TOWARDS A SPLITTING OF THE $K(2)$-LOCAL STRING BORDISM SPECTRUM

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Abstract. We show that $K(2)$-locally, the smash product of the string bordism spectrum and the spectrum T_2 splits into copies of Morava E-theories. Here, T_2 is related to the Thom spectrum of the canonical bundle over $\Omega SU(4)$.

1. Introduction and statement of results

In the late 80s [Wit88], Witten gave an interpretation of the level one elliptic genus as the S^1-equivariant index of the Dirac operator on the free loop space LM of a manifold. He needed the loop space of the manifold to carry a spin structure. This is guaranteed if the classifying map of its stable tangent bundle lifts to $BString$, the 7-connected cover of BO. This classifying space and its associated Thom spectrum $MString$ have been extensively studied by homotopy theorists already in the 70s. At that time, Anderson, Brown and Peterson had just succeeded in calculating the spin bordism groups via a now famous splitting of the Thom spectrum $MSpin$ [ABP67]. The string bordism groups however have not been calculated yet. It is known, that they have torsion at the primes 2 and 3, and the hope is to get new insights using the Witten genus and the arithmetic of modular forms.

At the prime 3, Hovey and Ravenel [HR95, Corollary 2.2] have shown that the product $DA(0) \wedge MString$ is a wedge of suspensions of the Brown-Peterson spectrum BP. Here, $DA(0)$ is the 8-skeleton of BP; it is a 3-cell complex which is free over $P(0)$, the sub-Hopf algebra of the the mod 3 Steenrod algebra $\mathcal{A}$ generated by P^1.

At the prime 2, they have shown that $DA(1) \wedge MU\langle 6\rangle$ splits into a wedge of suspensions of BPs, where $MU\langle 6\rangle$ is the Thom spectrum of the 5-connected cover $BU\langle 6\rangle$ of BU and $DA(1)$ is an eight cell complex whose cohomology is free over $P(1)$, the double of $A(1)$.

This paper is concerned with a similar decomposition of $MString$ at the prime 2. Starting point is the spectrum of topological modular forms *tmf* which was introduced by Goerss, Hopkins and Miller, see [Goe10] or [DFHH14]. Its mod 2 cohomology is the quotient $\mathcal{A}//\mathcal{A}(2)$ which is known to be a direct summand in the cohomology of $MString$ (cf. [BM80]). There even is a ring map from $MString$ to *tmf* which induces the Witten genus mentioned earlier (cf. [AHR]). Moreover, the 2-local equivalence

$$DA(1) \wedge tmf \simeq BP\langle 2\rangle$$

(cf. [Mat16]) looks encouraging when looking for new splittings of products of known spectra with $MString$ into complex orientable spectra at the prime 2.

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When dealing with objects from derived algebraic geometry, it is convenient to work with ring spectra rather than spectra. Ravenel introduced an important filtration $\{X(n)\}_{n\in\mathbb{N}}$ of MU by ring spectra which arise from the thomification of the filtration

$$\Omega SU(1) \subset \Omega SU(2) \cdots \subset \Omega SU = BU\,.$$

A multiplicative map from $X(n)$ to a complex orientable spectrum E corresponds to a complex orientation of E up to degree n . Locally at a prime p, $X(n)$ is equivalent to a wedge of suspensions of spectra $T(m)$ with $p^m \leq n < p^{m+1}$. We refer to [Rav86, Section 6.5] for a full report on these spectra. They can be regarded as p-typical versions of the spectra $X(n)$: they filter the Brown-Peterson spectrum BP in the same way as the spectra $X(n)$ filter the complex bordism spectrum MU. In addition, they satisfy

$$\pi_* E_* \wedge T(m) \cong E_*[t_1, t_2, \ldots, t_m]$$

for all complex orientable E. A multiplicative map from $T(m)$ to E corresponds to a p-typical orientation up to degree m. The homotopy groups of $T(m)$ coincide with the homotopy groups of BP up to dimension $2(p^m - 2)$.

For $p = 2$, the homotopy groups of $T(2)$ coincide with those of BP up to dimension 12. There is a map from the twelve dimensional even complex $DA(1)$ to BP which induces an isomorphism in π_0 (see for instance [Mat16]). It is unique in mod 2 homology and lifts to a map

$$DA(1) \longrightarrow T(2)$$

by what was just said. Hence, when looking for splitting results we may replace the finite spectrum $DA(1)$ with the reasonably small spectra $T(2)$ or $X(4)$. Along these lines, we mention that

$$\pi_* X(4) \wedge tmf \cong \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$$

carries the Weierstrass Hopf algebroid (cf. [DFHH14, Chapter 9]).

In order to state the main theorem, we work in the $K(2)$-local category and omit the localization functor from the notation. In this category BP splits into a sum of Johnson-Wilson spectra $E(2)$ (see [HS99a]). We shall write T_2 for the even periodic version of $T(2)$ by which we mean the following: the class v_2 of degree 6 is a unit in the local $T(2)$ to which we may associate a root by setting

$$T_2 \quad = \quad T(2)[u^{\pm}]/(u^3 - v_2). \tag{1.1}$$

Note that $T(2)$ is related to $E(2)$ in the same way as T_2 to the Morava E-theory spectrum E_2 so that the notation fits well.

Theorem 1.1. *$K(2)$-locally at the prime 2, there is a splitting of $T_2 \wedge MString$ into a wedge of copies of E_2.*

The proof of the theorem is in spirit of Thom's and Wall's original proof for the splitting of MO and MSO (or even the refinements by Anderson-Brown-Peterson for $MSpin$). It uses a generalized Milnor-Moore argument which we believe to be of independent interest. One version of the original Milnor-Moore theorem states that a graded connected Hopf algebra is free as a module over any of its sub-Hopf algebras. There are dual versions for surjective coalgebra maps and Hopf algebroids [Rav86, A1.1.17], but they all use connectivity and the grading. We will show a version which only uses the coradical filtration of pointed coalgebras. The assumptions needed to make it work are automatically satisfied if the comodules

are graded and connected. This generalization applies in particular to the case of the Hopf algebra $\Sigma = \pi_0(K(n) \wedge E_n)$. We mention that that for $n = 1$ it can be used to give a new elementary proof of the Anderson-Brown-Peterson splitting of $MSpin$ into sums of KOs in the $K(1)$-local setting.
In the $K(2)$-local setting, we prove a key lemma which allows us to switch between the complex and the Witten orientation for $MU\langle 6\rangle$ by a comodule algebra automorphism. It relies on results of Ando-Hopkins-Strickland [AHS01] and supplies the requirements of the generalized Milnor-Moore theorem. Finally, we show that in the $K(2)$-local category a spectrum already splits into sums of copies of E_2 if its $K(2)$-homology splits as a comodule over Σ.

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2. A Milnor-Moore type theorem

Let C be a coalgebra over a field $\mathbb{F}$. Assume that C is pointed, that is, all simple subcoalgebras are 1-dimensional. (Recall that a coalgebra is simple if any proper subcoalgebra is trivial.) The coradical R of a pointed coalgebra is generated by the set G of grouplike elements (cf. [Swe69, p.182]) and

$$R = \mathbb{F}[G]$$

is a subcoalgebra of C. The iterated coproduct defines an increasing filtration

$$F_k = \ker(C \longrightarrow C^{\otimes(k+1)} \longrightarrow (C/R)^{\otimes(k+1)})$$

which is called the *coradical filtration*. By definition, $F_0 = R$. We call a filtration *exhaustive* if every element of C lies in some F_k. The following result is [Swe69, Theorem 9.1.6].

Lemma 2.1. *The coradical filtration satisfies*

$$\Delta(F_n) \subset \sum_{i=0}^{n} F_i \otimes F_{n-i}\,.$$

□

The following two lemmas are well known for irreducible coalgebras.

Lemma 2.2. *For $g \in G$ define the subspace of g-primitives by*

$$P_g = \{c \in C \mid \Delta(c) = c \otimes g + g \otimes c\}\,.$$

Then the inclusion map

$$R \oplus \Big(\bigoplus_{g\in G} P_g\Big) \longrightarrow F_1$$

is an isomorphism.

Proof. By the previous lemma we can write the diagonal of $c \in F_1$ in the form

$$\Delta(c) = \sum a_g \otimes g + g \otimes b_g$$

for suitable a_g, b_g in F_1. Coassociativity and linear independence of grouplike elements yield for each $g \in G$ the equality

$$\Delta a_g \otimes g + g \otimes g \otimes b_g = a_g \otimes g \otimes g + g \otimes \Delta b_g\,.$$

Applying $\epsilon \otimes 1 \otimes \epsilon$ to this equation we get

$$c = a_g + \epsilon(b_g)g = \epsilon(a_g)g + b_g\,.$$

Set $\tilde{c} = c - \sum \epsilon(a_g + b_g)g$. Then

$$\Delta(\tilde{c}) = \sum (a_g - \epsilon(a_g)g) \otimes g + g \otimes (b_g - \epsilon(b_g)g) = \sum \tilde{c} \otimes g + g \otimes \tilde{c}$$

and we conclude $\tilde{c}$ lies in $\bigoplus_{g \in G} P_g$. Hence the map is surjective. Injectivity follows from the linear independence of the grouplike elements. $\square$

Convention 2.3. In the sequel we suppose that Σ is a pointed bialgebra over $\mathbb{F}$ with an exhaustive coradical filtration.

Let G be a group and suppose we are given an injective algebra map

$$\mathbb{F}[G] \longrightarrow \Sigma$$

whose image is the set of grouplike elements. Then one verifies that the canonical map

$$\varphi : \mathbb{F}[G] \otimes P_1 \longrightarrow \bigoplus_{g \in G} P_g, \quad g \otimes \sigma \longmapsto g\sigma$$

is an isomorphism.
Next, suppose M is a right Σ-comodule with coaction ψ. Define a filtration F_k for $k \geq 0$ on M by

$$F_k(M) = \psi^{-1}(M \otimes F_k)\,.$$

Note that this filtration is preserved by maps $f : M \to M'$ of Σ-comodules. Indeed, if m has filtration k in M and if ψ' denotes the diagonal of M' then

$$\psi' f(m) = (f \otimes \mathrm{id})\psi(m) \in f(M) \otimes F_k \subset M' \otimes F_k$$

and hence $f(m) \in F_k(M')$.

Lemma 2.4. *For $g \in G$ define the space of g-primitives by*

$$P_g(M) = \{m \in M |\, \psi(m) = m \otimes g\}$$

and let the space of primitives $P(M)$ be generated by all $P_g(M)$. Then the maps $\oplus_{g \in G} P_g(M) \to P(M)$ and $P(M) \to F_0(M)$ induced by the inclusions are isomorphisms.

Proof. The first isomorphism follows from the linear independence of the grouplike elements. It remains to show surjectivity of the second map. Suppose $m \in F_0(M)$. Then we can write $\psi(m)$ in the form

$$\psi(m) = \sum_g m_g \otimes g$$

for some $m_g \in M$. When applying $1 \otimes \epsilon$ to this equation we obtain

$$m = \sum_g m_g\,.$$

Hence, it suffices to show that m_g lies in $P_g(M)$. Coassociativity implies

$$\sum_g \psi(m_g) \otimes g = \sum_g m_g \otimes g \otimes g$$

and thus $\psi(m_g) = m_g \otimes g$. $\square$

There is another way to think of g-primitives. Recall from [Rav86, A.1.1.4] that the cotensor product $M\Box_\Sigma M'$ of a right comodule (M,ψ,ϵ) with a left comodule (M',ψ',ϵ') is defined as the equalizer of $\psi\otimes \mathrm{id}_{M'}$ and $\mathrm{id}_M\otimes\psi'$. One readily verifies that the maps

$$\begin{array}{rcll} P_g(M) & \longrightarrow & M\Box_\Sigma \mathbb{F}g; & m\mapsto m\otimes g \\ P(M) & \longrightarrow & M\Box_\Sigma R; & (\sum_g m_g)\mapsto (\sum_g m\otimes g) \end{array}$$

are isomorphisms.

Lemma 2.5. $\psi(F_n(M))\subset P(M)\otimes F_n + M\otimes F_{n-1}$.

Proof. Choose a set of representatives σ for a basis in F_n/F_{n-1}. For $m\in F_n(M)$ write $\psi(m)$ in the form

$$\psi(m)=\sum_\sigma m_\sigma\otimes\sigma + \text{ terms in } M\otimes F_{n-1}\,.$$

Then coassociativity and Lemma 2.1 yield modulo terms in $M\otimes F_{n-1}$

$$\sum_\sigma \psi(m_\sigma)\otimes\sigma = \sum_\sigma m_\sigma\otimes\psi(\sigma)=\sum_{\sigma,g} m_\sigma\otimes g\otimes \sigma_g.$$

Let $\langle\sigma_g,\tau\rangle$ be the coefficient of σ_g with respect to the basis element τ. Then the last equation gives

$$\psi(m_\sigma)=\sum_{\tau,g} m_\tau\otimes\langle\tau_g,\sigma\rangle\, g\,\subset M\otimes F_0\,.$$

Thus m_σ has filtration 0, whence the claim. □

In case $M=\Sigma$ we have the two filtrations F_k and $F_k(\Sigma)$. There is yet another filtration given by

$$\bar F_k(\Sigma)=\psi^{-1}(F_k\otimes\Sigma)\,.$$

Fortunately, Lemma 2.1 says that all three filtrations agree.

Definition 2.6. Suppose the comodule M is equipped with maps of comodules $\eta:\mathbb{F}\longrightarrow M$ and $\epsilon\colon M\longrightarrow\mathbb{F}$ which satisfy $\epsilon\eta=\mathrm{id}$. We write 1 for the image of 1 under η as well. Define the graded left primitives by

$$\bar P_1 Gr_k(M)=\{m\in M|\,\psi(m)=1\otimes\sigma \bmod M\otimes F_{k-1} \text{ for some } \sigma\in F_k\}\,.$$

We say a map of comodules f is $\star$-*surjective* if it is surjective and the induced map on graded left primitives is surjective for all k.

Lemma 2.7. *Suppose M is a Σ-comodule $\mathbb{F}[G]$-algebra. Then the map*

$$\varphi:\mathbb{F}[G]\otimes P_1(M)\longrightarrow P(M);\quad g\otimes m\mapsto gm$$

is an isomorphism.

Proof. For $m\in P_1(M)$ the calculation

$$\psi(gm)=(g\otimes g)(m\otimes 1)=gm\otimes g$$

shows $gm\in P_g(M)$. An inverse map is given by

$$\varphi^{-1}(m)=\varphi^{-1}(\sum_g m_g)=\sum_g g\otimes(g^{-1}m_g)\,.$$

□

A version of the classical Milnor-Moore theorem is the statement that a graded connected Hopf algebra is free over each of its sub Hopf algebras. A generalization of the dual statement for comodules is [MM65, Proposition 2.6]. For graded connected Hopf algebroids the result can be found in [Rav86, A1.1.17]. The following result is a generalization to comodules over pointed coalgebras with an exhaustive coradical filtration.

Theorem 2.8. *Let M be a right Σ-comodule $\mathbb{F}[G]$-algebra. Suppose that G is grouplike in M. Let $f\colon M \longrightarrow \Sigma$ be a Σ-comodule and $\mathbb{F}[G]$-algebra map which is $\star$-surjective. Then there is an isomorphism of Σ-comodules*

$$h : M \longrightarrow P_1(M) \otimes \Sigma\,.$$

Proof. Choose a linear left inverse r of the inclusion map $i : P(M) \longrightarrow M$. Define h as the composite

$$M \xrightarrow{\psi} M \otimes \Sigma \xrightarrow{r\otimes\mathrm{id}} P(M) \otimes \Sigma \xrightarrow{\varphi^{-1}\otimes\mathrm{id}} \mathbb{F}[G] \otimes P_1(M) \otimes \Sigma \xrightarrow{\epsilon\otimes\mathrm{id}\otimes\mathrm{id}} P_1(M) \otimes \Sigma$$

with φ as in Lemma 2.7. First we show that h is injective. Let $m \in F_n(M)$ be in the kernel of h. As in Lemma 2.5 write

$$\psi(m) = \sum m_\sigma \otimes \sigma + \text{ terms in } M \otimes F_{n-1}$$

with $m_\sigma \in P(M)$. Write $m_\sigma = \sum_g m_{\sigma,g}$, $m_{\sigma,g} \in P_g(M)$. Again, coassociativity implies modulo terms of lower filtration the equality

$$\sum_\sigma m_{\sigma,g} \otimes \sigma = \sum_\sigma m_{\sigma,g} \otimes \sigma_g\,.$$

Here, σ_g is the term which comes up in

$$\psi(\sigma) = \sum_g g \otimes \sigma_g + \text{ terms of lower filtration.}$$

Calculating modulo $M \otimes F_{n-1}$

$$h(m) = (\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(\varphi^{-1} r \otimes \mathrm{id})\psi(m) = \sum_{\sigma,g} (\epsilon \otimes \mathrm{id} \otimes \mathrm{id})\varphi^{-1}(m_{\sigma,g})) \otimes \sigma_g$$

$$= \sum_{\sigma,g} (\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(g \otimes g^{-1} m_{\sigma,g}) \otimes \sigma_g = \sum_{\sigma,g} g^{-1} m_{\sigma,g} \otimes \sigma_g.$$

Set

$$Gr_{n,g} = \{\sigma \in F_n/F_{n-1} \mid \psi(\sigma) = g \otimes \sigma \bmod M \otimes F_{n-1}\}\,.$$

The map

$$\bigoplus_{g\in G} P_g(M) \otimes Gr_{n,g} \longrightarrow P_1(M) \otimes \bigoplus_{g\in G} Gr_{n,g}$$
$$m_g \otimes \sigma_g \longmapsto g^{-1} m_g \otimes \sigma_g$$

is an isomorphism. Thus the calculation of $h(m)$ implies that $\psi(m)$ vanishes modulo terms in $M \otimes F_{n-1}$. This means that m has filtration m and an obvious induction shows that h is injective.

It remains to show that h is onto. For an element $\sigma \in F_n$ write modulo terms in $\Sigma \otimes F_{n-1}$

$$\psi(\sigma) = \sum g \otimes \sigma_g$$

with $\sigma = \sum_g \sigma_g$ and $\psi(\sigma_g) = g \otimes \sigma_g$. It suffices to show that $n \otimes \sigma_g$ for $n \in P_1(M)$ lies in the image of h modulo terms in $P_1(M) \otimes F_{n-1}$. Choose an inverse m of $g^{-1}\sigma_g$ in $\bar{P}_1 Gr_n(M)$. Then modulo those terms we have $\psi(gm) = g \otimes \sigma_g$ and hence

$$h(gmn) = (\epsilon \otimes \mathrm{id})\varphi^{-1}(gn) \otimes \sigma_g = n \otimes \sigma_g \,.$$

The claim now follows by induction. □

3. The spectrum $T_2 \wedge MString$.

Let p be a prime and let $E = E_n$ be the height n Morava E-theory spectrum at the prime p. Let $\mathfrak{m} = (p, u_1, u_2, \ldots)$ be the unique homogeneous maximal ideal in the coefficient ring

$$E_* \cong W\mathbb{F}_{p^n}[\![u_1, u_2, \ldots u_{n-1}]\!][u, u^{-1}] \,.$$

Then $K = E/\mathfrak{m}$ is a variant of the Morava K-theory spectrum with the same Bousfield localization as $K(n)$. The group Γ of ring spectrum automorphisms of E is a version of the Morava stabilizer group.

Theorem 3.1. [Hov04] *The inclusion of Γ as a subgroup of E^0E induces an isomorphism of the completed group ring*

$$E^*[\![\Gamma]\!] \longrightarrow E^*E \,.$$

Dually, we have the isomorphism

$$E^\vee_* E \longrightarrow C(\Gamma, E_*)$$

between the homotopy groups of the $K(2)$-localized product $E \wedge E$ and the ring of continuous maps from the profinite group Γ to E_*.

The objects $(E_*, E^\vee_* E)$ and $C(\Gamma, E_*)$ are graded formal Hopf algebroids. In fact, they are evenly graded and since E_* has a unit in degree 2 we may as well restrict our attention to the degree 0 part. Let V be BP_* as an ungraded ring and let VT be the ungraded ring BP_*BP. Then Landweber exactness furnishes the equivalence

$$E^\vee_0 E \cong E_0 \otimes_V VT[t_0, t_0^{-1}] \otimes_V E_0.$$

The algebra K_0E is the reduction modulo the maximal ideal $\mathfrak{m}$ and hence itself carries the structure of a Hopf algebroid. (cf. [Hov04, Proposition 3.8]).
The paper at hand deals with the case $p = 2$ and $n = 2$. Here, it will prove useful to employ a version of the Morava E-theory which comes from the deformation of the elliptic curve

$$C: \quad y^2 + y = x^3 \,.$$

The Hopf algebroid $\Sigma = K_0E$ is not pointed when we consider the curve over $\mathbb{F}_4$. Hence, we will only consider it over $\mathbb{F}_2$ so that

$$E_* = \mathbb{Z}_2[\![u_1, u_2, \ldots u_{n-1}]\!][u, u^{-1}] \,.$$

The results stated above hold without changes. Since now left and right unit coincide modulo the ideal $\mathfrak{m}$ the ring $\Sigma = K_0E$ actually carries the structure of a Hopf algebra. Explicitly we have (see [Hov04])

$$\Sigma = \mathbb{F}_2[t_0, t_1, \ldots]/(t_0^3 - 1, t_1^4 - t_1, t_2^4 - t_2, \ldots) \,.$$

Lemma 3.2. *The Hopf algebra Σ is pointed and has an exhaustive coradical filtration.*

Proof. The elements $1, t_0, t_0^2$ are grouplike. All classes t_i with $i \geq 1$ come from the graded Hopf algebroid BP_*BP. (More precisely, the generators t_i in Σ are obtained from those of BP_*BP by multiplication with a suitable power of the periodicity element to place them in degree zero.) A complete formula for the diagonal is given in [ABP67, A2.1.27], but from the grading it is already clear that

$$\Delta(t_i) \quad = \quad 1 \otimes t_i + t_i \otimes 1 + \text{ terms involving only } t_1, t_2, \ldots t_{i-1}. \tag{3.1}$$

Consider the surjection of coalgebras

$$\rho\colon \mathbb{F}_2[\mathbb{Z}/3][t_1, t_2, \ldots] \longrightarrow \Sigma$$

which sends the generator 1 of $\mathbb{Z}/3$ to t_0. Note that the source is in fact a connected graded coalgebra. Suppose S is a non trivial simple subcoalgebra of Σ. Then $\rho^{-1}S$ is a subcoalgebra of the polynomial algebra. From formula 3.1 it is clear that it contains a grouplike element and so does S. Since S is simple it has to coincide with the one-dimensional subcoalgebra generated by this element. It follows that Σ is pointed.

To show exhaustiveness one argues similarly: formula 3.1 implies that for each monomial in the t_i, the maximal degree of a tensor factor, modulo elements in the group ring, decreases each time the diagonal Δ is applied. Hence the polynomial ring is coradically exhaustive and so is Σ. $\square$

Recall from [Rav86, Section 6.5] the ring spectrum $T(2)$ which is part of a filtration of BP and which was already mentioned in the introduction. Equation (1.1) defines the ring spectrum T_2. Note that T_2 comes with a canonical map

$$can\colon T_2 \longrightarrow (L_{K(2)}BP)[u^{\pm}]/(v_2 - u^3) \longrightarrow E\,.$$

Set

$$M = K_0(T_2 \wedge MString)$$
$$M^{\mathbb{C}} = K_0(T_2 \wedge MU\langle 6\rangle).$$

Then M and $M^{\mathbb{C}}$ are right Σ-comodule $\mathbb{F}_2[\mathbb{Z}/3]$-algebras. Moreover, we have the Witten orientation

$$\tau_W : MString \longrightarrow E$$

and the complex orientation induced by the standard coordinate on the elliptic curve C

$$\tau_U : MU\langle 6\rangle \longrightarrow MU \longrightarrow E$$

(for details see e.g. [LO16]). The composite

$$\tau_W^{\mathbb{C}} : MU\langle 6\rangle \longrightarrow MString \xrightarrow{\tau_W} E$$

gives another $MU\langle 6\rangle$-orientation on E. The difference class

$$r_U = \tau_W^{\mathbb{C}}/\tau_U \in E^0(BU\langle 6\rangle)$$

plays an important role in the theory of string characteristic classes (loc. cit.). The orientations induce maps

$$\tau_{W*}: \quad M = K_0(T_2 \wedge MString) \xrightarrow{(can\wedge\tau_W)_*} K_0(E_2 \wedge E_2) \xrightarrow{\mu_*} K_0(E) = \Sigma$$
$$\tau_{U*}: \quad M^{\mathbb{C}} = K_0(T_2 \wedge MU\langle 6\rangle) \xrightarrow{(can\wedge\tau_W^{\mathbb{C}})_*} K_0(E_2 \wedge E_2) \xrightarrow{\mu_*} K_0(E) = \Sigma$$

Lemma 3.3. *There is a Σ-comodule algebra automorphism α of $M^{\mathbb{C}}$ with the property that*

$$\begin{tikzcd} M^{\mathbb{C}} \arrow[r, "\alpha"] \arrow[d] & M^{\mathbb{C}} \arrow[d, "\tau_{U*}"] \\ M \arrow[r, "\tau_{W*}"] & \Sigma \end{tikzcd}$$

commutes.

Proof. Ando-Hopkins-Strickland have shown in [AHS01] that a ring map from $BU\langle 6\rangle_+$ to a complex oriented ring spectrum coincides with a cubical structure on the associated formal group. In particular, such a map is determined by its restriction to

$$P = \mathbb{C}P^\infty \wedge \mathbb{C}P^\infty \wedge \mathbb{C}P^\infty\,.$$

The class $r_U \in E^0 BU\langle 6\rangle$ corresponds to such a cubical structure and hence satisfies the cocycle condition when restricted to P. We claim that the composite

$$r\colon BU\langle 6\rangle_+ \xrightarrow{r_U} E \xrightarrow{\eta_R} K \wedge E$$

already maps to $K \wedge T_2$. This is clear when it is restricted to P because the coefficients of the power series already lie in $\eta_R(\pi_* E)$ and hence in the subring

$$\eta_R(\pi_* T_2) \subset \pi_*(K \wedge T_2)$$

where η_R denotes the right unit. Since the restriction map satisfies the cocycle condition it comes from a map $\hat{r}$ in $(K \wedge T_2)^0 BU\langle 6\rangle$.
Consider the commutative diagram

$$\begin{tikzcd} K \wedge T_2 \wedge MU\langle 6\rangle \arrow[r] \arrow[d, "1\wedge\Delta"] & K \wedge E \wedge MU\langle 6\rangle \arrow[d, "1\wedge\Delta"] \\ K \wedge T_2 \wedge BU\langle 6\rangle_+ \wedge MU\langle 6\rangle \arrow[r] \arrow[d, "1\wedge\hat{r}\wedge 1"] & K \wedge E \wedge BU\langle 6\rangle_+ \wedge MU\langle 6\rangle \arrow[d, "1\wedge r\wedge 1"] \\ K \wedge T_2 \wedge K \wedge T_2 \wedge MU\langle 6\rangle \arrow[r] \arrow[d, "\mu\wedge 1"] & K \wedge E \wedge K \wedge E \wedge MU\langle 6\rangle \arrow[d, "\mu\wedge 1"] \\ K \wedge T_2 \wedge MU\langle 6\rangle \arrow[r] & K \wedge E \wedge MU\langle 6\rangle \end{tikzcd}$$

in which Δ is the Thom diagonal. The automorphism α is induced by the composite of the vertical maps on the left. Since the bottom horizontal map is injective in homotopy it suffices to prove commutativity of the claimed square for the map induced by r instead of $\hat{r}$. This in turn follows immediately from the equality

$$\tau_W = r_U \tau_U^{\mathbb{C}}\,.$$

It remains to show that α is a Σ-comodule automorphism which again is obvious for the the map induced by r instead of $\hat{r}$. □

Proposition 3.4. *The maps τ_{U*} and τ_{W*} are $\star$-surjective. In particular, there are isomorphisms of Σ-comodules*

$$h^{\mathbb{C}} : M^{\mathbb{C}} \longrightarrow P_1(M^{\mathbb{C}}) \otimes \Sigma$$
$$h : M \longrightarrow P_1(M) \otimes \Sigma$$

Proof. As already mentioned in the introduction, Ravenel and Hovey show in [HR95, Corollary 2.2(2)] that at $p = 2$ the spectrum $DA(1) \wedge MU\langle 6\rangle$ is a wedge of suspensions of BP. A closer inspection of the proof reveals that the splitting isomorphism can be chosen to make the diagram

$$\begin{tikzcd} DA(1) \wedge MU\langle 6\rangle \arrow[r, "\simeq"] \arrow[d] & \bigvee_{i\in I} \Sigma^{n_i} BP \arrow[d] \\ MU \wedge BP \arrow[r] & BP \end{tikzcd}$$

commutative. Here the right vertical map is the projection onto the summand which contains the unit in homotopy. Since $K(2)$-locally each BP-summand splits further into sums of Johnson-Wilson spectra $E(2)$ it follows that there is a section of the composite

$$g : DA(1) \wedge MU\langle 6\rangle \longrightarrow BP \longrightarrow E(2)\,.$$

Let j be the map from $E(2)$ to E. Then jg factors over $T_2 \wedge MU\langle 6\rangle$ and we obtain a section s of the canonical map

$$T_2 \wedge MU\langle 6\rangle \longrightarrow E\,.$$

Now it is clear that τ_{U*} is $\star$-surjective. Moreover, Lemma 3.3 implies the same is true for the map τ_{W*}. Hence, the second claim is a corollary of Theorem 2.8. □

Proposition 3.5. *The module $K_*(T_2 \wedge MString)$ is concentrated in even dimensions.*

Proof. It is clear that K_*T_2 is concentrated in even dimensions. For the module

$$K_*MString \cong K_*BString,$$

this is the main result of [KLW04] (see also [Lau16, Remark 3.1, Theorem 1.3] for this version of Morava K-Theory). The claim follows from the Künneth isomorphism. □

Proposition 3.6. *Let X be a $K(n)$-local spectrum whose Morava K-homology is concentrated in even degrees. Then X is a wedge of copies of E if and only if $K_0(X)$ is cofree as a comodule over $\Sigma = K_0(E)$.*

Proof. Obviously, the cofreeness as a comodule is a necessary condition for a splitting. In order to show the converse, we first observe with [HS99b, Proposition 8.4(e)(f)] that the Morava K-homology of X being even implies the profreenes of $E^\vee_*(X)$ and

$$K_*(X) \cong E^\vee_*(X)/\mathfrak{m}.$$

Choose a wedge F of copies of E and an isomorphism of Σ-comodules $\bar\alpha$ from $K_0(X)$ to $K_0(F)$. Also, choose a lift of $\bar\alpha$ in the diagram

$$\begin{tikzcd} E^\vee_0(X) \arrow[r, dashed, "\alpha"] \arrow[d, "\bmod \mathfrak{m}"] & E^\vee_0(F) \arrow[d, "\bmod \mathfrak{m}"] \\ K_0(X) \arrow[r, "\bar\alpha"] & K_0(F) \end{tikzcd}\quad.$$

The universal coefficient theorem for E-module spectra yields an isomorphism

$$\alpha^*\colon F^*(F) \cong \mathrm{Hom}_{E_*}(E^\vee_*F, F_*) \xrightarrow{\cong} \mathrm{Hom}_{E_*}(E^\vee_*(X), F_*) \cong F^*(X).$$

Let $f\colon X \to F$ be the image of 1 under this map. In other words, f corresponds to the composite

$$E^\vee_*(X) \xrightarrow{\alpha_*} E^\vee_*(F) \xrightarrow{\mu_*} F_*$$

where μ is the multiplication. We claim that f induces the map $\bar\alpha$ in K-homology and hence is a K-local isomorphism. Let $p\colon F \to E$ be the projection onto a summand. It suffices to show the equality

$$p_*f_* = p_*\bar\alpha : K_0X \longrightarrow K_0E$$

of Σ-comodule maps or, dually, the equality of $K^0(E)$-module maps from K^0E to K^0X. By construction, the maps p_*f_* and $p_*\bar\alpha$ coincide when composed with the augmentation μ_* to K_*. Hence, the dual maps coincide on the generator $1 \in K^0(E)$ and the result follows. □

Proof of Theorem 1.1. The theorem is a consequence of the Propositions 3.4, 3.5 and 3.6. □

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