$.
- We work under $E\|X_1\|^4 < \infty$, but when p is *fixed*, we can use a **truncation** argument to relax it to $E\|X_1\|^{2+\epsilon} < \infty$. Hence, we have extended the result of Davis and Mikosch (1999).
- If p is *fixed*, we have considered the **dependent** case, i.e. of multivariate linear processes $X_t = \sum_{\ell=1}^{\infty} \Psi_\ell(Z_{t-\ell})$, where the Ψ_ℓ's are matrices and $(Z_t)_{t\in\mathbb{Z}}$ is an i.i.d. sequence in H. In that case, we rather work with

$$L_N = \max_{j=1,\dots,q} \|\widehat{\mathcal{F}}_{\omega_j}^{-\frac{1}{2}} D_N(\omega_j)\|^2$$

Uniform test

Consider a functional time series

$$X_t \;=\; s_t \;+ \varepsilon_t$$

where $t \mapsto s_t$ is **periodic** of **unknown** frequency ω and $(\varepsilon_t)_{t\in\mathbb{Z}}$ is some iid noise in H. We wish to test whether

$$\begin{cases} \mathcal{H}_0 & : \; \|s_t\| = 0 \quad \forall t, \\ \mathcal{H}_1 & : \; \exists t \text{ such that } \|s_t\| > 0 \end{cases}$$

From the previous result, we deduce that *under* $\mathcal{H}_0$,

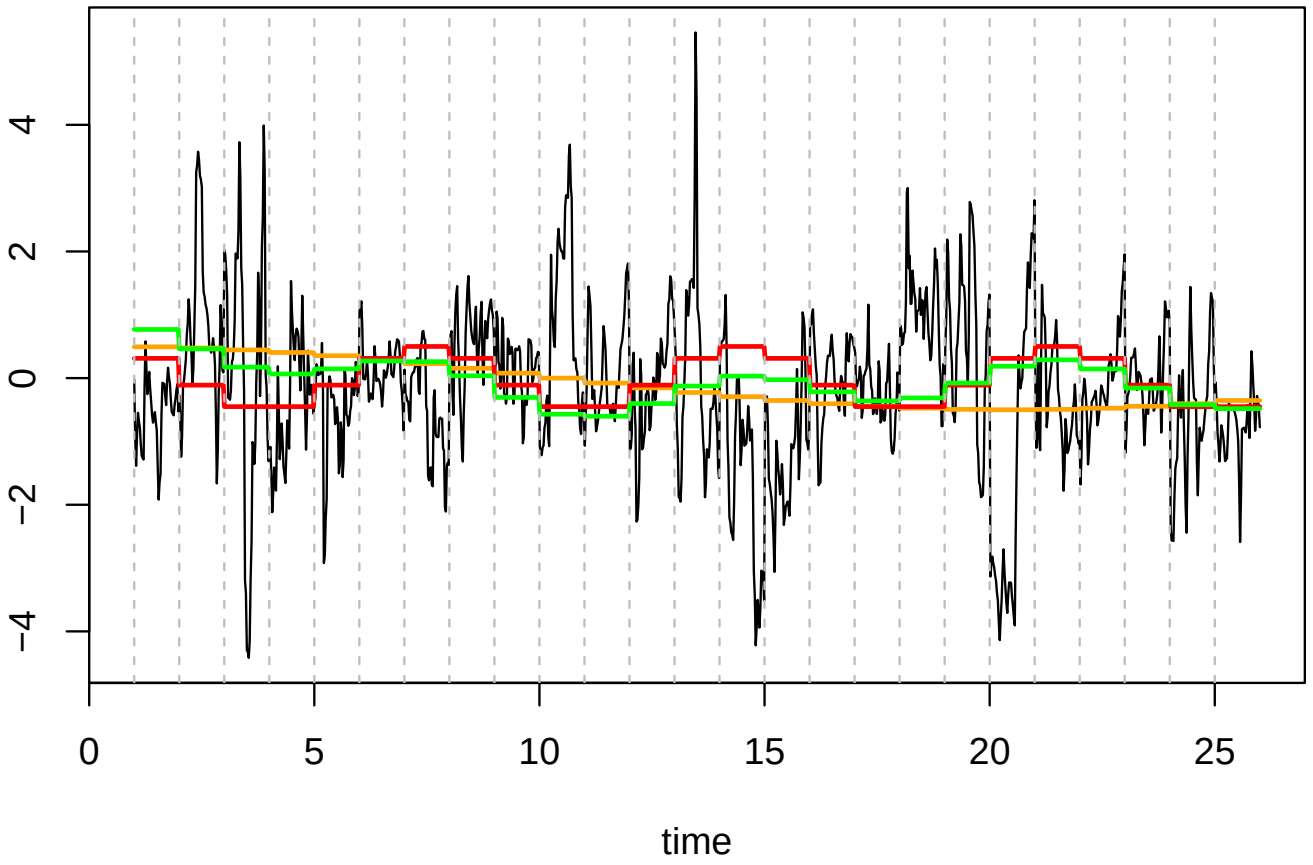
$$\mathbf{T} \;=\; \frac{M_N - b_N}{\lambda_1} \;\xrightarrow{d}\; \mathcal{G}.$$

On the other hand, we have that $\mathbf{T} \xrightarrow{P} +\infty$ *under* $\mathcal{H}_1$.

Simulation

We consider 3 different *alternative hypothesis*:

- **low frequency**, $s_t^{(1)} = .5\cos(t \cdot \frac{2\pi}{40})$
- **high frequency**, $s_t^{(2)} = .5\cos(t \cdot \frac{2\pi}{7})$
- **mixture**, $s_t^{(3)} = .3\cos(t \cdot \frac{2\pi}{40}) + .3\cos(t \cdot \frac{2\pi}{20}) + .3\cos(t \cdot \frac{2\pi}{7})$



Simulations

Rejection rates under the *null* and 3 different *alternatives* at level $\alpha = 0.05$
with increasing sample size, computed with 1000 replications.

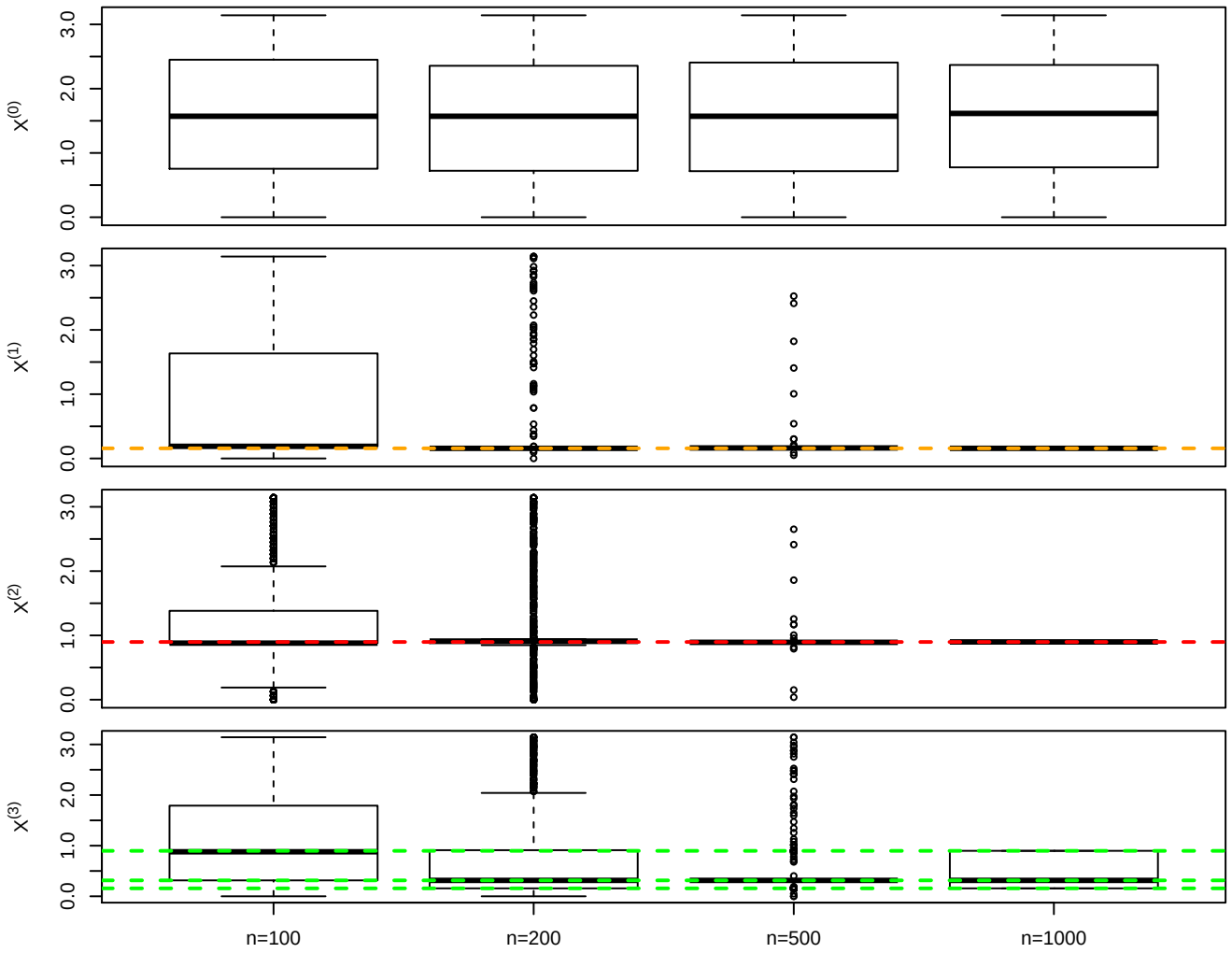
	$N = 100$	$N = 200$	$N = 500$	$N = 1000$
$X^{(0)}$	0.06	0.05	0.06	0.06
$X^{(1)}$	0.18	0.87	0.97	1.00
$X^{(2)}$	0.31	0.49	0.97	1.00
$X^{(3)}$	0.14	0.39	0.87	1.00

$$X_t^{(i)} = s_t^{(i)} + \varepsilon_t, \qquad t = 1, \ldots, n$$

with $s_t^{(0)} := 0$, and the curves ε_t are bootstrapped from the PM_{10} data set.

Simulations

We set $\omega^* = \underset{j=1,\ldots,q}{\operatorname{argmax}} \|D_N(\omega_j)\|^2$



Summary

- We have proposed several tests for periodicity of functional data.
- We distinguish between projection based approach and fully functional approach.
- Fully functional approach theoretically and practically appealing.
- Projection based approach very powerful, but needs tuning.
- We developed our tests from LR approach which improves also multivariate theory.
- Extension to dependent data.
- Extension to unknown length of period.