CHARACTERISTIC CLASSES IN TMF OF LEVEL $\Gamma_1(3)$

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ABSTRACT. Let $TMF_1(n)$ be the spectrum of topological modular forms equipped with a $\Gamma_1(n)$-structure. We compute the $K(2)$-local $TMF_1(3)$-cohomology of $BString$ and $BSpin$: both are power series rings freely generated by classes that we explicitly construct and which generalize the classical Pontryagin classes. As a first application of this computation, we show how to construct $TMF(3n)$-cohomology classes from stable positive energy representations of the loop groups $LSpin$.

1. INTRODUCTION AND STATEMENT OF RESULTS

Characteristic classes are cohomology classes which are naturally associated to principal G-bundles on topological spaces. Examples include Stiefel-Whitney classes for $O(n)$-bundles or KO-Pontryagin classes for $SO(n)$-bundles. If the cohomology theory is oriented with respect to G-bordism, the characteristic classes can be evaluated on fundamental classes of manifolds with G-structure and yield characteristic numbers. It is well known that oriented bordism is determined by Stiefel-Whitney numbers and rational Pontryagin numbers, whereas spin bordism is determined by Stiefel-Whitney numbers and KO-Pontryagin numbers.

In this paper we consider characteristic classes for spin and string bundles. Here, $String$ is the 6-connected cover of the $Spin$-group. The cohomology theory is $TMF_1(3)$, the cohomology of topological modular forms of level $\Gamma_1(3)$. It is possible that string bordism is determined by Stiefel-Whitney and TMF-characteristic numbers for some level structures. At the prime 2, the connective version of TMF should split off of the Thom spectrum $MString$ and there is some evidence that another summand is given by the 16th suspension of the connective cover of $TMF_0(3)$ (compare [MR09]). However, in order to get a map, a better understanding of the characteristic classes is necessary. (The relation between $TMF_1(3)$ and $TMF_0(3)$ is described in section 2.)

The homology groups $E_*BU\langle 6 \rangle$ for complex oriented theories E have been described in [AHS01] in terms of cubical structures on the formal group of E. Here, $BU\langle 6 \rangle$ is the complex analogue of $BString$. One may hope that the real case $E_*BString$ admits a similar description in terms of "real" cubical structures. In [LO] the relation between cubical structures and the results of the current article has been further investigated.

The cohomology groups $E^*BString$ for various complex oriented versions of TMF have been considered in [Lau99]. When 2 is inverted these groups have been computed by the topological q-expansion principle. The difficulty appears at the prime 2 since here a splitting result for the string bordism spectrum along the lines of Hovey-Ravenel [HR95] is missing. This paper deals with this remaining prime.

We first look at characteristic classes for spin bundles and show the following q-expansion principle.

Theorem 1.1. *The diagram*

$$\begin{array}{ccc} TMF_1(3)^*BSpin & \xrightarrow{\ \lambda\ } & K^*_{Tate}BSpin \\ \downarrow & & \downarrow{ch} \\ H^*(BSpin, TMF_1(3)^*_{\mathbf{Q}}) & \longrightarrow & H^*(BSpin, K_{Tate}{}^*_{\mathbf{Q}}) \end{array}$$

is a pullback.

The horizontal map is Miller's elliptic character which corresponds to the evaluation at the Tate curve on the moduli stack of elliptic curves ([Mil89].) On coefficients this map is just the traditional q-expansion map for modular forms. The theory K_{Tate} is the power series ring $K[1/3]((q))$ of K-theories as in [Lau99]. The right vertical map is the Chern character and the left vertical map is the Dold character, that is, the map to rational cohomology induced by the exponential of the formal group law (see ibid.)

The theorem determines the ring of $TMF_1(3)$-characteristic classes for spin bundles as follows. An element of $K^*_{Tate}BSpin$ is a K_{Tate}-characteristic class for spin bundles, that is, a formal series of virtual vector bundles which is naturally defined for spin bundles. If its Chern character is invariant under the appropriate Möbius transformations then it gives rise to a unique $TMF_1(3)$-characteristic class. This property allows the construction of many natural classes. An application to representations of loop groups will be given later.

The proof of the theorem is given in Section 3. Locally at the prime 2 the cohomology theory $TMF_1(3)$ is a generalized Johnson-Wilson spectrum $E(2)$ by Proposition 2.4. Hence it relates to the Morava K-theory $K(2)$ by a Bockstein spectral sequence. A splitting principle which is based on the Kitchloo-Laures computation [KL02] of the Morava $K(2)$-homology of $BSpin$ forces the universal coefficient spectral sequence to collapse. Its E_2-term is studied via a chromatic resolution. Note that this part of the work does not involve the theory of elliptic curves but only the study of formal group laws of height at most 2. However, the statement of the theorem is a transchromatic property which finds a natural formulation in the q-expansion of modular forms once the generalized $E(2)$-theory has been chosen to be $TMF_1(3)$. The pullback property is reduced via the universal coefficient isomorphisms to the classical q-expansion principle.

The theorem allows us to the construct Pontryagin classes for $TMF_1(3)$. The pullback diagram relates these to the classical Pontryagin classes of K-theory and rational Pontryagin classes in a nice way.

Theorem 1.2.   (i) *There are unique classes $p_i \in TMF_1(3)^{4i}BSpin$ with the following property: the formal series $p(t) = 1 + p_1 t + p_2 t^2 \ldots$ is given by*

$$\prod_{i=1}^{m}(1 - t\rho^*(x_i\overline{x}_i))$$

when restricted to the classifying space of each maximal torus of $Spin(2m)$. Here, ρ is the map to the maximal torus of $SO(2m)$ and the x_i (and $\overline{x}_i$) are the first $TMF_1(3)$-Chern classes of the canonical line bundles L_i (resp. $\overline{L}_i$) over the classifying spaces of the tori.

(ii) *The classes p_i freely generate the $TMF_1(3)$-cohomology of $BSpin$, that is,*

$$TMF_1(3)^* BSpin \cong TMF_1(3)^*[\![p_1, p_2, \ldots]\!].$$

Next we turn to the calculation of the characteristic classes for $BString$. Here it turns out that the q-expansion principle only holds for odd primes. A string characteristic class may not even be determined by its characters. However, we have the following result.

Theorem 1.3. *Let $\widehat{TMF}_1(3)$ be the $K(2)$-localization of $TMF_1(3)$ at the prime 2. Then there is an isomorphism of algebras*

$$\widehat{TMF}_1(3)^* BString \cong \widehat{TMF}_1(3)^*[\![r, p_1, p_2, \ldots]\!]$$

where $p_1, p_2, \ldots$ are the Pontryagin classes coming from $BSpin$ and r restricts to a topological generator of degree 6 in the $K(2)$-cohomology of $K(\mathbb{Z}, 3)$.

Theorems 1.2 and 1.3 solve the problem stated in [KLW04a]: they give explicit generators for the cohomology rings of $BSpin$ and $BString$. Their proofs are given in Section 4. Theorem 1.3 relies on a description of the Morava $K(2)$-homology of $BString$ given by Kitchloo-Laures-Wilson in [KLW04a, KLW04b]. We remark that the completed $TMF_1(3)$ can be described as a fixed point spectrum of Lubin-Tate theory (see [MR09].)

As a first consequence we give a proof of the stable version of a conjecture by Brylinski which was stated in [Bry90]. The precise formulation is given in the last section. For sufficiently large n divisible by 3 as in Theorem 5.2, there is a group homomorphism

$$\varphi : (P_m)_{\Gamma(n)} \longrightarrow TMF(n)^* BString$$

from a stable group of positive energy representations V of the free loop group $LSpin$ to the TMF-cohomology with level n-structure. Here, the congruence group is larger than the one considered before since the character of the representation is only known to be invariant under the action of $\Gamma(n)$ by a theorem of Kac and Wakimoto [KW88, Theorem A].

We will describe the map φ in terms of its elliptic character. Suppose P is a $String$-principal bundle over X. Let $\tilde{L}Spin$ be the universal central extension of $LSpin$. Then by [Wit88], LX carries a $\tilde{L}Spin$-principal bundle $\tilde{L}P$ whose associated $LSpin$-bundle is LP. In particular, this holds for the universal $String$-bundle $EString$ over $BString$. The elliptic character of φ gives the bundle

$$\lambda\varphi(V) = (\tilde{L}ESpin \times_{\tilde{L}Spin} V)_{|BString}$$

(when V is suitably normalized with a character of the rotation circle). In this formula, the right hand side is considered as a formal power series of virtual bundles by decomposing the bundle as a representation of the circle group which reparameterizes the loops. The evaluation of this class on the fundamental class of a string manifold is the formal index of the Dirac operator on LM with coefficient in the bundle associated to the representation V (see [Bry90]).

2. Topological modular forms of level $\Gamma_1(3)$

In this section we describe the spectrum of topological modular forms with level structures. Most of the results are well known and are collected for later reference. We start more generally than actually needed in order to put the main theorem in a broader picture.

Let $\mathcal{M}$ denote the stack of smooth elliptic curves. A morphism $f : S \longrightarrow \mathcal{M}$ determines an elliptic curve C_f over the base scheme S (see Deligne and Rapoport [DR73].) The cotangent spaces of the identity section of each elliptic curve give rise to the line bundle ω of invariant differentials on $\mathcal{M}$. A section of $\omega^{\otimes k}$ is called a modular form of weight k. We write M_k for the group of modular forms of weight k.

Let C be an elliptic curve over a ring R. Let $C[n]$ denote the kernel of the self-map which multiplies by n on C. If n is invertible in R, then étale-locally, $C[n]$ is of the form $\mathbb{Z}/n \times \mathbb{Z}/n$. A choice of an isomorphism is called a $\Gamma(n)$-structure on C. A monomorphism $\mathbb{Z}/n \longrightarrow C[n]$ is a $\Gamma_1(n)$-structure. It corresponds to a choice of a point of exact order n. A choice of a subgroup scheme of $C[n]$ which is isomorphic to $\mathbb{Z}/n$ is called a $\Gamma_0(n)$-structure. There are maps of associated moduli stacks with n inverted

$$(1) \qquad\qquad \mathcal{M}_{\Gamma(n)} \longrightarrow \mathcal{M}_{\Gamma_1(n)} \longrightarrow \mathcal{M}_{\Gamma_0(n)} \longrightarrow \mathcal{M}$$

and the sections of the associated line bundles are called modular forms with the corresponding level structures. For example, a $\Gamma_1(n)$-modular form f of weight k associates an element of the ring R to each triple $(C/R, \omega, P)$ where C is an elliptic curve over $R = R[1/n]$, ω is a translation invariant nowhere vanishing differential and P is a point of exact order n. This association should only depend on the isomorphism class of the triple, it should be invariant under base change, and it should satisfy

$$f(C, a\omega, P) = a^{-k} f(C, \omega, P)$$

for all $a \in R^\times$.

The case $n = 3$ can be made more explicit. Locally, for any such triple (C, ω, P) the curve can uniquely be written in the form

$$(2) \qquad\qquad C : \quad y^2 + a_1 xy + a_3 y = x^3$$

in a way that P is the origin $(0,0)$ and ω has the standard form $\omega = dx/(2y + a_1 x + a_3)$. A proof of this fact can be found in [MR09, 3.2]. It means that there is universal triple $(C, \omega, (0,0))$ over the ring $\mathbb{Z}[1/3, a_1, a_3, \Delta^{-1}]$ with the property that locally any other triple is obtained by base change. Hence a $\Gamma_1(3)$-modular form is determined by its value on the universal triple and gives an element in this ring. Moreover, any element in the ring gives a modular form with level structure. We get

$$(3) \qquad\qquad M_{\Gamma_1(3)_*} \quad \cong \quad \mathbb{Z}[1/3, a_1, a_3, \Delta^{-1}].$$

One should mention that all moduli problems for $\Gamma_1(n)$ and $\Gamma(n)$-structures with $n \geq 3$ are representable this way but $\Gamma_0(n)$ is not representable. A good reference for these classical results on modular forms and level structures is the book [KM85].

There is a derived version of these concepts as follows.

Theorem 2.1. [Goe10, HL, AHR] *There is a sheaf $\mathcal{O}_{TMF}$ of E_∞-ring spectra over $\mathcal{M}$ in the étale topology. This sheaf satisfies:*

(i) *The spectrum $TMF = \Gamma\mathcal{O}_{TMF}$ only has 2 and 3 torsion in homotopy. Away from 6 it is concentrated in even degrees and we have an isomorphism*

$$\pi_{2k}TMF[1/6] \cong M_k[1/6].$$

(ii) *There is an orientation map $MString \longrightarrow TMF$ which induces the Witten genus in homotopy. In fact, its image coincides with the homotopy groups of a connective version of TMF.*

(iii) *The sequence (1) of moduli stacks gives a sequence of spectra*

$$TMF(n) \longleftarrow TMF_1(n) \longleftarrow TMF_0(n) \longleftarrow TMF[1/n]$$

and induced equivalences (with n inverted)

$$TMF[1/n] \cong TMF(n)^{h\,Gl_2(\mathbb{Z}/n)}$$
$$TMF_0(n) \cong TMF_1(n)^{h\,Gl_1(\mathbb{Z}/n)}.$$

(iv) *For all étale $f : Spec(R) \longrightarrow \mathcal{M}$ the spectrum $E = \Gamma f^*\mathcal{O}_{TMF}$ is a complex orientable ring spectrum whose formal group $E^0 BS^1$ is equipped with an isomorphism to the formal completion of C_f.*

The spectrum $TMF_1(3)$ can be described much more elementarily if one is only interested in its associated cohomology theory. Since the moduli problem is representable, its homotopy coincides with the ring $M_{\Gamma_1(3)}$ by property (iv) as we now explain. (The ring $M_{\Gamma_1(3)}$ has been determined in (3).) When we choose a coordinate on the formal group of the curve (2) we obtain a formal group law. These are classified by a map from the Lazard ring L, which coincides with the homotopy groups of complex bordism MU. For instance, the 2-typicalization of the standard coordinate has Hazewinkel generators (see [Lau04, Lemma 1]) at $p = 2$

$$(4) \qquad\qquad\qquad v_1 = a_1$$
$$(5) \qquad\qquad\qquad v_2 = a_3.$$

The Hazewinkel generators form a regular sequence and hence satisfy the Landweber exactness conditions. Thus for finite complexes X we have natural isomorphisms

$$(6) \qquad\qquad TMF_1(3)^*X \cong MU^*X \otimes_{MU^*} M_{\Gamma_1(3)}.$$

(In the older literature for example [Bry90, Fra92, Bak94] this theory carried the names $Ell^{\Gamma_1(3)}$ or $E^{\Gamma_1(3)}$.)

Lemma 2.2. *The map*

$$\mathbb{Z}_{(2)}[v_1, v_2^{\pm 1}, (v_1^3 - 27v_2)^{-1}] \longrightarrow TMF_1(3)_{(2)*}$$

is an isomorphism.

Proof. The discriminant of the universal curve has the form

$$\Delta = a_3^3(a_1^3 - 27a_3).$$

The result follows from (4) and (5). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let $w_k \in \pi_*MU$ denote any element which projects to a generator of the indecomposables in $\pi_{2(p^k-1)}BP$. Recall from [Mat, Definition 3.9] that an MU-module spectrum M is a form of $BP\langle n \rangle$ if there are $w_{n+1}, w_{n+2}, \ldots$ such that

$$M \cong BP/(w_{n+1}, w_{n+2}, \ldots).$$

In the sequel, we will call these theories generalized $BP\langle n\rangle$ and will use the same notation.

Definition 2.3. An MU-module spectrum M is a generalized $E(n)$ if there exist a generalized $BP\langle n\rangle$ and an element $w \in \pi_*MU$ such that

$$BP\langle n\rangle[w^{-1}] \cong M$$

and the element w coincides with a power of v_n modulo the ideal $(p, v_1, \ldots, v_{n-1})$.

Proposition 2.4. *Locally at the prime 2, the spectrum $TMF_1(3)$ is a generalized Johnson-Wilson theory $E(2)$.*

Proof. Use [LN12] or [Mat, Example 3.11] to get a connective version of $TMF_1(3)$ which is a generalized $BP\langle 2\rangle$. The claim then follows from Lemma 2.2. □

Theorem 1.1 involves the elliptic character map. This map originates from the Tate curve

$$y^2 + xy = x^3 + B(q)x + C(q)$$

where

$$B(q) = \frac{-1}{48}(E_4(q) - 1) \text{ and } C(q) = \frac{1}{496}(E_4(q) - 1) - \frac{1}{864}(E_6(q) - 1)$$

with the Eisenstein series E_4 and E_6. The series B and C are integral power series in q. The evaluation of an ordinary modular form on the Tate curve with its canonical differential corresponds to its q-expansion. The formal group associated to the Tate curve is the multiplicative formal group. In order to get a $\Gamma_1(n)$-structure, that is, a point of order n, one can use the extension of scalars $\mathbb{Z}((q)) \to \mathbb{Z}((q))$; $q \mapsto q^n$. The resulting curve is usually denoted by $Tate(q^n)$ and its multiplicative reduction furnishes the Miller character map [Mil89][Lau99]

$$\lambda : TMF_1(n) \longrightarrow K[1/n]((q)).$$

In homotopy this map is the classical q-expansion.

Lemma 2.5. *Locally at $p = 2$, the map*

$$\lambda_*X : TMF_1(3)[a_1^{-1}]_*X \longrightarrow K[1/3]((q))_*X$$

is a monomorphism for all X.

Proof. Note that the q-expansion of a_1 starts with 1 and hence is invertible in the target. We will use Equation (6) and replace MU with BP. Every comodule is the inductive limit of its finitely generated subcomodules. Hence, we may assume that BP_*X has a finite Landweber filtration (F_k) with subsequent quotients of the form BP_*/I_t with I_t the invariant prime ideal $(p, v_1, \ldots, v_{t-1})$. By the q-expansion principle the character map λ_* is injective and injective mod p. Hence it is injective when tensored with each of the quotients BP_*/I_t. The claim follows from the obvious inductive argument. □

Remark 2.6. It would be interesting to set up character maps for finer structures like $\Gamma_0(n)$. We will come back to this question in a subsequent work.

3. THE $TMF_1(3)$-COHOMOLOGY OF $BSpin$

In this section we will show the universal coefficient isomorphism for the $TMF_1(3)$-cohomology of the space $BSpin$ and the pullback diagram of Theorem 1.1. The main ingredient is the Morava $K(2)$-homology which has been computed by Kitchloo-Laures in [KL02]. We can use this result to obtain information about the $E(k, n)$-cohomology by methods of Ravenel-Wilson-Yagita. A chromatic argument enables us to compute the universal coefficients spectral sequence for $TMF_1(3)^* BSpin$. We will show that it collapses at the E_2-term and obtain Theorem 1.1 from the classical q-expansion principle. Some of the results of this section apply to other situations and hence are formulated more generally than actually needed.

From now on we fix a prime p. In a first step we do not want to deal with $\lim^1$-questions and hence work with p-completed spectra. Let $E(n)$ be a p-completed generalized Johnson-Wilson spectrum. Let I_k be the invariant prime ideal $(p, v_1, \ldots, v_{k-1})$ and let $E(k, n)$ be the spectrum

$$E(k, n) = E(n)/I_k.$$

By definition we have $E(n, n) \cong K(n)$, $E(0, n) = E(n)$ and there are cofibre sequences

(7) $$E(k - 1, n) \xrightarrow{v_{k-1}} E(k - 1, n) \longrightarrow E(k, n) .$$

Remark 3.1. Recall from Hovey and Sadofsky [HS99a, Theorem 3.4] that for all generalized $K(n)$, there is a faithfully flat extension of its coefficient ring over which its formal group law becomes strictly isomorphic to the Honda formal group law. This allows us to carry over results from the classical to the generalized $K(n)$.

In the following, for a fixed $0 < k \leq n$, let X be a space with even $E(k, n)$-cohomology. The exact sequence induced by (7)

(8)
$$0 \longrightarrow E(k - 1, n)^{ev} X \xrightarrow{v_{k-1}} E(k - 1, n)^{ev} X \longrightarrow E(k, n)^{ev} X$$
$$\longrightarrow E(k - 1, n)^{odd} X \xrightarrow{v_{k-1}} E(k - 1, n)^{odd} X \longrightarrow 0$$

tells us that each element in $E(k - 1, n)^{odd} X$ is infinitely divisible by v_{k-1}. The following result applies:

Theorem 3.2. *If x is infinitely divisible by v_k in $E(k, n)^* X$, then it is zero.*

Proof. The proof for [RWY98, Corollary 4.11] verbatim carries over to the generalized $E(k, n)$. In more detail, one first observes that x is infinitely divisible when restricted to each finite subcomplex of X. Hence one can assume that X is finite. Then for $P(k) = BP/I_k$ one tensors a Landweber filtration of $P(k)^* X$ with the generalized $E(k, n)^*$ to reduce the claim to modules of the form $E(k, n)^*/(v_k, \ldots v_s)$ for some $s < n$. Here, x has to be zero. $\square$

Corollary 3.3. *If X is a space with even Morava $K(n)$-cohomology then $E(k, n)^* X$ is even for all k and the exact sequences (8) are short exact.*

Proof. By an inductive argument we may assume that all elements of $E(k - 1, n)^{odd} X$ are infinitely divisible by v_{k-1}. Then the claim follows from Theorem 3.2. $\square$

Next we look at the universal coefficient spectral sequence for $E = E(n)$

$$\text{Ext}_{E_*}^{s,t}(E_* X, E_*) \Longrightarrow E^{s+t} X$$

Lemma 3.4. *The global dimension of $E_* = E(n)_*$ in the category of graded modules equals n.*

Proof. The corresponding result is well known in the ungraded setting. Its graded version is harder to find in the literature but the proof given in [Eis95, 19.5] carries over: Let k_* be the graded field $\mathbb{F}_p[v_n^{\pm}]$. Since $(p, v_1, \ldots, v_{n-1})$ is a regular sequence the Koszul complex provides a free graded resolution of length n. This implies the vanishing of $\text{Tor}_{i+1}^{E_*}(k_*, M_*)$ for all $i \geq n$ and for all M_*. Next let $F = (F_{n*}, \varphi_n)$ be a graded minimal free resolution of of a finitely generated module M_*. (The minimality condition means that for each n a basis of F_n maps to a minimal set of generators of $\ker \varphi_{n-1}$.) Since all differentials in $k_* \otimes F$ are 0 we have

$$\text{Tor}_{i+1}^{E_*}(k_*, M_*) \cong k_* \otimes_{E_*} F_{i+1,*}.$$

This vanishes iff $F_{i+1,*}$ vanishes because the resolution is free. Thus we have shown the claim for finitely generated modules. The general result follows from the graded version of Auslander's Theorem [Eis95, 19.1]. $\qquad\square$

Lemma 3.5. *Let E be $E(2)$ and let E/p^∞ be the cofibre of $E \to p^{-1}E$. Suppose $\text{Ext}_{E_*}^{0,t+1}(E_* X, E_*/p^\infty)$ vanishes for all even t. Then for even t there is an isomorphism*

$$\text{Ext}_{E_*}^{2,t}(E_* X, E_*) \cong (E/p^\infty)^{t+1} X.$$

Proof. Suppose F is a ring theory with rational coefficients. Then Serre's result and the comparison theorem for homology theories imply that the F_*-linear extension of the F-Hurewicz map

$$F_* \otimes \pi_*^{st}(X) \longrightarrow F_*(X)$$

is an isomorphism. It follows that $p^{-1}E_* X$ is a free $p^{-1}E_*$ module. Hence

$$\text{Ext}_{E_*}^s(E_* X, p^{-1}E_*) \cong \text{Ext}_{p^{-1}E_*}^s(p^{-1}E_* X, p^{-1}E_*)$$

vanishes for all $s > 0$, and the short exact sequence

$$E_* \rightarrowtail p^{-1}E_* \twoheadrightarrow E_*/p^\infty$$

induces isomorphisms

$$\text{Ext}_{E_*}^{s+1}(E_* X, E_*) \cong \text{Ext}_{E_*}^s(E_* X, E_*/p^\infty).$$

By 3.4 these groups vanish for $s \geq 2$ and so does $\text{Ext}_{E_*}^{0,t+1}(E_* X, E_*/p^\infty)$ for even t. Hence the claim follows from the universal coefficient spectral sequence

$$\text{Ext}_{E_*}^{s,t}(E_* X, E_*/p^\infty) \Longrightarrow (E/p^\infty)^{s+t} X.$$

$\qquad\square$

We now specify to the case $X = BSpin$ and prove a splitting principle using the following computation.

Theorem 3.6. [KL02, 1.2] *Let n be 1 or 2. Let $b_i \in K(n)_{2i}BS^1$ be the dual to the power c_1^i of the first Chern class of the canonical line bundle. Denote its image under the map induced by the inclusion of the maximal torus*

$$BS^1 \longrightarrow BSpin(3) \longrightarrow BSpin$$

by the same name. Then we have

$$K(n)_* BSpin \cong \pi_* K(n)[b_{2^n \cdot 2}, b_{2^n \cdot 4}, b_{2^n \cdot 6}, \ldots].$$

Corollary 3.7. *Let $T(\lfloor m/2 \rfloor)$ be the maximal torus of $Spin(m)$ and set $BT^\infty = \mathrm{colim}\, BT(\lfloor m/2 \rfloor)$. Then the restriction map from $K(n)^* BSpin$ to $K(n)^* BT^\infty$ is injective for $n \leq 2$.*

Proof. It suffices to show that the dual map $K(n)_* BT^\infty \longrightarrow K(n)_* BSpin$ is surjective. This is immediate from the theorem since each monomial $b_{i_1} b_{i_2} \cdots b_{i_k}$ comes from the classifying space of the k-dimensional torus. $\square$

Corollary 3.8. *For $E = E(k, n)$ with $n \leq 2$ the restriction map from $E^* BSpin$ to $E^* BT^\infty$ is injective.*

Proof. By descending induction on $k \leq n$, by the previous corollary and by Corollary 3.3 we have a map of short exact sequences

$$
\begin{array}{ccccc}
E(k-1,n)^* BSpin & \overset{v_{k-1}}{\rightarrowtail} & E(k-1,n)^* BSpin & \longrightarrow\!\!\!\rightarrow & E(k,n)^* BSpin \\
\downarrow & & \downarrow & & \downarrow \\
E(k-1,n)^* BT^\infty & \overset{v_{k-1}}{\rightarrowtail} & E(k-1,n)^* BT^\infty & \longrightarrow\!\!\!\rightarrow & E(k,n)^* BT^\infty
\end{array}
$$

for which the last vertical map can be assumed to be injective. Hence, any element in $E(k-1,n)^* BSpin$ which restricts trivially to the torus is divisible by v_{k-1} and the quotient again restricts trivially to the torus. Continuing this way, we see that it must be infinitely divisible by v_{k-1} and thus has to vanish by Theorem 3.2. $\square$

Theorem 3.9. *Let E be $E = E(1)$ or the 2-completed $TMF_1(3)$. Then the universal coefficient isomorphism*

$$E^* BSpin \cong \mathrm{Hom}_{E_*}(E_* BSpin, E_*)$$

holds.

Proof. First note that we have the isomorphism

$$E^* BT^\infty \cong \mathrm{Hom}_{E_*}(E_* BT^\infty, E_*)$$

because $E_* BT^\infty \cong \mathrm{colim}\, E_* T^k$ is free: a basis is given by arbitrary products of the form

$$\beta_{i_1} \otimes \beta_{i_2} \otimes \cdots \otimes \beta_{i_k}$$

where β_i is dual to c_1^i.

For the space $X = BSpin$ and for $E = E(1)$ the universal coefficient spectral sequence degenerates to the short exact sequence

$$\mathrm{Ext}^{1,*}_{E_*}(E_{*-1}X, E_*) \rightarrowtail E^* X \longrightarrow\!\!\!\rightarrow \mathrm{Ext}^{0,*}_{E_*}(E_* X, E_*)$$

by Lemma 3.4. Furthermore, the second map is injective because it factors through the restriction map into $E^* BT^\infty \cong \mathrm{Ext}^{0,*}_{E_*}(E_* BT^\infty, E_*)$ and this map is injective by Corollary 3.8.

For the case $E = TMF_1(3)$ at $p = 2$, we already know from Corollary 3.3 that E^*X is concentrated in even degrees. Since E_* is torsion free and $p^{-1}E_*X$ is in even degrees, the group $\mathrm{Hom}_{E_*}(E_*X, E_*)$ must be even, too. Hence, we can restrict our attention to even degrees.

We would like to apply Lemma 3.5, and therefore we have to show that $\mathrm{Ext}_{E_*}^{0,t+1}(E_*X, E_*/p^\infty)$ vanishes for all even t. By Lemma 2.5 or simply by the q-expansion principle, the group E_*/p^∞ injects into K_{Tate*}/p^∞ and so does the induced map on Ext^0-groups. The isomorphisms at $p = 2$

$$
\begin{aligned}
\mathrm{Ext}_{E_*}^{0,t+1}(E_*X, K_{Tate*}/p^\infty) &\cong \mathrm{Ext}_{BP_*}^{0,t+1}(BP_*X, K_{Tate*}/p^\infty) \\
&\cong \mathrm{Ext}_{E(1)_*}^{0,t+1}(E(1)_*X, K_{Tate*}/p^\infty)
\end{aligned}
$$

follow from the Landweber exactness of E and K_{Tate}. The latter group coincides with $\mathrm{Ext}_{E(1)_*}^{1,t+1}(E(1)_*X, K_{Tate*})$ which is a product of groups of the form $\mathrm{Ext}_{E(1)_*}^{1,t+1}(E(1)_*X, E(1)_*)$. Thus it is trivial.

The even dimensional Ext^2-term of the universal coefficient spectral sequence has been identified with the odd part of $(E/p^\infty)^*X$ in Lemma 3.5. Hence, for its vanishing it is enough to show the injectivity of the first map in the exact sequence

$$
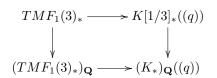E^*X \longrightarrow (p^{-1}E)^*X \longrightarrow (E/p^\infty)^*X \ .
$$

This follows from the observation that its composite with the restriction map to $(p^{-1}E)^*BT^\infty$ is injective by Corollary 3.8. In even degrees we obtain the short exact sequence

$$
\mathrm{Ext}_{E_*}^{1,*}(E_{*-1}X, E_*) \rightarrowtail E^*X \twoheadrightarrow \mathrm{Ext}_{E_*}^{0,*}(E_*X, E_*) \ .
$$

Once more by Corollary 3.8 the kernel of the second map vanishes. $\qquad\square$

Remark 3.10. We do not know much about the generalized $E(2)$-homology of $BSpin$, not even if it is concentrated in even degrees. This is what makes the proof of the universal coefficient isomorphism difficult.

Proof of Theorem 1.1: We first show the pullback property for the 2-completed theories. Recall from [Lau99, Lemma 1.5] that there is a pullback of rings

$$
\begin{array}{ccc}
TMF_1(3)_* & \longrightarrow & K[1/3]_*((q)) \\
\downarrow & & \downarrow \\
(TMF_1(3)_*)_{\mathbf{Q}} & \longrightarrow & (K_*)_{\mathbf{Q}}((q))
\end{array}
$$

and each corner is the coefficient ring of a Landweber exact theory. For each such theory E, and for all spaces X, we have the natural isomorphism

$$
\mathrm{Hom}_{BP_*}(BP_*X, E_*) \cong \mathrm{Hom}_{E_*}(E_*X, E_*).
$$

Hence, when applying the left exact functor $\mathrm{Hom}_{BP_*}(BP_*BSpin, -)$ to the diagram we still have a pullback. By Theorem 3.9 each corner satisfies the universal coefficient isomorphism and hence we get the desired pullback diagram.

It remains to show the integral result. For odd primes we still have the appropriate pullback diagram by [Lau99, Theorem 1.12]. Moreover, for all spectra Y we

have an arithmetic pullback by [Bou79, Proposition 2.9]

$$
\begin{array}{ccc}
Y & \longrightarrow & \prod_p L_{S\mathbb{Z}/p} Y \\
\downarrow & & \downarrow \\
L_{S\mathbf{Q}} Y & \longrightarrow & L_{S\mathbf{Q}}(\prod_p L_{S\mathbb{Z}/p} Y)
\end{array}
\tag{9}
$$

where SG denotes the Moore spectrum. Mapping $X = BSpin$ to the diagram for $Y = TMF_1(3)$ yields a pullback of cohomology groups by the previous results. In this pullback diagram the lower right corner has the form

$$
H^*\left(X, \left(\prod_p TMF_1(3)_p^\wedge\right)_{\mathbf{Q}}^*\right) \cong \left(\prod_p TMF_1(3)_p^\wedge\right)_{\mathbf{Q}}^* X,
$$

which we can replace with

$$
H^*\left(X, \prod_p \left(TMF_1(3)_p^\wedge\right)_{\mathbf{Q}}^*\right) \cong \prod_p \left(TMF_1(3)_p^\wedge\right)_{\mathbf{Q}}^* X.
$$

Then the enlarged diagram

$$
\begin{array}{ccccc}
TMF_1(3)^* X & \longrightarrow & \prod_p \left(TMF_1(3)_p^\wedge\right)^* X & \longrightarrow & \prod_p \hat{K}_p^* X((q)) \\
\downarrow & & \downarrow & & \downarrow \\
TMF_1(3)_{\mathbf{Q}}^* X & \longrightarrow & \prod_p \left(TMF_1(3)_p^\wedge\right)_{\mathbf{Q}}^* X & \longrightarrow & \prod_p (\hat{K}_p^*)_{\mathbf{Q}} X((q))
\end{array}
$$

is a pullback. Since the upper composite factors through $K_{Tate}^* X$ the result follows. □

Corollary 3.11. *The integral universal coefficient isomorphism*

$$
TMF_1(3)^* BSpin \quad \cong \quad \mathrm{Hom}_{TMF_1(3)_*}(TMF_1(3)_* BSpin, TMF_1(3)_*)
$$

holds.

Proof. The right hand side satisfies the pullback property in the diagram of the main theorem. □

4. PONTRYAGIN CLASSES AND THE COHOMOLOGY OF $BString$

In this section we construct explicit generators in the $TMF_1(3)$-cohomology rings of $BSpin$ and $BString$ with the help of Theorem 1.1. We start with a reminder of the KO-Pontryagin classes. We will see that for line bundles they are defined by the same formula as the Pontryagin classes in singular homology. Using Theorem 1.1 we then obtain a construction of $TMF_1(3)$-Pontryagin classes.

Recall from [ABP66, Proposition 4.4(b)] that the restriction maps from $KO^0(BSO(n))$ to the maximal torus $K^0(BT(\lfloor n/2 \rfloor))$ are monomorphisms with images the invariants of the Weyl groups. The KO-Pontryagin classes are defined by the preimage of the series

$$
\prod_{i=1}^{\lfloor n/2 \rfloor} (1 - t(x_i \bar{x}_i)) \in K^0(BT(\lfloor n/2 \rfloor)).
\tag{10}
$$

Here, $x_i = 1 - L_i$ are the first Chern classes of the canonical line bundles L_i over $BT(\lfloor n/2 \rfloor)$ in K-theory.

The Pontryagin classes freely (topologically) generate the ring $KO^0(BSO)$. Since we do not know a reference for this fact we give a short argument: first note that in K-theory we have for all $x = 1 - L$ the equality

$$x + \bar{x} = x\bar{x}.$$

Hence a power series which is invariant under the map which interchanges x and $\bar{x}$ can be written as a power series in $x\bar{x}$. We conclude that each class in $KO^0(BSO)$ is a symmetric power series in $x_i\bar{x}_i$ when restricted to K^0BT for all tori T. Thus it is a power series in the Pontryagin classes.

We also note that

$$KO^0(BSpin) \cong KO^0(BSO).$$

This fact follows from the arithmetic square (9) since the map from $BSpin$ to BSO is a $K(1)$-local and rational equivalence (see [KL02, Theorem 1.2(ii)]).

Proof of Theorem 1.2: The K-Pontryagin classes freely generate the K^*_{Tate}-algebra $K^*_{Tate}(BSpin)$. We also know that the classical Pontryagin classes in rational singular cohomology freely generate $H^*(BSpin; TMF_1(3)^*_{\mathbf{Q}})$ as a $TMF_1(3)^*_{\mathbf{Q}}$-algebra. They are defined in the same way except that in Formula (10), the x_i are the ordinary first Chern classes.

The multiplicative formal group law over K_{Tate*} is strictly isomorphic to the one coming from the Tate curve, that is, the q-expansion of the curve (2) together with the standard coordinate. By [Mil89] there is a natural automorphism of K_{Tate} which exchanges the corresponding two orientations. Hence, when we replace the x_i in the formula for the K-Pontryagin classes by the first Chern classes with respect to the new orientation we still have free generators. The same argument holds for rational singular homology because here all formal group laws are strictly isomorphic.

These Pontryagin classes define elements in each corner of the pullback diagram of Theorem 1.1 and hence free generators of $TMF_1(3)^*BSpin$. Moreover, these classes are determined by their restrictions to the maximal tori by Corollary 3.8 and are here given by the displayed formula. $\square$

Next we consider the cohomology of $BString$. The space $BString$ is defined as the homotopy fibre of the map $BSpin \to K(\mathbb{Z}, 4)$ which kills the lowest homotopy group. In particular, we have a sequence of infinite loop spaces

(11) $K(\mathbb{Z}/2, 2) \longrightarrow K(\mathbb{Z}, 3) \longrightarrow BString \longrightarrow BSpin$.

Theorem 4.1 ([KLW04a] [KLW04b])**.** *The sequence (11) induces an exact sequence of Hopf algebras in Morava $K(2)$-homology at the prime 2*

$$K(2)_* \longrightarrow K(2)_*K(\mathbb{Z}/2, 2) \longrightarrow K(2)_*K(\mathbb{Z}, 3) \longrightarrow K(2)_*BString$$

$$\longrightarrow K(2)_*BSpin \longrightarrow K(2)_*.$$

Algebraically, there is short exact sequence of Hopf algebras

$$K(2)_*K(\mathbb{Z}, 3) \longrightarrow K(2)_*BString \longrightarrow K(2)_*BSpin$$

which splits. (Here, the first map is not the one which is induced from the second map of (11).) In particular, the module $K(2)_*BString$ is concentrated in even dimensions.

We will use this information for the computation of the cohomology ring of $BString$ with respect to

$$E = L_{K(2)}E(2)$$

for a generalized $E(2)$. The coefficients of E are given by $E_* = (E(2)_*)^\wedge_{I_2}$.

Proposition 4.2 ([HS99b] Proposition 2.5). *Suppose X is a space with even Morava $K(2)$-homology. Then E^*X is the completion with respect to I_2 of a free E_*-module.*

Proposition 4.3. *There is an isomorphism of algebras*

$$E^*K(\mathbb{Z}, 3) \cong E^*[\![r]\!]$$

with r in degree 6.

Proof. The $K(2)$-cohomology of $K(\mathbb{Z}, 3)$ has been computed in [RW80] and [JW85] (see also Su [Su07] for its $E(1,2)$-cohomology.) It is topologically free on a generator of degree 6. Lift this generator to an E-cohomology class. This is possible because the algebra $E^*K(\mathbb{Z}, 3)$ is concentrated in even degrees with the Morava $K(2)$-cohomology as its I_2-reduction. (This can be seen as before with the exact sequence (8).) Clearly, when restricted to a finite subcomplex of $K(\mathbb{Z}, 3)$ every reduced class becomes nilpotent. Hence, we obtain an algebra map from $E^*[\![r]\!]$ to $E^*K(\mathbb{Z}, 3)$. The result follows from Proposition 4.2 and the following version of Nakayama's Lemma. $\square$

Lemma 4.4. *Let R be a graded ring with a unique maximal homogeneous ideal m and let P and Q be pro-free R modules. Then $f : P \to Q$ is an isomorphism if and only if it is so modulo m.*

Proof. Tensor the short exact sequences

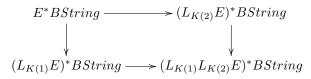$$m^k/m^{k+1} \rightarrowtail R/m^{k+1} \twoheadrightarrow R/m^k$$

with f and use the fact that m^k/m^{k+1} is a free module over R/m. $\square$

Proof of Theorem 1.3: By Theorem 4.1, we find a lift r of the generator of $K(2)^*K(\mathbb{Z}, 3)$ to $K(2)^*BString$. Since $K(2)^*BString$ is concentrated in even degrees it is the quotient of $E^*BString$ by the ideal I_2. Hence we can lift r further to a class in $E^*BString$ and obtain an algebra map

$$f : \; E^*BSpin \hat{\otimes} E^*[\![r]\!] \longrightarrow E^*BString \; .$$

This map is an isomorphism by Proposition 4.2 and the previous lemma. $\square$

Remark 4.5. Let E be the 2-complete $TMF_1(3)$. Then there is a pullback square of cohomology rings

$$
\begin{array}{ccc}
E^*BString & \longrightarrow & (L_{K(2)}E)^*BString \\
\downarrow & & \downarrow \\
(L_{K(1)}E)^*BString & \longrightarrow & (L_{K(1)}L_{K(2)}E)^*BString
\end{array}
$$

for which we have computed the three corners: $K(1)$-locally the map from $BSpin$ to $BString$ is an equivalence because the $K(1)$-homology of $K(\mathbb{Z}, 3)$ vanishes. Hence, the lower horizontal map is the inclusion

$$(L_{K(1)}E)^*[\![p_1, p_2, \ldots]\!] \longrightarrow (L_{K(1)}L_{K(2)}E)^*[\![p_1, p_2, \ldots]\!]$$

but for the right vertical map the image of the class r is unclear. We will address this question in a subsequent work.

5. Applications to representations of loop groups

A principal $Spin(d)$-bundle P over X and a representation V of $Spin(d)$ give rise to a vector bundle over X by associating V to each fibre of P. Hence P and V define an element in the K-theory ring $K^0(X)$. If X is a spin manifold and P is the principal bundle associated to its tangent bundle, the pushforward of this K-theory class to a point is the index of the Dirac operator twisted by V (see [AS68]). The construction induces a ring map

$$(12) \qquad\qquad Rep(Spin(d)) \longrightarrow K^0(X) .$$

It factors through the map from $K^0(BSpin(d))$ to $K^0(X)$ which classifies P (see [AS69]).

In [Bry90] Brylinski conjectured a similar connection for loop space representations and the elliptic cohomology of the 7-connected cover of $BSpin(d)$ which is usually denoted by $BString(d)$. Recall from [PS86]) that a positive energy representation V of the free loop space $LSpin(d)$ means that V is a representation of the semi-direct product $\tilde{L}Spin(d) \rtimes S^1$ where $\tilde{L}Spin(d)$ is the universal central extension of $LSpin(d)$ and S^1 acts on $\tilde{L}Spin(d)$ by reparameterizing the loops. 'Positive energy' means that for the action of $R_t = e^{2\pi it} \in S^1$ the vector space

$$V_n = \{v \in V : \ R_t(v) = e^{2\pi int}v \text{ for all } t\}$$

is finite dimensional for all n and vanishes for all $n < n_0$ for some n_0. (Note that we can always multiply the rotation action with a character of S^1 to get $n_0 = 0$.) When identifying the group $Spin(d)$ with a subgroup of $\tilde{L}Spin(d) \rtimes S^1$, we obtain a formal power series

$$\sum_{n \in \mathbb{Z}} [V_n]q^n \in Rep(Spin(d))((q))$$

and hence via the map (12) a class in $K^0_{Tate}(X)$ for every $Spin(d)$-bundle P over X. If P comes from a string bundle then the loop space LX carries a $\tilde{L}Spin(d)$-principal bundle (see ibidem). Hence, the class in $K^0_{Tate}(X)$ associated to V can be viewed as

$$(\tilde{L}P \times_{\tilde{L}Spin(d)} V)_{|X}.$$

In case X is a string manifold and P comes from its tangent bundle, the pushforward of this K_{Tate}-class to a point is the formal index of the S^1-equivariant Dirac operator on the free loop space of X with coefficient in the bundle associated to the representation V (see [Wit88]). The fact that this index is a modular form of some level leads to the hope that the $K_{Tate}(M)$-cohomology class refines to a class in topological modular forms (c.[Bry90]). More precisely, let m be an integer and let $P_m(d)$ be the free abelian group generated by the isomorphism classes of

irreducible positive energy representations of $\tilde{L}Spin(d)$ of level m (see [PS86, 9.3] for the meaning of 'level').

Conjecture 5.1. There is an integer n depending on d and m and an additive map
$$\varphi_d : P_m(d) \longrightarrow TMF(n)^0 BString(d)$$
whose elliptic character λ (defined as before in [Mil89]) coincides with the bundle

(13) $$\lambda\varphi_d(V) \quad = \quad (\tilde{L}ESpin(d) \times_{\tilde{L}Spin(d)} V)_{|BString(d)}.$$

Let $n = 24(m+g)$, with $g = d-2$ the Coxeter number. Theorem A in [KW88] states that the character J of a positive energy representation V is a formal Jacobi modular form of level n, weight $w = 0$ and index $\frac{m}{2}$. This means that it is invariant under the action of $\Gamma(n)$ defined by

(14) $$\left(J \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)(z,\tau) = (c\tau+d)^{-w} e^{-2\pi imc(\sum_j z_j^2)/(c\tau+d)} J\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)$$

when suitably normalized with a character of the rotation group. (In this formula the Chern roots x_i are replaced by $2\pi i z_i$ (c.[Bry90]).) Since for string bundles the first Pontryagin class $p_1 = \sum_j z_j^2$ vanishes we have the following result.

Theorem 5.2. [Bry90, Remark 3.10] *Conjecture 5.1 holds rationally for* $n = 24(m+g)$.

We will show a stable version of the integral conjecture. Let P_m be the inverse limit of all $P_m(d)$'s. For n divisible by 3, let $(P_m)_{\Gamma(n)}$ be the subgroup of P_m consisting of representations V with character J invariant under the action (14) of $\Gamma(n)$.

Theorem 5.3. *There is an additive map*
$$\varphi : (P_m)_{\Gamma(n)} \longrightarrow \widehat{TMF(n)}^0 BString$$
whose elliptic character is the class described in Equation (13).

The proof will be given at the end of the section.

Lemma 5.4. *Let 3 be inverted in the following rings of modular forms and let n be divisible by 3. Then the following ring extensions are flat:*
$$\begin{array}{ccc} M_{\Gamma_1(3)} & \longrightarrow & M_{\Gamma(3)} \\ M_{\Gamma(3)} & \longrightarrow & M_{\Gamma(n)} \end{array}$$

Proof. This follows from [KM85, 5.5.1] since the moduli problem for $\Gamma_1(3)$ structures is representable. $\square$

Lemma 5.5. *Let R be coherent, M be a finitely generated R module and N be a flat R-module. Then we have the isomorphism*
$$\mathrm{Hom}_R(M,R) \otimes_R N \cong \mathrm{Hom}_R(M,N).$$

Proof. This result should be standard. Choose a finitely generated free presentation F_* of M. Since N is flat the left hand side of the claim is the kernel of
$$\mathrm{Hom}_R(F_0,R) \otimes_R N \longrightarrow \mathrm{Hom}_R(F_1,R) \otimes_R N.$$

By finiteness it coincides with the kernel of

$$\mathrm{Hom}_R(F_0, N) \longrightarrow \mathrm{Hom}_R(F_1, N).$$

which is the right hand side. □

Proposition 5.6. *The diagram*

$$
\begin{array}{ccc}
TMF(n)^*BSpin & \xrightarrow{\ \ \lambda\ \ } & K^*_{Tate}BSpin \\
\downarrow & & \downarrow \\
H^*(BSpin, TMF(n)^*_{\mathbf{Q}}) & \longrightarrow & H^*(BSpin, K_{Tate}{}^*_{\mathbf{Q}})
\end{array}
$$

is a pullback for all n divisible by 3.

Proof. This follows as before from the universal coefficient isomorphism:

$$
\begin{aligned}
TMF(n)^*BSpin & \cong & TMF_1(3)^*BSpin \hat{\otimes}_{M_{\Gamma_1(3)}} M_{\Gamma(n)} \\
& \cong & \mathrm{Hom}(TMF_1(3)_*BSpin, M_{\Gamma_1(3)}) \hat{\otimes}_{M_{\Gamma_1(3)}} M_{\Gamma(n)} \\
& \cong & \mathrm{Hom}(TMF_1(3)_*BSpin, M_{\Gamma(n)}).
\end{aligned}
$$

Here, the first isomorphism holds for all finite spectra by Lemma 5.4. (The completion of the tensor product is then unnecessary). Taking inverse limits over all finite subspectra and using Corollary 3.11 we see that the isomorphism holds for $BSpin$ for the completed tensor product. The second isomorphism follows again from Corollary 3.11. The last one is a consequence of Lemmas 5.4 and 5.5 when considering finite subspectra of $BSpin$. □

Proof of 5.3. Set $E = TMF_1(3)$. We first construct a specific invertible element S in $E^0 BString$ whose character is a Jacobi form of weight 0 and index $\frac{1}{2}$. Let $x : MU \to E$ be the orientation considered earlier. Its exponential is given in terms of the Weierstrass Φ-function by

$$\frac{\Phi(\tau, z)\Phi(\tau, -\omega)}{\Phi(\tau, z - \omega)}$$

for the standard division point $\omega = 2\pi i/3$ (see [HBJ92, 5.3, 6.4]). Let $BU\langle 6\rangle$ be the 5-connected cover of BU and $MU\langle 6\rangle$ be the corresponding Thom spectrum. Then E admits two ring maps from $MU\langle 6\rangle$: the one which factors through x and the one which factors through the Witten orientation (Theorem 2.1(ii)). Using the Thom isomorphism we obtain a class

$$\frac{\sigma}{x} \in E^0 BU\langle 6\rangle$$

whose augmentation is 1. Let S be the image of $\frac{\sigma}{x}$ under the complexification map from $BString$ to $BU\langle 6\rangle$. The character of S is the function

$$\frac{\Phi(\tau, z - \omega)}{\Phi(\tau, -\omega)}.$$

One can check that this is a formal Jacobi function of weight 0 and index $\frac{1}{2}$ either by direct calculation or one uses the fact that the Witten orientation σ comes from a formal Jacobi modular form of index $\frac{1}{2}$ and x is one of index 0 (see [Bry90, p.469]). Since $\frac{\sigma}{x}$ has the inverse $\frac{x}{\sigma}$ the class S is invertible.

Next, let V be an irreducible positive energy representation in $(P_m)_{\Gamma(n)}$ and let

$$\hat{V} = \sum_n [V_n]q^n$$

be the the class in $K^0_{Tate}(BSpin)$ constructed above. Without loss of generality let E be $L_{S\mathbb{Z}/2}TMF(n)$. Consider S as a class in $E^0 BString$. Since the $K(1)$-homology of $K(\mathbb{Z}, 3)$ vanishes (c.[RW80]) the 2-completed K_{Tate}-cohomologies of $BString$ and $BSpin$ coincide. Hence, we may consider the element

$$\hat{V}\lambda(S)^{-m} \in K^0_{Tate}(BSpin).$$

Since its character is a Jacobi form of index 0 and weight 0 we can apply Proposition 5.6 to obtain a unique class in $E^0 BSpin$ and hence in $E^0 BString$. Define $\varphi[V]$ as the product of this class with S^m.

$\square$

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