ON THE MORAVA K-THEORY OF M_{12}

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Abstract. We show that for all n and all primes p, the Morava K-theory of the classifying space of the Mathieu group M_{12} is generated by transfers of Euler classes.

1. Introduction

In this note, we add to the growing list of calculations of the Morava K-theory of classifying spaces of finite groups by considering the 2-Sylow subgroup of the Mathieu group M_{12}. We mainly work with mod 2 Morava K-theory; this is a periodic, complex oriented cohomology theory with coefficients $K(n)^* \cong \mathbb{F}_2[v_n, v_n^{-1}]$ with v_n of degree $-2(2^n-1)$. It has a complex orientation x such that the 2-series of the asociated formal group law has the form $[2](x) = v_n x^{2^n}$. At some point we may refer to the 'integral' version $\widetilde{K}(n)$ with coefficients $\widetilde{K}(n)^* \cong W\mathbb{F}_{2^n}[v_n, v_n^{-1}]$ where $W\mathbb{F}_{2^n}$ is the ring of Witt vectors over $\mathbb{F}_{2^n}$.

The Mathieu group M_{12} is a simple group of order $96\,040 = 2^6 \cdot 3^3 \cdot 5 \cdot 11$; a presentation of a 2-Sylow subgroup is e.g.

$$G = \langle a, b, c, d \mid a^4 = b^4 = c^2 = d^2 = [a,b] = [c,d] = 1, cac = b, dad = a^3, dbd = b^3 \rangle \,.$$

Thus G is a semidirect product of $\langle a, b\rangle \cong C_4 \times C_4$ with $\langle c, d\rangle \cong C_2 \times C_2$. Our main result is as follows. Recall from [2] that a finite group is called $K(n)$-*good* (at the prime p) if its mod p Morava K-theory is generated by transfers of Euler classes of G.

Theorem. *Let G be a 2-Sylow subgroup of M_{12}. Then G is $K(n)$-good.*

In particular, this implies that $K(n)^*(BG)$ is concentrated in even degrees, so that the $K(n)$ Euler characteristic

$$\chi_{n,2}(G) = \mathrm{rank}_{K(n)^*} K(n)^{\mathrm{even}}(BG) - \mathrm{rank}_{K(n)^*} K(n)^{\mathrm{odd}}(BG)$$

coincides with the rank.

The other p-Sylow subgroups of M_{12} are the extraspecial group of order 27 and exponent 3 and the cyclic groups of order 5 and 11, all of which have good Morava K-theory at the respective prime. As the classifying space of M_{12} is a p-local stable summand of BG, one obtains

Corollary. *M_{12} is $K(n)$-good for any prime p.*

In the next section we shall describe the method of calculation. Section 3 will provide the necessary prerequisites about the Morava K-theory of a certain subgroup

2010 *Mathematics Subject Classification.* Primary 55R35, 55N20; Secondary 57T25.
Key words and phrases. Morava K-theory, finite 2-groups.

of G; this material is not new (see [9]). The main theorem is then proved in Section 4. In an appendix we calculate the Morava K-theory Euler characteristic of G, which by then has been seen to coincide with the rank.

2. Outline of the calculation

We propose to calculate the Morava K-theory of G in two stages: Since the Morava K-theories of the groups of order 32 are known, at least additively ([9]), one can attempt to compute the Serre spectral sequence for a suitable extension with a C_2 on top. G contains several index 2 subgroups; for no particular reason we chose $H := \langle a, b, d\rangle = N \rtimes d$ with $N = \langle a, b\rangle$. Then one has a split extension

$$1 \longrightarrow H \longrightarrow G \longrightarrow \langle c\rangle \longrightarrow 1 \tag{2.1}$$

with $\langle c\rangle \cong C_2$ whose associated Serre spectral sequence we want to compute. We might equally well have started with either $K := \langle a, b, c\rangle$ or $L := \langle a, b, cd\rangle$, both of which are isomorphic to the wreath product $C_4 \wr C_2$, which was among the first nonabelian groups whose Morava K-theory was computed ([3, 2]).
To begin with, we have to know $K(n)^*(BH)$, as a module for $K(n)^*[C_2]$. The calculation of the additive structure of $K(n)^*(BH)$ was already outlined in [9]: the Serre spectral sequence for the extension $1 \to N \to H \to \langle d\rangle \to 1$ is as simple as one might hope, in the sense that $E_2^{0,*} = H^0(\langle d\rangle; K(n)^*(BN))$ consists of permanent cycles. To see that, note that $K(n)^*(BN)$ decomposes as a $K(n)^*[C_2]$-module as a direct sum $F \oplus T$ of free and trivial summands whose corresponding invariants are transfers of Euler classes. Thus $E_2 \cong F^{\langle d\rangle} \oplus T[u]$ with $u \in H^1$, and the only differential is $d_{2^{n+1}-1}u = v_n u^{2^{n+1}}$. Such a behaviour of the spectral sequence has sometimes been called 'simple', e.g. in [3]. We shall recall this calculation in the next section.
From the group structure it is evident how c acts on the E_2-page, and hence on the E_∞-page of this spectral sequence. Thus we obtain the action not quite on $K(n)^*(BH)$, but at least on the associated graded. This module again decomposes into free an trivial summands, and if we can show that all invariants are transfers of Euler classes, the same will hold for the unfiltered module, since unfiltering can only result in combining two trivial modules to form a free module. Thus the analysis of the graded module will prove sufficient to show that the Serre spectral of (2.1) is again 'simple', finishing the proof. A calculation of the Euler characteristic, which is deferred to an appendix, will then show that the associated graded is indeed isomorphic to $K(n)^*(BH)$ as a $\langle c\rangle$-module.

Notational conventions. For the sake of leaner formulas we set $v_n = 1$, which means that degrees have to be taken modulo $2(2^n - 1)$. We also assume $n \geq 2$ throughout.

3. The Morava K-theory of H

We begin this section with a detailed analysis of $K(n)^*(BC_4)$ as a module for the action by inversion, i.e., $s \mapsto s^{-1}$ for s a generator of C_4. Recall that $K(n)^*(BC_4) \cong K(n)^*[z]/(z^{2^{2n}})$ where z is the Euler class of a generator of the complex representation ring of C_4. On the level of Morava K-theory, the action translates to z being mapped to $[-1](z)$, the formal group law inverse of z. Using the formula

$$x_1 +_{K(n)} x_2 = x_1 + x_2 + \Big(x_1x_2 + (x_1+x_2)(x_1x_2)^{2^{n-1}}\Big)^{2^{n-1}} \bmod \Big(\big((x_1+x_2)x_1x_2\big)^{2^{2n-2}}\Big)$$

for the formal group law, given as Lemma 2.2 (ii) in [1] (also see [8, Lemma 3.1 (ii)], one obtains, calculating modulo $[4](z) = z^{2^{2n}}$,

$$(3.1)\quad [-1](z) = [3](z) = z +_{K(n)} [2](z) = z +_{K(n)} z^{2^n} = z + z^{2^n} + z^{2^{2n-1}+2^{n-1}}$$

which immediately implies

$$(3.2)\qquad z + [-1](z) = \big(z\cdot[-1](z)\big)^{2^{n-1}} \mod z^{2^{2n}}.$$

Remark. Equation (3.2) reflects part of the multiplicative structure of the Morava K-theory of the dihedral group D_8: if c_1 and c_2 denote the first and second Chern classes of the irreducible two-dimensional representation of D_8, then c_1 and c_2 restrict to $z+[-1](z)$ respectively $z\cdot[-1](z)$, and $c_1 - c_2^{2^{n-1}}$ lies in the kernel of restriction to C_4; see e.g. [8, Section 4].

The next result describes $K(n)^*(BC_4)$ as a module for $K(n)^*[C_2]$.

Lemma 3.1. (a) *$K(n)^*(BC_4)$ decomposes into $2^{2n-1}-2^{n-1}$ free and 2^n trivial summands.*

(b) *The free summands are given by $\langle z^k, [-1](z^k)\rangle$ for $1 \le k < 2^{2n-1}$ and $k \not\equiv 0\ (2^n)$. The invariants of the free summands are powers of $z\cdot[-1](z)$.*

(c) *The trivial summands are generated by $\big(z\cdot[-1](z)\big)^k$, $0 \le k < 2^{n-1}$, and the transfer classes $\mathrm{Tr}_{C_2}^{C_4}(u^{2\ell+1})$, $0 \le \ell < 2^{n-1}$, where $u = \mathrm{Res}_{C_2}^{C_4}(z)$.*

Proof. (a) The decomposition can be recovered from the rank of $K(n)^*(BD_8)$: dihedral groups are 'good' (see e.g. [7] or [10]), from which one easiliy deduces that the Serre spectral sequence for the extension $1 \to C_4 \to D_8 \to C_2 \to 1$ with $E_2 = H^*(C_2; K(n)^*(BC_4)) \cong F^{C_2} \oplus T[t]$, where F stand for the free summands, T for the trivial summands, and $t \in E_2^{0,1}$, has only one differential $d_{2^{n+1}-1}t = t^{2^{n+1}}$. Thus $E_\infty \cong F^{C_2} \oplus T[t^2]/(t^{2^{n+1}})$. The rank of $K(n)^*(BD_8)$ being $3\cdot 2^{2n-1} - 2^{n-1}$ [7], one obtains the numbers claimed.

(b) For k as described, the modules $\langle z^k, [-1](z^k)\rangle$ are $2^{2n-1} - 2^{n-1}$ distinct free summands. Their invariants are symmetric functions in $z+[-1](z)$ and $z\cdot[-1](z)$, whence the second claim follows from (3.2).

(c) These classes are certainly invariant. For the given range of k, $\big(z\cdot[-1](z)\big)^k$ does not belong to any summand as in (a), and using Frobenius reciprocity, Quillen's formula $\mathrm{Tr}_1^{C_p}(1) = [p](x)/x$ (see [6]), and naturality, the transfer classes

$$\mathrm{Tr}_{C_2}^{C_4}(u^{2\ell+1}) = z^{2\ell+1}\,\mathrm{Tr}_{C_2}^{C_4}(1) = z^{2\ell+1}\frac{[4](z)}{[2](z)} = z^{2^{2n}-2^n+2\ell+1}$$

are odd powers of z and thus not in the subring generated by $z\cdot[-1](z)$. □

This describes the invariants as either in the image of restriction or transfers. Since the transfer induces a map of spectral sequences [5] and the spectral sequence of $C_2 \to C_4 \to C_2$ is 'simple', these transfer classes are permanent cycles, giving independent confirmation for the behaviour of the spectral sequence asserted in the proof of (a). In fact all invariants in the image of restriction, but in the present form they are easier to work with.

The group $H = \langle a, b, d\rangle = N \rtimes \langle d\rangle$ is the group number 34 in the Hall-Senior list; its Morava K-theory was seen to be 'good' in [9]. The easiest way to see that it is concentrated in even degrees is by appealing to the calculation of $K(n)^*(BD_8)$.

The group H can be mapped injectively into a product of two copies $\langle a, d_1\rangle \times \langle b, d_2\rangle$ of D_8 (with $d_1 a d_1 = a^3$, etc.), by sending d to (d_1, d_2). Since D_8 is good, a theorem of Kriz [4, Theorem 2.1] implies that the integral Morava K-theory $\widetilde{K}(n)^*(BC_4) \cong \widetilde{K}(n)^*[z]/([4](z))$ decomposes as a direct sum of free and trivial modules for the action of the quotient C_2. Thus the same is true for $\widetilde{K}(n)^*(BN)$ as a module for $\langle d\rangle$, which by using Kriz's theorem again implies that $K(n)^*(BH)$ is concentrated in even degrees. Our analysis from the previous section even yields that $K(n)^*(BH)$ is generated by transfers of Euler classes, i.e., H is 'good'. The Serre spectral sequence of the extension

$$1 \longrightarrow N \longrightarrow H \longrightarrow \langle d\rangle \longrightarrow 1$$

has $E_2 = H^*(\langle d\rangle; K(n)^*(BN))$, with $K(n)^*(BN) \cong K(n)^*(BC_4) \otimes_{K(n)^*} (BC_4)$ decomposing as $F \oplus T$, a sum of free and trivial modules as before. All invariants are either restrictions under the composite $N \to H \to D_8 \times D_8$ or transfers from $\langle a^2, b^2\rangle$, as the following application of Frobenius reciprocity and naturality of the transfer shows:

Lemma 3.2. *Let z be a generator of $K(n)^*(BC_4)$ and $u = \mathrm{Res}^{C_4}_{C_2}(z)$. Then*

$$\mathrm{Tr}^N_{\langle a^2,b^2\rangle}(u^k \otimes u^l) = \mathrm{Tr}^{\langle a\rangle}_{\langle a^2\rangle}(u^k) \otimes \mathrm{Tr}^{\langle b\rangle}_{\langle b^2\rangle}(u^\ell)\,.$$

Proof. In the following calculation, let $\pi \times \pi\colon \langle a, b\rangle \to \langle \bar{a}, \bar{b}\rangle$ denote the obvious projection and pr_2 the projection onto the second factor. Then $\pi^*(\bar{z}) = [2](z)$ for the generator $\bar{z}$ of $K(n)^*(BC_2)$, and

$$\begin{aligned}
\mathrm{Tr}^N_{\langle a^2,b^2\rangle}(u^k \otimes u^l) &= \mathrm{Tr}^N_{\langle a^2,b^2\rangle}\big(\mathrm{Res}^N_{\langle a^2,b^2\rangle}(z^k \otimes z^l)\big)\\
&= z^k \otimes z^l \cdot \mathrm{Tr}^N_{\langle a^2,b^2\rangle}(1)\\
&= z^k \otimes z^l \cdot (\pi\times\pi)^* \,\mathrm{Tr}^{\langle \bar{a},\bar{b}\rangle}_1(1)\\
&= z^k \otimes z^l \cdot (\pi\times\pi)^* \,\mathrm{Tr}^{\langle \bar{a},\bar{b}\rangle}_{\langle \bar{a}\rangle}\Big(\frac{[2](\bar{z})}{\bar{z}}\Big)\\
&= z^k \otimes z^l \cdot (\pi\times\pi)^* \bigg(\frac{[2](\bar{z})}{\bar{z}} \otimes 1 \cdot \mathrm{Tr}^{\langle \bar{a},\bar{b}\rangle}_{\langle \bar{a}\rangle}(1)\bigg)\\
&= z^k \otimes z^l \cdot (\pi\times\pi)^* \bigg(\frac{[2](\bar{z})}{\bar{z}} \otimes 1 \cdot \mathrm{pr}_2^* \,\mathrm{Tr}^{\langle \bar{b}\rangle}_1(1)\bigg)\\
&= z^k \otimes z^l \cdot (\pi\times\pi)^* \bigg(\Big(\frac{[2](\bar{z})}{\bar{z}} \otimes 1\Big) \cdot \Big(1 \otimes \frac{[2](\bar{z})}{\bar{z}}\Big)\bigg)\\
&= (z^k \otimes z^l) \cdot \Big(\frac{[4](z)}{[2](z)} \otimes \frac{[4](z)}{[2](z)}\Big)\\
&= z^k \,\mathrm{Tr}^{\langle a\rangle}_{\langle a^2\rangle}(1) \otimes z^l \,\mathrm{Tr}^{\langle b\rangle}_{\langle b^2\rangle}(1)\\
&= \mathrm{Tr}^{\langle a\rangle}_{\langle a^2\rangle}(u^k) \otimes \mathrm{Tr}^{\langle b\rangle}_{\langle b^2\rangle}(u^\ell)\,.
\end{aligned}$$

□

Again, all invariants are permanent cycles. Thus the spectral sequence is 'simple' and one obtains an additive isomorphism

$$K(n)^*(BH) \cong F^{\langle d\rangle} \oplus T[x]/(x^{2^n}) \tag{3.3}$$

where x represents the Euler class of the nontrivial complex character of $\langle d\rangle$. Thus $K(n)^*(BH)$ is generated by transfers of Euler classes.

There is no a priori reason for this to be an isomorphism of modules for the action of c. We note though that c acts trivially on x.

Remark. From this description we can recover the rank of $K(n)^*(BH)$: each trivial module contributes 2^n, and each free module 1, so that

$$\operatorname{rank}_{K(n)^*} K(n)^*(BH) = \frac{1}{2}16^n + 8^n - \frac{1}{2}4^n .$$

4. Proof of the theorem

In order to prove the theorem, it suffices to show that the vertical edge in the Serre spectral sequence associated to extension (2.1) consists of permanent cycles. This vertical edge is given by the invariants of the action of c on the Morava K-theory of H,

$$E_2^{0,*} = H^0(\langle c\rangle; K(n)^*(BH)) \cong K(n)^*(BH)^{\langle c\rangle} .$$

For the moment, we shall pretend that the isomorphism (3.3) is as $\langle c\rangle$-modules. As remarked earlier, this need only be true up to extensions, but if we can show that all the invariants pertaining to this module structure are permanent cycles, we are done: the only thing that could happen is that two trivial modules of this graded structure combine to a free module, whose invariant is already taken care of.

We now consider the action of c on the $\langle d\rangle$-invariants. Since x is represented by an Euler class which is clearly invariant under c, it is enough to analyse the restrictions of the invariants to $K(n)^*(BN)$. As above, let F and T denote the free and trivial parts of $K(n)^*(BN)$ as $\langle d\rangle$-module. Since c acts by interchanging the two factors of N, it appears convenient to rearrange the summands as follows. Let F^i and T^j denote the trivial and free modules in the decomposition of $K(n)^*(BC_4)$, then

$$F = \bigoplus_{i<j} A_{ij} \oplus \bigoplus_{k} B_{kk} \oplus \bigoplus_{\ell,m} C_{\ell m}$$
$$T = \bigoplus_{i<j} D_{ij} \oplus \bigoplus_{k} E_{kk}$$

where

$$A_{ij} = F^i \otimes F^j \oplus F^j \otimes F^i\ , \quad B_{kk} = F^k \otimes F^k\ , \quad C_{\ell m} = F^\ell \otimes T^m \oplus T^m \otimes F^\ell$$
$$D_{ij} = T^i \otimes T^j \oplus T^j \otimes T^i\ , \quad E_{kk} = T^k \otimes T^k$$

Each $C_{\ell m}^{\langle d\rangle}$ is a free $\langle c\rangle$-module, as is D_{ij}, whereas $A_{ij}^{\langle d\rangle}$ is a sum of two free $\langle c\rangle$-modules.

Lemma 4.1. *The invariants belonging to the free summands are in the image of restriction.*

Proof. Indeed, they are in the image of $\operatorname{Res}_H^G \operatorname{Tr}_H^G$. □

We are left with the summands of type B and E. Each E_{kk} ia a trivial $\langle c\rangle$-module, and $B_{kk}^{\langle d\rangle}$ is the sum of two trivial modules .

Lemma 4.2. *The modules E_{kk} consist of permanent cycles.*

Proof. According to Lemma 3.1, we have to consider two types of classes:

(i) $z^k \cdot [-1](z^k) \otimes z^k \cdot [-1](z^k)$, where z is the Euler class of a generator ρ of the complex representation ring of C_4, and

(ii) $\operatorname{Tr}(u^k) \otimes \operatorname{Tr}(u^k)$, with $u = \operatorname{Res}^{C_4}_{C_2}(z)$.

For classes of type (ii), consider the subgroup $M := \langle a^2, b^2, c\rangle \times \langle d\rangle \cong D_8 \times C_2$ of G. Let Δ be the two-dimensional irreducible complex representation of D_8 and η the nontrivial complex character of C_2, then Δ restricts to $\eta \otimes 1 + 1 \otimes \eta$ on $\langle a^2, b^2\rangle$, hence its Euler class $e(\Delta)$ to $e(\eta) \otimes e(\eta) = u \otimes u$. Thus

$$\begin{aligned}\operatorname{Res}^G_N \operatorname{Tr}^G_M\bigl(e(\Delta)^k \otimes 1\bigr) &= \operatorname{Tr}^N_{\langle a^2,b^2\rangle} \operatorname{Res}^M_{\langle a^2,b^2\rangle}\bigl(e(\Delta)^k \otimes 1\bigr)\\ &= \operatorname{Tr}^N_{\langle a^2,b^2\rangle}\bigl((u \otimes u)^k\bigr)\\ &= \operatorname{Tr}^{\langle a\rangle}_{\langle a^2\rangle}(u^k) \otimes \operatorname{Tr}^{\langle b\rangle}_{\langle b^2\rangle}(u^k)\end{aligned}$$

using the double coset formula and Lemma 3.2. For (i), observe that G has a complex representation χ that restricts to $\rho \otimes \rho + \rho^{-1} \otimes \rho^{-1} + \rho^{-1} \otimes \rho + \rho \otimes \rho^{-1}$ on N. The representation $\rho \otimes \rho$ of N is fixed under the action of c, hence extends to a representation κ of the wreath product $K = \langle a, b, c\rangle$; for the same reason $\rho^{-1} \otimes \rho^{-1}$ extends to a representation $\kappa' = \kappa^d$ of K. Similarly, $\rho^{-1} \otimes \rho$ and $\rho \otimes \rho^{-1}$ are invariant under cd, whence they extend to representations λ and $\lambda' = \lambda^c$ of $L := \langle a, b, cd\rangle \cong C_4 \wr C_2$. Now let

$$\chi := \operatorname{Ind}^G_K(\kappa) + \operatorname{Ind}^G_L(\lambda)\,;$$

since $N\backslash G/K = N1K \sqcup NdK$ and $N\backslash G/L = N1L \sqcup NcL$,

$$\begin{aligned}\operatorname{Res}^G_N(\chi) &= \operatorname{Res}^K_N(\kappa) + \operatorname{Res}^K_N(\kappa^d) + \operatorname{Res}^L_N(\lambda) + \operatorname{Res}^L_N(\lambda^c)\\ &= \rho \otimes \rho + \rho^{-1} \otimes \rho^{-1} + \rho^{-1} \otimes \rho + \rho \otimes \rho^{-1}\\ &= (\rho \otimes 1 + \rho^{-1} \otimes 1)(1 \otimes \rho + 1 \otimes \rho^{-1})\,.\end{aligned}$$

The Euler class of χ restricts to

$$\bigl(z \cdot [-1](z)\bigr) \otimes \bigl(z \cdot [-1](z)\bigr)$$

as required, and similarly for the kth powers. □

Lemma 4.3. *The $\langle c\rangle$-invariants of the modules $B^{\langle d\rangle}_{kk}$ consist of permanent cycles.*

Proof. By Lemma 3.1 again, two types of elements have to be considered.

(i) $y_k = z^k \otimes z^k + [-1](z^k) \otimes [-1](z^k)$,
(ii) $w_k = z^k \otimes [-1](z^k) + [-1](z^k) \otimes z^k$.

For (i), let ρ as above denote a generator of the representation ring of C_4. Then $\rho\otimes 1+1\otimes\rho \in RN$ is invariant under the action of c, hence extends to a representation σ of $K = \langle a, b, c\rangle$, with

$$\operatorname{Res}^K_N e(\sigma) = e\bigl(\operatorname{Res}^K_N(\sigma)\bigr) = e(\rho \otimes 1 + 1 \otimes \rho) = e(\rho \otimes 1)e(1 \otimes \rho) = z \otimes z$$

where $e(\ )$ stands for the Euler class. Since $N\backslash G/K = N1K \sqcup NdK$, the double coset formula gives

$$\begin{aligned}\operatorname{Res}^G_N \operatorname{Tr}^G_K\bigl(e(\sigma)\bigr) &= \operatorname{Res}^K_N\bigl(e(\sigma)\bigr) + \operatorname{Res}^K_N\bigl(d \cdot e(\sigma)\bigr)\\ &= (1 + d)(z \otimes z)\\ &= z \otimes z + [-1](z) \otimes [-1](z) = y_1\,.\end{aligned}$$

For $k > 1$, replacing ρ by a sum of k copies of ρ yields the desired result.

In case (ii), we consider the subgroup $L = \langle a, b, cd\rangle \cong C_4 \wr C_2$. Here $\rho^{-1} \otimes \rho$ is invariant under cd and thus extends to a representation τ of L. Just as above, $\mathrm{Res}^L_N\big(e(\tau)\big) = [-1](z) \otimes z$ and

$$\mathrm{Res}^G_N \mathrm{Tr}^G_L\big(e(\tau) = (1+d)([-1](z) \otimes z) = [-1](z) \otimes z + z \otimes [-1](z) = w_1 \,.$$

For $k > 1$, argue as above with $k\rho$. □

We have thus seen that all elements of the vertical edge of the Serre spectral sequence associated to the extension (2.1) are permanent cycles, the spectral sequence must therefore be 'simple'. Since all generators of E_∞ are represented by transfers of Euler classes, we are done.

Remark. The decomposition of the E_∞ page of the spectral sequence converging to $K(n)^*(BH)$ as a $\langle c\rangle$-module into free and trivial modules does indeed coincide with the decomposition of $K(n)^*(BH)$, not just up to filtration: based on this (hypothetical) decomposition, one can calculate the rank of E_∞ by the same method as used in the concluding remark of the previous section. As there are $\frac{1}{4}16^n + \frac{1}{2}8^n - \frac{5}{4}4^n + \frac{1}{2}2^n$ free and $2 \cdot 4^n - 2^n$ trivial summands, on obtains

$$\frac{1}{4}16^n + \frac{5}{2}8^n - \frac{9}{4}4^n + \frac{1}{2}2^n \tag{4.1}$$

which coincides with the Euler characteristic computed in the appendix. Were there more free modules, the rank of E_∞ would be smaller.

Finally, an analysis of the degrees of the generators shows $K(n)^*(BG)$ to be 'equidistributed' in the sense that

$$\mathrm{rank}_{K(n)^*} K(n)^{2i}(BG) = \mathrm{rank}_{K(n)^*} K(n)^0(BG) - 1$$

for $i = 1, 2, \ldots, 2^n - 2$ (recall hat we are grading cyclically). Thus the rank determines the additive structure, as is the case in all known examples of good p-groups.

Appendix: The $K(n)$-Euler characteristic

We use the method of computing the Euler characteristic based on [2, Theorem D]:

$$\chi_{n,2}(G) = \sum_{A \leq G} \frac{\mu(A)}{[G : A]} \chi_{n,2}(A)$$

where the sum ranges over all abelian subgroups of G and the Möbius function μ is defined recursively by $\mu(A) = 1$ if A is maximal and

$$\sum_{A \leq A'} \mu(A') = 1 \,.$$

Thus we only need to consider the maximal subgroups and their intersections.

Maximal subgroups. There is a unique maximal subgroup isomorphic to $C_4 \times C_4$, namely $N = \langle a, b\rangle$. Secondly, there are four cyclic groups of order eight, coming in two conjugacy classes:

$$A_1 = \langle ac\rangle\,, \quad A_2 = A_1^c = \langle bc\rangle \qquad \text{both containing } \langle ab\rangle \cong C_4,$$
$$A_3 = \langle acd\rangle\,, \quad A_4 = A_3^d = \langle a^3cd\rangle \qquad \text{both containing } \langle ab^3\rangle \cong C_4,$$

Next, there are eight subgroups isomorphic to $C_4 \times C_2$, in four conjugacy classes; we list them together with their subgroups C_4:

$$
\begin{aligned}
B_1 &= \langle ab, c\rangle \supset \langle ab\rangle, \langle abc\rangle\,, & B_2 &= \langle ab, a^3bc\rangle \supset \langle ab\rangle, \langle a^2c\rangle\,,\\
B_3 &= \langle ab^3, cd\rangle \supset \langle a^3b\rangle, \langle ab^3cd\rangle\,, & B_4 &= \langle ab^3, abcd\rangle \supset \langle a^3b\rangle, \langle a^2cd\rangle\,,\\
B_5 &= \langle a^2c, d\rangle \supset \langle a^2c\rangle, \langle a^2cd\rangle\,, & B_6 &= \langle abc, b^2d\rangle \supset \langle abc\rangle, \langle a^3bcd\rangle\,,\\
B_7 &= \langle a^2c, abd\rangle \supset \langle a^2c\rangle, \langle a^3bcd\rangle\,, & B_8 &= \langle abc, ab^3d\rangle \supset \langle abc\rangle, \langle a^2cd\rangle\,,
\end{aligned}
$$

with $B_1 \sim B_2$, $B_3 \sim B_4$, $B_5 \sim B_6$, $B_7 \sim B_8$ (all via b).

Finally, there are eight elementary abelian aubgroups of rank three, coming in five conjugacy classes:

$$
\begin{aligned}
E_1 &= \langle a^2, b^2, d\rangle\,, & E_2 &= \langle a^2, b^2, abd\rangle\,,\\
E_3 &= \langle a^2, b^2, ad\rangle & E_4 &= \langle a^2, b^2, bd\rangle\,,\\
E_5 &= \langle a^2b^2, c, d\rangle\,, & E_6 &= \langle a^2b^2, a^3bc, b^2d\rangle\,,\\
E_7 &= \langle a^2b^2, c, abd\rangle\,, & E_8 &= \langle a^2b^2, a^3bc, ab^3d\rangle\,,
\end{aligned}
$$

with $E_3 \sim E_4$, $E_5 \sim E_6$, $E_7 \sim E_8$.

Intersections of maximal subgroups. There are six cyclic subgroups of order four and nine rank two elementary abelians appearing as intersections:

$$
\begin{aligned}
&\langle ab\rangle \subset A_1, A_2, B_1, B_2, N & &\langle a^3b\rangle \subset A_3, A_4, B_3, B_4, N\\
&\langle abc\rangle \subset B_1, B_6, B_8 & &\langle a^2c\rangle \subset B_2, B_5, B_7\\
&\langle ab^3cd\rangle \subset B_3, B_6, B_7 & &\langle a^2cd\rangle \subset B_4, B_5, B_8
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle a^2, b^2\rangle \subset E_1, E_2, E_3, E_4, N & &\\
&\langle a^2b^2, c\rangle \subset B_1, E_5, E_7 & &\langle a^2b^2, a^3bc\rangle \subset B_2, E_6, E_8\\
&\langle a^2b^2, cd\rangle \subset B_3, E_5, E_8 & &\langle a^2b^2, abcd\rangle \subset B_4, E_6, E_7\\
&\langle a^2b^2, d\rangle \subset B_5, E_1, E_5 & &\langle a^2b^2, b^2d\rangle \subset B_6, E_1, E_6\\
&\langle a^2b^2, abd\rangle \subset B_7, E_2, E_7 & &\langle a^2b^2, ab^3d\rangle \subset B_8, E_2, E_8
\end{aligned}
$$

Finally, the centre is contained in all of the above. This completes the description of the poset of abelian subgroups.

Möbius function and Euler characteristic. The maximal subgroups N, A_i, B_j, E_k all have $\mu = 1$. The groups $\langle ab\rangle$, $\langle a^3b\rangle$, $\langle a^2, b^2\rangle$ are each contained in five maximal subgroups, hence have $\mu = -4$. All other intersections in above list (except Z) are contained in three maximal subgroups and thus have $\mu = -2$; there are 12 such subgroups. Finally, from

$$1 = \mu(Z) + 12 \cdot (-2) + 3 \cdot (-4) + 21$$

one obtains $\mu(Z) = 16$. Thus

$$\chi_{n,2}(G) = \frac{1}{4}16^n + \frac{5}{2}8^n - \frac{9}{4}4^n + \frac{1}{2}2^n\,.$$

References

[1] M. Bakuradze and V. Vershinin. Morava K-Theory rings for the dihedral, semidihedral and generalized quaternion groups. *Proc. Amer. Math. Soc.* **134** (2006), 3707–3714.

[2] M. J. Hopkins, N. J. Kuhn, and D. C. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc.* **13** (2000), 553–594.

[3] J. R. Hunton. The Morava K-theories of wreath products. *Math. Proc. Camb. Phil. Soc.* **107** (1990), 309–318.

[4] I. Kriz. Morava K-theory of classifying spaces: some calculations. *Topology* **36** (1997), 1247–1273.

[5] C. Prieto. Transfer and the spectral sequence of a fibration. *Trans. Amer. Math. Soc.* **271** (1982), 133–142.

[6] D. G. Quillen. Elementray proofs of some results in cobordism theory using Steenrod operations. *Advances in Math.* **7** (1971), 29–56.

[7] B. Schuster. On the Morava K-theory of some finite 2-groups. *Math. Proc. Camb. Phil. Soc.* **121** (1997), 7–13.

[8] B. Schuster. $K(n)$ Chern approximations of some finite groups Preprint, 2010. Available at `www.math.uni-wuppertal.de/~schuster/research/pdf/CBGex.pdf`

[9] B. Schuster. On the Morava K-theory of groups of order 32. Preprint, 2010. Available at `www.math.uni-wuppertal.de/~schuster/research/pdf/ord32.pdf`

[10] B. Schuster and N. Yagita. Morava K-theory of extraspecial 2-groups. *Proc. Amer. Math. Soc.* **132** (2004), no. 4, 1229–1239.

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