TRANSFERS OF CHERN CLASSES IN BP-COHOMOLOGY AND
CHOW RINGS

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Abstract. The $BP^\ast$-module structures of $BP^\ast(BG)$ for extraspecial 2-groups are studied using transfer and Chern classes. These give rise to $p$-torsion elements in the kernel of the cycle map from the Chow ring to ordinary cohomology first obtained by Totaro.

1. Introduction

Let $G$ be a compact Lie group, e.g. a finite group, and $BG$ its classifying space. For complex oriented cohomology theories $h$ one can define in $h^\ast(BG)$ Chern classes of complex representations of $G$, and also transfer maps. We are interested in the Mackey closure $\overline{Ch}_h(G)$ of the ring of Chern classes in $h^\ast(BG)$, namely the subring of $h^\ast(BG)$ recursively generated by transfers of Chern classes. By [H-K-R], this is equal to the $h^\ast$-module generated by transfers of Euler classes.

For ordinary mod $p$ cohomology, Green-Leary [G-L] showed that the inclusion map $i:\overline{Ch}_{H\mathbb{Z}/p} \hookrightarrow H^\ast(BG;\mathbb{Z}/p)$ is an $F$-isomorphism, i.e., the induced map of varieties is a homeomorphism. Green-Minh [G-M] however noticed that $i/\sqrt[p]{0}$ need not be an isomorphism in general. Next consider $h=BP$ or $h=K(n)$, the $n$-the Morava K-theory, at a fixed prime $p$. Following Hopkins-Kuhn-Ravenel [H-K-R], we shall call a group $G$ "good" for $h$-theory if $h^\ast(BG)$ is generated (as an $h^\ast$-module) by transferred Euler classes of representations of subgroups of $G$. It is clear that if the Sylow $p$-subgroup of $G$ is good, then so is $G$ and one has an isomorphism $h^\ast(BG) \cong \overline{Ch}_h(G)$. Furthermore, it follows from [R-W-Y] that $G$ is good for $BP$ if it is good for $K(n)$ for all $n$. Examples for groups that are $K(n)$-good for all $n$ are the finite symmetric groups. Another typical case are $p$-groups of $p$-rank at most 2 and $p \geq 5$: in [Y1] it is shown that the Thom map $\rho:BP^\ast(-) \rightarrow H^\ast(-)_{(p)}$ induces an isomorphism $BP^\ast(BG) \otimes_{BP^\ast}\mathbb{Z}_{(p)} \cong H^{even}(BG)$. Note however that I. Kriz claimed that $K(n)^{odd}(BG) \neq 0$ for some $p$-groups $G$.

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On the other hand, B. Totaro [T1] found a way to compare \(BP\)-theory to the Chow ring. For a complex algebraic variety \(X\), the groups \(CH^i(X)\) of codimension \(i\) algebraic cycles modulo rational equivalence assemble to the Chow ring \(CH^*(X) = \sum_i CH^i(X)\). Totaro constructed a map \(\rho : CH^*(X) \to BP^*(X) \otimes_{BP^*} \mathbb{Z}(p)\) such that the composition \(^{\rho} \circ \rho : CH^i(X) \otimes_{BP^*} \mathbb{Z}(p) \to BP^* \otimes_{BP^*} \mathbb{Z}(p) \to H^*(X)(p)\) coincides with the cycle map. One of the main results of [T1] is that there exists a group \(G\) for which the kernel of \(\rho\) contains \(p\)-torsion elements. To prove this, Totaro defined the Chow ring of a classifying space \(BG\) as \(\lim_{m \to \infty} CH^*((\mathbb{C}^m - S)/G)\) where \(G\) acts on \(\mathbb{C}^m - S\) freely and \(\text{codim}(S) \to \infty\) as \(m \to \infty\). He then constructed a non-zero element \(x\) in \(\text{Ker}(\rho)\) such that

\[
x \in \text{Im}(CH_{BP}(BG) \to (BP^*(BG) \otimes_{BP^*} \mathbb{Z}(p))).
\]

Since transfers and Chern classes also exist in the Chow ring \(CH^*(BG)\), there is an element \(\tilde{x} \in CH_{BP}(G)\) that also lies in \(\text{Ker}(\rho)\). The group Totaro uses is \(G = \mathbb{Z}/2 \times D^1_{1+4}\), where \(D^1_{1+4} = D(2)\) is the extraspecial 2-group of order 32, which is isomorphic to the central product of two copies of the dihedral group \(D_8\) of order 8. He first proves that there exists an element \(x \in BP^*(BD(2))\) satisfying (1.1) but which restricts to zero under the map \(\rho_{\mathbb{Z}/2} : BP^*(-) \to H^*(-; \mathbb{Z}/2)\), where he uses the computation of \(BP^*(BSO(4))\) from [K-Y].

Let \(D(n) = 2^{1+2n}\) denote the extraspecial 2-group of order \(2^{2n+1}\); it is isomorphic to the central product of \(n\) copies of \(D_8\). In this paper, we construct non-zero elements \(x \in BP^*(BD(n))\) satisfying (1.1) but with \(\rho_{\mathbb{Z}/2}(x) = 0\) directly for each \(n\).

Let \(\tilde{W}\) be a maximal elementary abelian 2-subgroup and \(N\) the center of \(D(n)\). For a one-dimensional real representation \(e\) of \(\tilde{W}\) restricting non-trivially to the center, set \(\Delta = \text{Ind}_{\tilde{W}}^{D(n)}(e)\). This is the unique irreducible representation which acts non-trivially on \(N\). Then the \(i\)-th Stiefel-Whitney class \(w_i(\Delta)\) for \(i < 2^n\) can be written as a polynomial in variables \(w_1(e_j), 1 \leq j \leq 2n\), for 1-dimensional representations \(e_j\) of \(D(n)/N\) ([Q], Remark 5.13), i.e. \(w_1(\Delta) = w_1(w_1(e_1), \ldots, w_1(e_{2n}))\). Let \(e'_C\) denote the complex representation induced from the real representation \(e'\). Then we can prove that

\[
(1.2) \quad x = e_{2^n-1}(\Delta_C) - w_{2^n-1}(e_1C, \ldots, e_1(e_{2n}C))
\]

satisfies (1.1) together with \(\rho_{\mathbb{Z}/2}(x) = 0\), and furthermore conclude \(\text{Ker}(\rho) \neq 0\) for \(G = \mathbb{Z}/2 \times D(n)\).
Secondly, we construct a non-nilpotent element $x \in BP^*(BG)$ which is not in $\overline{CH}_{BP}(BG)$ such that

$$x \in \text{Ker}(\rho) \text{ and } 0 \neq x \in (BP^*(BG) \otimes_{BP_* \mathbb{Z}} \mathbb{Z}(p))$$

However we do not know whether $x$ comes from the Chow ring or not, and we only obtain the result for $n = 3, 4$. Set

$$x = [v_1 \otimes w_{2^n}(\Delta)].$$

to be the class represented by $v_1 \otimes w_{2^n}(\Delta)$ in the $E_\infty$-page of the Atiyah-Hirzebruch spectral sequence. If this element exists, then restricting to the center of $D(n)$ we see that $x$ is not in $bch_{BP}(BG)$. However, it seems difficult to prove that this cycle is permanent. For the case $n = 3, 4$, we use $BP$-theory of $B\text{Spin}(7)$ and $B\text{Spin}(9)$ computed in [K-Y] to see that $x$ is a permanent cycle.

These arguments do not seem to work for other extraspecial 2-groups nor 2-groups that have a cyclic maximal normal subgroup [S].

In Section 2, we recall the mod 2 cohomology of extraspecial 2-groups following [Q]. In particular, $w_{2^n-2^i}(\Delta)$ is represented by the Dickson invariant $D_i$, and we study the action of the Milnor primitives $Q_j$ on $D_i$. To see $\rho(x) \neq 0$ in $H^*(BD(n); \mathbb{Z})$, we recall the integral cohomology in Section 3. In Section 4, we show that $x$ satisfies (1.1). In Section 5, we study how elements in Ker$(\rho)$ are represented in the Atiyah-Hirzebruch spectral sequence, assuming some technical conditions which are satisfied in the cases $n = 3$ and $n = 4$. The element $x$ in (1.4) is proved not to be in $\overline{CH}_{BP}(BD(n))$ in Section 6. In Section 7 the element $x$ in (1.4) is proved to be a permanent cycle in the Atiyah-Hirzebruch spectral sequence for $n = 3, 4$ by comparing the spectral sequence to the corresponding spectral sequence for $H^*(B\text{Spin}(2n+1))$. The last section gives more examples of $p$-torsion elements in the kernel of the cycle map using spinor groups and the exceptional group $F_4$.

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2. Extraspecial 2-groups

The extraspecial 2-group $D(n) = 2^{1+2n}$ is the central product of $n$ copies of the dihedral group $D_8$ of order 8. So there is a central extension

$$0 \rightarrow N \rightarrow D(n) \xrightarrow{\pi} V \rightarrow 0$$
with \( N \cong \mathbb{Z}/2 \) and \( V \) elementary abelian of rank \( 2n \). Take a set of generators \( c, \tilde{a}_1, \ldots, \tilde{a}_{2n} \) of \( D(n) \) such that \( c \) is a generator of \( N \), the elements \( a_i = \pi(\tilde{a}_i) \) form a \( \mathbb{Z}/2 \)-basis of \( V \), and

\[
[\tilde{a}_j, \tilde{a}_{2i}] = \begin{cases} 
  c & \text{if } j = 2i - 1 \\
  0 & \text{else}
\end{cases}
\]

Using the Hochschild-Serre spectral sequence associated to extension (2.1), Quillen [Q] determined the mod 2 cohomology of \( D(n) \). Let \( e_i \) denote the real 1-dimensional representation of \( D(n) \) given as the projection onto \( \langle a_i \rangle \) followed by the nontrivial character \( \langle a_i \rangle \rightarrow \{ \pm 1 \} \subset \mathbb{R} \), and \( e : \hat{V}^{\text{odd}} \rightarrow N \rightarrow \{ \pm 1 \} \subset \mathbb{R} \) where \( \hat{V}^{\text{odd}} = \langle e, \tilde{a}_{2i-1} \mid 1 \leq i \leq n \rangle \) is a maximal elementary abelian 2-subgroup of \( D(n) \). Define classes \( x_i \in H^1(D(n); \mathbb{Z}/2) \), \( w_2 \in H^2(D(n); \mathbb{Z}/2) \) as the Euler classes of the \( e_i \) and of \( \Delta = \text{Ind}^{D(n)}_{\hat{V}^{\text{odd}}} (e) \), respectively. The extension (2.1) is represented by the class \( f = x_1 x_2 + \cdots + x_{2n-1} x_{2n} \), and one has

\[
H^*(BD(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_{2n}] \otimes \mathbb{Z}/2[x_1, \ldots, x_{2n}]/(f, Q_0 f, \ldots, Q_{n-2} f)
\]

where the \( Q_i \) are Milnor’s operations recursively defined by \( Q_0 = Sq^1 \) and \( Q_i = [Sq^2, Q_{i-1}] \). The extension class \( f \) defines a quadratic form \( q : V \rightarrow \mathbb{Z}/2 \) on \( V \). A subspace \( W \subset V \) is said to be \( q \)-isotropic if \( q(x) = 0 \) for all \( x \in W \). The maximal (elementary) abelian subgroups of \( D(n) \) are in one-to-one correspondence with the maximal isotropic subspaces of \( V \). Indeed, if \( W \) is maximal isotropic, then \( \tilde{W} := \pi^{-1}(W) \cong N \oplus W \) is maximal (elementary) abelian. Quillen also proved that the mod 2 cohomology of \( D(n) \) is detected on maximal elementary abelian subgroups, i.e. the restrictions define an injective map

\[
H^*(BD(n); \mathbb{Z}/2) \hookrightarrow \prod H^*(\tilde{W}; \mathbb{Z}/2)
\]

where the product ranges over conjugacy classes of maximal elementary abelian subgroups. Since the restriction of \( \Delta \) to any such \( \tilde{W} \) is the real regular representation (see [Q], Section 5), we have

\[
\text{Res}_{\tilde{W}}(w_{2n}) = \prod_{x \in H^1(W; \mathbb{Z}/2)} (z + x)
\]

where \( z \) denotes the generator of \( H^*(N; \mathbb{Z}/2) \) dual to \( c \). For simplicity, write \( w' = \text{Res}_{\tilde{W}}w_{2n} \), and choose generators of \( H^*(W; \mathbb{Z}/2) \cong \mathbb{Z}/2[x'_1, \ldots, x'_n] \). It is well-known that the right hand side of (2.4) can be written in terms of Dickson invariants,

\[
w' = z^{2^n} + D_1 z^{2^{n-1}} + \cdots + D_n z
\]
where $D_i$ has degree $2^n - 2^{n-i}$ and $H^*(W; \mathbb{Z}/2)^{\text{GL}_2(\mathbb{Z}/2)} \cong \mathbb{Z}/2[D_1, \ldots, D_n]$. Using that the product of all the $x_i'$s is clearly invariant and that the Milnor primitives are derivations, it is easy to see that the Dickson invariants may be written in terms of the $Q_i$ as follows:

\begin{align*}
D_n &= Q_0Q_1 \cdots Q_{n-2}(x'_1 \cdots x'_n) \\
D_i &= (Q_0 \cdots \hat{Q}_i \cdots Q_{n-1}(x'_1 \cdots x'_n))/D_n
\end{align*}

**Lemma 2.1.** The Milnor operations $Q_1, \ldots, Q_{n-1}$ act by

1. $Q_{n-1}D_i = D_nD_i$;
2. $Q_{n-j-1}D_j = D_n$;
3. $Q_iD_j = 0$ for $i < n-1$ and $i \neq n - j - 1$.

**Proof.** First note that from (2.6) and $Q_k^2 = 0$ we immediately get $Q_k(D_n) = 0$ for $k \neq n - 1$ and $Q_{n-1}D_n = Q_0 \cdots Q_{n-1}(x'_1 \cdots x'_n) = D_n^2$. Thus, for each $1 \leq i \leq n - 1$,

\begin{align*}
0 &= Q_{n-1}(Q_0 \cdots \hat{Q}_i \cdots Q_{n-1}(x'_1 \cdots x'_n) = Q_{n-1}(D_iD_n) \\
&= (Q_{n-1}D_i)D_n + D_iQ_{n-1}D_n = (Q_{n-1}D_i)D_n + D_iD_n^2
\end{align*}

whence (1). Similarly, (2) is implied by

\begin{align*}
D_n^2 &= Q_{n-1} \cdots Q_0(x'_1 \cdots x'_n) = Q_{n-1}(D_iD_n) \\
&= (Q_{n-i-1}D_i)D_n + D_iQ_{n-i-1}D_n = (Q_{n-i-1}D_i)D_n.
\end{align*}

Finally, for $k \neq n - i - 1$ we get $0 = Q_k(D_iD_n) = (Q_kD_i)D_n + D_iQ_kD_n = (Q_kD_i)D_n$. $\square$

**Corollary 2.2.** $Q_{n-1}w' = D_nw'$ and $Q_kw' = 0$ for $k < n - 1$.

**Proof.** For $j \neq n - 1$, we have $Q_jw' = \sum_{i=1}^{n-1} (Q_jD_i)z^{2^{n-i}} + Q_j(D_nz) = D_nz^{2^{j+1}} + D_nz^{2^{j+1}} = 0$. For $j = n - 1$, we get $Q_{n-1}w' = 0 + D_nD_1z^{2^{n-1}} + \cdots + D_nD_{n-1}z^2 + Q_{n-1}(D_nz)$. The last term equals $D_n^2z + D_nz^{2^n}$; the claim follows. $\square$

**Corollary 2.3.** $Q_kw2^n = 0$ for $0 \leq k \leq n - 2$, but $Q_{n-1}w2^n \neq 0$.

For future reference we note:

**Lemma 2.4.** $Q_n(D_nw') = D_n^2w'^2$, $Q_n(D_nD_n) = D_n^3$, $Q_n(D_{n-1}w') = D_n^4w' + D_nD_{n-1}w'^2$, $Q_{n+1}Q_n(D_{n-1}w') = D_n^4w'^4$.

**Proof.** From (2.5) we see that $Q_n(D_nw') = (Q_nD_n)z^{2^n} + \sum_{i=1}^{n-1} Q_n(D_nD_i)z^{2^{n-i}} + D_n^2z^{2^{n+1}}$. The coefficients of $z^{2^{n+1}}$ and $z$ tell us that this is equal to $D_n^2w'^2$. Comparing coefficients
further shows \( Q_n(D_n) = D_n^2D_n \) and \( Q_n(D_nD_i) = D_n^2D_{i+1} \), in particular \( Q_n(D_nD_{n-1}) = D_n^4 \). Thus we have

\[
Q_n(D_nD_{n-1}D_nw') = Q_n(D_nD_{n-1}D_nw') + D_nD_{n-1}Q_n(D_nw') = D_n^5w' + D_n^3D_{n-1}w'^2.
\]

Hence we get \( Q_n(D_{n-1}w') = D_n^3w' + D_nD_{n-1}w'^2 \). Next consider

\[
Q_{n+1}(D_nw') = (Q_{n+1}D_n)z^{2n} + \sum_{i=1}^{n-1} Q_{n+1}(D_nD_i)z^{2n-i} + D_n^2z^{2n+2}.
\]

From the coefficients of \( z^{2n+2} \) and \( z^{2n+1} \), we see that \( Q_{n+1}(D_nw') = D_n^2w'^4 + D_n^2D_1^4w'^2 \) and hence \( Q_{n+1}(D_nD_{n-1}) = D_n^4D_1^1 \). Therefore

\[
Q_{n+1}(D_nD_{n-1}D_nw') = D_n^4D_1^1D_nw' + D_nD_{n-1}(D_n^2w'^4 + D_n^2D_1^4w'^2).
\]

Thus \( Q_{n+1}(D_{n-1}w') = D_n^4w'D_1^1 + D_nD_{n-1}w'^4 + D_nD_{n-1}D_1^4w'^2 \). Hence

\[
Q_{n+1}Q_n(D_{n-1}w') = Q_nQ_{n+1}(D_{n-1}w') = D_n^4w'^2D_1^1 + D_n^2w'^4 + D_n^2D_1^4w'^2 = D_4w'^4.
\]

\]

\[ \square \]

3. The integral cohomology

The integral cohomology of \( D(n) \) is studied by Harada-Kono ([H-K], also see [B-C]) by means of the Bockstein spectral sequence

\[
E_1 = H^*(BG; \mathbb{Z}/2) \Rightarrow \mathbb{Z}/2 \otimes H^*(BG)/(2\text{-torsion}).
\]

The \( E_2 \)-page of this spectral sequence is the \( Q_0 \)-homology of \( H^*(BG; \mathbb{Z}/2) \), and \( E_\infty \approx \mathbb{Z}/2 \) for a finite group \( G \). For \( 0 \leq i \leq n-2 \), let \( R(i) = H^*(BV; \mathbb{Z}/2)/(f, Q_if, \ldots, Q_if) \). Using the long exact sequence associated to the short exact sequence

\[
0 \to R(i-1) \xrightarrow{Q_if} R(i-1) \to R(i) \to 0,
\]

Harada-Kono computed the \( E_2 \)-page for \( D(n) \) as follows:

\[
H(H^*(BD(n); \mathbb{Z}/2); Q_0) \cong \Lambda(a, b_1, \ldots, b_{n-1}) \otimes \mathbb{Z}/2[w_2,]
\]

where \( |a| = 3 \) and \( |b_i| = 2^i \). Since \( E_\infty \cong \mathbb{Z}/2 \), the first non-trivial differential must be \( da = b_1 \), and there have to be subsequent differentials \( d(ab_i) = b_{i+1} \). Thus there appear exactly \( n \) non-zero differentials in this spectral sequence. On the other hand, using corestriction arguments it is easy to see that the exponent of \( H^*(BD(n)) \) is at most \( n+1 \). Based on these facts, Harada-Kono proved the following.
Theorem 3.1. [H-K] Let $C(n)^* = H^*(BD(n))/J_V$, where $J_V$ is the ideal generated by the image of $H^*(BV)$ in $H^*(BD(n))$. Then $C(n)^* \subset H^*(BD(n))$, and there is an additive isomorphism

$$C(n)^k = \begin{cases} 
\mathbb{Z}/2^{v_2(k)} & \text{if } v_2(k) \leq n - 1, \\
\mathbb{Z}/2^{n+1} & \text{if } v_2(k) = n
\end{cases}$$

where $v_2(k)$ denotes the 2-adic valuation of $k$. \qed

Let $c_k(n)$ denote a $\mathbb{Z}_2$-module generator of $C(n)^{2^k}$. Then $c_k(n)$ reduces to $w_{2^n}$ modulo $H^*(BV; \mathbb{Z}/2)$. Consider the restriction map $i : C(n)^* \to C(n-1)^*$. Now $c_{n-1}(n-1) = w_{2^n-1}$ mod $H^*(BV; \mathbb{Z}/2)$ implies $i^*c_n(n) = c_{n-1}(n-1)^2$. Since the order of $c_{n-1}(n)$ is $2^{n-1}$ and the order of $c_{n-1}(n-1)$ is $2^n$, we know that $i^*c_{n-1}(n) = 2^sc_{n-1}(n-1)$ for some $s > 0$. A corestriction argument now implies $s = 1$, since the index of $D(n-1)$ in $D(n)$ is 2.

The elements $a$ and $b_j$ are natural in the sense that $i^*(a) = a$ and $i^*(b_j) = b_j$ for $1 \leq j \leq n-2$, abusing notation. Thus $i^*c_j(n) = c_j(n-1)$ for $j < n-1$, and we obtain

Corollary 3.2. If $n \geq 2$, there is an additive isomorphism

$$C(n)^* \cong \mathbb{Z}\{1, 2{\bar{w}}_2^{i_2}, \ldots, 2{\bar{w}}_2^{i_n-1} | \epsilon_i = 0 \text{ or } 1 \} / (2^{i+1}{\bar{w}}_2, = 0 \text{ if } 2 \leq i \leq n)$$

where the $\bar{w}_2$, are the reductions of the elements $w_2$, in $H^2(BD(i))$. \qed

Remark. When $n = 1$, the element $w_2 \in H^*(BD_1; \mathbb{Z}/2)$ does not lift to the integral cohomology and $C(1)^* \cong \mathbb{Z}\{w_2^2\}/(4w_2^2)$.

4. Brown-Peterson Cohomology of $BD(n)$

Let $BP^*(-; \mathbb{Z}/2)$ denote $BP$-theory mod 2 with coefficients $BP^*/(2) = \mathbb{Z}/2[v_1, v_2, \ldots]$. We consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*} = H^*(BD(n); \mathbb{Z}/2) \otimes BP^* \implies BP^*(BD(n); \mathbb{Z}/2). \quad (4.1)$$

Lemma 4.1. The elements $x_1^2$ and $w_{2^n}^2$ are permanent cycles in the spectral sequence (4.1).

Proof. These elements are the top Chern classes of the representations $c_{iC}$ and $\Delta_C$, respectively. \qed

It is well-known that some of the differentials of (4.1) are given by

$$d_{2^{i+1}-1}(x) = v_1 \otimes Qix \mod (v_1, \ldots, v_{i-1}). \quad (4.2)$$
Since $Q_{n-1}w_{2n} \neq 0$ by Corollary 2.3, we know that $w_{2n}$ cannot be a permanent cycle, which implies $w_{2n} \notin \text{Im}[\rho_{\mathbb{Z}/2} : BP^*(BD(n)) \to H^*(BD(n); \mathbb{Z}/2)]$. Thus the integral lift $\bar{w}_{2n}$ of $w_{2n}$ does not lie in the image of $\rho : BP^*(BD(n)) \to H^*(BD(n))$, either.

Let as above $\tilde{W}$ denote a maximal elementary abelian subgroup of $D(n)$, and $w(\Delta)$ the total Stiefel-Whitney class of $\Delta$. Then

$$\text{Res}^D_{\tilde{W}}(w(\Delta)) = \prod (1 + x + z) = (1 + z)^{2^n} + D_1(1 + z)^{2^{n-1}} + \cdots + D_n(1 + z)$$

$$= 1 + D_1 + \cdots + D_n + \text{Res}^D_{\tilde{W}}(w_{2n})$$

in particular,

$$\text{Res}^D_{\tilde{W}}(w_{2n-2n-1}(\Delta)) = D_i.$$  \hspace{2cm} (4.3)

Hence, by (2.2), we can choose polynomials $\tilde{D}_i \in \mathbb{Z}/2[x_1, \ldots, x_{2n}] \cong H^*(BV; \mathbb{Z}/2)$ with $w_{2n-2n-1} = \tilde{D}_i$. Recall that $J_V$ stands for the image of $H^*(BV)$ in $H^*(BD(n))$.

**Theorem 4.2.** There is an element in $BP^*(BD(n))$,

$$x = c_{2n-1}(\Delta_C) - \tilde{D}_1(c_1(e_{1C}), \ldots, c_1(e_{2nC}))$$

which is non-zero in $BP^*(BD(n)) \otimes_{BP^*} \mathbb{Z}/2$, and such that

1. $\rho(x) = 2\bar{w}_{2n} \mod J_V$,
2. $\rho_{\mathbb{Z}/2}(x) = 0$ in $H^*(BD(n); \mathbb{Z}/2)$.

**Proof.** Since $x$ is defined via Chern classes, it is an element of $BP^*(BD(n))$. Assertion (2) is immediate from (4.3). Since $\bar{w}_{2n} \notin \text{Im}(\rho)$, it suffices to prove (1) to show that $x$ is a $BP^*$-module generator. Let $F = \langle a_1, a_2 \rangle \subset D(n)$; this is cyclic of order 4. By the double coset formula,

$$\text{Res}^D_{\tilde{W}}(\text{Ind}^D_{V^{\text{odd}}}(e_C)) = \bigoplus_{FgV^{\text{odd}}} \text{Ind}^F_{FgV^{\text{odd}}} \text{Res}_{FgV^{\text{odd}}}(g^*e_C)$$

$$= \bigoplus_{FgV^{\text{odd}}} \text{Res}_{FgV^{\text{odd}}}(g^*e_C)$$

since the elements $g = a_2^\epsilon \cdots a_{2n}^\epsilon$, $\epsilon = 0$ or 1, form a complete set of double coset representatives. Notice that $\text{Ind}^V_{\tilde{W}}(e_C)$ decomposes as $e_F \oplus -e_F$ where $e_F$ is a faithful 1-dimensional complex representation of $\mathbb{Z}/4$. Thus the total Chern class of $\Delta_C$ restricts to $F$ as

$$\text{Res}_F(e(\Delta_C)) = ((1 + u)(1 - u))^{2^{n-1}} = (1 - u^2)^{2^{n-1}}$$

with $H^*(BF) \cong \mathbb{Z}[u]/(4u)$. 


Consequently, we have \( \text{Res}_F(c_{2n-1}(\Delta_c)) = 2u^{2n-1} \) in \( H^*(F) \). Since \( \text{Res}_F(c_1(c_c)) = 2\lambda_1u \) for some \( \lambda_1 \in \mathbb{Z}/4 \), we immediately obtain \( \text{Res}_F(D_i) = 0 \) and therefore (1). \( \Box \)

Now recall the following lemma of Totaro.

**Lemma 4.3.** ([T1]) Let \( p \) be a prime and \( X \) any space. If \( \rho_{\mathbb{Z}/p} : BP^*(X) \otimes_{BP^*} \mathbb{Z}(p) \to H^*(X; \mathbb{Z}/p) \) is not injective, then \( \rho : BP^{*+2}(X \times B\mathbb{Z}/p) \otimes_{BP^*} \mathbb{Z}(p) \to H^{*+2}(X \times B\mathbb{Z}/p) \) is also not injective.

**Proof.** We have \( BP^*(B\mathbb{Z}/p) \otimes_{BP^*} \mathbb{Z}(p) \cong H^*(B\mathbb{Z}/p)(p) \cong \mathbb{Z}(p)[u]/(pu) \) with \( u \) in degree two. If \( \rho_{\mathbb{Z}/p}(x) = 0 \), then \( \rho(x \otimes u) = 0 \) in \( H^*(X \times B\mathbb{Z}/p)(p) \). On the other hand it is well-known that \( BP^*(B\mathbb{Z}/p) \) is \( BP^* \)-flat and thus \( BP^{*+2}(X \times B\mathbb{Z}/p) \cong BP^*(X) \otimes_{BP^*} BP^*(B\mathbb{Z}/p) \). Hence if \( 0 \neq x \in BP^*(X)_{BP^*} \mathbb{Z}(p) \), then \( x \otimes u \) is also non-zero in \( BP^{*+2}(X \times B\mathbb{Z}/p) \otimes_{BP^*} \mathbb{Z}(p) \). \( \Box \)

Let \( \rho' : CH^*(-) \to H^*(-) \) denote the cycle map respectively \( \rho'_{\mathbb{Z}/2} \) the cycle map followed by reduction modulo 2. Since Chow rings have Chern classes, we easily deduce

**Corollary 4.4.** There is a non-zero element \( x' \) in \( CH^{2n}(BD(n)) \) satisfying

(1) \( \rho'(x') = 2\bar{w}_{2n} \mod J_V \);

(2) \( \rho'_{\mathbb{Z}/2}(x') = 0 \).

Hence \( \rho' : CH^{2n+2}(B(D(n) \times \mathbb{Z}/2)) \to H^{2n+2}(B(D(n) \times \mathbb{Z}/2)) \) is not injective. \( \Box \)

**Remark.** First note that the above argument does not hold for \( n = 1 \). Indeed, in that case \( H^*(BD_8) \subseteq \text{Im}(\rho) \) modulo \( H^*(BV) \). Similar facts hold for 2-groups \( G \) which have a cyclic maximal normal subgroup \([S] \), i.e. dihedral, semidihedral, quasidihedral, and generalized quaternion groups of order a power of 2. Moreover \( BP^*(BG) \) is generated by Chern classes for these groups. The extraspecial 2-groups of order \( 2^{2n+1} \) are of two types. Quillen calls them the real and the quaternionic type, where the real type corresponds to the groups \( D(n) \) considered above, and the quaternionic group of order \( 2^{n+1} \) is the central product of \( D(n-1) \) with the quaternion group \( Q_8 \) of order 8. Consider now this second case, and denote this group by \( D'(n) \); it also has center \( \mathbb{Z}/2 \) with quotient \( V \cong (\mathbb{Z}/2)^{2n} \). In Quillen’s notation \([Q] \), this corresponds to \( h = n + 1 \) and \( r = 2 \). The quadratic form (extension class) is \( f = x_1^2 + x_1x_2 + x_2^2 + \sum_{i=2}^{n} x_{2i-1}x_{2i} \), and the cohomology is given by

\[
H^*(BD'(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_{2n+1}] \otimes \mathbb{Z}/2[x_1, \ldots, x_{2n}]/(f, Q_0f, \ldots, Q_{n-1}f).
\]

Here the \( x_i \) are as before the generators of \( H^*(BV; \mathbb{Z}/2) \) inflated to \( D'(n) \), and \( w_{2n+1} \) is the Euler class of the \( 2^{n+1} \)-dimensional irreducible representation \( \Delta \). The cohomology of \( D'(n) \)
is also detected on subgroups $\tilde{W} \cong Q_8 \times W$ in one-to-one correspondence with maximal isotropic subspaces, i.e. there is an injection

$$H^*(BD'(n); \mathbb{Z}/2) \hookrightarrow \prod_W H^*(B(Q_8 \times W); \mathbb{Z}/2)$$

where $W$ ranges over the maximal isotropic subspaces of $V$ (which have dimension $n - 1$). The Stiefel-Whitney classes $w_j(\Delta)$ are zero except for the following values of $j$ ([Q], (5.6)):

$$\text{Res}_{Q_8 \times W}(w_j(\Delta)) = \begin{cases} 
(D')^4_i & \text{for } j = 2^n - 2^{h-i}, \ 1 \leq i \leq n - 1 \\
\sum_{i=0}^{n-2} e^{2i}(D'_{n-i-1})^4 & \text{for } j = 2^{n+1}
\end{cases}$$

where $e \in H^4(Q_8; \mathbb{Z}/2)$ is the Euler class of the obvious 4-dimensional irreducible representation of $Q_8$ and $D'$ is the degree $(2^{n-1} - 2^{n-1-i})$ Dickson invariant for rank $n - 1$. Thus almost all arguments for $D(n)$ work in this case, too, except for $Q_m w_j(\Delta) = 0$. For example, we can define $x = c_{2n}(e \otimes (\bar{D}'))^4$ in $BP^*(BD'(n))$; this class satisfies $\rho(x) = 2\bar{w}_{2n+1}$ and $\rho_{\mathbb{Z}/2}(x) = 0$. However, it seems that we can not prove that $x$ is a $BP^*$-module generator of $BP^*(BD'(n))$ because $\text{Res}_N(c_{2n}(\text{Ind}_{Z/4 \mathbb{Z} \otimes W}(e \otimes F))) = u^{2^n}$ and $w_{2n+1}(\Delta) \in \text{Im}(\rho)$ mod $(H^*(BV))$.

5. Permanent cycles

This section deals with the Atiyah-Hirzebruch spectral sequence converging to $BP^*(BD(n))$.

In the course of the section, we shall make several technical assumptions on the behaviour of this spectral sequence. These will be verified for $n = 3, 4$ in section 7.

Given a space $X$, each non-zero element $x \in BP^*(X)$ with $\rho(x) = 0 \in H^*(X)_p$ is represented by a non-zero element in $E^2_{\infty,a}$ with $a < 0$ in the Atiyah-Hirzebruch spectral sequence converging to $BP^*(X)$.

**Assumption 5.1.** Let $n \geq 3$. In the Atiyah-Hirzebruch spectral sequence converging to $BP^*(BD(n))$, every nonzero element in the ideal $(2, v_1, \ldots, v_{n-2}) \otimes \bar{w}_{2n}$ is a nonzero permanent cycle.

The outer automorphism group of $D(n)$ is the orthogonal group $O(V)$ of $V$ associated to the quadratic form $q$ ([B-C], p. 216). Since $\Delta$ is the unique irreducible representation which acts non-trivially on the center, the element $w_{2n}$ is invariant under the orthogonal group ([Q], Remark 4.7). Moreover the invariant ring generated by the Stiefel-Whitney classes of
\( \Delta ([Q], \text{Corollary 5.12 and Remark 5.14}), \)

\[ H^*(BD(n); \mathbb{Z}/2)^{(V)} = \mathbb{Z}/2[\bar{D}_1, \ldots, \bar{D}_n, w_{2^n}] \text{ with } \bar{D}_i = w_{2^{n-i} - 2}(\Delta). \]

To consider the Atiyah-Hirzebruch spectral sequence

\[ (5.1) \quad E_2^{*,*}(X) = H^*(X) \otimes BP^* \Rightarrow BP^*(X) \]

for the spaces \( X = BD(n) \) or \( B\text{Spin}(m) \), we need the integral version of the above invariant ring. Note \( \beta(\bar{D}_{n-1}) = \bar{D}_n \); let

\[ W(\Delta) = \mathbb{Z}(2)[\bar{D}_1, \ldots, \bar{D}_{n-2}, \bar{D}_{n-1}^2, \bar{D}_n, w_{2^n}] / (2\bar{D}_n). \]

Suppose \( X \) is a space such that

\[ (5.2) \quad \text{there is a map } f : W(\Delta) \to H^*(X)_{(2)} \text{ such that } W(\Delta) / (2) \subset H^*(X; \mathbb{Z}/2) \]

as \( \Lambda(Q_0, \ldots, Q_n) \)-algebras.

In section 7 we shall see that we may take \( X = B\text{Spin}(7) \) and \( X = B\text{Spin}(9) \) for the cases \( n = 3 \) and \( n = 4 \), respectively. In the spectral sequence for \( BD(n) \), let \( W(\Delta)_r \) be the subalgebra of \( E_\infty^{*,*} \) which is the subquotient algebra of \( BP^* \otimes f(W(\Delta)) \). In general, the invariants \( (E_\infty^{*,*})^{O(V)} \) are not equal to \( W(\Delta)_r \). Below we consider the case where \( W(\Delta)_r \) is nevertheless closed under the differentials.

**Lemma 5.2.** Let \( n \geq 3 \). Suppose that \( X \) satisfies \( (5.2) \) and \( d_r(W(\Delta)_r) \subset W(\Delta)_r \) for all \( r \geq 2 \) in the spectral sequence \( (5.1) \). Then each element in the ideal \( (2, v_1, \ldots, v_{n-2}) \otimes w_{2^n} \) and the ideal \( (2, v_1, \ldots, v_{n-2-i}) \otimes \bar{D}_i, 1 \leq i \leq n - 2 \) is a permanent cycle. Moreover, if \( w_{2^n} \) (resp. \( \bar{D}_i \)) is not in the image of \( Q_k \), then \( [v_k \otimes w_{2^n}] \) (resp. \( [v_k \otimes \bar{D}_i] \), \( i \leq n - k - 2 \)) is a non-zero element in \( E_\infty^{*,*} \).

Before beginning with the proof, recall the cohomology theory \( P(m)^*(-) \) with coefficients \( BP^*/(2, v_1, \ldots, v_{m-1}) \cong \mathbb{Z}/2[v_m, \ldots] \). In particular, the theory \( P(1)^*(-) \) is mod 2 \( BP \)-theory \( BP^*(-; \mathbb{Z}/2) \).

**Proof.** First note that \( |\bar{D}_i| = 2^n - 2^{n-1}, \) which is even except for the case \( i = n \). Hence

\[ W(\Delta)_2^{odd} = P(1)^*[w_{2^n}, \bar{D}_1, \ldots, \bar{D}_{n-2}, \bar{D}_{n-1}^2, \bar{D}_n] / \{ \bar{D}_n \}. \]

By induction, we assume that for \( 2^r \leq i \leq 2^{r+1} - 1 \)

\[ W(\Delta)_i^{odd} = P(r)^* \otimes A \otimes B_r \otimes C_r \{ \bar{D}_n \} \quad \text{with} \]

\[ A = \mathbb{Z}/2[w_{2^n}], \quad B_r = \mathbb{Z}/2[\bar{D}_1, \ldots, \bar{D}_{n-r-1}], \quad C_r = \mathbb{Z}/2[\bar{D}_{n-r}, \ldots, \bar{D}_n]. \]
Let \( E(P(m))^* \) be the Atiyah-Hirzebruch spectral sequence converging to \( P(m)^*(X) \) and \( \rho_i : E_i^{*,*} \to E(P(m))_i^{*,*} \) be the map of spectral sequences induced from the natural transformation \( \rho : BP^*(X) \to P(m)^*(X) \) of cohomology theories. Since \( |v_r| = -2^{r+1} + 2 \), we see that \( E(P(r))^{*}_{2r+1} \cong E(P(r))^{*}_{2} \) for degree reasons. Now \( W(\Delta)_i^{odd} \) is \( P(r)^* \)-free, so the restriction map \( \rho_i|W(\Delta)_i \) is injective for \( i < 2^{r+1} \). Hence there is no element \( x \) with \( 0 \neq d_r(x) \in W(\Delta)_i^{odd} \), and we have \( W(\Delta)_i^{odd} \cong W(\Delta)_i^{odd} \). Except for \( \tilde{D}_{n-r-1} \), generators in \( W(\Delta)_i \) are annihilated by \( Q_r \) and thus by \( d_{2r+1} \). The non-zero differential is

\[
d_{2r+1-1}(\tilde{D}_{n-r-1}) = v_r \otimes Q_r(\tilde{D}_{n-r-1}) = v_r \otimes \tilde{D}_n.
\]

Therefore (5.3) also holds for \( i = 2^{r+1} \). Since \( W(\Delta)_i \) is a \( P(r)^* = BP^*/(2, \ldots, v_{r-1}) \)-module, every element in the ideal \( (2, \ldots, v_{r-1}) \otimes \tilde{D}_{n-r-1} \) is a cycle in \( E_{2r+1}^{*,*} \). Consequently we get

\[
W(\Delta)^{odd}_{2r+1} = P(n-1)^* \otimes A \otimes C_n-1 \{ \tilde{D}_n \}.
\]

The next differential is

\[
d_{2r+1}(w_{2r}) = v_{n-1} \otimes w_{2r} \tilde{D}_n \quad \text{and} \quad d_{2r-1}(\tilde{D}_n) = v_{n-1} \otimes \tilde{D}_n^2.
\]

Hence we have

\[
W(\Delta)^{odd}_{2r+1} = P(n)^* \otimes C_n \{ \tilde{D}_n w_{2r} \} \quad \text{where} \quad C_n = \mathbb{Z}/2[w_{2r}] \otimes C_{n-1}.
\]

Here we note that each element in the ideal \( (2, v_1, \ldots, v_{n-2}) \otimes w_{2r} \) is a cycle in \( E_{2r+1}^{*,*} \) because \( w_{2r} \tilde{D}_n \) generates a \( P(n-1)^* \)-module in \( E_{2r+1}^{*,*} \).

Lastly we consider the differential

\[
d_{2r+1-1}(w_{2r} \tilde{D}_n) = v_n \otimes Q_{n+1}(w_{2r} \tilde{D}_n) = v_n \otimes (w_{2r}^2 \tilde{D}_n^2).
\]

Thus \( W(\Delta)^{odd}_{2r+1} = 0 \), and each element in the above ideals is a permanent cycle. If \( w_{r} \notin \text{Im}(Q_r) \), then considering the map \( \rho_{2r-1} \) we see that \( [v_r \otimes w_{2r}] \) is non-zero. \( \Box \)

Next we consider the mod 2-version of the above arguments. We study the Atiyah-Hirzebruch spectral sequence

\[
E_2 = H^*(X; \mathbb{Z}/2) \otimes P(1)^* \implies P(1)^*(X).
\]

Denote the invariant ring by

\[
W(\Delta; \mathbb{Z}/2) = H^*(BD(n); \mathbb{Z}/2)^{O(V)} \cong W(\Delta)/(2) \otimes \Lambda(D_{n-1}).
\]
Thus we consider the following situation:

\[(5.5) \quad \text{there is an injection } W(\Delta; \mathbb{Z}/2) \subset H^*(X; \mathbb{Z}/2) \text{ as } \Lambda(Q_0, \ldots, Q_{n+1})\text{-algebras.}\]

In the spectral sequence \((5.4)\), let \(W(\Delta; \mathbb{Z}/2)_r\) be the subalgebra of \(E_{r,*}^*\) which is the subquotient algebra of \(P(1)^* \otimes (W(\Delta); \mathbb{Z}/2)\). Of course the mod 2 reductions of the permanent cycles in \((5.1)\) are also permanent cycles in \((5.4)\). Moreover we have

**Lemma 5.3.** Let \(n \geq 3\). Suppose that \(X\) satisfies \((5.5)\) and \(d_r(W(\Delta; \mathbb{Z}/2)_r) \subset W(\Delta; \mathbb{Z}/2)_r\) for all \(r \geq 2\) in the spectral sequence \((5.4)\). Then every element in the ideal \((v_1, \ldots, v_{n-1}) \otimes w_{2^n} D_{n-1}\) and the ideal \((v_1, \ldots, v_n) \otimes D_{n-1}\) is a permanent cycle.

**Proof.** The proof is similar to the \(BP^*\)-case. In particular, for \(i \leq 2^n - 1\), we have \(W(\Delta; \mathbb{Z}/2)_i^{odd} = W(\Delta)_i^{odd}/(2) \otimes \Lambda(D_{n-1})\). The difference starts with

\[d_{2^n-1}(x) = v_{n-1} \otimes D_n x \quad \text{for } x = w_{2^n}, D_{n-1}, D_n.\]

Hence we get

\[W(\Delta; \mathbb{Z}/2)_{2^n+1-1}^{odd} = P(n)^* \otimes C_n \{D_{n} w_{2^n}, D_n D_{n-1}\}.\]

Here note that \(D_{n-1} w_{2^n}\) is a cycle in \(E_{2^n+1-1}^{*,*}\). We also know that each element in the ideal \((v_1, \ldots, v_{n-1}) \otimes D_{n-1}\) is a cycle in \(E_{2^n+1-1}^{*,*}\).

From Lemma 2.4 and (2.3), the image of \(w_{2^n} D_n\) (resp. \(D_{n-1} D_n\)) under the differential \(d_{2^n+1}\) is \(v_n \otimes (w_{2^n} D_n^2)\) (resp. \(D_n^2\)). Hence we see that

\[
\text{Ker}(W(\Delta; \mathbb{Z}/2)_{2^n+1-1}^{odd}) = P(n)^* \otimes C_n \{a\} \quad \text{where } a = D_{n-1} D_n w_{2^n} + D_n^1 w_{2^n}.\]

Since \(d_{2^n+1}(w_{2^n} D_{n-1}) = v_n \otimes a\), we get \(W(\Delta; \mathbb{Z}/2)_{2^n+2-1}^{odd} = P(n+1)^* \otimes C_n \{a\}\). Here note that \(v_{n-1} \otimes w_{2^n} D_n\) is a cycle in \(E_{2^n+2-1}^{odd}\). The last nonzero differential is, again by Lemma 2.4,

\[d_{2^n+2-1}(a) = v_{n+1} \otimes D_n^2 w_{2^n}.\]

Thus \(W(\Delta; \mathbb{Z}/2)_{2^n+2-2}^{odd} = 0\). Hence we get the permenancy of elements in the lemma.

If \(BP^*(X)\) is 2-torsion free, e.g., \(BP^*(X)\) and \(P(1)^*(X)\) are generated by even dimensional elements, then the Bockstein exact sequence induces an isomorphism \(BP^*(X)/(2) \cong P(1)^*(X)\). In particular, \(BP^*(X) \otimes BP^* \mathbb{Z}/2 \cong P(1)^*(X) \otimes P(1)^*, \mathbb{Z}/2\). These facts hold for \(X = B\text{Spin}(m)\) for \(m = 7, 9\) (see section 7 below). The following assertion seems reasonable for dimensional reasons.
Assumption 5.4. Suppose that (5.2) holds and $BP^*(X)/(2) \cong P(1)^*(X)$. The element $[2 \otimes \tilde{D}_{n-1}]$ (resp. $[2w_n^*]$) in the spectral sequence (5.1) corresponds to the element $[v_i \otimes \tilde{D}_{n-1}]$ (resp. $[v_{n-1} \otimes w_2 \cdot D_{n-1}]$) in the spectral sequence (5.4), e.g. the element $x$ with $\rho_{2/3}(x) = 0$ in Theorem 4.2 is represented by $[v_{n-1} \otimes w_2 \cdot D_{n-1}]$.

Remark. Let $M_p$ be the Moore space such that $H^2(M_p) \cong \mathbb{Z}/p$. Then there is an isomorphism $P(1)^*(X) \cong BP^{*+2} (X \wedge M_p)$ if $BP^*(X)$ is $p$-torsion free. Hence we can deduce the behaviour of the Atiyah-Hirzebruch spectral sequence converging to $BP^*(D(n) \times B\mathbb{Z}/2)$ from that converging to $P(1)^*(BD(n))$.

6. Transfers of Chern classes

To study Chern classes, we consider the restriction to the center $N \cong \mathbb{Z}/2$ of $D(n)$. Let $I$ denote the ideal $(2, v_1, v_2, \ldots)$ in $BP^*$. Then

$$\rho_{2/3} : BP^*(BN)/I \cong \mathbb{Z}/2[z^2] \subset H^*(BN; \mathbb{Z}/2).$$

Since the image of the restriction $H^*(BD(n); \mathbb{Z}/2) \rightarrow H^*(BN; \mathbb{Z}/2)$ is generated by $w_2 \in \text{Im}(\rho_{2/3})$, we see that

$$(6.1) \quad \text{Im}[BP^*(BD(n)) \rightarrow BP^*(BN)/I] = \mathbb{Z}/2[u^2],$$

where $u$ denotes the obvious generator in degree 2. Let $\xi$ be a complex representation of $D(n)$; it restricts to $N$ as the sum of $m$ copies (say) of the nontrivial character $e_C$ plus some trivial representations. Then there is an element $u' \equiv u \mod I$ in $BP^*(BN)$ with

$$(6.2) \quad \text{Res}_N(c(\xi)) = (1 + u')^m$$

where $c(\xi)$ denotes as usual the total Chern class of $\xi$. Then $u^m \in \text{Im}[BP^*(BD(n)) \rightarrow BP^*(BN)/I]$, whence $m$ has to be divisible by $2^n$.

Proposition 6.1. Suppose Assumption 5.1 holds and $n \geq 3$. Then the permanent cycles $[v_1 w_2^*], \ldots, [v_{n-1} w_2^*]$ are not represented by $BP^*$-linear combinations of products of Chern classes.

Proof. Let $\xi$ be a representation satisfying (6.2) for some $m = 2^n m'$. The restriction of the total Chern class of $\xi$ is given by

$$\text{Res}_N(c(\xi)) \equiv 1 + 2m'(u')^{2n-1} \mod (I^2, u^2)$$

$$= 1 + m'(v_1 u^{2n-1} + \cdots + v_i u^{2n-1+2^i-1} + \cdots) \mod (I^2, u^2)$$
which does not contain the term \(v_i u^{2n-1}\). But \(\text{Res}_N([v_i w_{2n}]) = v_i u^{2n-1} \mod (u^{2n-1} + 1)\). Hence no \(BP^*\)-linear combination of products of Chern classes can represent \([v_i w_{2n}]\).

**Theorem 6.2.** Suppose Assumption 5.1 holds and \(n \geq 3\). Then \([v_1 w_{2n}], \ldots, [v_{n-2} w_{2n}]\) are not represented by transfers of \(BP^*\)-linear combinations of product of Chern classes.

**Proof.** Let \(H\) be a subgroup of \(D(n)\), and suppose \([v_j w_{2n}] = \text{Tr}^D_{H}(n)(x)\) for some \(x \in BP^*(BH)\). By the double coset formula,

\[
\text{Res}_N^D(n) \text{Tr}_H^D(n)(x) = \sum_{HgN} \text{Tr}_g^N \text{Res}_{g^{-1}Hg \cap N}^g(g^*x)
\]

where the sum ranges over double coset representatives \(g\) of \(H \backslash D(n) \cap N\). If \(H\) intersects \(N\) trivially, then so does any conjugate of \(H\). Hence we need only consider subgroups \(H\) containing the center, and the double coset formula evaluates to \(|D(n)/H| \cdot \text{Res}_N(x)\). Since this element is represented by

\[
\text{Res}_N[v_i w_{2n}] = v_i u^{2n-1} \not\equiv 0 \mod I^2,
\]

we get \(|D(n)/H| = 2\) and thus \(H \cong D(n-1) \times \mathbb{Z}/2\) or \(H \cong D(n-1) \times_N \mathbb{Z}/4\).

The total Chern class \(c(\zeta)\) of any representation \(\zeta\) of \(D(n-1)\) restricts as

\[
\text{Res}_N(c(\zeta)) = (1 + u')^{2n-1}m = 1 + mu^{2n-1} \mod (I, u^n).
\]

Hence we have

\[
\text{Res}_N(2c(\zeta)) = 2 + 2mu^{2n-1}
\]

\[
= (v_1 u^2 + \cdots + v_i u^{2i} + \cdots) + m(v_1 u^{2n-1} + \cdots + v_i u^{2n-1+2i-1} + \cdots) \mod (I^2, u^n),
\]

which does not contain \(v_i u^{2n-1}\). Therefore \([v_j w_{2n}]\) is not represented by any \(BP^*\)-linear combination of product of Chern classes.

**7. \(BP^*(B\text{Spin}(7))\) and \(BP^*(B\text{Spin}(9))\)**

The mod 2 cohomology of \(B\text{Spin}(n)\) was computed by Quillen [Q]:

\[
H^*(B\text{Spin}(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_{2h}(\Delta)] \otimes \mathbb{Z}/2[w_2, \ldots, w_n]/(w_2, Q_0 w_2, \ldots, Q_{h-1} w_2)
\]

where \(\Delta\) is a spin representation of \(\text{Spin}(n)\) and \(2^h\) the Radon-Hurwitz number (see [Q] §6). This is proved by calculating the Serre spectral sequence of the fibration

\[
B\mathbb{Z}/2 \longrightarrow B\text{Spin}(n) \longrightarrow B\text{SO}(n).
\]
We consider the case $n = 7$. Then $h = 3$ and the mod 2 cohomology of $B\text{Spin}(n)$ is a polynomial algebra on the Stiefel-Whitney classes $w_4, w_6, w_7, w_8$ of a spin representation, i.e.

\begin{equation}
H^*(B\text{Spin}(7); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_4, w_6, w_7, w_8].
\end{equation}

Recall that $\text{Spin}(7)$ has the exceptional Lie group $G_2$ as a subgroup. $G_2$ contains a rank three elementary abelian 2-subgroup, and its mod 2 cohomology is isomorphic to the rank three Dickson invariants, i.e. $H^*(BG_2; \mathbb{Z}/2) \cong \mathbb{Z}/2[D_1, D_2, D_3]$. Here we may identify the Dickson invariants with the Stiefel-Whitney classes of the restriction of the spin representation to $G_2$, namely $D_1 = w_4, D_2 = w_6$, and $D_3 = w_7$. In particular, we have $H^*(B\text{Spin}(7); \mathbb{Z}/2) \cong \mathbb{Z}/2[D_1, D_2, D_3] \otimes \mathbb{Z}/2[w_8]$.

Thus $H^*(B\text{Spin}(7)) = W(\Delta)$, and the technical assumptions of Section 5 are satisfied with $X = B\text{Spin}(7))$. Hence all results from that section hold in this case.

Indeed, the Brown-Peterson cohomology of $B\text{Spin}(7)$ is given in [K-Y]. In the Atiyah-Hirzebruch spectral sequence converging to $BP^*(B\text{Spin}(7))$, all non-zero differentials are of the form $d_{2m-1} = v_{m-1} \otimes Q_{m-1}$:

\[d_3 w_4 = v_1 w_7, \quad d_7 w_7 = v_2 w_7^2, \quad d_7 w_8 = v_2 w_7 w_8, \quad d_{15}(w_7 w_8) = v_3 w_7^2 w_8^2.\]

Thus

\[E^{*,*}_\infty = E^{*,*}_{16} \cong BP^*\{1, 2w_4, 2w_6, 2w_4 w_8, v_1 w_8\} \otimes A \oplus (P(3)^*[w_7^2](w_7^2) \otimes (v_3 w_7^2 w_8^2))\]

with $A = \mathbb{Z}/2[w_4^2, w_6^2, w_8^2]$.

For the spectral sequence converging to $P(1)^*(B\text{Spin}(7))$, the arguments from Section 5 give

\[E^{*,*}_\infty = E^{*,*}_{32} \cong P(1)^*\{1, v_1 w_6, v_1 w_8, v_1 w_6 w_8, v_2 w_6 w_8\} \otimes A \oplus P(3)^*[w_7^2](w_7^2) \otimes (v_3 w_7^2, v_3 w_7^2 w_8^2, v_4 w_7^2 w_8^2)\]

Note that $BP^*(B\text{Spin}(7))/2$ is isomorphic to $P(1)^*(B\text{Spin}(7))$, which is also implied by the relations

\[2[w_7^2] + v_3[w_7^2] + \cdots = 0 \quad \text{and} \quad 2[w_7^2 w_8^2] + v_4[w_7^2 w_8^2] + \cdots = 0\]

which follow from the fact that if $\sum v_i x_i = 0$ in $BP^*(X)$, then there exist classes $y \in H^*(X; \mathbb{Z}/p)$ with $p_{\mathbb{Z}/p}(x_i) = Q_i y ([Y1])$. 

Theorem 7.1. The element $[v_1 w_8]$ is not represented by a transfer of a $BP^*$-linear combination of products of Chern classes.

Proof. This follows from Proposition 6.1 by looking at the commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \text{Spin}(7) & \longrightarrow & SO(7) & \longrightarrow & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & D(3) & \longrightarrow & (\mathbb{Z}/2)^6 & \longrightarrow & 0
\end{array}
$$

whose rows are central extensions.

Similar arguments work for Spin(9) and $D(4)$; in this case the Radon-Hurwitz number is 16.

The mod 2 cohomology is

$$
H^*(BS\text{Spin}(9); \mathbb{Z}/2) \cong H^*(BS\text{Spin}(7); \mathbb{Z}/2) \otimes \mathbb{Z}/2[w_{16}].
$$

Since $D(4) \subset Spin(9)$, the above cohomology ring contains the rank four Dickson algebra $\mathbb{Z}/2[D_1, \ldots, D_4]$. The invariant $D_1$ is equal to $w_8 + w_2^3$, from (2.5). Since $Sq^4 D_1 = D_2$, we get $D_2 = w_8 w_4 + w_2^3$. Similarly $D_3 = w_8 w_6 + w_2^3$ and $D_4 = w_8 w_7$. We consider the Atiyah-Hirzebruch spectral sequence converging to $P(1)^*(BS\text{Spin}(9))$. The odd degree part of $E^*_2$-page is additively

$$
E^*_2,\text{odd} \cong P(1)^*[w_{16}] \otimes B \otimes \Lambda(w_4, w_6, w_8)\{w_7\} \text{ with } B = \mathbb{Z}/2[w_1^2, w_5^2, w_7^2, w_8^2].
$$

Using the calculations in Section 5, we can compute

$$
\begin{align*}
E^*_4,\text{odd} & \cong P(2)^*[w_{16}] \otimes B \otimes \Lambda(w_6, w_8)\{w_7\} \\
E^*_8,\text{odd} & \cong P(3)^*[w_{16}] \otimes B \otimes \{w_6 w_7, w_8 w_7\} \\
E^*_16,\text{odd} & \cong P(4)^*[w_{16}^2] \otimes B \otimes \{w_8 w_7^3 + w_7 w_6 w_8^2 = D_3 D_4, w_8 w_7 w_16 = D_4 w_{16}\}.
\end{align*}
$$

The next term is $E^*_32,\text{odd} \cong P(5)^*[w_{76}] \otimes B\{a\}$, and finally $E^*_64,\text{odd} = 0$. Therefore all differentials have the form $d_{2r+1}(x) = v_r \otimes Q_r(x)$ and the assumptions needed in the lemmas of Section 5 hold. The integral case can also be proved to satisfy these assumptions by similar but easier arguments. Indeed, $BP^*(BS\text{Spin}(9))$ is also computed in [K-Y].

Theorem 7.2. In $BP^*(BD(4))$, the elements $[v_1 \otimes w_{16}]$ and $[v_2 \otimes w_{16}]$ are not transfers of $BP^*$-linear combinations of products of Chern classes.
8. A 4-DIMENSIONAL PERMANENT CYCLE

In this section, we show that the class $2w_4$ in $H^*(B\text{Spin}(n))(2)$ is represented by a Chern class. A similar statement holds for the exceptional group $F_4$ and $p = 3$.

Suppose that $G$ is a simply connected simple Lie group having $p$-torsion in $H^*(G)$. Then it is known that $G$ is 2-connected and there is an element $x_3 \in H^3(G;\mathbb{Z}/p)$ with $Q_1x_3 \neq 0$. Consider the classifying space $BG$ and its cohomology. Denote by $y_4$ the transgression of $x_3$ in $H^4(BG;\mathbb{Z}/p)$, so that $Q_1(y_4) \neq 0$. We shall denote the integral lift of $y_4$ to $H^4(BG;p)$ also by $y_4$. Then $y_4$ is not in the image from $BP^*(BG)$ and the following lemma is immediate.

**Lemma 8.1.** If $py_4 \in H^4(BG)(p)$ is represented by a Chern class, then the kernel of the map $\bar{\rho} : CH^2(BG)/p \to H^4(BG;\mathbb{Z}/p)$ is not injective.

First, we consider the case $G = \text{Spin}(2n+1)$ and $p = 2$. The complex representation ring is

$$R(\text{Spin}(2n+1)) \cong \mathbb{Z}[\lambda_1, \ldots, \lambda_{n-1}, \Delta_C]$$

where $\lambda_i$ is the $i$-th elementary symmetric function in variables $z_1^2 + z_1^{-2}, \ldots, z_n^2 + z_n^{-2}$ in $R(T) \cong \mathbb{Z}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$ for the maximal torus $T$ in $\text{Spin}(2n+1)$. Consider the restriction to $R(S^1) \cong \mathbb{Z}[z_1, z_1^{-1}]$. Since $\text{Res}_{S^1}(\lambda_1) = z_1^2 + z_1^{-2} + 2(n - 1)$, the total Chern class of this representation is

$$\text{Res}_{BS^1}(c(\lambda_1)) = (1 + 2u)(1 - 2u).$$

Therefore $4u^2 \in H^*(BS^1)$ is the restriction of a Chern class in $H^*(B\text{Spin}(n+1))(2)$.

On the other hand, consider the diagram

$$\begin{array}{ccc}
H^*(BT;\mathbb{Z}/2) & \leftarrow & H^*(B\text{Spin}(2n+1);\mathbb{Z}/2) \\
\uparrow p^*_T & & \uparrow p^* \\
H^*(BT;\mathbb{Z}/2) & \leftarrow & H^*(B\text{SO}(2n+1);\mathbb{Z}/2)
\end{array}$$

Here $p^*_T(u_i) = 2u_i$ and we see $\text{Res}_{BT}(w_4) = 0$ in $H^*(BT;\mathbb{Z}/2)$. Thus $\text{Res}_{BS^1}(H^4(B\text{Spin}(2n+1))(2)) \subset \mathbb{Z}(2)\{2u^2\}$. Therefore we see that for $G = \text{Spin}(2n+1)$ and $p = 2$ the assumptions of Lemma 8.1 are satisfied. By naturality, all the groups $\text{Spin}(n)$ for $n \geq 7$ satisfy the assumptions.
Next consider the case $G = F_4$ and $p = 3$. The exceptional Lie group $F_4$ contains Spin(8) as a subgroup and

$$R(F_4) \cong R(\text{Spin}(8)) \Sigma_3$$

in $R(T) \cong \mathbb{Z}[z_1, z_1^{-1}, \ldots, z_4, z_4^{-1}]$ (for the details of the action of $\Sigma_3$, see [A], Chapter 14).

There is a 26-dimensional irreducible representation $U$ of $F_4$, whose restriction to Spin(8) is $2 + \lambda_1 + \Delta^+ + \Delta^-$, where $\Delta^\pm$ are the half spin representations of dimension 8. The weight of $\Delta^\pm$ is $\frac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4)$ with an even number of minus signs for $\Delta^+$ and an odd number for $\Delta^-$. Thus

$$\text{Res}_{S^1}(\Delta^+) = \sum_{\epsilon_1, \ldots, \epsilon_4} z_1^{\epsilon_1} z_2^{\epsilon_2} z_3^{\epsilon_3} z_4^{\epsilon_4},$$

and similarly for $\Delta^-$. Restricting further to $S^1$, we obtain

$$\text{Res}_{S^1}(U) = z_1^2 + z_2^2 + 8 + 8z_1 + 8z_1^{-1}.$$

Therefore its total Chern class is

$$\text{Res}_{BS^1}(c(U)) = (1 + 2u)(1 - 2u)(1 + u)^8(1 - u)^8 = (1 - 4u^2)(1 - u^2)^8 = 1 - 12u^2 + \cdots.$$ 

Hence $3y_4 \in H^4(BF_4) \cong \mathbb{Z}(3)$ is represented by a Chern class. Thus we get the following theorem.

**Theorem 8.2.** Let $G = \text{Spin}(n)$, $n \geq 7$ and $p = 2$ or $G = F_4$ and $p = 3$. The kernels of the maps

$$CH^2(BG)/(p) \to H^4(BG; \mathbb{Z}/p)$$

$$CH^3(BG \times B\mathbb{Z}/p)/(p) \to H^6(BG \times B\mathbb{Z}/p)/(p)$$

are both non-zero.

**References**


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