Notes on Cobordism

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Contents

CHAPTER 1

Unoriented Bordism

1. Steenrod's Question

Let M be a closed, smooth n-manifold. Then $H_n(M;\mathbb{F}_2)$ contains a fundamental class $[M]$ which is characterized by the fact that it restricts to the non-zero element in

$$H_n(M, M \setminus \{x\};\mathbb{F}_2) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\};\mathbb{F}_2) \cong \mathbb{F}_2$$

for each $x \in M$.

N.E. Steenrod asked the following question. Given a space X and a homology class $\alpha \in H_n(X;\mathbb{F}_2)$ can α be represented by a singular manifold (M, f)? That is, is there a closed n-manifold M and a continuous map $f : M \to X$ such that $f_*[M] = \alpha$?

An oriented manifold carries a fundamental integral homology class and we may ask the analogous question in this case as well. We will only discuss the first question and for the rest of this section all homology groups are taken with $\mathbb{F}_2$ coefficients unless otherwise indicated.

Let us first investigate when a singular manifold (M, f) represents the zero homology class. Suppose that our manifold M is the boundary of an $(n+1)$-manifold W and that f extends to a map $F : W \to X$. Then the left arrow in the commutative diagram

$$\begin{array}{ccc} H_{n+1}(W, M) & \xrightarrow{F_*} & H_{n+1}(X, X) \\ \partial \downarrow & & \partial \downarrow \\ H_n(M) & \xrightarrow{f_*} & H_n(X) \end{array}$$

sends the fundamental class of W to the fundamental class of M, and since $H_{n+1}(X, X) = 0$, we have $f_*[M] = 0$.

Definition 1.1. *We say that a closed manifold M is* **null-bordant** *if there exists a manifold W whose boundary is M. A singular manifold (M, f) is* **null-bordant** *if there exists an $F : W \to X$ extending f as above.*

Note that two singular manifolds (M, f) and (N, g) determine the same class in $H_n(X)$ if $(f, g) : M \amalg N \to X$ is null-bordant. When this is so we will say that (M, f) and (N, g) are **bordant**.

The concept of bordism was first introduced by R. Thom in [**24**]. Bordism is an equivalence relation. The only non-trivial point to check is transitivity, which requires some knowledge of differential topology.

Definition 1.2. *We define the* **unoriented bordism group** *of X, denoted $\mathfrak{N}_n(X)$, to be the set of all isomorphism classes of singular n-manifolds $M \to X$ modulo bordism. (There are no set-theoretic problems, as any n-manifold can be embedded in $\mathbb{R}^N$ for large enough N.) This is an abelian group under disjoint union, and since everything has order two, it is an $\mathbb{F}_2$-vector space.*

In fact, $\mathfrak{N}_n$ is a covariant functor from topological spaces to vector spaces. Given $k : X \to Y$ one defines $\mathfrak{N}_n(k) : \mathfrak{N}_n(X) \to \mathfrak{N}_n(Y)$ by $\mathfrak{N}_n(k)(M, f) = (M, kf)$. It is not difficult to check that the bordism relation is preserved under composition.

Moreover, there are product maps $\mathfrak{N}_m(X) \otimes \mathfrak{N}_n(Y) \to \mathfrak{N}_{m+n}(X \times Y)$ sending $(M, f, X) \otimes (N, g, Y)$ to $(M \times N, f \times g, X \times Y)$. Again, one must check that this is well-defined. An interesting special case occurs when X and Y consist of a single point. Then we find that $\mathfrak{N}_*(*)$ is a graded-commutative $\mathbb{F}_2$-algebra. It is denoted $\mathfrak{N}_*$ and is called the **bordism ring**.

Notice that we have a well-defined map $\mathfrak{N}_n(X) \to H_n(X)$ which sends $f : M \to X$ to $f_*[M]$. With these definitions, Steenrod's question can be phrased, *is this map surjective?*

Similarly, we can play the same game with oriented manifolds. In this case we get the **oriented bordism group** $\Omega_n(X)$, when we define M to be bordant to N if there is a W with $\partial W = M - N$. (Here $M - N$ denotes the disjoint union of M and N with the orientation of N reversed.) Now the second question becomes, *is the map $\Omega_n(X) \to H_n(X; \mathbb{Z})$ surjective?*

In the next few sections we develop the machinery we need to attack these problems.

2. Thom Spaces and Stable Normal Bundles

Let M be an n-manifold and consider an embedding i of M in $\mathbb{R}^{n+k}$. (Putting $k = n$ would suffice, but we will see below that it is better to let k be arbitrary.) Such an embedding defines a k-plane bundle ν over M, called the **normal bundle**, by requiring that the sequence

$$0 \longrightarrow \tau M \xrightarrow{i_*} i^*\tau\mathbb{R}^{n+k} \longrightarrow \nu \longrightarrow 0$$

be exact. Equivalently, one could put a metric on the tangent bundle $\tau\mathbb{R}^{n+k}$ and define $E(\nu)$ to be those vectors in the restriction $i^*\tau\mathbb{R}^{n+k}$ which are orthogonal to τM.

A word on notation may be in order. When we refer to a vector bundle as a whole, we denote it by a symbol such as ξ. The total space, base space, and projection map will be denoted $E(\xi)$, $B(\xi)$ and π_ξ, with the ξ omitted when the context makes it clear.[1] As usual, $f^*\xi$ is the pullback of the bundle ξ by a map $f : X \to B(\xi)$. A good reference on vector bundles is Milnor and Stasheff [**15**].

The normal bundle ν depends on the choice of embedding. An embedding i into $\mathbb{R}^{n+k}$ can always be extending to an embedding i' into $\mathbb{R}^{n+m}$ for $m > k$ by composing it with the usual embedding of $\mathbb{R}^{n+k}$ into $\mathbb{R}^{n+m}$. These two embeddings are easy to compare. We find that $\nu' = \nu + \boldsymbol{(m-k)}$, where $\boldsymbol{(m-k)}$ denotes the trivial rank $m-k$ bundle over M.

To compare arbitrary embeddings, we need the following two theorems, whose proofs may be found in the books of Hirsch [**5**] and Lang [**9**].

THEOREM 2.1. *Given an embedding i of M in $\mathbb{R}^{n+k}$ with normal bundle ν, there is an embedding of $E(\nu)$ onto an open set in $\mathbb{R}^{n+k}$ extending the*

[1] An exception to this rule occurs when one of the spaces already has a name. For example, the base space of the tangent bundle τM of a manifold M is of course M, and the total space is usually written TM.

embedding of M:

$$\begin{array}{ccc} E(\nu) & & \\ {\scriptstyle \text{0-section}}\big\uparrow & \searrow & \\ M & \xhookrightarrow{i} & \mathbb{R}^{n+k} \end{array} .$$

This can be pictured as a tubular neighborhood of M in $\mathbb{R}^{n+k}$.

THEOREM 2.2. *Suppose i and j are embeddings of M into $\mathbb{R}^{n+k}$ and $\mathbb{R}^{n+l}$ respectively, with extensions to embeddings of $E(\nu_i)$ and $E(\nu_j)$. Then for m large enough, the composite embeddings of $E(\nu_i)$ and $E(\nu_j)$ into $\mathbb{R}^{n+m}$ are isotopic. Roughly speaking, this means that they are homotopic through embeddings.*

Therefore, $\nu_i + \boldsymbol{(m-k)}$ and $\nu_j + \boldsymbol{(m-l)}$ are isomorphic. In other words, ν_i and ν_j are **stably isomorphic**. So any manifold has a well-defined **stable normal bundle**.

Maps from spheres are always interesting to topologists, and since the one-point compactification of $\mathbb{R}^{n+k}$ is S^{n+k} one is led to study compactifications. Compactification is a contravariant functor in the sense that if $U \hookrightarrow S$ is an open inclusion, we get a pointed map $S_+ \to U_+$, where S is a locally compact Hausdorff space and $+$ denotes the one-point compactification. Note that we use the convention that the basepoint of S_+ is the point at infinity, and if S is compact, then S_+ is S with a disjoint basepoint.

Applying this to the above, we get a pointed map

$$S^{n+k} = \mathbb{R}^{n+k}_+ \longrightarrow E(\nu)_+ =: \operatorname{Th}\nu. \tag{2.1}$$

In general, given a vector bundle ξ over a compact space X, $E(\xi)_+$ is called the **Thom space** of ξ and is denoted $\operatorname{Th}\xi$. There is an alternate definition, which gives the same answer for compact X, but works in general. Choose a metric on ξ and consider $D(\xi)$, the closed disk bundle, and $S(\xi)$, the unit sphere bundle. Define $\operatorname{Th}\xi$ to be $D(\xi)/S(\xi)$.

The Thom construction is a covariant functor. A bundle map $\xi \to \eta$ which is an injective on each fibre gives rise to a map $\operatorname{Th}\xi \to \operatorname{Th}\eta$ in the obvious way. Also, for locally compact spaces, $(S \times T)_+ = S_+ \wedge T_+$. Therefore

$\mathrm{Th}(\xi \times \eta) = \mathrm{Th}\,\xi \wedge \mathrm{Th}\,\eta$, where $\xi \times \eta$ denotes the bundle $E\xi \times E\eta \to B\xi \times B\eta$. There is a homeomorphism $\mathrm{Th}(\xi \times \eta) = \mathrm{Th}\,\xi \wedge \mathrm{Th}\,\eta$ even when $B\xi$ and $B\eta$ are not compact.

3. The Pontrjagin–Thom Construction

An Invariant of a Singular Manifold. The classifying space $BO(k)$ carries a k-plane bundle ξ_k and for any k-plane bundle ξ over a paracompact space X, there exists a map $\eta : X \to BO(k)$, unique up to homotopy, such that $\eta^*\xi_k \cong \xi$. Because of this direct correspondence, the map η will be denoted by ξ as well.

Let (M, f) be a singular n-manifold in X and fix an embedding $i : M \hookrightarrow \mathbb{R}^{n+k}$. For the normal bundle ν we get a map $\nu : M \to BO(k)$. That $\nu^*\xi_k$ is isomorphic to ν is equivalent to saying that $\nu : M \to BO(k)$ is covered by a bundle map $\overline{\nu}$ which is an isomorphism on each fibre:

$$\begin{array}{ccc} E(\nu) & \xrightarrow{\overline{\nu}} & E(\xi_k) \\ \big\downarrow & & \big\downarrow \\ M & \xrightarrow{\nu} & BO(k) \end{array} .$$

Now we may form the following bundle map:

$$\begin{array}{ccc} E(\nu) & \xrightarrow{(\overline{\nu}, f\pi)} & E(\xi_k) \times X = E(\xi_k \times \mathbf{0}) \\ \big\downarrow & & \big\downarrow \\ M & \xrightarrow{(\nu, f)} & BO(k) \times X \end{array} .$$

Here $\mathbf{0}$ denotes the rank 0 bundle over X. The Thom construction is functorial, so we get an induced map

$$\mathrm{Th}\,\nu \longrightarrow \mathrm{Th}(\xi_k \times \mathbf{0}).$$

Combining this with the map from the sphere (2.1) gives us a map

$$S^{n+k} \longrightarrow \mathrm{Th}\,\nu \longrightarrow \mathrm{Th}\,\xi_k \wedge X_+,$$

where we have used the fact that $\mathrm{Th}\,\mathbf{0} = X_+$. (The quotient $D(\mathbf{0})/S(\mathbf{0})$ is by definition X with a disjoint basepoint.)

The Thom space $\operatorname{Th}\xi_k$ will occur so often that we will denote it $MO(k)$. The fruit of our labor is thus the homotopy class in

$$\pi_{n+k}(MO(k) \wedge X_+)$$

defined by the above map. How does this homotopy class depend on the choices we've made along the way?

To determine this we examine the effect of stabilization. Extend our embedding i to an embedding i' into $\mathbb{R}^{n+k+1}$. We claim that the following diagram commutes

$$\begin{array}{ccccc} S^{n+k+1} & \longrightarrow & \operatorname{Th}(\mathbf{1}+\nu) & \longrightarrow & MO(k+1) \wedge X_+ \\ \uparrow \cong & & \uparrow \cong & & \uparrow \alpha\wedge 1 \\ \Sigma S^{n+k} & \longrightarrow & \Sigma \operatorname{Th}\nu & \longrightarrow & \Sigma MO(k) \wedge X_+ \end{array} ,$$

where the top line comes from i' and the bottom line is the suspension of the maps from i. The leftmost vertical arrow is clear, and the middle vertical arrow comes from the fact that

$$\operatorname{Th}(\mathbf{n}_X + \xi) \cong \operatorname{Th}(\mathbf{n}_* \times \xi) \cong S^n \wedge \operatorname{Th}\xi,$$

where we distinguish between the trivial bundles over X and $*$ by subscripts. The map α is induced by the bundle map that classifies $\mathbf{1} + \xi_k$:

$$\begin{array}{ccc} E(\mathbf{1}+\xi_k) & \longrightarrow & E(\xi_{k+1}) \\ \downarrow & & \downarrow \\ BO(k) & \longrightarrow & BO(k+1) \end{array} .$$

The moral of all of this is that a singular manifold (M, f) determines an element of

$$MO_n(X) := \varinjlim_k \pi_{n+k}(MO(k) \wedge X_+),$$

where the maps in the colimit are induced by the suspension homomorphism and the $\alpha \wedge 1$'s.

The Dependence on the Bordism Class. To obtain a natural homomorphism $\mathfrak{N}_n(X) \to MO_n(X)$ we also have to explain why the equivalence relation is respected by our construction. That is, we must determine what happens if our singular manifold is null-bordant.

Suppose $F : W \to X$ restricts to f on $\partial W = M$. A theorem from differential topology says that any embedding of M in S^{n+k} can be extended to a 'nice' embedding of W in D^{n+k+1} such that ∂W is precisely $W \cap S^{n+k}$. By 'nice' we mean that W is not tangent to S^{n+k} at any point $x \in \partial W$: $T_xW \not\subseteq T_xS^{n+k}$. Then any tubular neighbourhood ν of M in the sphere is the intersection with S^{n+k} of a tubular neighbourhood $\tilde{\nu}$ of W in the disk [**5**]. Thus applying the above construction to $\mathrm{Th}\,\tilde{\nu}$ we get a commutative diagram

$$\begin{array}{ccccc} S^{n+k} & \longrightarrow & \mathrm{Th}\,\nu & \longrightarrow & MO(k) \wedge X_+ \\ \big\downarrow & & \big\downarrow & \nearrow & \\ D^{n+k+1} & \longrightarrow & \mathrm{Th}\,\tilde{\nu} & & \end{array}$$

and have verified that the upper row is nullhomotopic.

The Inverse Homomorphism. Next we sketch the construction of a two-sided inverse to the map $\mathfrak{N}_n(X) \to MO_n(X)$. First observe that the classifying space $BO(k)$ can be approximated by the Grassmann manifolds $Gr_k(\mathbb{R}^{n+k})$. $Gr_k(\mathbb{R}^{n+k})$ is the space of all k-dimensional subspaces of $\mathbb{R}^{n+k}$ and comes equipped with a canonical bundle $\xi_{k,n}$ with total space

$$E(\xi_{k,n}) = \{(V, x) \,:\, x \in V\} \subset Gr_k(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}.$$

Classifying $\xi_{k,n}$ we obtain maps $Gr_k(\mathbb{R}^{n+k}) \to BO(k)$ for all $n \in \mathbb{N}$. Actually, $BO(k)$ can be considered as the union of all these $Gr_k(\mathbb{R}^{n+k})$ under the natural inclusions $Gr_k(\mathbb{R}^{n+k}) \hookrightarrow Gr_k(\mathbb{R}^{n+1+k})$ and $E(\xi_k)$ as the union of the $E(\xi_{k,n})$.

We also have the notion of transversality, the topological version of 'general position'. Let

$$\begin{array}{ccc} U \times_X V & \xrightarrow{\bar{f}} & V \\ \Big\downarrow{\scriptstyle \bar{g}} & & \Big\downarrow{\scriptstyle g} \\ U & \xrightarrow{f} & X \end{array}$$

be a pullback diagram where U, V and X are smooth manifolds without boundary and f and g are differentiable maps. When does $U \times_X V$ carry the structure of a manifold?

DEFINITION 3.1. *f and g are said to be* **transverse** *if for every (u, v) in $U \times_X V$*

$$f_* T_u(U) + g_* T_v(V) = T_x X,$$

where $x = f(u) = g(v)$.

If in particular the sum of the dimensions of U and V is lower than the dimension of X, transversality cannot hold unless the images of f and g are disjoint. In case of complementary dimensions the images must meet in a 0-dimensional manifold. The following can be found in any book on differential topology.

THEOREM 3.2. *(i) If f and g are transverse, then $U \times_X V$ is a submanifold of $U \times V$ of codimension* $\dim X$.
(ii) For any smooth f and g there is a map f_1 'near' f which is transverse to g. Nearby maps are homotopic.
(iii) If f and g are transverse, and g is an embedding with normal bundle ν, then $\bar{g}$ is an embedding of $U \times_X V$ in U with normal bundle $\bar{f}^*\nu$.

Let us now be given a map $g : S^{n+k} \to MO(k) \wedge X_+$. Its composition with the projection

$$MO(k) \wedge X_+ = \frac{MO(k) \times X}{* \times X} \xrightarrow{pr_1} MO(k) = \mathrm{Th}\, \xi_k$$

compresses to some $\mathrm{Th}\, \xi_{k,n} \subset \mathrm{Th}\, \xi_k$, since S^{n+k} is compact. Though the space $\mathrm{Th}\, \xi_{k,n}$ has a singularity at the basepoint, the complement of the basepoint is a manifold. Since the zero section $G_k(\mathbb{R}^{n+k}) \hookrightarrow \mathrm{Th}\, \xi_{k,n}$ stays far from the singularity the above theorem applies. Thus we can deform $pr_1 g$ to a transverse map g_1. The pullback diagram

$$\begin{array}{ccc} M & \xrightarrow{\bar{g}_1} & Gr_k(\mathbb{R}^{n+k}) \\ \big\downarrow & & \big\downarrow \\ S^{n+k} & \xrightarrow{g_1} & \mathrm{Th}\, \xi_{k,n} \end{array}$$

defines a manifold M. The restriction of $pr_2\, g$ to M makes sense and gives an element in $\mathfrak{N}_n(X)$.

Note that the normal bundle of $M \subset S^{n+k}$ is classified by $\bar{g}_1$. The reader may check that we have indeed defined an inverse homomorphism.

4. Spectra

In these notes we take a cavalier approach to spectra. An excellent and much more thorough exposition of the category of spectra can be found in Part III of [**2**].

Definition 4.1. *A* **spectrum** *is a sequence of pointed spaces X_k for $k \in \mathbb{Z}$ together with maps*

$$\Sigma X_k \longrightarrow X_{k+1}.$$

Why do we care about spectra? In section 3 we were forced to look at the spaces $MO(k)$ and the maps $\alpha_k : \Sigma MO(k) \to MO(k+1)$, and we defined

$$MO_n(X) := \varinjlim_k \pi_{n+k}(MO(k) \wedge X_+)$$

(for $k < 0$ take $MO(k) = *$). However this is only the first example of a spectrum.

Let X be a pointed space. The **suspension spectrum** $\Sigma^\infty X$ has k^{th} space $\Sigma^k X$ and structure maps $\Sigma(\Sigma^k X) \xrightarrow{\text{id}} \Sigma^{k+1} X$. We denote the suspension spectrum $\Sigma^\infty S^0$ of the zero-sphere by S and sometimes write X for $\Sigma^\infty X$. Another example arises from an abelian group G. The Eilenberg–Mac Lane complexes $K(G, n)$ are characterized up to homotopy equivalence by the property

$$\pi_i(K(G,n)) = \begin{cases} G & i = n \\ 0 & i \neq n \end{cases}.$$

These spaces represent singular cohomology with coefficients in G. That is,

$$\bar{H}^n(X; G) \cong [X, K(G,n)]$$

for all pointed CW-spaces X. Let HG denote the spectrum with $HG_n := K(G,n)$ and connecting maps $\Sigma K(G,n) \to K(G,n+1)$ given by the image of the identity map of $K(G,n)$ in

$$[K(G,n),K(G,n)] = \bar{H}^n(K(G,n);G) \cong \bar{H}^{n+1}(\Sigma K(G,n);G) = [\Sigma K(G,n),K(G,n+1)].$$

THEOREM 4.2 (G.W. Whitehead). *If E is spectrum, then on pointed spaces the functor*

$$X \longmapsto \varinjlim_k \pi_{n+k}(E_k \wedge X) =: \bar{E}_n X$$

is a reduced generalized homology theory. That is, for every map $f : A \to X$ we have a long exact sequence

$$\cdots \longrightarrow \bar{E}_k(A) \xrightarrow{f_*} \bar{E}_k(X) \longrightarrow \bar{E}_k(C_f) \xrightarrow{\partial} \bar{E}_{k-1}(A) \longrightarrow \cdots .$$

By C_f we mean the cofibre $X \cup_f CA$ of the map f. If f is a CW-inclusion or more generally a cofibration then C_f is homotopy equivalent to the quotient X/A. The boundary map above is simply the map induced by the map $C_f \to \Sigma A$ that collapses X to a point composed with the natural map $\bar{E}_k(\Sigma A) \to \bar{E}_{k-1}(A)$. As an aside, note that this last map is an isomorphism. This follows, for example, from using the long exact sequence coming from the inclusion of A into CA.

Thus $\mathfrak{N}_*(-)$ is a homology theory and we can use many tools known from singular homology. As an exercise the reader may check the exactness property directly from the definition of $\mathfrak{N}_*(X)$.

The Eilenberg–Mac Lane spectrum HG gives us a homology theory $\overline{HG}_*(-)$ with coefficients

$$HG_n(*) = \overline{HG}_n(S^0) = \varinjlim_k \pi_{n+k}(K(G,k) \wedge S^0) = \begin{cases} G & n = 0 \\ 0 & n \neq 0 \end{cases} .$$

By the uniqueness theorem this is ordinary homology with coefficients in G:

$$\bar{H}_n(X;G) \cong \pi_{n+k}(K(G,k) \wedge X) \text{ for } k > n.$$

Thus all difficulties in computing the homotopy groups of a space X are removed by smashing X with an Eilenberg–Mac Lane complex.

We may extend the definition of the homology of a space to spectra by defining

$$H_n E := \varinjlim_k \bar{H}_{n+k} E_k,$$

where the colimit is taken over

$$\bar{H}_{n+k}(E_k) \cong \bar{H}_{n+k+1}(\Sigma E_k) \longrightarrow \bar{H}_{n+k+1}(E_{k+1}).$$

In fact spectra form a category. We don't want to go into a discussion of the morphisms, but suffice it to say that they should be thought of as stable homotopy classes of maps. This category also has a smash product, and its definition takes some care. The important point is that $H_n E$ is equal to $\pi_n(H\mathbb{Z} \wedge E)$, where the latter is defined to be $[S^n, H\mathbb{Z} \wedge E]$. This leads to the general definition

$$F_n E = [S^n, F \wedge E]$$

for any two spectra E and F.[2] We also define

$$F^n E = [E, S^n \wedge F].$$

The spectrum MO has some additional structure. For example, we have maps

$$MO(m) \wedge MO(n) \longrightarrow MO(m+n)$$

induced from the classifying map

$$\begin{array}{ccc} E(\xi_m) \times E(\xi_n) & \longrightarrow & E(\xi_{m+n}) \\ \downarrow & & \downarrow \\ BO(m) \times BO(n) & \longrightarrow & BO(m+n) \end{array} .$$

[2]We adopt the rule of thumb that symbols should not be interchanged when unnecessary. Hence $F_* E$ is $\pi_*(F \wedge E)$ and not $\pi_*(E \wedge F)$.

These fit together to give a map $MO \wedge MO \to MO$. We also have maps $S^n \to MO(n)$ such that

$$\begin{array}{ccc} \Sigma S^n & \longrightarrow & \Sigma MO(n) \\ \Big\| & & \Big\downarrow{\scriptstyle \alpha_n} \\ S^{n+1} & \longrightarrow & MO(n+1) \end{array}$$

commutes. These are simply the maps induced from the inclusion of the fibre $\mathbb{R}^n \to E(\xi_n)$. This gives a map from the sphere spectrum S to MO.

If we were willing to take the time to define the smash product of spectra, we would find that this product has the sphere spectrum as a unit and could make the following definition.

Definition 4.3. *A* **ring spectrum** *is a spectrum E and maps of spectra $\eta : S \to E$ and $\mu : E \wedge E \to E$ such that*

$$\begin{array}{ccccc} S \wedge E & \xrightarrow{\eta \wedge 1} & E \wedge E & \xleftarrow{1 \wedge \eta} & E \wedge S \\ & {\scriptstyle \cong}\searrow & \Big\downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \cong} & \\ & & E & & \end{array}$$

commutes. When appropriate we speak of an **associative** *or* **commutative** *ring spectrum.*

When E is an associative ring spectrum, $\pi_* E$ has a natural ring structure. The above maps make MO into an associative, commutative ring spectrum and

$$\mathfrak{N}_* \longrightarrow \pi_*(MO)$$

is a ring homomorphism. See Adams' book for more details.

5. The Thom Isomorphism

Suppose ξ is an n-plane bundle over a paracompact space X. Choose a metric on ξ and consider the disk bundle $D(\xi)$ and the sphere bundle $S(\xi)$. Recall that the Thom space $\mathrm{Th}\,\xi$ is defined to be $D(\xi)/S(\xi)$. Therefore,

$$\bar{H}^*(\mathrm{Th}\,\xi) \cong H^*(D(\xi), S(\xi)).$$

There is a relative form of Serre's spectral sequence with E_2 term $H^*(X;\tilde{R}^*)$ converging to $H^*(D(\xi),S(\xi);R)$, where R is a ring and $\tilde{R}^k$ is the local coefficient system with value above the point $x \in X$ given by

$$H^k(D(\xi)_x, S(\xi)_x; R) \cong H^k(D^n, S^{n-1}; R) \cong \begin{cases} R & k = n \\ 0 & k \neq n \end{cases}.$$

Since the coefficients are non-zero only in one dimension, the spectral sequence collapses and we have that

$$H^q(X;\tilde{R}^n) \cong \bar{H}^{q+n}(\operatorname{Th}\xi; R).$$

Now, if $R = \mathbb{F}_2$ the local system is canonically trivialized, but in general it may not be trivial at all. If it is trivial, then a choice of trivialization is called an ***R*-orientation** of ξ. Such a trivialization gives for each x an isomorphism

$$R \cong H^n(D(\xi)_x, S(\xi)_x; R) \tag{5.1}$$

which depends continuously on x. An orientation gives us an isomorphism $H^*(X;R) \cong \bar{H}^{*+n}(\operatorname{Th}\xi;R)$, the so-called **Thom isomorphism**.

We can say more. Using the fact that the spectral sequence is a spectral sequence of modules over H^*X (we omit the coefficients now for brevity) we see that the Thom isomorphism is an isomorphism of H^*X-modules, where the H^*X-module structure on $H^*(\operatorname{Th}\xi)$ is defined in the following way. From the pullback diagram

$$\begin{array}{ccc} E(\xi) & \longrightarrow & E(\mathbf{0}) \times E(\xi) \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

we get an induced map $\operatorname{Th}\xi \to X_+ \wedge \operatorname{Th}\xi$, which in cohomology gives

$$\bar{H}^*(\operatorname{Th}\xi) \longleftarrow H^*X \otimes \bar{H}^*(\operatorname{Th}\xi).$$

We denote the image of $x \otimes y$ by $x \cdot y$.

So we have that $\bar{H}^*(\operatorname{Th}\xi)$ is free of rank 1 as an H^*X-module. We denote the generator by u and call this the **Thom class**. It is determined by the choice of R-orientation, which, as mentioned above, is no choice at all when $R = \mathbb{F}_2$. The Thom class is characterized by the fact that for any x, its restriction to

$D(\xi)_x/S(\xi)_x$ is the distinguished generator of $H^n(D(\xi)_x/S(\xi)_x)$ given by the orientation. That is, it is the image of 1 under the isomorphism (5.1).

NOTE 5.1. If ξ and η are oriented vector bundles with Thom classes u_ξ and u_η, then $\xi \times \eta$ is naturally oriented and has corresponding Thom class $u_{\xi\times\eta} = u_\xi \wedge u_\eta$ under the identification $\mathrm{Th}(\xi \times \eta) \cong \mathrm{Th}\,\xi \wedge \mathrm{Th}\,\eta$.

The Thom isomorphism says that $\bar{H}^*(\mathrm{Th}\,\xi) \cong \bar{H}^*(\Sigma^n X_+)$ as H^*X-modules. Thus we can think of $\mathrm{Th}\,\xi$ as a 'twisted suspension' of X_+. When ξ is trivial, $\mathrm{Th}\,\xi$ is in fact the suspension of X_+. When ξ is simply R-orientable this might not be the case, but the twisting is mild enough that singular cohomology groups cannot distinguish between the two spaces. If we are able to use other means to distinguish between $\mathrm{Th}\,\xi$ and $\Sigma^n(X_+)$ we will have detected the non-triviality of the bundle. We will use Steenrod operations to do this.

For a slightly different approach to the topics in this section, see Milnor and Stasheff [**15**].

6. Steenrod Operations

In this section our coefficient ring will always be $\mathbb{F}_2$.

THEOREM 6.1. *There is a unique family of natural transformations of functors from the category* Top *of spaces and continuous maps to the category* Ab *of abelian groups*

$$\mathrm{Sq}^k_n : H^n(X) \longrightarrow H^{n+k}(X) \quad (n \geq 0, k \geq 0)$$

such that

(i) The 0^{th} square is the identity: $\mathrm{Sq}^0_n = \mathrm{id}$.
(ii) The n^{th} square is the cup square on classes of dimension n: $\mathrm{Sq}^n_n\, x = x^2$.
(iii) The 'Cartan formula' holds: for $x \in H^m X$ and $y \in H^n X$

$$\mathrm{Sq}^k_{m+n}(x \smile y) = \sum_{i+j=k} (\mathrm{Sq}^i_m\, x) \smile (\mathrm{Sq}^j_n\, y).$$

A construction of the squares may be found in Appendix A. From the above axioms we can deduce several consequences.

Proposition 6.2 (External Cartan formula). *For* $x \in \bar{H}^m X$ *and* $y \in \bar{H}^n Y$ *we have*

$$\operatorname{Sq}^k_{m+n}(x \wedge y) = \sum (\operatorname{Sq}^i_m x) \wedge (\operatorname{Sq}^j_n y)$$

in $\bar{H}^{m+n+k}(X \wedge Y)$.

To prove this, first prove the analogous result for the cartesian product in unreduced cohomology, by considering $(x \times 1) \smile (1 \times y)$. Then use that the natural map $H^*(X \wedge Y) \to H^*(X \times Y)$ is monic.

As an example of the proposition, take $Y = S^1$ and $y = \sigma \in \bar{H}^1(S^1)$ the generator. Then we have

$$\operatorname{Sq}^k_{m+1}(x \wedge \sigma) = (\operatorname{Sq}^k_m x) \wedge \sigma.$$

That is, the diagram

$$\begin{array}{ccc} \bar{H}^m X & \xrightarrow{\operatorname{Sq}^k_m} & \bar{H}^{m+k} X \\ \downarrow \cong & & \downarrow \cong \\ \bar{H}^{m+1}(\Sigma X) & \xrightarrow{\operatorname{Sq}^k_{m+1}} & \bar{H}^{m+k+1}(\Sigma X) \end{array}$$

commutes. This property of the squares is referred to as **stability**. Because of stability, one usually omits the bottom subscript on the squares.

Another fact that follows from the axioms is that $\operatorname{Sq}^k x = 0$ for $k > \dim x$. We leave this as an exercise. [I don't know how to prove this from the axioms; Spanier includes this as one of the axioms. So does Whitehead, with the Cartan formula replaced by the assumption of stability. JDC]

Note that if we write $\operatorname{Sq} x = x + \operatorname{Sq}^1 x + \operatorname{Sq}^2 x + \cdots$ (a finite sum) the Cartan formula takes the form $\operatorname{Sq}(x \smile y) = \operatorname{Sq} x \smile \operatorname{Sq} y$. Also, we have that $\operatorname{Sq} 1 = 1$, so Sq is in fact a ring homomorphism.

7. Stiefel-Whitney Classes

Continuing to work with coefficient ring $\mathbb{F}_2$, we have that any rank n vector bundle ξ has a canonical Thom class u in $\bar{H}^n(\operatorname{Th}\xi)$. Since $\bar{H}^*(\operatorname{Th}\xi)$ is freely generated by u, we have that $\operatorname{Sq}^k u = w_k(\xi) \cdot u$ for a unique class $w_k(\xi) \in H^k X$. We call $w_k(\xi)$ the k^{th} **Stiefel-Whitney class** of ξ. It is easy to see that if $f : Y \to X$, then $w_k(f^*\xi) = f^* w_k(\xi)$, since everything in sight is natural.

This is what it means to have a **characteristic class**. In particular, we may consider the Stiefel-Whitney classes of the canonical bundle ξ_n over $BO(n)$. By naturality, these $w_k(\xi_n) \in H^k(BO(n))$ determine the characteristic class w_k entirely.

Note that $w_k(\xi) = 0$ for $k > \operatorname{rank}(\xi)$, since $\operatorname{Sq}^k u = 0$ for $k > \dim u = \operatorname{rank} \xi$.

EXAMPLE 7.1. In order to study the canonical bundle λ over $BO(1) = \mathbb{RP}^\infty$, we consider the canonical bundle λ_n over finite-dimensional real projective space $\mathbb{RP}^n$. We have a natural embedding $\mathbb{RP}^{n-1} \hookrightarrow \mathbb{RP}^n$ and with respect to this embedding λ_{n-1} is the restriction of λ_n.

What is the normal bundle ν of this embedding? Given a line L in $\mathbb{R}^n$, a nearby line in $\mathbb{R}^{n+1}$ in the direction of the new coordinate is determined by giving a linear map $L \to \mathbb{R}$. Therefore

$$\nu \cong \operatorname{Hom}(\lambda_{n-1}, \mathbf{1}) = \lambda_{n-1}^*$$

canonically, and by choosing a metric on λ_{n-1} we have that $\lambda_{n-1}^* \cong \lambda_{n-1}$. (For details and a picture, see Milnor and Stasheff.) These 'nearby' lines contain everything in $\mathbb{RP}^n$ except the 'line at infinity' L_∞. So $\mathbb{RP}^n \setminus \{L_\infty\} = E(\nu)$. Thus

$$\operatorname{Th} \lambda_{n-1} \cong \operatorname{Th} \nu = E(\nu)_+ \cong \mathbb{RP}^n .$$

To summarize, *the Thom space of the canonical bundle over* $\mathbb{RP}^{n-1}$ *is* $\mathbb{RP}^n$*.*

This allows us to inductively compute the cohomology of $\mathbb{RP}^n$. Given that $H^*(\mathbb{RP}^{n-1}) = \mathbb{F}_2[x]/(x^n)$, we know that $\bar{H}^*(\mathbb{RP}^n)$ is free of rank 1 over $\mathbb{F}_2[x]/(x^n)$ and so

$$\bar{H}^*(\mathbb{RP}^n) =< u, xu, x^2u, \dots , x^{n-1}u > .$$

But under the inclusion $\mathbb{RP}^{n-1} \hookrightarrow \mathbb{RP}^n$, u restricts to x. Therefore $H^*(\mathbb{RP}^n) = \mathbb{F}_2[u]/(u^{n+1})$. Since the cohomology in dimension k only depends on the $(k+1)$-skeleton, we get that $H^*(\mathbb{RP}^\infty) = \mathbb{F}_2[u]$.

As a final conclusion, notice that $u^2 = \operatorname{Sq}^1 u = w_1(\lambda) \cdot u$ and so $w_1(\lambda) = u$.

The following proposition follows directly from the Cartan formula and an earlier remark about the Thom class of a product bundle.

PROPOSITION 7.2 (Whitney Sum Formula). *Define* $w(\xi) = 1 + w_1(\xi) + \cdots + w_n(\xi)$. *Then*

$$w(\xi \times \eta) = w(\xi) \times w(\eta).$$

This implies that $w(\xi \oplus \eta) = w(\xi) \cdot w(\eta)$. The bundle $\xi \oplus \eta$ is the **Whitney sum** of ξ and η and is defined to be the pullback of $\xi \times \eta$ along the diagonal $\Delta : X \to X \times X$.

EXAMPLE 7.3. Consider the bundle $\xi = \lambda \times \overset{(n)}{\cdots} \times \lambda$ over $(\mathbb{R}\mathrm{P}^\infty)^n$. We know that $H^*((\mathbb{R}\mathrm{P}^\infty)^n) = \mathbb{F}_2[x_1, \ldots, x_n]$ with each x_i of dimension 1, and

$$w(\xi) = \prod_{i=1}^{n}(1 + x_i) = \sum_{k=0}^{n} \sigma_k(x_1, \ldots, x_n),$$

where σ_k is the k^{th} elementary symmetric function.

By the naturality of the Stiefel-Whitney classes,

$$\begin{array}{ccc} \mathbb{F}_2[w_1, \ldots, w_n] & \xrightarrow{\beta} & H^*(BO(n)) \\ \downarrow & & \downarrow \\ \mathbb{F}_2[x_1, \ldots, x_n] & = & H^*((\mathbb{R}\mathrm{P}^\infty)^n) \end{array}$$

commutes, where in the top-left corner the w_k are formal symbols, the map β sends w_k to $w_k(\xi_n)$, the map on the left-hand side sends w_k to $\sigma_k(x_1, \ldots, x_n)$, and the right-hand map is induced from the classifying map for ξ. Since the symmetric polynomials are algebraically independent, the left-hand map is injective. Therefore, the map β is injective.

THEOREM 7.4. *β is an isomorphism.*

One proof of surjectivity constructs a CW-structure for $BO(n)$ such that the number of cells in dimension l is equal to the rank of the polynomial algebra $\mathbb{F}_2[w_1, \ldots, w_n]$ in dimension l. See Milnor and Stasheff for details.

EXAMPLE 7.5. The stable tangent bundle τ of $\mathbb{R}\mathrm{P}^{n-1}$ has a description in terms of the canonical line bundle. Suppose we are given a line L in $\mathbb{R}^n$. A tangent vector in $T_L\mathbb{R}\mathrm{P}^{n-1}$ can be pictured as an infinitesimal movement along a normal vector $v \in \mathbb{R}^n$ at a non-zero point $x \in L$; for non-zero $\lambda \in \mathbb{R}$, λv at

λx represents the same tangent vector. Thus there is a natural isomorphism from the tangent space $T_L\mathbb{R}\mathrm{P}^{n-1}$ to $\mathrm{Hom}(L, L^\perp)$. This implies that

$$n\,\lambda \cong n\,\lambda^* = \mathrm{Hom}(\lambda, \mathbf{n}) = \mathrm{Hom}(\lambda, \lambda^\perp \oplus \lambda)$$
$$\cong \mathrm{Hom}(\lambda, \lambda^\perp) \oplus \mathrm{Hom}(\lambda, \lambda) \cong \tau \oplus \mathbf{1}.$$

For the last equation observe that $\mathrm{Hom}(\lambda, \lambda)$ has a nowhere vanishing cross section given by the identity of λ. Since a trivial bundle is the pullback under a constant map, naturality tells us that $w(\mathbf{n}) = 1$. It follows that

$$w(\tau) = w(\tau \oplus \mathbf{1}) = w(n\lambda) = w(\lambda)^n = (1+x)^n = \sum_{k=0}^{n-1} \binom{n}{k} x^k,$$

with x the non-trivial element in $H^1(\mathbb{R}\mathrm{P}^{n-1})$.

The following application illustrates the usefulness of characteristic classes. Since we will not refer to it later, the reader may skip to the next section.

Given an m-dimensional manifold M we may ask how many everywhere linearly independent vector fields exist on M. That is, we are looking for a trivial subbundle of the tangent bundle τ of maximal dimension. Now a subbundle in τ furnishes a decomposition $\tau = \mathbf{k} \oplus \mathbf{k}^\perp$. Thus the Stiefel-Whitney classes

$$w(\tau) = w(\mathbf{k})\, w(\mathbf{k}^\perp) = w(\mathbf{k}^\perp)$$

have to vanish in dimensions greater than $m - k$. Therefore, in the case of $\mathbb{R}\mathrm{P}^{n-1}$ we are interested in the binomial coefficients

$$\binom{n}{k} \equiv \begin{cases} 1 \mod 2 & \text{if } k \subseteq n \\ 0 \mod 2 & \text{otherwise} \end{cases}.$$

Here $k \subseteq n$ means that whenever the l^{th} binary digit of k is 1, then so is the l^{th} digit of n. Let $\nu(n)$ be the largest power of 2 in n. Then

$$n - 2^{\nu(n)} = \max\{k < n : \binom{n}{k} \equiv 1 \mod 2\}$$

is a lower bound for the codimension of any trivial subbundle in τ. Thus we are led to the following conclusion.

Proposition 7.6. *There are at most* $2^{\nu(n)} - 1$ *independent vector fields on* $\mathbb{R}\mathrm{P}^{n-1}$.

Corollary 7.7. *If* $\mathbb{R}\mathrm{P}^{n-1}$ *is parallelizable,* i.e., *has a trivial tangent bundle, then* n *is a power of 2.*

In fact F. Adams has shown that $\mathbb{R}\mathrm{P}^1$, $\mathbb{R}\mathrm{P}^3$ and $\mathbb{R}\mathrm{P}^7$ are the only parallelizable real projective spaces.

Exercise 7.8. Show that $\mathbb{R}\mathrm{P}^{n-1}$ does not immerse in codimension $2^d - n - 1$, where 2^d is the smallest power of 2 with $2^d \geq n$. In particular $\mathbb{R}\mathrm{P}^{2^l}$ does not immerse in $\mathbb{R}^{2^{l+1}-2}$.

An old result of Whitney says that every smooth manifold of dimension greater than 1 can be immersed in $\mathbb{R}^{2n-1}$. Thus his result is optimal in a certain sense.

Exercise 7.9. Let $n = \sum_{i=1}^{k} 2^{e_i}$. Then $\mathbb{R}\mathrm{P}^{2^{e_1}} \times \mathbb{R}\mathrm{P}^{2^{e_2}} \times \cdots \times \mathbb{R}\mathrm{P}^{2^{e_k}}$ does not immerse in $\mathbb{R}^{2n-\alpha(n)-1}$, where $\alpha(n)$ denotes the sum of the bits e_i.

Whitney conjectured that any n-dimensional manifold can be immersed in $\mathbb{R}^{2n-\alpha(n)}$.

8. The Euler Class

Definition 8.1. *The* **Euler class** $e \in H^n(X; R)$ *of an* R*-oriented vector bundle* ξ *over* X *is the characteristic class obtained as the image of the Thom class* $u \in \bar{H}^n(\mathrm{Th}\,\xi; R)$ *under the projection*

$$\mathrm{Th}\,\xi \xleftarrow{\pi} D\xi \simeq X,$$

or, if you like, under the zero section

$$\mathrm{Th}\,\xi \xleftarrow{s} X.$$

Lemma 8.2. *If* ξ *has a nowhere vanishing section then* $e(\xi) = 0$.

Proof. Let σ be a normalized nowhere vanishing section. Then as t runs from 0 to 1 the maps $t\sigma$ provide a homotopy between the zero section s and σ itself. But as a map to $\mathrm{Th}\,\xi$, σ is constant, so s is nullhomotopic. □

Thus in contrast to the Stiefel-Whitney classes the Euler class is unstable.

LEMMA 8.3. *The equality $e(\xi) = w_n(\xi)$ holds for $R = \mathbb{F}_2$.*

PROOF. The diagonal $X \xrightarrow{\Delta} X \times X$ is covered by two bundle maps

$$\begin{array}{ccc} \xi & \longrightarrow & \mathbf{0} \times \xi \\ & {\scriptstyle\Delta}\searrow & \downarrow \\ & & \xi \times \xi \end{array}$$

which are homotopic to each other through vector bundle maps; a homotopy is given by $h_t(v) := (tv, v)$. Applying the Thom space construction we obtain a homotopy commutative diagram

$$\begin{array}{ccc} \operatorname{Th} \xi & \longrightarrow & \operatorname{Th} \mathbf{0} \wedge \operatorname{Th} \xi \\ & \searrow & \downarrow {\scriptstyle s \wedge 1} \\ & & \operatorname{Th} \xi \wedge \operatorname{Th} \xi \end{array}$$

such that in cohomology $u \wedge u$ is mapped to $e \cdot u = u^2 \in H^{2n}(\operatorname{Th} \xi)$. On the other hand we have $u^2 = \operatorname{Sq}^n u = w_n(\xi) \cdot u$ and the lemma follows by the Thom isomorphism. □

Since $s^* : \bar{H}^*(\operatorname{Th} \xi) \to \bar{H}^* X$ is an H^*X-module homomorphism we see that $\operatorname{im} s^* = H^*X \cdot e$. This implies that

$$\bar{H}^* MO(n) \cong H^* BO(n) \cdot w_n(\xi),$$

since $H^*BO(n)$ is a polynomial algebra on the Stiefel-Whitney classes (7.4). Moreover, the isomorphism respects the squares. Thus to understand $\bar{H}^*MO(n)$ as a module over the Steenrod algebra we just have to focus our attention on the element $w_n(\xi)$.

9. The Steenrod Algebra

In Section 6 we introduced certain natural transformations

$$\operatorname{Sq}^i : H^n(X; \mathbb{F}_2) \longrightarrow H^{n+i}(X; \mathbb{F}_2).$$

These generate the algebra of stable operations, which we want to investigate now.

LEMMA 9.1 (Yoneda). *Let* $F : \mathcal{C} \to \mathrm{Set}$ *be any contravariant functor and let* $K \in \mathcal{C}$. *Then the map*

$$\begin{array}{rcl} \mathrm{Nat}([-,K],F) & \longrightarrow & F(K) \\ \theta & \longmapsto & \theta_K(\mathrm{id}_K) \end{array}$$

is a bijection.

The proof is easy and left as an exercise. It follows that

$$\mathrm{Nat}(H^k, H^n) \cong H^n(K_k) = [K_k, K_n],$$

where K_k is short for the Eilenberg–Mac Lane space $K(\mathbb{F}_2, k)$. Since the squares are not only natural transformations but homomorphisms, we would like to know which cohomology classes correspond to additive natural transformations.

The addition in H^k corresponds to a product $K_k \times K_k \xrightarrow{\mu} K_k$; if $y, z \in H^*X$ are represented by $\hat{y}, \hat{z} : X \to K_k$ respectively, then $\widehat{y+z}$ is the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{\hat{y}\times\hat{z}} K_k \times K_k \xrightarrow{\mu} K_k.$$

Thus the additivity of the operation represented by $x \in H^n K_k$ is equivalent to the commutativity of

$$\begin{array}{ccc} K_k \times K_k & \xrightarrow{\mu} & K_k \\ {\scriptstyle \hat{x}\times\hat{x}}\downarrow & & \downarrow{\scriptstyle \hat{x}} \\ K_n \times K_n & \xrightarrow{\mu} & K_n \end{array} .$$

If $\psi : H^*K_k \to H^*(K_k \times K_k) \cong H^*K_k \otimes H^*K_k$ is the map induced by μ, then the composite $\hat{x}\mu$ represents ψx. On the other hand, since

$$K_k \times K_k \xrightarrow{\Delta} (K_k \times K_k) \times (K_k \times K_k) \xrightarrow{pr_1 \times pr_2} K_k \times K_k$$

is the identity and $1 \otimes x$ and $x \otimes 1$ are represented by the appropriate projection followed by $\hat{x}$, one sees that the composite $\mu(\hat{x} \times \hat{x})$ represents $1 \otimes x + x \otimes 1$. Therefore, the commutativity of the square is equivalent to requiring that $\psi x = 1 \otimes x + x \otimes 1$. Now we have proved the following result.

PROPOSITION 9.2. *Additive natural transformations $H^k \to H^n$ are in one-to-one correspondence with the* **primitive elements**

$$PH^n(K_k) := \{x \in H^n K_k : \psi x = x \otimes 1 + 1 \otimes x\}.$$

Note that in low dimensions everything is primitive; since the multiplication μ has a two sided unit, we always have

$$\psi x = 1 \otimes x + x \otimes 1 + \text{other terms in} \bigoplus_{p+q=n} \bar{H}^p K_k \otimes \bar{H}^q K_k.$$

But $\bar{H}^i K_k = 0$ for $i < k$ and so there are no other possible terms in ψx if $n < 2k$.

The change of the behaviour in this dimension also becomes apparent by looking at the spectral sequence of the path space fibration

$$\begin{array}{ccc} K_{k-1} \simeq \Omega K_k & \hookrightarrow & (K_k, *)^{(I,0)} \\ & & \downarrow \\ & & K_k \end{array} .$$

Since the total space is contractible, all transgressions

$$d^q : H^{q-1} K_{k-1} \cong E_q^{0,q-1} \longrightarrow E_q^{q,0} \cong H^q K_k$$

are isomorphisms for $q < 2k$. Note that an inverse to these is given by

$$H^q K_k \cong [\, K_k, K_q \,] \xrightarrow{\Omega} [\, K_{k-1}, K_{q-1} \,] \cong H^{q-1} K_{k-1}.$$

Therefore, an element in the inverse limit

$$\mathcal{A}^n := \varprojlim_q H^{q+n} K_q$$

can be considered as a **stable cohomology operation** that shifts dimensions by n. Stable cohomology operations are additive and we have just seen the following.

PROPOSITION 9.3. *$\mathcal{A}^n \cong H^{q+n} K_q$ for $q \geq n$.*

The sum $\mathcal{A} := \bigoplus_{n \geq 0} \mathcal{A}^n$ forms a graded algebra, the so called **Steenrod algebra**. For the following we refer to the book of Mosher and Tangora [**16**].

THEOREM 9.4. *The squares* Sq^i *generate* $\mathcal{A}$. *More precisely, a basis for* $\mathcal{A}$ *is given by*

$$\{\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_k} \,:\, i_{j-1}\geq 2i_j,\ k\geq 0\}.$$

Sequences $(i_1, i_2, \ldots, i_k)$ *with* $i_{j-1}\geq 2i_j$ *are called* **admissible**.

The relations among the squares are called the Adem relations (see [**22**]). Let us ignore the relations for a minute and think about the free, associative $\mathbb{F}_2$-algebra $\mathcal{T}$ generated by Sq^0, Sq^1, $\mathrm{Sq}^2, \ldots$ modulo $\mathrm{Sq}^0 = 1$, with $|\mathrm{Sq}^i| = i$. Note that there is a natural surjection $\mathcal{T}\to\mathcal{A}$ sending Sq^i to itself. With the Cartan formula in mind we can write down an algebra map

$$\mathcal{T}\xrightarrow{\psi}\mathcal{T}\otimes\mathcal{T} \qquad \mathrm{Sq}^k\longmapsto\sum_{i+j=k}\mathrm{Sq}^i\otimes\mathrm{Sq}^j\,.$$

LEMMA 9.5. ψ *extends to a map* $\psi:\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$. *That is, the Adem relations are respected by* ψ.

PROOF. Unfortunately the map $\phi:\mathcal{A}\to H^*(K_n\times K_n)$ sending θ to $\theta(\iota_n\times\iota_n)$ is not an algebra map. However, it fits into the diagram

$$\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\ \psi\ } & \mathcal{T}\otimes\mathcal{T} \\
\downarrow & & \downarrow \\
\mathcal{A} & \overset{\psi}{\dashrightarrow} & \mathcal{A}\otimes\mathcal{A} \\
{\scriptstyle\phi}\downarrow & & \downarrow \\
H^*(K_n\times K_n) & \xleftarrow{\ \cong\ } & H^*K_n\otimes H^*K_n
\end{array}$$

whose commutativity is equivalent to the iterated Cartan formula

$$\mathrm{Sq}^I(\iota_n\times\iota_n) = \sum_{J+K=I}\mathrm{Sq}^J\iota_n\times\mathrm{Sq}^K\iota_n.$$

Here Sq^I denotes the product $\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_k}$ for a multiindex $I = (i_1, i_2, \ldots, i_k)$. To finish the argument observe that $\mathcal{A}\otimes\mathcal{A}\to H^*K_n\otimes H^*K_n$ is an isomorphism in degree lower than n. □

The 'comultiplication' ψ makes $\mathcal{A}$ into a 'Hopf algebra'. Before continuing our study of the Steenrod algebra, we learn a bit about Hopf algebras.

10. Hopf Algebras

If we describe algebras in terms of commutative diagrams, the definition looks as follows. Let k be a field. A graded vector space A over k together with maps

$$A \otimes A \xrightarrow{\mu} A \xleftarrow{\eta} k$$

is an **algebra** if the multiplication μ is associative and η is a unit. That is, if the diagrams

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes 1} & A \otimes 1 \\ {\scriptstyle 1\otimes\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array} \qquad \begin{array}{ccccc} k \otimes A & \xrightarrow{\eta \otimes 1} & A \otimes A & \xleftarrow{1 \otimes \eta} & A \otimes k \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1} & \\ & & A & & \end{array}$$

are commutative.

Definition 10.1. *A graded vector space C over k together with maps*

$$C \otimes C \xleftarrow{\psi} C \xrightarrow{\epsilon} k$$

is a **coalgebra** *if the same diagrams commute with all arrows reversed. ψ is the* **comultiplication** *or* **diagonal map**, *and ϵ is the* **counit** *or* **augmentation**. *(Unless otherwise stated, maps of graded objects preserve degree. Hence $\epsilon(c) = 0$ for c of non-zero degree, since k is graded with the trivial gradation: k in degree 0, 0 in other degrees.)*

Example 10.2. H_*X becomes a coalgebra by

$$\begin{aligned} \psi &: H_*(X;k) \xrightarrow{\Delta_*} H_*(X \times X;k) \xrightarrow{\cong} H_*(X;k) \otimes H_*(X;k) \\ \epsilon &: H_*(X;k) \longrightarrow H_*(*;k) = k. \end{aligned}$$

In fact $C = H_*X$ is even **cocommutative**, *i.e.*, the diagram

$$\begin{array}{ccc} & & C \otimes C \\ & {\scriptstyle \psi}\nearrow & \downarrow{\scriptstyle T} \\ C & \xrightarrow[\psi]{} & C \otimes C \end{array}$$

commutes. Here T denotes the switch map $T(x \otimes y) = (-1)^{|x|\,|y|} y \otimes x$. We say C is **connected** if $\epsilon : C \to k$ is an isomorphism in degrees 0 and below.

For example, H_*X is connected iff X is path connected. In this case we write $1 \in C_0$ for the unique element with $\epsilon(1) = 1$. We will also use the notation

$$\psi x = \sum_{i=1}^{n} x_i' \otimes x_i'' = \sum x' \otimes x''$$

following the Einstein convention halfway. The elements $x' \otimes x''$ are not uniquely determined. But since

$$\begin{array}{ccccc} & & C & & \\ & {\scriptstyle 1}\swarrow & \downarrow{\scriptstyle \psi} & \searrow{\scriptstyle 1} & \\ C \otimes k & \xleftarrow{1\otimes\epsilon} & C \otimes C & \xrightarrow{\epsilon\otimes 1} & k \otimes C \end{array}$$

is commutative, or in other words

$$\sum x' \otimes \epsilon x'' = x \otimes 1, \quad 1 \otimes x = \sum \epsilon x' \otimes x'',$$

we may write

$$\sum x' \otimes x'' = 1 \otimes x + x \otimes 1 + \sum_{|x'|,|x''|>0} x' \otimes x''$$

in the connected case. As before an element $x \in C$ is **primitive** if $\psi x = 1 \otimes x + x \otimes 1$.

The maps

$$\mathcal{A} \otimes \mathcal{A} \xleftarrow{\psi} \mathcal{A} \xrightarrow{\epsilon} k$$

with $\epsilon(\mathrm{Sq}^0) = 1$ define a cocommutative coalgebra structure on $\mathcal{A}$. But more is true: ψ is also an algebra map.

Definition 10.3. *A* **Hopf algebra** *is a vector space A with maps μ, η, ψ and ϵ such that (A, μ, η) is an algebra, (A, ψ, ϵ) is a coalgebra, and ψ and ϵ are algebra maps. (Drawing the diagrams shows that the last condition is equivalent to saying that μ and η are coalgebra maps. Thus the definition is symmetric.)*

We want to develop some general facts about Hopf algebras. The reader will find a more thorough treatment in the article of Milnor and Moore [**14**].

Let A be a Hopf algebra and let M and N be A-modules (this uses only the algebra structure of A). Then $M \otimes_k N$ is an A-module with diagonal action

$$\begin{array}{ccc} A \otimes (M \otimes N) & \xrightarrow{\theta_{M\otimes N}} & M \otimes N \\ \downarrow{\scriptstyle \psi\otimes 1} & & \uparrow{\scriptstyle \theta_M \otimes \theta_N} \\ (A \otimes A) \otimes (M \otimes N) & \xrightarrow{1\otimes T \otimes 1} & A \otimes M \otimes A \otimes N \end{array} ,$$

or, if you like,

$$a\,(x \otimes y) := \sum (-1)^{|a''|\,|x|} a'x \otimes a''y.$$

For instance, this structure turns $H^*X \otimes H^*Y$ into an $\mathcal{A}$-module and the product

$$H^*X \otimes H^*Y \xrightarrow{\times} H^*(X \times Y)$$

into an $\mathcal{A}$-module map. Moreover H^*X is an **$\mathcal{A}$-module algebra**, *i.e.,* an $\mathcal{A}$-module and an algebra such that the algebra structure maps are $\mathcal{A}$-module maps.

An A-module coalgebra is defined similarly, and for use in Section 12 we prove the following result.

PROPOSITION 10.4 (Milnor-Moore). *Let k be a field, let A be a connected Hopf algebra over k, and let M be a connected A-module coalgebra. Assume that $i : A \to M : a \mapsto a \cdot 1$ is monic. Then M is a free A-module.*

PROOF. Let $\bar{M} = M/IM$, where $I = \ker(\epsilon : A \to k)$ is the augmentation ideal, and let $\pi : M \to \bar{M}$ be the projection. Choose a k-linear splitting $\sigma : \bar{M} \to M$, so we have $\pi\sigma = 1_{\bar{M}}$. Let $e = \sigma\pi : M \to M$; it is an idempotent and $(1 - e)M = IM$. Define $\phi : A \otimes \bar{M} \to M$ to be the composite

$$A \otimes \bar{M} \xrightarrow{1\otimes\sigma} A \otimes M \xrightarrow{\text{action}} M.$$

$A \otimes \bar{M}$ is a free A-module, and we claim that ϕ is an A-linear isomorphism. (Unless otherwise adorned, all tensor products are over the field.) Note that the action of A on $A \otimes \bar{M}$ is defined using the coproduct in A, but that in this case it amounts to acting by multiplication on the left factor. One can easily check that ϕ is A-linear and that the following equations

$$\begin{aligned}\sigma(x) &= \phi(1\otimes x)\\ e(m) &= \phi(1\otimes \pi m)\end{aligned}$$

hold for $x\in\bar{M}$ and $m\in M$.

Now we prove surjectivity of ϕ by induction on the degree of an element m in M. Assume that ϕ hits every element of M of degree less than n, and let $m\in M_n$. Then

$$\begin{aligned}m &= em+(1-e)m\\ &= \phi(1\otimes\pi m)+\sum a_i m_i \quad \text{for } a_i\in I, m_i\in M, |m_i|<n\\ &= \phi(1\otimes\pi m)+\phi(\sum a_i y_i),\end{aligned}$$

where y_i is chosen so that $m_i=\phi(y_i)$, and we have used the A-linearity of ϕ. Therefore, ϕ is surjective.

Next we prove that ϕ is monic by showing that the composite γ

$$A\otimes\bar{M}\xrightarrow{\phi}M\xrightarrow{\psi}M\otimes M\xrightarrow{1\otimes\pi}M\otimes\bar{M}$$

is monic. First notice that γ sends $1\otimes x$ to

$$\sigma x\otimes\pi 1+\cdots+1\otimes\pi\sigma x=\sigma x\otimes 1+\cdots+1\otimes x,$$

where we denote $\pi 1$ by 1. Since each of the maps making up γ is A-linear, so is γ. Thus $a\otimes x$ maps to

$$a\sigma x\otimes 1+\cdots+ia\otimes x.$$

Filter $\bar{M}$ by degree: $(F_j\bar{M})_n=\bar{M}_n$ if $n\le j$ and 0 otherwise. Since γ is filtration preserving, it induces a map $A\otimes F_j/F_{j-1}\to M\otimes F_j/F_{j-1}$. From the above description of γ, it is clear that the induced map is simply $i\otimes 1$ which is injective. Working by induction on j one can use the diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ A \otimes F_{j-1} & \rightarrowtail & M \otimes F_{j-1} \\ \downarrow & & \downarrow \\ A \otimes F_j & \longrightarrow & M \otimes F_j \\ \downarrow & & \downarrow \\ A \otimes F_j/F_{j-1} & \overset{i\otimes 1}{\rightarrowtail} & M \otimes F_j/F_{j-1} \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

to prove the inductive step. □

11. Return of the Steenrod Algebra

In this section we explore $\mathcal{A}$ using its representation on $H^*\mathbb{R}P^\infty = \mathbb{F}_2[x]$. We calculate

$$\operatorname{Sq} x^n = (\operatorname{Sq} x)^n = x^n(1+x)^n,$$

and thus

$$\operatorname{Sq}^i x^n = \binom{n}{i} x^{n+i}.$$

Since the Frobenius map is linear modulo 2 we have that

$$\operatorname{Sq}(x^{2^n}) = (x + x^2)^{2^n} = x^{2^n} + x^{2^{n+1}}.$$

That is, the only non-zero squares on x^{2^n} are $\operatorname{Sq}^0 x^{2^n} = x^{2^n}$ and $\operatorname{Sq}^{2^n} x^{2^n} = x^{2^{n+1}}$. Thus we obtain an interesting $\mathcal{A}$-submodule

$$\mathcal{A}x = < x, x^2, x^4, x^8, \dots >,$$

which is sometimes denoted $F(1)$. We also may look at the bigger representation $H^*((\mathbb{R}P^\infty)^n) = \mathbb{F}_2[x_1, x_2, \dots, x_n]$, $|x_i| = 1$, and at the orbit of the Euler class $x_1 x_2 \cdots x_n$.

LEMMA 11.1. *$\mathcal{A}$ acts freely on $x_1x_2\cdots x_n$ up to degree n. That is, the map*

$$\begin{array}{rcl}\mathcal{A}^q & \longrightarrow & H^{n+q}((\mathbb{R}P^\infty)^n)\\ \theta & \longmapsto & \theta(x_1x_2\cdots x_n)\end{array}$$

is monic for $q \leq n$.

PROOF. We want to show that the admissible squares act independently in dimensions $q \leq n$. To calculate their action on the Euler class we may use the Cartan formula. However, the large sums force us to isolate a certain summand.

Therefore, let us order the monomials $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ of a fixed degree left-lexicographically. We say that $x_1^{i_1}\cdots x_n^{i_n} < x_1^{j_1}\cdots x_n^{j_n}$ if and only if $i_1 = j_1$, $\dots$, $i_{k-1} = j_{k-1}$ and $i_k < j_k$ for some k. Given an admissible monomial $\mathrm{Sq}^I = \mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_k}$ we see from

$$\begin{aligned}\mathrm{Sq}^r(x_1x_2\cdots x_n) &= \sum_{s_1+\cdots+s_n=r} \mathrm{Sq}^{s_1}x_1\,\mathrm{Sq}^{s_2}x_2\,\cdots\,\mathrm{Sq}^{s_n}x_n\\ &= x_1^2\cdots x_r^2\,x_{r+1}\cdots x_n + \text{smaller terms}\end{aligned}$$

that the largest term in $\mathrm{Sq}^I(x_1\cdots x_n)$ is

$$x_1^{2^k}\cdots x_{e_k}^{2^k}\,x_{e_k+1}^{2^{k-1}}\cdots x_{e_k+e_{k-1}}^{2^{k-1}}\,x_{e_k+e_{k-1}+1}^{2^{k-2}}\cdots x_n.$$

Here the indices $(e_1, e_2, \dots, e_k)$ are defined by

$$e_s := \begin{cases} i_s - 2i_{s+1} & \text{for } \quad 0 < s < k\\ i_k & \text{for } \qquad s = k\end{cases}.$$

We are now able to recover I from the largest term of $\mathrm{Sq}^I(x_1\cdots x_n)$ because the sequences $(e_1, e_2, \dots, e_k)$ with $e_i \geq 0$ are in one-to-one correspondence with admissible sequences $(i_1, i_2, \dots, i_k)$. □

We can also use our representation to construct linear functionals on $\mathcal{A}$. Let ξ_i be the element of the dual Hopf algebra $\mathcal{A}_* := \mathrm{Hom}_{\mathbb{F}_2}(\mathcal{A}, \mathbb{F}_2)$ whose value on $\theta \in \mathcal{A} = \mathcal{A}^*$ is the coefficient of x^{2^i} in θx. Thus

$$\theta x = \sum_{i=0}^{\infty} \langle\, \theta, \xi_i\,\rangle x^{2^i}.$$

Note that ξ_i has degree $2^i - 1$.

$\mathcal{A}_*$ is commutative as an algebra and has a very beautiful structure.

THEOREM 11.2 (Milnor). *The natural map* $\mathbb{F}_2[\xi_1, \xi_2, \dots] \xrightarrow{\beta} \mathcal{A}_*$ *is an isomorphism.*

PROOF. Consider $H^*((\mathbb{R}\mathrm{P}^\infty)^n) = \mathbb{F}_2[x_1, \dots, x_n]$ and recall that $\mathcal{A}$ acts faithfully on $x_1 \cdots x_n$ through dimension n in $\mathcal{A}$. Let's calculate the action of an arbitrary $\theta \in \mathcal{A}$ on this element.

$$\begin{aligned}
\theta(x_1 \cdots x_n) &= \sum \theta^{(1)} x_1 \cdots \theta^{(n)} x_n \\
&= \sum \sum_{i_1, \dots, i_n} \langle\, \theta^{(1)}, \xi_{i_1} \,\rangle\, x_1^{2^{i_1}} \cdots \langle\, \theta^{(n)}, \xi_{i_n} \,\rangle\, x_n^{2^{i_n}} \\
&= \sum \sum_{i_1, \dots, i_n} \langle\, \theta^{(1)} \otimes \cdots \otimes \theta^{(n)}, \xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \,\rangle\, x_1^{2^{i_1}} \cdots x_n^{2^{i_n}} \\
&= \sum_{i_1, \dots, i_n} \langle\, \theta, \xi_{i_1} \cdots \xi_{i_n} \,\rangle\, x_1^{2^{i_1}} \cdots x_n^{2^{i_n}}.
\end{aligned}$$

The last equality holds because the multiplication in $\mathcal{A}_*$ is the dual of the diagonal map in $\mathcal{A}^*$. (We are using a variation of the notation introduced earlier to write the iterated coproduct of θ as $\sum \theta^{(1)} \otimes \cdots \otimes \theta^{(n)}$.) Now we prove that β is surjective. If it is not, there exists a non-zero $\theta \in \mathcal{A}$ such that $\langle\, \theta, \xi_{i_1} \cdots \xi_{i_n} \,\rangle = 0$ for all monomials in the ξ's. The calculation then shows that $\theta(x_1 \cdots x_n) = 0$. But $\mathcal{A}$ acts faithfully in low dimensions, so by choosing n large enough we get a contradiction. Therefore β is surjective.

To show that β is injective, we count dimensions in each degree. Recall that we have a bijection between admissible sequences $I = (i_1, \dots, i_n)$ and sequences $E = (e_1, \dots, e_n)$ with each e_i non-negative. One can check that if I and E correspond under this bijection, then Sq^I and ξ^E have the same degree. Thus $\mathcal{A}_*$ and $\mathbb{F}_2[\xi_1, \dots, \xi_n]$ have the same dimension in each degree. □

So far the dual Steenrod algebra $\mathcal{A}_*$ seems simpler than $\mathcal{A}$. One may wonder whether the complications are hidden in the diagonal map. Let's calculate it. But first, a lemma.

LEMMA 11.3. *For* $\theta \in \mathcal{A}$ *homogeneous of degree* $2^n(2^k - 1)$,

$$\theta x^{2^n} = \langle\, \theta, \xi_k^{2^n} \,\rangle x^{2^{n+k}},$$

where x *is the generator of* $H^*(\mathbb{R}\mathrm{P}^\infty)$.

Proof. Consider $\Delta : \mathbb{R}\mathrm{P}^\infty \to (\mathbb{R}\mathrm{P}^\infty)^{2^n}$; to simplify the notation, we let $r = 2^n$. Then

$$\begin{aligned}\theta x^r &= \theta(\Delta^*(x_1 \cdots x_r)) \\ &= \Delta^*(\theta(x_1 \cdots x_r)) \\ &= \Delta^* \sum_I \langle\, \theta, \xi_{i_1} \cdots \xi_{i_r} \,\rangle x_1^{2^{i_1}} \cdots x_r^{2^{i_r}} \\ &= \sum_I \langle\, \theta, \xi_{i_1} \cdots \xi_{i_r} \,\rangle x^{2^{n+k}}.\end{aligned}$$

Now let $\xi = \xi_1 + \xi_2 + \cdots$. Then $\xi^r = \sum_I \xi_{i_1} \cdots \xi_{i_r}$. But we are working over a field of characteristic two, and $r = 2^n$, so $\xi^r = \xi_1^r + \xi_2^r + \cdots$. Thus

$$\sum_I \langle\, \theta, \xi_{i_1} \cdots \xi_{i_r} \,\rangle x^{2^{n+k}} = \sum_i \langle\, \theta, \xi_i^r \,\rangle x^{2^{n+k}} = \langle\, \theta, \xi_k^r \,\rangle x^{2^{n+k}}$$

as required. □

With this we may calculate the diagonal map on $\mathcal{A}_*$. Let θ and ϕ be homogeneous elements of $\mathcal{A}$. We will calculate $\theta\phi x$ in two ways. For this to have a chance of being non-zero, it is necessary that $|\theta\phi| = 2^n - 1$ for some n and that $|\phi| = 2^j - 1$ for some j. Hence $|\theta| = 2^j(2^i - 1)$, where $i = n - j$. Now

$$(\theta\phi)x = \langle\, \theta\phi, \xi_n \,\rangle x^{2^n}$$

and (using the lemma)

$$\begin{aligned}\theta(\phi x) &= \theta\langle\, \phi, \xi_j \,\rangle x^{2^j} \\ &= \langle\, \theta, \xi_i^{2^j} \,\rangle\langle\, \phi, \xi_j \,\rangle x^{2^n} \\ &= \langle\, \theta \otimes \phi, \xi_i^{2^j} \otimes \xi_j \,\rangle x^{2^n}.\end{aligned}$$

So we conclude that

$$\langle\, \theta \otimes \phi, \psi\xi_n \,\rangle = \langle\, \theta\phi, \xi_n \,\rangle = \langle\, \theta \otimes \phi, \xi_i^{2^j} \otimes \xi_j \,\rangle$$

if $|\theta| = 2^j(2^i - 1)$ and $|\phi| = 2^j - 1$ for some i, j with $i + j = n$; it is zero otherwise. Therefore,

$$\psi\xi_n = \sum_{i+j=n} \xi_i^{2^j} \otimes \xi_j,$$

where $\xi_0 = 1$. For example, $\psi\xi_1 = 1\otimes\xi_1 + \xi_1\otimes 1$ and $\psi\xi_2 = 1\otimes\xi_2 + \xi_1^2\otimes\xi_1 + \xi_2\otimes 1$.

12. The Answer to the Question

In this section we finally are able to answer the question that motivated this chapter. The answer will appear as Theorem 12.5.

THEOREM 12.1. *The Steenrod algebra $\mathcal{A}$ acts freely through degree n on the Thom class of $MO(n)$.*

PROOF. We have

$$\begin{array}{ccccc} H^*MO(n) & \xrightarrow{s^*} & H^*BO(n) & \xrightarrow{f^*} & H^*((\mathbb{R}\mathrm{P}^\infty)^n) \\ u & \longmapsto & e & \longmapsto & x_1\cdots x_n \ , \end{array}$$

where s is the zero section and f is the map that classifies the bundle $\lambda\times\cdots\times\lambda$. These maps respect the action of the Steenrod algebra, and in the last section we showed that this action is free on $x_1\cdots x_n$, so it must be free on u. □

The Thom class u in $H^n(MO(n))$ provides us with a map $MO(n)\to K(\mathbb{F}_2,n)$. These piece together to give a map $MO\to H$ of spectra which represents a class u in H^0MO. The next corollary follows directly from the above theorem and motivates the subsequent theorem.

COROLLARY 12.2. *$\mathcal{A}$ acts freely on the Thom class in H^0MO.*

THEOREM 12.3. *In fact, H^*MO is free as an $\mathcal{A}$-module.*

PROOF. Recall that MO is a ring spectrum, and therefore H^*MO is a coalgebra. Since the map $H^*MO\to H^*(MO\wedge MO)$ comes from a map of spectra, it commutes with the Steenrod operations, and one can show that the isomorphism $H^*(MO\wedge MO)\cong H^*MO\otimes H^*MO$ does so as well. Therefore the coproduct on H^*MO is an $\mathcal{A}$-module map. The counit is also induced by a map of spectra, so H^*MO is what is called an $\mathcal{A}$-module coalgebra. The theorem now follows from the proposition of Milnor and Moore (10.4). □

Pick generators $\{v_\alpha\}$ for H^*MO as a free $\mathcal{A}$-module, and denote the degree of v_α by $|\alpha|$. So we have maps

$$v_\alpha : MO\longrightarrow \Sigma^{|\alpha|}H$$

which induce a map

$$v : MO\longrightarrow \prod_\alpha \Sigma^{|\alpha|}H = \bigvee_\alpha \Sigma^{|\alpha|}H. \tag{12.1}$$

The theorem just proved says that v is an isomorphism in mod 2 singular cohomology. More is true.

THEOREM 12.4. *v is a weak homotopy equivalence, and so MO is a graded Eilenberg–Mac Lane spectrum.*

PROOF. The difficulty here is that we only have mod 2 information at this point. We remedy this in the following way. To each abelian group G there is a (-1)-connected spectrum M with trivial singular homology in all dimensions except zero, in which it is G. For example, M is the sphere spectrum when $G = \mathbb{Z}$, and M is the suspension spectrum of $S^{n-1} \cup_{p^k} e^n$ suitably desuspended when $G = \mathbb{Z}/p^k$. For a general abelian group G we pick a free resolution

$$0 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 0$$

and, after choosing bases for R and F, we can write down a map $\vee S \xrightarrow{f} \vee S$ whose effect in 0-dimensional homology is simply the map $R \to F$. The cofibre of f is then the spectrum M and is called the **Moore spectrum** for the group G.

In fact, M is unique up to homotopy equivalence so it makes sense to define the (stable) homotopy groups of X with coefficients in G to be the M homology of X:

$$\pi_n(X; G) := [S^n, M \wedge X].$$

This coincides with the usual stable homotopy groups when the group is the integers. More generally, if E is a spectrum we denote by EG the spectrum $E \wedge M$ where M is a Moore space for the group G. Then $SG = S \wedge M = M$, so we use the notation SG from now on.

Because $\pi_n(-; G)$ is a homology theory, it satisfies the usual properties. Moreover, we get a long exact sequence

$$\cdots \longrightarrow \pi_n(X; \mathbb{Z}/p^{k-1}) \longrightarrow \pi_n(X; \mathbb{Z}/p^k) \longrightarrow \pi_n(X; \mathbb{Z}/p) \longrightarrow \cdots$$

from the obvious short exact coefficient sequence. This implies that if a map $f : X \to Y$ of spectra is an isomorphism in mod p homotopy, then it is an isomorphism in mod p^k homotopy for all k, and therefore is an isomorphism in $\mathbb{Z}/p^\infty$ homotopy, where $\mathbb{Z}/p^\infty = \varinjlim \mathbb{Z}/p^k$. Moreover, there is also a Whitehead theorem. If f is an isomorphism in mod p homology and X and Y are connected spectra, then f is an isomorphism in mod p homotopy. Combining the above, we see that v is an isomorphism in $\mathbb{Z}/2^\infty$ homotopy.

Now recall that $\pi_*(MO) \cong \mathfrak{N}_*$ and that 2 kills $\mathfrak{N}_*$. The long exact sequence coming from the short exact sequence $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p$ implies that $\pi_*(MO; \mathbb{Z}/p)$ is trivial for $p \neq 2$. Also, $\pi_*(MO; \mathbb{Q}) = \pi_*(MO) \otimes \mathbb{Q} = 0$. Similarly, $\pi_*(\vee \Sigma^{|\alpha|} H; G) = 0$ for $G = \mathbb{Z}/p$, $p \neq 2$, and for $G = \mathbb{Q}$. So the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \bigoplus_p \mathbb{Z}/p^\infty \longrightarrow 0$$

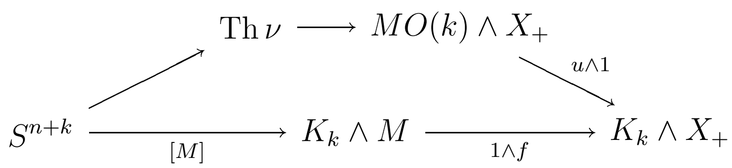
gives rise to a long exact sequence that implies that v is an isomorphism in integral homotopy. This is what we were trying to prove. □

Consider now the diagram

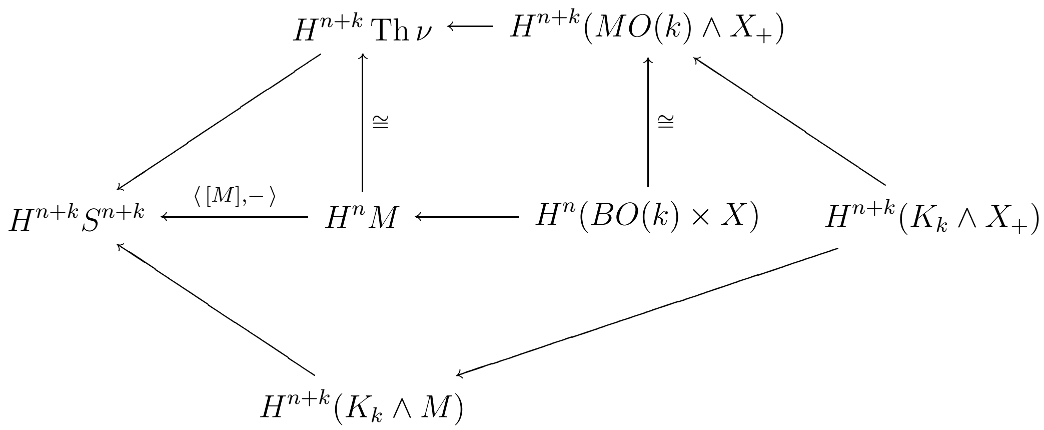
$$\begin{array}{ccc} \mathfrak{N}_*(X) & \xrightarrow{\cong} & MO_*(X) \\ & \searrow & \downarrow{\scriptstyle u_*} \\ & & H_*(X) \end{array} ,$$

where the top arrow is the geometrically defined isomorphism we studied many sections ago, the diagonal arrow is the map sending a singular manifold (M, f) to $f_*[M]$, and the vertical arrow is induced by the Thom class $u : MO \to H$. For the moment we assume that this triangle commutes. Steenrod wondered whether the diagonal arrow is surjective, and this is equivalent to the surjectivity of the vertical arrow. We can assume that u is among the chosen generators v_α of H^*MO. Then the decomposition (12.1) shows that we have a map $H \to MO$ such that the composite $H \longrightarrow MO \xrightarrow{u} H$ is the identity and hence that u_* is surjective.

All that remains to be shown is the commutativity of the above triangle. Starting with a singular manifold (M, f) in $\mathfrak{N}_n X$, its image in $H_n X$ under the roundabout route is represented by the top composite in the following diagram, while its image under the direct route (*i.e.*, $f_*[M]$) is represented by the bottom composite.



(The unlabelled maps come from the Pontrjagin–Thom construction in Section 3.) We need to show that this commutes. Using the fact that $H^*Y \cong \mathrm{Hom}(H_*Y, \mathbb{F}_2)$ we see that it suffices to check the commutativity in cohomology. In particular, we just need to apply H^{n+k}. This results in the following diagram, in which we've included a few more arrows.



To see that the regions in this diagram commute is labour that we leave to the reader.

We can finally state the following:

THEOREM 12.5. *For each homology class $\alpha \in H_n(X; \mathbb{F}_2)$ there is a closed n-manifold M and a continuous map $f : M \to X$ such that $f_*[M] = \alpha$.*

13. Further Comments on the Eilenberg–Mac Lane Spectrum

We have shown that MO is a graded Eilenberg–Mac Lane spectrum; it follows that there is an isomorphism $H_*MO \cong H_* \bigvee \Sigma^{|\alpha|}H \cong \mathcal{A}_* \otimes \pi_*MO$. Since $H_*MO = \mathbb{F}_2[x_1, x_2, \dots : |x_i| = i]$ and $\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \dots : |\xi_i| = 2^i - 1]$ it appears that the unoriented bordism ring looks like

$$\mathfrak{N}_* = \pi_*MO = \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, \dots] = \mathbb{F}_2[x_n : n \neq 2^i - 1].$$

This is actually true but we will omit the proof here.

Instead, we can ask if the mod 2 Eilenberg–Mac Lane spectrum H is itself a Thom spectrum. That is, we are looking for a stable vector bundle $\xi : X \to BO$ such that $\mathrm{Th}\,\xi \cong H$. Here we associate to a stable bundle ξ a spectrum $\mathrm{Th}\,\xi$ defined in the following way. Any cover of X by finite subcomplexes X_α gives a system of vector bundles compatible with the inclusions

$$\begin{array}{ccc} X_\alpha & \xrightarrow{\xi_\alpha} & BO(n_\alpha) \\ \downarrow & & \downarrow \\ X_\beta & \xrightarrow{\xi_\beta} & BO(n_\beta) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\xi} & BO \end{array} \qquad\qquad \begin{array}{ccc} E(\xi_\alpha) \oplus (\boldsymbol{n_\beta - n_\alpha}) & \longrightarrow & E(\xi_\beta) \\ \downarrow & & \downarrow \\ X_\alpha & \hookrightarrow & X_\beta \end{array} .$$

Let us desuspend each $\mathrm{Th}\,\xi_\alpha$ as a spectrum so that its Thom class lies in dimension 0. Then form the colimit of the spectra $\Sigma^{-n_\alpha}\,\mathrm{Th}\,\xi_\alpha$ (using the maps induced from the right-hand diagram above) to obtain a spectrum $\mathrm{Th}\,\xi$. $\mathrm{Th}\,\xi$ is independent of the cellular structure on X.

NOTE 13.1. Given a spectrum E, we can desuspend each space E_n as a spectrum. We claim that the colimit of $\Sigma^{-n}E_n$ (using the obvious maps induced from the structure maps) is simply E. Using this fact one can show that the Thom spectrum of the stable vector bundle represented by the identity map $BO \to BO$ is the Thom spectrum MO. This is good.

A Thom spectrum even obtains a ring structure $\mathrm{Th}\,\xi \wedge \mathrm{Th}\,\xi \to \mathrm{Th}\,\xi$ if $X \to BO$ is a map of H-spaces. Now, if H were in fact a Thom spectrum $\mathrm{Th}\,\xi$ as a ring spectrum, then we would have algebra isomorphisms

$$H_*X \cong H_*\,\mathrm{Th}\,\xi \cong \mathcal{A}_* \cong \mathbb{F}_2[x_1, x_3, x_7, \ldots].$$

Therefore, we should try to find an H-space with the above homology. A possible candidate is the following.

Recall that BO is an infinite loop space

$$BO = X_0 \simeq \Omega X_1 \simeq \Omega^2 X_2 \simeq \Omega^3 X_3 \cdots .$$

(Bott periodicity says that $\Omega^8 BO \simeq BO \times \mathbb{Z}$, and so we may take for X_8 the space $BO\langle 9\rangle$ obtained from BO by killing π_i for $1 \le i \le 8$. For information on killing homotopy groups, see [**16**].) Let $S^1 \to BO(1) \hookrightarrow BO = \Omega^2 X_2$ be the generator of $\pi_1 BO = \mathbb{F}_2$. Applying Ω^2 to its adjoint we obtain a map

$$\Omega^2 S^3 \longrightarrow \Omega^2 X_2 = BO.$$

To calculate the homology of $\Omega^2 S^3$ we first look at the Serre spectral sequence for the fibration $PS^3 \to S^3$ with fibre ΩS^3. Observe that the homology of the fibre acts on the whole spectral sequence and the differentials are linear over $H_*\Omega S^3$. With this we immediately verify that

$$H_*\Omega S^3 = \mathbb{F}_2[x_2],\ x_2 = d_3[S^3].$$

Now the Serre spectral sequence of $P\Omega S^3 \to \Omega S^3$ has E_2 term

$$\bigotimes \mathbb{F}_2[x_{2^i-1}] \otimes E[x_2^{2^i}]$$

and $d^{2^{i+1}}$ maps the exterior algebra generator $x_2^{2^i}$ to $x_{2^{i+1}-1}$. This is a fun calculation and we highly recommend it to the reader.

So $\Omega^2 S^3$ is an H-space with the correct homology. We assert without proof that it is in fact the space we were searching for. For details the reader may have a look at [**17**].

THEOREM 13.2 (Mahowald). *The Thom spectrum of* $\Omega^2 S^3 \to BO$ *is the mod 2 Eilenberg–Mac Lane spectrum H.*

This may be used to give another proof that MO is a graded Eilenberg–Mac Lane spectrum since we obtain a ring spectrum map $H \to MO$.

CHAPTER 2

Complex Cobordism

1. Various Bordisms

We generalize the bordism relation to any fibration $\pi : B \to BO$.

DEFINITION 1.1. *A* **π-structure** *on a closed manifold M is a homotopy class of lifts $\tilde{\nu}$ of a map ν classifying its stable normal bundle:*

$$\begin{array}{ccc} & & B \\ & \overset{\tilde{\nu}}{\nearrow} & \big\downarrow \\ M & \xrightarrow{\ \nu\ } & BO \end{array} .$$

This is independent of the choice of ν in its homotopy class in the following sense. Given a homotopy h from ν_0 to ν_1, there is a one-to-one correspondence between structures $(M, \tilde{\nu_0})$ over ν_0 and $(M, \tilde{\nu_1})$ over ν_1: the homotopy h can be lifted

$$\begin{array}{ccc} M \times 0 & \xrightarrow{\ \tilde{\nu_0}\ } & B \\ \big\downarrow & \overset{H}{\nearrow} & \big\downarrow \pi \\ M \times I & \xrightarrow{\ h\ } & BO \end{array}$$

to get a π-structure $\tilde{\nu}_1 = H|_{M \times 1}$ over ν_1.

Let $\{G_n\}$ be a sequence of topological groups together with maps $G_n \to G_{n+1}$ and orthonormal representations $G_n \to O_n$ such that

$$\begin{array}{ccc} G_n & \longrightarrow & G_{n+1} \\ \big\downarrow & & \big\downarrow \\ O_n & \longrightarrow & O_{n+1} \end{array}$$

commutes, where $O_n \to O_{n+1}$ is the standard inclusion. After applying the functor B we obtain an example of a fibration over BO:

$$\begin{array}{ccccc} BG_n & \longrightarrow & BG_{n+1} & \longrightarrow & BG \\ \downarrow & & \downarrow & & \downarrow \pi \\ BO_n & \longrightarrow & BO_{n+1} & \longrightarrow & BO \end{array} .$$

A π-structure is called a ***G*-structure** in this case and we say that M is a ***G*-manifold**. Good references on classifying spaces are Milnor [**12**] and Segal [**21**].

Note that an SO-structure on a manifold can be regarded as an orientation of its normal bundle. We can expand this example by successively killing homotopy groups. Recall that the homotopy groups of O are as follows

i	0	1	2	3	4	5	6	7
$\pi_i O$	$\mathbb{F}_2$	$\mathbb{F}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

and that $\pi_i BO = \pi_{i-1} O$ for positive i while $\pi_0 BO = 0$. More precisely, the fibrations

$$\begin{array}{ccc} BO\langle 8\rangle & & \\ \downarrow & & \\ B\,\mathrm{Spin} = BO\langle 4\rangle & \longrightarrow & K(\mathbb{Z},4) \\ \downarrow & & \\ BSO = BO\langle 2\rangle & \longrightarrow & K(\mathbb{F}_2,2) \\ \downarrow & & \\ BO & \longrightarrow & K(\mathbb{F}_2,1) \end{array}$$

give an interesting tower of G-structures and corresponding Thom spectra

$$MO\langle 8\rangle \to M\,\mathrm{Spin} \to MSO \to MO.$$

Further examples of bordisms are

$$\begin{array}{rcll}
U_n & \hookrightarrow & O_{2n} & \text{Complex Bordism} \\
Sp_n & \hookrightarrow & O_{4n} & \text{Symplectic Bordism} \\
\Sigma_n & \hookrightarrow & O_n & \text{Symmetric Bordism} \\
Br_n \to \Sigma_n & \hookrightarrow & O_n & \text{Braid Bordism}
\end{array}$$

THEOREM 1.2. *There is a map $B(Br) \to \Omega^2 S^3$ which is an isomorphism in homology. (However, it is not a homotopy equivalence.) Here $Br = \cup Br_n$.*

Hence, using the theorem of Mahowald we can represent every mod 2 homology class by a manifold whose normal bundle has a reduction to the braid group.

We may focus our attention on fibrations $\pi : BG \to BO$. In fact this is hardly any restriction since a result of Milnor says that every connected space X is weakly homotopy equivalent to $B(\Omega X)$.

DEFINITION 1.3. *A G-manifold $(M, \tilde{\nu})$ is* **null-bordant** *if there is a manifold W embedded in $\mathbb{R}^{n+k} \times \mathbb{R}_+$ such that the following holds: $\partial W = W \cap (\mathbb{R}^{n+k} \times 0)$, W meets $\mathbb{R}^{n+k}$ transversely with $M \cong \partial W$, and the classifying map of the normal bundle of W lifts as in the following diagram:*

$$\begin{array}{ccc}
M & \xrightarrow{\tilde{\nu}} & BG \\
\Big\downarrow & \overset{N}{\nearrow} & \Big\downarrow \\
W & \longrightarrow & BO
\end{array} .$$

We also may define a negative to $(M, \tilde{\nu})$: The product $M \times I$ embeds in $\mathbb{R}^{n+k} \times \mathbb{R}_+$ as above. A G-structure on $W = M \times I$ is obtained by using that $BG \to BO$ is a fibration and simply choosing a lift N. The negative G-structure is now

$$M \overset{i_1}{\hookrightarrow} M \times I \xrightarrow{N} BG.$$

Obviously this represents an inverse under the addition which is the disjoint union of G-manifolds. We denote the G-bordism group by Ω_*^G. The diligent reader may also define the group of singular G-manifolds $\Omega_*^G(X)$ for a topological space X and prove the following theorem along the lines of Chapter 1.

THEOREM 1.4 (Thom).

$$\Omega_*^G(X) \xrightarrow{\cong} MG_*(X) = \pi_*(MG \wedge X_+).$$

We are especially interested in the case $G = U$, which gives “almost almost complex manifolds”.

2. Complex Oriented Cohomology Theories

We want to expand the notion of orientation to generalized cohomology theories. Therefore, first consider $S^1 \subset \mathbb{C}$ with its orientation given by $[S^1] \in H_1(S^1)$. Then $[S^1] \wedge \cdots \wedge [S^1]$ defines an orientation $[S^n] \in H_n(S^n)$ for S^n. We may also construct a canonical generator $\sigma_n \in \bar{E}^n(S^n) = [S^n, E \wedge S^n]$ for any ring spectrum E. We simply have to suspend the unit $\eta : S \to E$ n times. Next look at the tautological line bundle λ over $\mathbb{C}\mathrm{P}^{n-1}$ given by $E(\lambda) = \{(l, v) \ : \ v \in l\} \subset \mathbb{C}\mathrm{P}^{n-1} \times \mathbb{C}^n$. The total space of its dual λ^* is canonically homeomorphic to $\mathbb{C}\mathrm{P}^n - \{*\}$ in such a way that

$$\begin{array}{ccc} & \mathbb{C}\mathrm{P}^{n-1} & \\ \swarrow & & \searrow{\scriptstyle s} \\ \mathbb{C}\mathrm{P}^n & \cong & \mathrm{Th}\,\lambda^* \\ \uparrow & & \uparrow{\scriptstyle i} \\ \mathbb{C}\mathrm{P}^1 & \cong & S^2 \end{array}$$

commutes. Here s denotes the zero section and i the fibre inclusion. So the evident Thom class

$$u(\lambda^*) \in \bar{H}^2(\mathrm{Th}\,\lambda^*) = [\,\mathrm{Th}\,\lambda^*, \mathbb{C}\mathrm{P}^\infty\,]$$

evaluated on $[\mathbb{C}\mathrm{P}^1]$ has a positive sign. Equivalently we can consider the Euler class

$$< e(\lambda^*), [\mathbb{C}\mathrm{P}^1] >= 1.$$

Now since for line bundles $e(\alpha \otimes \beta) = e(\alpha) + e(\beta)$, we get $\left\langle\, e(\lambda), [\mathbb{C}\mathrm{P}^1] \,\right\rangle = -1$. In the stable case this means that

$$\begin{array}{ccccc} BU(1) & \xrightarrow[\cong]{s} & MU(1) & \xrightarrow{u} & E \wedge S^2 \\ \uparrow & & & \nearrow{\scriptstyle \sigma_2} & \\ \mathbb{C}\mathrm{P}^1 & \xrightarrow{deg(-1)} & S^2 & & \end{array}$$

commutes.

DEFINITION 2.1. *Let E be commutative and associative ring spectrum. An* ***E*-orientation** *of an oriented vector bundle ξ is a class $u \in \bar{E}^n(\mathrm{Th}\,\xi)$ which restricts to $\sigma_n \in E^n(S^n)$ in each fibre:*

$$\begin{array}{ccc} S^n & \xrightarrow{i} & \mathrm{Th}\,\xi \\ & \searrow^{\sigma_n} & \downarrow u \\ & & E \wedge S^n \end{array} .$$

We also call u the **Thom class***, and its pullback e under the zero section is called the* **Euler class***. Note that even for a complex vector bundle an E-orientation need not exist. We say a ring spectrum E is* **complex oriented** *if there is an E-orientation for the canonical bundle over $\mathbb{C}\mathrm{P}^\infty$. In this case all complex bundles are E-oriented by naturality and the splitting principle.*

Our previous considerations can now be summarized in the diagram

$$\begin{array}{ccc} \mathbb{C}\mathrm{P}^1 & \hookrightarrow & \mathbb{C}\mathrm{P}^\infty \\ \downarrow \cong & & \downarrow e \\ S^2 & \xrightarrow{-\sigma_2} & E \wedge S^2 \end{array} .$$

EXAMPLE 2.2. A universal example of an oriented cohomology theory is $E = MU$ given by $(MU)_{2n} = MU(n)$ and $(MU)_{2n+1} = \Sigma MU(n)$. The structure maps $\Sigma(MU)_{2n+1} = \Sigma^2 MU(n) \to MU(n+1) = MU_{2n+2}$ are furnished by the classifying maps $\xi_n \oplus \mathbf{1} \to \xi_{n+1}$ after passing to Thom spaces. MU is canonically complex oriented since a map $\Sigma^\infty MU(1) \to MU \wedge S^2$ can be defined by $MU(1) \xrightarrow{1} MU_2$ using the desuspension theorem.

Complex oriented theories are calculable. The Atiyah-Hirzebruch spectral sequence is a fundamental tool for computing a generalized cohomology theory E. We have to assume that E satisfies the axiom of strong additivity, *i.e.,*

$$\bar{E}^*(\bigvee_{\alpha \in I} X_\alpha) \cong \prod_{\alpha \in I} \bar{E}^* X_\alpha,$$

which is fulfilled when E comes from a spectrum. If X is a CW-complex, its skeleta X_i fit into cofibration sequences

$$\begin{array}{ccccccccc} \cdots & \hookrightarrow & X_{i-1} & \hookrightarrow & X_i & \hookrightarrow & X_{i+1} & \hookrightarrow & \cdots \\ & & & & \downarrow & & \downarrow & & \\ & & & & \bigvee S^i & & \bigvee S^{i+1} & & . \end{array}$$

Thus there is an exact couple in cohomology $E^*(-)$, *i.e.,* a spectral sequence. The E_1 term consists of the cellular cochains with coefficients in E^* and so

$$E_2^{s,t} = H^s(X, E^t).$$

Unfortunately, the Atiyah-Hirzebruch spectral sequence doesn't converge very well because E^* is typically non-trivial for infinitely many positive and negative dimensions. If X is finite-dimensional it converges. (The above works for X a bounded spectrum as well.)

Let us try to compute $E^*\,\mathbb{C}\mathrm{P}^{n-1}$. For oriented E the element $x \otimes 1 \in H^2(\mathbb{C}\mathrm{P}^{n-1}) \otimes E^0$ must survive to $e(\lambda) \in E^2(\mathbb{C}\mathrm{P}^{n-1})$. Thus all differentials have to be zero and the spectral sequence collapses. We obtain a well-defined homomorphism

$$E^*[x]/(x^n) \to E^*(\mathbb{C}\mathrm{P}^{n-1})$$

sending x to the Euler class $e(\lambda) \in \bar{E}^*(\mathbb{C}\mathrm{P}^{n-1})$. (For this recall that any n-fold product of classes $\alpha_1, \alpha_2, \ldots, \alpha_n \in \bar{E}^*(X)$ vanishes if X can be covered by n contractible open sets.) Now filter both sides by $|x| = 2$ and get an isomorphism of graded rings.

To calculate $E^*\mathbb{C}\mathrm{P}^\infty$ we introduce the **Milnor sequence**. It applies to any topological space X which is the direct limit of well-pointed spaces

$$X_0 \xrightarrow{i} X_1 \xrightarrow{i} X_2 \xrightarrow{i} \cdots .$$

If each X_n is a CW-subcomplex of X_{n+1} we may replace X by the (pointed) homotopy equivalent telescope T which is obtained by taking

$$\coprod_{n \geq 0} X_n \times [\,n\,, n+1\,]$$

and identifying $(x_n, n+1)$ with $(ix_n, n+1)$ and all $(*, t)$ to one point. T is filtered by $F_0T = \mathrm{im}(\coprod_{n\geq 0} X_n \times \{n\} \hookrightarrow T) = \bigvee_n X_n$ and $F_1T = T$. Observe that $F_0T \hookrightarrow F_1T$ is a cofibration. The quotient T/F_0T is homeomorphic to $\bigvee_{n\geq 0} \Sigma X_n$. Hence the filtration induces a long exact sequence in $\bar{E}^*$

$$\begin{array}{ccc} \prod_n \bar{E}^*(X_n) = \bar{E}^*(\bigvee_n X_n) & \longrightarrow & \bar{E}^*(\bigvee_n \Sigma X_n) = \prod_n \bar{E}^{*-1}(X_n) \\ & \nwarrow \quad \swarrow & \\ & \bar{E}^*(T) & \end{array}.$$

A picture may convince the reader that the above map

$$\prod_n \bar{E}^* X_n \longrightarrow \prod \bar{E}^* X_n$$

is given by $1 - \mathrm{shift}$, *i.e.*,

$$(b_0, b_1, \dots) \longmapsto (b_0 - i^*b_1, b_1 - i^*b_2, \dots).$$

Its kernel can be identified with $\varprojlim \bar{E}^n X_i$ and the cokernel is often denoted by $\varprojlim^1 \bar{E}^{n-1}X_i$. $\bar{E}^n T$ is now part of the short exact sequence

$$0 \longrightarrow \varprojlim{}^1 \bar{E}^{n-1}(X_i) \longrightarrow \bar{E}^n(T) \longrightarrow \varprojlim \bar{E}(X_i) \longrightarrow 0.$$

The calculation of $\varprojlim^1 A_n = \mathrm{coker}(1 - \mathrm{shift})$ for sequences

$$A_0 \xleftarrow{j} A_1 \xleftarrow{j} A_2 \cdots$$

of abelian groups is a purely algebraic problem. We want to derive a criterion for when it vanishes. Suppose the homomorphisms j are surjective. Then every $(b_0, b_1, \dots) \in \prod_i A_i$ lies in the image of $1 - \mathrm{shift}$, since the equation

$$(a_0 - ja_1, a_1 - ja_2, \dots) = (b_0, b_1, b_2, \dots)$$

can be solved inductively; so we have $\varprojlim^1 A_n = 0$. This situation applies to the case $A_n = \bar{E}^*(\mathbb{C}\mathrm{P}^{n-1})$ and we have the following result.

THEOREM 2.3. $E^*\mathbb{C}\mathrm{P}^\infty = \varprojlim E^*\mathbb{C}\mathrm{P}^n = E^*[\![x]\!]$.

This result needs some interpretation. For $E = H$ the inverse limit is polynomial in a given dimension. But most of the time $E^* = \pi_{-*}E$ is not bounded below. Hence a homogeneous element might have the form

$$\sum_{i=0}^{\infty} a_i x^i \in (E^*[\![x]\!])^n$$

with $|a_i| + 2i = n$ for infinitely many a_i in E^*.

We want to investigate the map $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \xrightarrow{\mu} \mathbb{C}\mathrm{P}^\infty$ which classifies the tensor product of line bundles. Or in other words it corresponds to the addition map $K(\mathbb{F}_2, 2) \times K(\mathbb{F}_2, 2) \to K(\mathbb{F}_2, 2)$. The Künneth isomorphism doesn't apply to $E^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty)$. So again we are forced to go through the Milnor sequence and obtain

$$E^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) = E^*[\![x, y]\!],$$

where $x = e(\lambda), x = pr_1^* x, y = pr_2^* x$. The ring homomorphism

$$E^*[\![x]\!] \xrightarrow{\mu^*} E^*[\![x, y]\!]$$

is completely determined by $\mu^*(x) =: F_E(x, y)$. Naturality implies

$$e(\alpha \otimes \beta) = F_E(e(\alpha), e(\beta))$$

for any line bundles α and β. $F_E(x, y)$ is what is called a formal group law.

3. Generalities on Formal Group Laws

DEFINITION 3.1. *A* **formal group law** *over a graded commutative ring R is a homogeneous power series $F(x, y) \in R[\![x, y]\!]$ of degree two (where $|x| = |y| = 2$) such that*

$$F(0, y) = y, \quad F(x, 0) = x,$$

$$F(x, F(y, z)) = F(F(x, y), z),$$

and

$$F(x, y) = F(y, x).$$

EXAMPLE 3.2. (i) In the case $E = H$ the Euler class is primitive for dimension reasons. That is, $e(\alpha \otimes \beta) = e(\alpha) + e(\beta)$. Thus we obtain the **additive formal group law** G_a:

$$F_H(x, y) = G_a(x, y) := x + y.$$

(ii) The coefficients of complex K-theory are Laurent series $K^* = \mathbb{Z}[\,v, v^{-1}\,]$ with $v \in K^{-2} = \pi_2(BU)$. A complex orientation is given by

$$e(\alpha) = \frac{1-[\alpha]}{v} \in K^2(X)$$

for line bundles α over X. Since

$$\begin{aligned} e(\alpha \otimes \beta) &= \frac{1-\alpha\beta}{v} \\ &= \frac{(1-\alpha)+(1-\beta)-(1-\alpha)(1-\beta)}{v} \\ &= e(\alpha)+e(\beta)-ve(\alpha)e(\beta) \end{aligned}$$

we have the **multiplicative formal group law** G_m:

$$F_K(x,y) = G_m(x,y) := x+y+vxy.$$

Generally, a formal group law has the form

$$F(x,y) = x+y+\sum_{i,j>0} a_{ij}x^iy^j,$$

where $|a_{ij}| = 2-2(i+j)$.

PROPOSITION 3.3. *The functor*

$$\mathcal{F} : \mathrm{GCR} \longrightarrow \mathrm{Set}$$

which sends a graded commutative ring R to the set of formal group laws over R is corepresentable. That is, there is a ring L and a formal group law G over L such that any f.g.l. F over R can be obtained from G by applying a unique ring homomorphism $L \to R$.

PROOF. Simply define L to be $\mathbb{Z}[a_{ij} : i,j > 0]$ with a_{ij} in dimension $2-2(i+j)$ modulo the relations forced by the unit, associativity and commutativity conditions, and take

$$G(x,y) = x+y+\sum_{i,j} a_{ij}x^iy^j.$$

□

We will prove that the **Lazard ring** L has the following structure.

THEOREM 3.4 (Lazard). $L \cong \mathbb{Z}[x_1, x_2, \ldots]$, *where* $|x_i| = -2i$.

However, we will not have the opportunity to prove the following result. See Section II.8 of the blue book [**2**] for a proof.

THEOREM 3.5 (Quillen). *The map $L \to MU^*$ classifying F_{MU} is an isomorphism. That is, F_{MU} is the universal formal group law and MU_* has a description as a polynomial algebra.*

$\mathcal{F}(R)$ is the class of objects in a category: We set

$$\mathrm{Hom}_R(F,G) = \{\theta(x) \in (R[\![x]\!])^2 \,:\, \theta(0)=0 \text{ and } \theta(F(x,y)) = G(\theta(x),\theta(y))\}.$$

Composition is the composition of power series. Observe that the coefficient a_i in

$$\theta(x) = a_0 x + a_1 x^2 + \cdots$$

has degree $-2i$ and that a homomorphism is invertible iff a_0 is a unit in R. We speak of a **strict isomorphism** if $a_0 = 1$. If $F \in \mathcal{F}(R)$ and $\theta(x) \in xR[\![x]\!]^\times$ is invertible and of degree two then

$${}^{\theta}F(x,y) := \theta F(\theta^{-1}(x), \theta^{-1}(y))$$

is a formal group law and $\theta : F \to {}^{\theta}F$. Similarly, $F^{\theta}(x,y) := \theta^{-1}F(\theta(x),\theta(y))$.

EXAMPLE 3.6. Over the ring $R = \mathbb{Z}[m_1, m_2, \ldots]$ with $|m_i| = -2i$, the formal group law $G_a^{\log}$ is universal for formal group laws which are strictly isomorphic to the additive one, where

$$\log(x) = x + m_1 x^2 + m_2 x^3 + \cdots .$$

Given a formal group law we may ask when there is a strict isomorphism $\varphi : F \to G_a$, *i.e.*, when $\varphi F(x,y) = \varphi(x) + \varphi(y)$ for some strictly invertible φ. Differentiating this equation with respect to y we find

$$\varphi'(F(x,y))F_2(x,y) = \varphi'(y)$$

and evaluating at $y=0$ we obtain the condition $\varphi'(x)F_2(x,0) = 1$. This last equation can be uniquely solved for φ

$$\varphi(x) = \int_0^x \frac{dt}{F_2(t,0)},$$

but the integral of polynomials involves denominators. That is, we have to assume that S is a $\mathbb{Q}$-algebra.

PROPOSITION 3.7. *Over a $\mathbb{Q}$-algebra any formal group law is uniquely strictly isomorphic to the additive formal group law G_a. We denote the isomorphisms by $\log_F$ and $\exp_F$ as in the following diagram:*

$$F \underset{\exp_F}{\overset{\log_F}{\rightleftarrows}} G_a$$

Since the ring homomorphism $L \to R \otimes \mathbb{Q}$ classifying $G_a^{\log}$ factors through $L \otimes \mathbb{Q}$ we have

COROLLARY 3.8. $L \otimes \mathbb{Q} \xrightarrow{\cong} R \otimes \mathbb{Q} = \mathbb{Q}[\, m_1, m_2, \ldots\,]$.

EXAMPLE 3.9. For the multiplicative formal group law $F(x, y) = x + y - vxy$ we compute $F_2(x, 0) = 1 - vx$ and find that

$$\log_F(x) = \int_0^x \frac{dt}{1 - vx} = \sum_{i=1}^{\infty} \frac{v^{i-1}x^i}{i} = \frac{\log(1 - vx)}{-v}$$

and

$$\exp_F(x) = \frac{1 - e^{vx}}{v}.$$

So we know that $L \otimes \mathbb{Q} = \mathbb{Q}[m_1, m_2, \ldots]$ and that the canonical map $L \to L \otimes \mathbb{Q}$ classifies the formal group law with logarithm given by

$$\log x = \sum m_i x^{i+1}, \quad m_0 = 1,$$

i.e., the formal group law

$$G_a^{\log}(x, y) = \log^{-1}(\log x + \log y).$$

Now we make a construction that will be used in the proof of Lazard's theorem. For any abelian group A and positive integer n consider the graded commutative ring $\mathbb{Z} \oplus A[2 - 2n]$, where the number in square brackets indicates that A is to be placed in degree $2 - 2n$. The product is given by $(m, a)(n, b) = (mn, mb + na)$, which is (honestly) commutative. We use rings of this form to determine the structure of L using the fact that

$$\mathcal{F}(\mathbb{Z} \oplus A) = \text{Rings}(L, \mathbb{Z} \oplus A).$$

There is a natural augmentation map $L \xrightarrow{\epsilon} \mathbb{Z}$ classifying the additive formal group law over $\mathbb{Z}$, and this map is an isomorphism in non-negative degrees.

Let $I = \ker \epsilon$ be the augmentation ideal, which is strictly negatively graded. The ring $\mathbb{Z} \oplus A$ has an obvious augmentation to $\mathbb{Z}$. It is clear that any ring homomorphism $L \to \mathbb{Z} \oplus A$ preserves the augmentations and hence induces a map $I \to A$. Since $A^2 = 0$, the map factors through $QL := I/I^2$:

$$\begin{array}{ccc} I & \longrightarrow & A[2-2n] \\ & \searrow \quad \nearrow & \\ & I/I^2 & \end{array} .$$

So $\mathrm{Rings}(L, \mathbb{Z} \oplus A[2-2n]) \cong \mathrm{Ab}((I/I^2)^{2-2n}, A)$. In fact, the functor

$$\begin{aligned} \mathrm{Ab} &\longrightarrow \mathrm{AugRings} \\ A &\longmapsto \mathbb{Z} \oplus A[2-2n] \end{aligned}$$

has a left adjoint which sends an augmented ring R to the abelian group QR^{2-2n}.

For $a \in A$ define $\theta_a(x) = x + ax^n$, a homogeneous power series of degree 2. Then $\theta_a^{-1} = \theta_{-a}$ and so

$$\begin{aligned} {}^{\theta_a}G_a(x,y) &= (x - ax^n + y - ay^n) + a(x - ax^n + y - ay^n)^n \\ &= x + y + a[(x+y)^n - x^n - y^n] \end{aligned}$$

is a formal group law over $\mathbb{Z} \oplus A$. It sometimes happens that $[(x+y)^n - x^n - y^n]$ is divisible by an integer.

LEMMA 3.10. $\gcd\{\binom{n}{i} : 1 \leq i \leq n-1\} = \epsilon_n$, *where*

$$\epsilon_n = \begin{cases} p & \text{if } n = p^s \text{ for } p \text{ prime and } s \geq 1 \\ 1 & \text{otherwise} \end{cases} .$$

PROOF. Left to the reader. ☐

Let

$$C_n(x,y) = \frac{1}{\epsilon_n}[(x+y)^n - x^n - y^n] \in \mathbb{Z}[x,y]$$

and consider the formal group law $x + y + aC_n(x,y)$ over $\mathbb{Z} \oplus A[2-2n]$. (An easy calculation shows that this is indeed a f.g.l.) Let α be the map $A \to \mathcal{F}(\mathbb{Z} \oplus A[2-2n])$ sending a to $x + y + aC_n(x,y)$; since $C_n(x,y)$ is a primitive polynomial, α is injective.

PROPOSITION 3.11 (Lazard). *In fact, α is a natural bijection.*

Before proving this, we show how it enables us to prove the structure theorem of Lazard.

PROOF. The proposition implies that $\mathrm{Ab}(QL^{2-2n}, A) \cong A$ as sets for any abelian group A. The structure theorem for finitely generated abelian groups then tells us that $QL^{2-2n} \cong \mathbb{Z}$. For each n, choose an element $x_{n-1} \in L^{2-2n}$ that projects to a generator of QL^{2-2n}. This gives us a surjective ring homomorphism $\mathbb{Z}[x_1, x_2, \ldots] \to L$ and thus the map $\mathbb{Q}[x_1, x_2, \ldots] \to L \otimes \mathbb{Q}$ is also surjective. In the commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}[x_1, x_2, \ldots] & \twoheadrightarrow & L & & \\ \downarrow & & \downarrow & & \\ \mathbb{Q}[x_1, x_2, \ldots] & \twoheadrightarrow & L \otimes \mathbb{Q} & = & \mathbb{Q}[m_1, m_2, \ldots] \end{array}$$

we have by a dimension count that the bottom arrow is an isomorphism, and since $\mathbb{Z}[x_1, x_2, \ldots]$ is torsion free, the left map is injective. Therefore the top map is an isomorphism. □

COROLLARY 3.12. *If $R \to S$ is a surjective graded ring homomorphism, then $\mathcal{F}(R) \to \mathcal{F}(S)$ is also surjective.*

PROOF. L is free commutative and so projective. □

Now we proceed to prove the proposition.

PROOF. Over $\mathbb{Z} \oplus A[2-2n]$ any formal group law has the form $F(x, y) = x + y + P(x, y)$, where

$$P(x, y) = \sum_{i+j=n} a_i x^i y^j, \quad a_0 = a_n = 0.$$

The commutativity condition implies that

$$P(x, y) = P(y, x),$$

while associativity implies that

$$P(y, z) + P(x, y + z) = P(x, y) + P(x + y, z).$$

This last condition is in fact a cocycle condition in a chain complex that arises quite naturally, and we denote the set of P satisfying the two conditions by $Z^{2,n}_{sym}$. So what we are trying to prove is that the map

$$\begin{aligned} A &\longrightarrow Z^{2,n}_{sym} \\ a &\longmapsto aC_n(x,y) \end{aligned}$$

is surjective.

Using the convention that sums are over non-negative i, j and k that sum to n with additional constraints as indicated, the associativity condition can be written

$$\sum_{i=0} a_j y^j z^k + \sum a_i (j,k) x^i y^j z^k = \sum_{k=0} a_i x^i y^j + \sum (i,j) a_{i+j} x^i y^j z^k,$$

where $(j,k) = \binom{j+k}{j}$. Looking at the coefficient of $xy^{m-1}z^{n-m}$ we find that

$$\binom{n-1}{m-1} a_1 = m a_m$$

for $1 \le m \le n-1$. We now exploit this relationship in a case by case analysis.

Case I: $A = \mathbb{Z}/p$, p prime. If $n = p$, then we claim that $P = a_1 C_p$. Indeed, consider $P(x,y) - a_1 C_p(x,y) = \sum a'_i x^i y^j$, which satisfies the associativity condition. Any m with $1 \le m \le p-1$ is a unit in $\mathbb{Z}/p$, and so we have that $a'_m = \frac{1}{m}\binom{n-1}{m-1} a'_1 = 0$, since $a'_1 = 0$. Thus $P = a_1 C_p$.

If $p \mid n$ but $p < n$, write $n = kp$. We use the following lemma, whose proof is omitted.

Lemma 3.13. *$C_{pk}(x,y) = uC_k(x^p, y^p)$ mod p for some unit u. In fact,*

$$u = \begin{cases} \epsilon_k & \text{if } k \neq p^s \\ 1 & \text{otherwise} \end{cases}.$$

In our relation above we take $m = p$ and find that

$$\binom{n-1}{p-1} a_1 = 0$$

and so $a_1 = 0$, since $\binom{n-1}{p-1}$ is a unit mod p. Thus, as above, $a_m = 0$ unless $p \mid m$. So we can write $P(x,y) = Q(x^p, y^p)$ for some Q which is again a cocycle. Therefore, by induction and the lemma,

$$P(x,y) = Q(x^p, y^p) = aC_k(x^p, y^p) = au^{-1} C_n(x,y).$$

Note that we have made no use of the symmetry condition as of yet.

Now we take the remaining subcase, namely the case when p does not divide n. Consider $P - a_1 \frac{\epsilon_n}{n} C_n$. This expression was constructed to have leading term zero, thus as before the only non-zero coefficients are the coefficients of $x^{p^k} y^{n-p^k}$. So, by symmetry, all coefficients are zero.

Case II: $A = \mathbb{Z}/p^s$. This is done by induction on s using the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & pA & \longrightarrow & A & \longrightarrow & A/pA & \longrightarrow & 0 \\ & & \downarrow{\cong} & & \downarrow{\alpha} & & \downarrow{\cong} & & \\ 0 & \longrightarrow & Z^{2,n}_{sym}(pA) & \longrightarrow & Z^{2,n}_{sym}(A) & \longrightarrow & Z^{2,n}_{sym}(A/pA) & \longrightarrow & 0 \end{array}$$

whose rows are easily seen to be exact.

Case III: $A = \mathbb{Q}$. We get surjectivity here because all formal groups laws are isomorphic to additive ones. (The $\mathbb{Z}$ summand doesn't interfere.)

Case IV: $A = \mathbb{Z}$. This follows from the rational case, since C_n is primitive.

Case V: A a finitely generated abelian group. This follows from the fact that

$$\begin{array}{ccc} & B \oplus C & \\ {\scriptstyle \alpha\oplus\alpha}\swarrow & & \searrow{\scriptstyle \alpha} \\ Z^{2,n}_{sym}(B) \oplus Z^{2,n}_{sym}(C) & \xrightarrow[\cong]{} & Z^{2,n}_{sym}(B \oplus C) \end{array}$$

commutes.

Case VI: A arbitrary. Let $P \in Z^{2,n}_{sym}(A)$. Then $P(x,y) = \sum a_i x^i y^j$. Let B be the subgroup of A generated by the a_i. It is finitely generated and $P \in Z^{2,n}_{sym}(B)$, so by the previous case $P = bC_n(x,y)$ for some $b \in B \subseteq A$. $\square$

The Lazard ring suffers from some defects: it is too big and has no specific generators. Therefore we restrict our study to a smaller class of formal group laws.

4. p-Typicality of Formal Group Laws

Let S be a graded ring and let

$$\Gamma(S) = \{\gamma \in (S[\![x]\!])^2 \, : \, \gamma(x) \equiv x \mod (x^2)\}$$

be the group of formal power series (of degree 2) with leading term x. $\Gamma(S)$ acts on $\mathcal{F}(S)$ on the right by

$$(F, \gamma) \longmapsto F^{\gamma},$$

where

$$F^{\gamma}(x, y) := \gamma^{-1} F(\gamma(x), \gamma(y)).$$

For a fixed commutative ring R let Γ_R and $\mathcal{F}_R$ be the restriction of these functors to R-algebras; for example the action of $\Gamma_{\mathbb{Q}}$ on $\mathcal{F}_{\mathbb{Q}}$ is simply transitive. That is, $F = G_a^{\log_F}$ for a unique power series $\log_F$.

In the more delicate case $R = \mathbb{Z}_{(p)}$ we can ask for formal group laws with a simple logarithm after rationalizing. Let us write $\Gamma_{(p)} = \Gamma_{\mathbb{Z}_{(p)}}$, $\mathcal{F}_{(p)} = \mathcal{F}_{\mathbb{Z}_{(p)}}$ and $\tilde{\gamma}(x) = \sum_{j=0}^{\infty} a_{p^j - 1} x^{p^j}$ if $\gamma(x) = \sum_{i=1}^{\infty} a_{i-1} x^i \in \Gamma_{(p)}(S)$.

Theorem 4.1 (Cartier). *There is a unique natural transformation*

$$\xi : \mathcal{F}_{(p)} \longrightarrow \Gamma_{(p)}$$

with the following property: for any $\mathbb{Q}$-algebra S and F in $\Gamma_{(p)}(S)$ we have

$$\log_{\tilde{F}} = \widetilde{\log_F},$$

where $\tilde{F} = F^{\xi_F}$.

Cartier's result gives for every $F \in \mathcal{F}_{(p)}(R)$ a power series $\xi_F \in \Gamma_{(p)}(R)$ and over $R \otimes \mathbb{Q}$ a commutative diagram

$$\begin{array}{ccc} qF^{\xi_F} = q\tilde{F} & & \\ \downarrow {\scriptstyle q\xi_F} & \searrow {\scriptstyle \widetilde{\log_{qF}}} & \\ qF & \xrightarrow{\log_{qF}} & G_a \end{array} ,$$

where q is the canonical map $R \to R \otimes \mathbb{Q}$. Since strict isomorphisms over $\mathbb{Q}$ are unique, we are led to the identity

$$q\xi_F = \exp_{qF} \widetilde{\log_{qF}}.$$

Example 4.2. It is easy to see that $L \otimes \mathbb{Z}_{(p)}$ carries a formal group law G which is universal over $\mathbb{Z}_{(p)}$-algebras; G is simply the image of the standard universal group law under the canonical map $L \to L \otimes \mathbb{Z}_{(p)}$. Let us have a closer look at this case. The theorem asserts that $\exp_G \widetilde{\log_G}$, which seems to lie in $(L\otimes\mathbb{Q})[\![x]\!]$, in fact lies in $(L\otimes\mathbb{Z}_{(p)})[\![x]\!]$. Therefore, we see that Cartier's theorem is an integrality statement, and so the uniqueness statement is clear. (Here we are using the fact that $q : L\otimes\mathbb{Z}_{(p)} \to L\otimes\mathbb{Q}$ is monic.)

Example 4.3. In the case of the multiplicative formal group law $G_m(x,y) = x+y-xy$ over $\mathbb{Q}$[1] we know that $\exp_{G_m}(x) = 1-e^{-x}$ and $\log_{G_m}(x) = -\ln(1-x) = \sum_{i=1}^{\infty}\frac{x^i}{i}$. Now we have just been saying that

$$\xi_{G_m}(x) = 1 - e^{-\sum \frac{x^{p^j}}{p^j}}$$

lies in $\mathbb{Z}_{(p)}[\![x]\!]$. This is an observation which had already been made by E. Artin and Hasse.

Exercise 4.4. Show that the natural transformation

$$\begin{array}{ccc} \mathcal{F}_{(p)} & \longrightarrow & \mathcal{F}_{(p)} \\ F & \longmapsto & \tilde{F} \end{array}$$

is idempotent. In fact, $\xi_{\tilde{F}}(x) = x$.

We postpone the proof of Cartier's theorem to set up some machinery. Let

$$C_F := TR[\![T]\!] = \{\gamma(T)\in R[\![T]\!] \,:\, \gamma(T) = a_0T + a_1T^2 + \cdots\}$$

denote the **curves** on F with operation

$$(\gamma +_F \gamma')(T) := F(\gamma(T), \gamma'(T)).$$

Lemma 4.5. *$(C_F, +_F)$ is an abelian group which is Hausdorff and complete with respect to the T-adic filtration.*

Proof. Clearly the 0 curve provides C_F with a unit. But why do inverses exist? Let us first look at the case $\gamma(T) = T$ and inductively construct $\iota(T) \in C_F$ such that $F(T,\iota(T)) = 0$. Setting $\iota_1(T) = -T$ we may suppose $\iota_{n-1}(T)$ satisfies $F(T,\iota_{n-1}(T)) = a_nT^n + \cdots \equiv 0 \quad (T^n)$. We can estimate $f +_F g$ by

[1] We are developing the theory in the graded case, but everything goes through in the ungraded case by replacing gradation arguments by filtration arguments. We will have occasion to work in the ungraded world below when discussing the Honda formal group law.

$f+g$ modulo the ideal (T^{m+n}) whenever $f \in (T^m)$ and $g \in (T^n)$. So we define $\iota_n(T)$ to be $\iota_{n-1}(T) +_F (-a_n T^n)$. Then

$$\begin{aligned} T +_F \iota_n(T) &= (T +_F \iota_{n-1}(T)) +_F (-a_n T^n) \equiv a_n T^n +_F (-a_n) T^n \\ &\equiv a_n T^n - a_n T^n = 0 \end{aligned}$$

completes the induction. Now since $\iota_n \equiv \iota_{n-1} \bmod (T^n)$ they tend to a limiting power series ι which is the required inverse to T. The inverse of an arbitrary curve γ is given by $\iota(\gamma)$.

That C_F is Hausdorff and complete is not hard to see. □

COROLLARY 4.6. *Any* $\gamma = a_0 T + a_1 T^2 + \cdots \in C_F$ *has a unique formal power expression*

$$\gamma(T) = \sum_{i=0}^{\infty}{}^{F} c_i T^{i+1}.$$

PROOF. Take $c_0 = a_0$ and suppose that

$$\gamma_{n-1}(T) = \sum_{i=0}^{n-1}{}^{F} c_i T^{i+1} \equiv \gamma(T) \mod (T^{n+1}).$$

Then there is a unique choice of c_n giving

$$\sum_{i=0}^{n}{}^{F} c_i T^{i+1} \equiv \gamma(T) \mod (T^{n+2}).$$

Indeed, define c_n to be the coefficient of T^{n+1} in

$$\gamma(T) -_F \gamma_{n-1}(T) = c_n T^{n+1} + \cdots .$$

□

The group of curves C_F enjoys three families of operators. The **homothety** operator $[a]$ is defined for any a in R and sends a curve $\gamma(T)$ to the curve

$$([a]\gamma)(T) := \gamma(aT).$$

For each positive integer n we have **verschiebung** and **Frobenius** operators which are defined by

$$(\mathrm{v}_n \gamma)(T) := \gamma(T^n)$$

and

$$(\mathrm{f}_n\,\gamma)(T) := \sum_{i=0}^{n-1}{}^{F}\ \gamma(\zeta^i T^{1/n})$$

respectively. The latter equation requires some interpretation. Let ζ be a primitive n^{th} root of unity and consider the power series $\sum^{F}\gamma(\zeta^i S)$ over $R[\zeta]$; one can easily see using

$$x^n - 1 = \prod_{i=1}^{n}(x - \xi^i)$$

that this in fact lies in $R[\![\, S^n \,]\!]$ and thus serves to define $\sum^{F}\gamma(\zeta^i T^{1/n})$ by replacing S^n with T.

For example, take F to be the additive formal group law over a ring R and let $\gamma(T) = \sum_{j=1}^{\infty} a_{j-1}T^j$. Then

$$(\mathrm{f}_n\,\gamma)(T) = \sum_i \gamma(\zeta^i T^{1/n}) = \sum_{i,j} a_{j-1}\zeta^{ij}T^{j/n} = \sum_j a_{j-1}\big(\sum_i \zeta^{ij}\big)T^{j/n}.$$

The inner sum in the last expression is n if n divides j and zero otherwise, so

$$(\mathrm{f}_n\,\gamma)(T) = \sum_{k=1}^{\infty} n a_{nk-1}T^k.$$

In the next proposition we collect together some immediate properties of our operators.

Proposition 4.7. *The homothety, verschiebung and Frobenius operators are continuous additive natural operators which satisfy the following relations:*

$$\begin{aligned}
[a][b] &= [ba]\\
\mathrm{v}_m\,\mathrm{v}_n &= \mathrm{v}_{mn}\\
[a]\,\mathrm{v}_n &= \mathrm{v}_n[a^n]\\
\mathrm{f}_n[a] &= [a^n]\,\mathrm{f}_n\\
\mathrm{f}_n\,\mathrm{v}_n &= n \quad (\textit{That is, } \mathrm{f}_n\,\mathrm{v}_n\,\gamma(T) = \gamma(T) +_F \cdots +_F \gamma(T).)\\
\mathrm{f}_n\,\mathrm{v}_m &= \mathrm{v}_m\,\mathrm{f}_n \quad \textit{when } (m,n) = 1.
\end{aligned}$$

DEFINITION 4.8. *A curve γ is called* **p-typical** *if $\mathrm{f}_q\gamma = 0$ for q prime and different from the prime p. The set of such curves is denoted $C_F^p \subset C_F$. We say that F is* **p-typical** *if γ_0 is p-typical, where $\gamma_0(T) = T$.*

EXAMPLE 4.9. Consider the additive formal group law over a $\mathbb{Z}_{(p)}$-algebra. Then by the calculation above, a curve $\gamma(T) = \sum a_{j-1}T^j$ is p-typical iff $a_{j-1} = 0$ for j not a power of p. That is, we must have $\gamma(T) = \sum a_{p^k-1}T^{p^k} = \tilde{\gamma}(T)$.

EXAMPLE 4.10. Now suppose F is a general formal group law which is a twisted version of the additive group law, so that there is an isomorphism

$$\theta : F \longrightarrow G_a.$$

(This happens, for example, for any formal group law over a rational algebra.) Then the induced map

$$\theta_* : C_F \longrightarrow C_{G_a}$$

which sends a curve $\gamma(T)$ to the composite $\theta(\gamma(T))$ is also an isomorphism, and takes γ_0 to θ. It is trivial to check that composition with θ commutes with the three types of operators, and therefore preserves p-typical curves. Thus F is p-typical if and only if θ is p-typical in G_a, *i.e.,* iff $\theta = \widetilde{\theta}$.

To state the next proposition properly, we need a category that includes as morphisms both ring homomorphisms and twisting by power series. The objects will be all formal group laws, *i.e.,* the disjoint union of $\mathcal{F}(R)$ over all rings. The category structure is that generated by both the twisting morphisms within each $\mathcal{F}(R)$ and ring homomorphisms. More specifically, the morphisms from $F \in \mathcal{F}(R)$ to $G \in \mathcal{F}(S)$ are all pairs (θ, f) where f is a ring homomorphism from R to S and θ is a morphism in $\mathcal{F}(S)$ from fF to G. That is, $\theta(x) \in (S[\![x]\!])$ is such that $\theta(0) = 0$ and $\theta(fF(x, y)) = G(\theta(x), \theta(y))$. The composite of $(\psi, g) \circ (\theta, f)$ is easily seen to be $(\psi(g\theta), gf)$.

C_F and C_F^p are functors on this category.

The following proposition will allow us to prove the theorem of Cartier.

PROPOSITION 4.11. *Restrict the functors C_F and C_F^p to formal group laws over $\mathbb{Z}_{(p)}$-algebras. Then there is a unique natural operator ε on C_F with image C_F^p such that if $F = G_a$ then $\varepsilon(\gamma) = \tilde{\gamma}$. Moreover, this operator is additive, continuous and idempotent.*

Before constructing ε we'll show how we can use it to construct ξ for the theorem. We simply take ξ_F to be $\varepsilon\gamma_0$. Naturality shows that $(\varepsilon\gamma_0)(T)$ has leading term T and degree two. ξ inherits its naturality from ε, so all that remains to be checked is that over a $\mathbb{Q}$-algebra $\log_{\tilde{F}} = \widetilde{\log}_F$, where, as before, $\tilde{F}$ denotes F^{ξ_F}. Well, by naturality we have that

$$\begin{array}{ccc} C_F & \xrightarrow{\varepsilon_F} & C_F \\ {\scriptstyle \log_{F*}}\Big\downarrow & & \Big\downarrow{\scriptstyle \log_{F*}} \\ C_{G_a} & \xrightarrow{\varepsilon_{G_a}} & C_{G_a} \end{array}$$

commutes. Following γ_0 along the two paths shows us that $\widetilde{\log}_F = \log_F(\xi_F)$. But by the uniqueness of the logarithm,

$$\begin{array}{ccc} \tilde{F} & \xrightarrow{\quad \xi_F \quad} & F \\ & {\scriptstyle \log_{\tilde{F}}}\searrow \qquad \swarrow{\scriptstyle \log_F} & \\ & G_a & \end{array}$$

commutes, and so $\widetilde{\log}_F = \log_{\tilde{F}}$. In fact, this argument shows that $\widetilde{G_a^{\theta}} = G_a^{\tilde{\theta}}$, even if we aren't over a rational algebra.

In fact, $\tilde{F}$ is p-typical and $F \mapsto \tilde{F}$ is a projection onto the subset of p-typical formal group laws. To see that $\tilde{F}$ is p-typical note that the bijection

$$\xi_{F*} : C_{\tilde{F}} \longrightarrow C_F$$

restricts to a bijection $C^p_{\tilde{F}} \to C^p_F$. Now $\xi_F = \varepsilon\gamma_0 \in C_F$ is p-typical, so $\gamma_0 \in C_{\tilde{F}}$ is, and so $\tilde{F}$ is.

Now we construct the idempotent ε. Let q be a prime different from p. Motivated by the fact that ε must be killed by f_q, we define e_q, a natural operator $C_F \to C_F$, by

$$\mathrm{e}_q\,\gamma = \gamma -_F \frac{1}{q}\,\mathrm{v}_q\,\mathrm{f}_q\,\gamma.$$

NOTE 4.12. Here we have made use of the $\mathbb{Z}_{(p)}$-module structure on C_F which is defined in the following way. For positive $n \in \mathbb{Z}$ define the power series $[n]_F(T)$ to be the n-fold sum $T +_F \cdots +_F T$. That is, $[n]_F = n\gamma_0$ using the $\mathbb{Z}$-module structure on C_F coming from the fact that it is an abelian group. For negative n, replace $\gamma_0(T)$ with $\iota(T)$. If q is a unit in $\mathbb{Z}_{(p)}$, then

$[q]_F(T) = qT + \cdots$ is an invertible power series; thus we may define $[1/q]_F(T)$ to be its inverse. Because of the commutativity of our formal group law, F is unchanged when twisted by these power series. Thus they induce maps $C_F \to C_F$, and this is how $\mathbb{Z}_{(p)}$ acts on C_F. (Don't confuse $[n]_F$ with the homothety operation $[n]$.)

The following straightforward calculation shows that we are on the right track:

$$\begin{aligned} \mathrm{f}_q\,\mathrm{e}_q &= \mathrm{f}_q -_F \mathrm{f}_q \frac{1}{q} \mathrm{v}_q\,\mathrm{f}_q \\ &= \mathrm{f}_q -_F \frac{1}{q} \mathrm{f}_q\,\mathrm{v}_q\,\mathrm{f}_q \\ &= \mathrm{f}_q -_F \frac{1}{q} q\,\mathrm{f}_q \\ &= \mathrm{f}_q -_F \mathrm{f}_q \;=\; 0. \end{aligned}$$

Moreover, it is easy to show that $\mathrm{e}_q\,\gamma \equiv \gamma \bmod x^q$, that $\mathrm{e}_q\,\mathrm{e}_r = \mathrm{e}_r\,\mathrm{e}_q$, and that each e_q is natural, idempotent, additive and continuous. Thus it makes sense to define

$$\varepsilon := \prod_{q \neq p} \mathrm{e}_q,$$

and this has the desired properties.

The uniqueness of ε follows from an argument similar to that used to show the uniqueness of ξ.

EXERCISE 4.13. Let F be a p-typical formal group law over a $\mathbb{Z}_{(p)}$-algebra. Any curve can be uniquely represented as a formal sum $\gamma(T) = \sum_{i=1}^{\infty}{}^{F}\, c_{i-1}T^i$. Show that $\gamma(T)$ is a p-typical curve if and only if $c_{i-1} = 0$ for i not a power of p. Moreover, $\mathrm{f}_p(\sum^F c_{p^i-1}T^{p^i}) = p\sum^F c_{p^{i+1}-1}T^{p^i}$.

EXERCISE 4.14. Consider the operator $C_F \to C_F$ defined by sending a curve $\gamma(T) = \sum_{i=1}^{\infty}{}^{F}\, c_{i-1}T^i$ to the curve $\sum_{k=0}^{\infty}{}^{F}\, c_{p^k-1}T^{p^k}$. Is this the operator ε? If not, which hypothesis of the uniqueness statement fails to hold? Which of the other properties of ε are shared by this operator?

5. The Universal p-Typical Formal Group Law

A possible candidate for a universal p-typical object is the p-typicalization $\tilde{G} = G^{\xi_G}$ of Lazard's formal group law G over the $\mathbb{Z}_{(p)}$-algebra $L_{(p)} := L \otimes \mathbb{Z}_{(p)}$. However, it turns out that while for any p-typical F over S there always *exists* a map $L_{(p)} \to S$ sending $\tilde{G}$ to F, it may not be *unique*. So we instead consider the factorization

$$\begin{array}{ccc} L & \xrightarrow{u} & L_{(p)} \\ \big\downarrow & & \big\uparrow{\scriptstyle i} \\ L_{(p)} & \xrightarrow{\tilde{u}} & L^p \end{array}$$

of the homomorphism $u : L \to L_{(p)}$ classifying $\tilde{G}$ through the canonical map $L \hookrightarrow L_{(p)}$ and through the $\mathbb{Z}_{(p)}$-subalgebra L^p generated by the coefficients of $\tilde{G}$, *i.e.*, generated by $\mathrm{im}(u)$. We claim that $(L^p, \tilde{G})$ is the universal p-typical formal group law. To see this, suppose $F \in \mathcal{F}^p(S)$ is p-typical, $v : L \to S$ is its classifying map, and $\tilde{v} : L_{(p)} \to S$ is the extension of v to $L_{(p)}$. From

$$\tilde{v}\,\tilde{G} = \tilde{v}\,G^{\xi_G} = F^{\xi_F} = F$$

we recognize that F is the pushforward of $\tilde{G}$ under $\tilde{v}i$ from L^p. Any map sending $\tilde{G}$ to F will fill in the dashed arrow of the diagram

$$\begin{array}{ccc} L_{(p)} & & \\ {\scriptstyle \tilde{u}}\big\downarrow & \searrow^{\tilde{v}} & \\ L^p & \dashrightarrow & S \end{array}$$

and is thus unique. In particular we may take S to be L^p, $F = \tilde{G}$ and $\tilde{v} = \tilde{u}$ and obtain the idempotency relation $\tilde{u}^2 = \tilde{u}$ (really $\tilde{u}i\tilde{u} = \tilde{u}$).

Next we want to show that L^p is a polynomial algebra. When dealing with formal group laws we should think about everything in terms of the logarithm. Let $R = \mathbb{Z}[\,m_1, m_2, \dots]$. We may use the power series

$$\log(T) = \sum_{i=0}^{\infty} m_i T^{i+1}, \qquad m_0 = 1$$

to define the formal group law $G_a^{\log}(x, y) = \log^{-1}(\log(x) + \log(y))$. Recall that the map $\phi : L \to R$ classifying $G_a^{\log}$ was shown to be a rational isomorphism

(3.8). Therefore, the induced map $L_{(p)} \to R_{(p)}$ which sends G to $G_a^{\log}$ is injective. We find that the following diagram commutes

$$\begin{array}{ccccccccc} G & & L_{(p)} & \xrightarrow{\tilde{u}} & L^p & \xrightarrow{i} & L_{(p)} & & \tilde{G} \\ \downarrow & & \downarrow \varphi & & \downarrow & & \downarrow \varphi & & \downarrow \\ G_a^{\log} & & R_{(p)} & \longrightarrow & R^p & \longrightarrow & R_{(p)} & & \widetilde{G_a^{\log}} = \widetilde{G_a^{\log}} \end{array},$$

$R^p = \mathbb{Z}_{(p)}[\, m_{p-1}, m_{p^2-1}, \ldots]$ and $R_{(p)} \to R^p$ is the natural projection. The center arrow is uniquely defined by requiring the diagram to be commutative, and is necessarily monic. Observe that for any choice of generators $L_{(p)} = \mathbb{Z}_{(p)}[x_1, x_2, \ldots]$ the x_i map under $\tilde{u}$ to decomposable elements unless $i = p^j - 1$, since they are injectively sent under $Q\varphi : QL^{2-2i} \to QR^{2-2i}$ to $\epsilon_{i+1} m_i$. On the other hand the x_{p^j-1} remain indecomposable. We have just proved

Proposition 5.1. *$L^p = \mathbb{Z}_{(p)}[\, \bar{x}_{p-1}, \bar{x}_{p^2-1}, \ldots]$, where $\bar{x}_{p-1} = \tilde{u} x_{p-1}$.*

Now we want to construct explicit generators for L^p. First observe that the group of p-typical curves C_F^p is closed under the operations $[p]$, v_p and f_p. In particular, $\mathrm{f}_p\, \gamma_0$ is in C_F^p if F is p-typical. Since any $\gamma \in C_F^p$ has a unique expansion

$$\gamma(T) = \sum_{i=0}^{\infty}{}^{F} c_{i+1} T^{p^i},$$

the equation

$$\mathrm{f}_p\, \gamma_0(T) = \sum_{i=0}^{\infty}{}^{F} v_{i+1} T^{p^i}$$

defines elements v_i in S (where $F \in \mathcal{F}(S)$). We claim that in the universal case these are generators for L^p. Let us abuse notation a little and write m_i for m_{p^i-1} and log for $\widetilde{\log}$. That is,

$$\log(T) = \sum_{i=0}^{\infty} m_i T^{p^i}, \qquad m_0 = 1.$$

Working over R^p we compute

$$\begin{aligned}\log \sum_{i=0}^{\infty}{}^{G_a^{\log}} v_{i+1}T^{p^i} &= \sum_{i=0}^{\infty} \log(v_{i+1}T^{p^i}) = \sum_{i,j=0}^{\infty} m_j v_{i+1}^{p^j} T^{p^{i+j}} \\ &= \sum_{n=0}^{\infty} (\sum_{i+j=n} m_j v_{i+1}^{p^j}) T^{p^n}\end{aligned}$$

and

$$\begin{aligned}\log \mathrm{f}_p\, \gamma_0(T) &= \log(\sum_{i=0}^{p-1}{}^{G_a^{\log}} \zeta^i T^{1/p}) = \sum_{i=0}^{p-1} \log(\zeta^i T^{1/p}) \\ &= \sum_{i,j} m_j \zeta^{ip^j} T^{p^{j-1}} = \sum_{j=0}^{\infty} (\sum_{i=0}^{p-1} \zeta^{ip^j}) m_j T^{p^{j-1}} \\ &= \sum_{j=1}^{\infty} p\, m_j T^{p^{j-1}} = \sum_{n=0}^{\infty} p\, m_{n+1} T^{p^n}.\end{aligned}$$

Thus we have an inductive formula

$$p\, m_{n+1} = \sum_{i+j=n} m_j v_{i+1}^{p^j}$$

for the v_i in R^p.

EXAMPLE 5.2. The first few instances of the formula are as follows:

$$\begin{aligned} p\, m_1 &= v_1 \\ p\, m_2 &= v_2 + m_1 v_1 \\ p\, m_3 &= v_3 + m_1 v_2^p + m_2 v_1^{p^2}. \end{aligned}$$

This shows $p\, m_n \equiv v_n$ modulo decomposables in R^p. (The usual convention is to set $v_0 = p$.)

PROPOSITION 5.3 (Hazewinkel). $L^p = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$.

From this we see that any p-typical formal group law is determined by $\mathrm{f}_p\, \gamma_0$. In general it is hard to describe the formal sum $\sum^F v_{i+1} T^{p^i}$. An exception occurs if it has only one summand, for example, if $v_n = 1$ and $v_i = 0$ for

$i \neq n$. In this case $L^p \to \mathbb{Z}_{(p)}$ defines a formal group law over $\mathbb{Z}_{(p)}$, the so called **Honda formal group law**[2] F_n. Here the inductive relation

$$p\, m_{k+1} = m_{k-n+1},\ m_0 = 1,\ m_i = 0 \text{ for } i < 0$$

has the solution

$$m_n = \frac{1}{p},\ m_{kn} = p^{-k},\ m_i = 0 \text{ for } n \not| \, i.$$

That is,

$$\log_{F_n}(T) = \sum_{k=0}^{\infty} \frac{T^{p^{nk}}}{p^k}.$$

Note that we could have just written down the power series $\sum \frac{T^{p^{ni}}}{p^i}$ and defined the Honda formal group law to be the formal group law with this logarithm. This would work over a rational algebra, and the point of what we have done is to show that this works over a $\mathbb{Z}_{(p)}$-algebra.

In addition to the Hazewinkel generators, there is another set of generators for L^p that we will now introduce. Both have their advantages and disadvantages.

Because f_q is an additive operation, C_F^p is a subgroup of C_F. Thus $[p]_F(T)$, the formal sum of γ_0 taken p times, is p-typical if F is. If we apply this to be the universal formal group law $\tilde{G}$ over L^p, then we can write

$$[p]_{\tilde{G}}(T) = \sum\nolimits^{\tilde{G}} w_i T^{p^i}$$

for unique w_i in L^p. It turns out that $w_1, w_2, \ldots$ serve as generators called the **Araki generators** and that $w_0 = p$. We prove this by calculating the

[2]We're in ungraded territory again.

logarithm of $[p]_{\tilde{G}}(T)$ in two ways. On the one hand

$$\begin{aligned}\log[p]_{\tilde{G}}(T) &= \log \sum_j^{\tilde{G}} w_j T^{p^j} \\ &= \sum_j \log w_j T^{p^j} \\ &= \sum_{i,j} m_i (w_j T^{p^j})^{p^i} \\ &= \sum_k \Big(\sum_{i+j=k} m_i w_j^{p^i} \Big) T^{p^k},\end{aligned}$$

and on the other hand

$$\begin{aligned}\log[p]_{\tilde{G}}(T) &= [p]_{G_a} \log(T) \\ &= p \log(T) \\ &= \sum_k p m_k T^{p^k}.\end{aligned}$$

Therefore, $pm_k = \sum_{i=0}^k m_i w_{n-i}^{p^i}$. So $p(1 - p^{p^k - 1}) m_k = w_k +$ decomposables. This shows that the w_k are generators.

Proposition 5.4 (Araki). $L^p = \mathbb{Z}_{(p)}[w_1, w_2, \ldots]$.

Exercise 5.5. Show that $v_k \equiv w_k \bmod p$ by showing that $\mathrm{v}_p \, \mathrm{f}_p \equiv [p]_{\tilde{G}} \bmod p$. (Here v_k is a Hazewinkel generator and v_p is a verschiebung operator.)

So a p-typical formal group law F is completely determined by its p-series $[p]_F(T)$, since this determines the w_i. Over $\mathbb{F}_p$ we get that $[p]_{F_n}(T) = T^{p^n}$ using the exercise.

6. Representing $\mathcal{F}(R)$

We have been able to corepresent $\mathcal{F}(-)$ as a set-valued functor as $\mathrm{Hom}(L, -)$, but this is only taking into account part of the picture. Consider the action of $\Gamma(R)$ on $\mathcal{F}(R)$ from the right, for example. $\Gamma(-)$ is corepresented by S, where $S = \mathbb{Z}[a_1, a_2, \ldots]$. Since $\Gamma(-)$ is naturally a group, S is a cogroup in the category of commutative rings, *i.e.,* it is a commutative Hopf algebra over $\mathbb{Z}$. In fact, L is a comodule-algebra over S, since the action of Γ on $\mathcal{F}$ is natural. We want to understand all of this better, and since L^p has nice generators, we will tackle this case. Unfortunately, the action of Γ on $\mathcal{F}$ doesn't restrict to $\mathcal{F}^p$; in fact, $\mathcal{F}^p(R) \cdot \Gamma(R) = \mathcal{F}(R)$. Instead we must consider $\mathcal{F}^p(R)$ as a full subgroupoid of the groupoid $\mathcal{F}(R)$. The groupoid structure on $\mathcal{F}(R)$ is obtained from the category structure on $\mathcal{F}(R)$ that was discussed earlier by considering only strictly invertible morphisms, *i.e.,* power series in $\Gamma(R)$. This structure arises from any set X endowed with the action of a group π in the following way. We let the objects of the groupoid be the elements of X, and we define

$$\mathrm{Mor}(y, x) = \{\alpha \in \pi \, : \, y = x\alpha\}.$$

A groupoid that can be formed in this way is called **split**. When corepresenting a split groupoid it suffices to corepresent the set, the group and the action, as we did above for $\mathcal{F}(R)$. But $\mathcal{F}^p(R)$ is not split, and this is why we must corepresent it as a groupoid.[3]

We corepresent the object set as $\mathrm{Hom}(L^p, -)$, of course. To give a morphism $F^f \to F$ we need to specify the target F and the power series f; the source is uniquely defined. But not any f in $\Gamma(R)$ will do, for we need that F^f is p-typical as well. It is easy to check that F^f is p-typical iff f is p-typical for F, and we know that this holds iff $f(T) = \sum^F t_i T^{p^i}$ for some t_i with $t_0 = 1$ (all under the assumption that F is p-typical of course). Thus the set of all morphisms in our groupoid is naturally equivalent to $\mathrm{Hom}(L^p[t_1, t_2, \dots], -)$. We denote the corepresenting ring by M.

Now we investigate how the structural properties of the groupoid translate into conditions on the generators of L^p and the t_i. Define

$$\eta_L : L^p \longrightarrow M$$

[3]What does it mean to corepresent something other than a set-valued functor? If the functor F takes values in a category whose objects are defined using only categorical notions in Set, then a corepresenting object naturally acquires the structure of an object of the opposite type in the domain category. For example, if F is group-valued or groupoid-valued, then a corepresenting object will be a cogroup or cogroupoid object in the domain category.

to be the obvious inclusion. This corepresents the map that sends a morphism to its target and also makes M into a left L^p-module. Define

$$\eta_R : L^p \longrightarrow M$$

to be the map classifying $\tilde{G}^\theta$, where $\theta(T) = \sum^{\tilde{G}} t_i T^{p^i}$. This corepresents the map that sends a morphism to its source and also makes M into a right L^p-module. Composition is a map from Comp to Mor where Comp is the set of composable pairs of morphisms and Mor is the set of all of morphisms. Comp is defined by the pullback diagram

$$\begin{array}{ccc} \text{Comp} & \longrightarrow & \text{Mor} \\ \Big\downarrow & & \Big\downarrow \text{target} \\ \text{Mor} & \xrightarrow{\text{source}} & \text{Obj} \end{array} .$$

Because this is natural we get a map ψ from M to $M \otimes_{L^p} M$, since $M \otimes_{L^p} M$ is the pushout of M and M over η_L and η_R. The unit map Obj $\to$ Mor is corepresented by the map $\epsilon : M \to L^p$ sending t_i to 0 for $i \geq 1$. And the inverse map Mor to Mor is corepresented by a map $c : M \to M$. So M is almost a Hopf algebra over L. Since M (and the other data) form a cogroupoid object in the category of commutative rings, we might call M a **Hopf algebroid**.

We want to understand the source map η_R by considering its composite with the embedding induced by the embedding of L^p in R^p:

$$L^p \xrightarrow{\eta_R} L^p[\,t_1, t_2, \ldots] \hookrightarrow R^p[\,t_1, t_2, \ldots].$$

We continue to use the notation $R^p = \mathbb{Z}_{(p)}[\,m_1, m_2, \ldots]$. Since the logarithm is unique we have the equation $\log_{\tilde{G}} f = \log_{\tilde{G}^f} = \log \eta_R(\tilde{G}) = \eta_R(\log_{\tilde{G}})$. Hence it follows that

$$\log_{\tilde{G}}(f(T)) = \log \sum_j^{\tilde{G}} t_j T^{P^j} = \sum_j \log(t_j T^{p^j}) = \sum_{i,j} m_i t_j^{p^i} T^{p^{i+j}}$$

and

$$\log(f(T)) = \eta_R \log(T) = \sum_k \eta_R(m_k) T^{p^k}$$

and we obtain a BP-formula.[4]

PROPOSITION 6.1. $\eta_R(m_k) = \sum\limits_{i+j=k} m_i t_j^{p^i}$.

As an exercise the reader can express the image of the Hazewinkel generator v_k in terms of the t_i.

We now derive a BP-formula due to Ravenel. As usual, $\tilde{G}$ denotes the universal p-typical formal group law, which lies over the ring L^p. Recall that the Araki generators are defined by

$$[p]_{\tilde{G}}(T) = \sum\nolimits^{\tilde{G}} w_i T^{p^i}$$

and that we have maps

$$L^p \xrightarrow{\eta_R} L^p[t_1, t_2, \ldots] \xleftarrow{\eta_L} L^p.$$

The map η_L is just the standard inclusion, and we denote $\eta_L \tilde{G}$ by $\tilde{G}$ as usual. The map η_R is the unique map such that $\eta_R \tilde{G} = \tilde{G}^\theta$, where $\theta(T) = \sum^{\tilde{G}} t_i T^{p^i}$. Notice that

$$\theta(\eta_R[p]_{\tilde{G}}(T)) = \theta([p]_{\tilde{G}^\theta}(T)) = [p]_{\tilde{G}}(\theta(T)).$$

But

$$\begin{aligned} \theta(\eta_R[p]_{\tilde{G}}(T)) &= \theta(\eta_R \sum_j\nolimits^{\tilde{G}} w_j T^{p^j}) \\ &= \theta(\sum_j\nolimits^{\tilde{G}^\theta} \eta_R(w_j) T^{p^j}) \\ &= \sum_j\nolimits^{\tilde{G}} \theta(\eta_R(w_j) T^{p^j}) \\ &= \sum_{i,j}\nolimits^{\tilde{G}} t_i \eta_R(w_j)^{p^i} T^{p^{i+j}} \end{aligned}$$

[4] We will soon see the origins of this terminology

and

$$\begin{aligned}[p]_{\tilde G}(\theta(T)) &= [p]_{\tilde G}\Big(\sum_j^{\tilde G} t_j T^{p^j}\Big)\\ &= \sum_j^{\tilde G} [p]_{\tilde G}(t_j T^{p^j})\\ &= \sum_{i,j}^{\tilde G} w_i t_j^{p^i} T^{p^{i+j}}.\end{aligned}$$

This proves

PROPOSITION 6.2. $\displaystyle\sum_{i+j=k}^{\tilde G} t_i \eta_R(w_j)^{p^i} = \sum_{i+j=k}^{\tilde G} w_i t_j^{p^i}$.

Next we want to understand the composition map $M \xrightarrow{\psi} M \otimes_L M$. Let $f(T) = \sum^F t_i T^{p^i}$ and $g(T) = \sum^{Gf} s_j T^{p^j}$ be p-typical curves. Then we have that

$$fg(T) = f\Big(\sum^{Gf} s_j T^{p^j}\Big) = \sum_j^G f(s_j T^{p^j}) = \sum_{i,j}^G t_i s_j^{p^i} T^{p^{i+j}}.$$

In terms of the tensor product we get

$$\psi t_k = \sum_{i+j=k}^G t_i \otimes t_j^{p^i} \qquad (t_0 = 1)$$

which should remind you of the antipode of the Milnor diagonal of Chapter 1.

7. Applications to Topology

We will use the above formulae to perform some calculations in E-homology for a complex oriented theory E. Recall that $E^*\mathbb{C}\mathrm{P}^\infty = E^*[\![x]\!]$ where x denotes the Euler class of the tautological bundle λ. Therefore, we find that

$$E_*\mathbb{C}\mathrm{P}^\infty = E_*\langle \beta_0, \beta_1, \dots \rangle \quad \text{and} \quad \langle x^i, \beta_j \rangle = \delta_{i,j}.$$

The notation here means the free E_*-module on the β's. (If you don't know how to prove this for yourself, wait until the end of the next page and try again.) The β_i produce elements (denoted again by β_i) in E_*BU under the inclusion $\mathbb{C}\mathrm{P}^\infty \hookrightarrow BU$. Observe that the last map is not a map of H-spaces.

A short look at the Atiyah-Hirzebruch spectral sequence may convince you that

$$E_*BU = E_*[\,\beta_1, \beta_2, \dots]$$

once you know this for singular homology. (See Switzer for the singular case.) We also want to know the E-cohomology of $X = BU$. Since it has only even-dimensional cells,

$$0 \longrightarrow E_*X_{j-1} \longrightarrow E_*X_j \longrightarrow E_*(X_j/X_{j-1}) \longrightarrow 0$$

is short exact; since the cokernel is free, it is split exact, and thus the bottom row of

$$\begin{array}{ccccc} E^*X_{j-1} & \longleftarrow & E^*X_j & \longleftarrow & E^*(X_j/X_{j-1}) \\ \downarrow & & \downarrow & & \downarrow \cong \\ \mathrm{Hom}_{E^*}(E_*X_{j-1}, E_*) & \longleftarrow & \mathrm{Hom}_{E^*}(E_*X_j, E_*) & \longleftarrow & \mathrm{Hom}(E_*(X_j/X_{j-1}), E_*) \end{array}$$

is also short exact. This allows us to proceed by induction using the strong additivity of E and the five-lemma to conclude that $E^*(X_j) \cong \mathrm{Hom}_{E^*}(E_*X_j, E_*)$ for each j. (This corresponds to the fact that the AH-spectral sequence collapses). Now the Milnor sequence finally gives

$$\begin{aligned} E^*X &= \varprojlim E^*X_j = \mathrm{Hom}(\varinjlim E_*X_j, E_*) = \mathrm{Hom}_{E_*}(E_*(X), E_*) \\ E^*BU &= H^*(BU; E^*) = \mathbb{Z}[c_1.c_2, \dots]\hat{\otimes} E^* = E^*[\![\, c_1, c_2, \dots]\!]. \end{aligned}$$

where

$$< c_m, \beta_{i_1}\beta_{i_2} \cdots \beta_{i_n} >= \begin{cases} 1 & m = n,\ i_j = 1 \\ 0 & \text{otherwise} \end{cases} .$$

In the case of $E = MU$ the c_i are called **Conner and Floyd classes**. We can play the same game with MU replacing BU to see

$$E_*MU = E_*[\, b_1, b_2, \dots].$$

But we want to be more precise about what the b_i's are. Applying the Thom construction to the inclusion $\mathbb{C}\mathrm{P}^\infty \hookrightarrow BU$ we get a map $\Sigma^{-2}MU(1) \to MU$, which is the MU-Euler class $x \in MU^2\mathbb{C}\mathrm{P}^\infty$ for the canonical bundle. In E-homology we have

$$\begin{array}{ccc} E_*(\Sigma^{-2}\mathbb{C}\mathrm{P}^\infty) & \longrightarrow & E_*MU \\ \beta_i & \longmapsto & b_{i-1}, \end{array}$$

where $b_0 = 1$. The isomorphism $E^*MU \xrightarrow{\cong} \mathrm{Hom}_{E_*}(E_*MU, E_*)$ can be used to define a map $u : MU \to E$ by demanding that $1 \mapsto 1$ and $b_{i_1} \cdots b_{i_n} \mapsto 0$. This u is called the **universal Thom class of** E.

A map of ring spectra $MU \to E$ gives rise to a map of E_*-algebras and vice versa. Therefore, the universal Thom class $u : MU \to E$ is a map of ring theories. Observe that the composition

$$\Sigma^{-2}\mathbb{C}\mathrm{P}^\infty \xrightarrow{x} MU \xrightarrow{u} E$$

is the orientation x_E of E. Hence, $f : L \cong MU^* \xrightarrow{u_*} E^*$ sends the universal formal group law to the formal group law for x_E.

We can ask what other orientations for E are possible. Recall that $y \in E^2(\mathbb{C}\mathrm{P}^\infty)$ is a complex orientation if and only if y is sent to 0 under $E^2(\mathbb{C}\mathrm{P}^\infty) \to E^2(*)$ and y is sent to the generator σ under $E^2(\mathbb{C}\mathrm{P}^\infty) \to E^2S^2$.

Let y be $f(x) = \sum_{i=-1}^\infty a_i x^{i+1}$ where $a_i \in E^{-2i}$. Then the first condition says that $a_{-1} = 0$ and the second says that $a_0 = 1$.

Proposition 7.1. *The collection of complex orientations for E is in one-to-one correspondence with the set of curves $\Gamma(E^*) = \{T + a_1T^2 + \cdots\}$.*

If $x(L)$ and $y(L)$ denote the Euler classes of a line bundle L according to the two orientations, the formal group law associated to y has the form $F_y = {}^fF_x$:

$$\begin{aligned} y(L \otimes L') &= f(x(L \otimes L')) = f(F_x(x(L), x(L'))) \\ &= fF_x(f^{-1}y(L), f^{-1}y(L')) = {}^fF_x(y(L), y(L')). \end{aligned}$$

8. Characteristic Numbers

Let M be an n-dimensional U-manifold with $\nu : M \to BU$ the lifting. Then M has a canonical orientation, so there is a well-defined class $[M] \in H_n(M; \mathbb{Z})$. Any class $c \in H^n(BU)$ gives a characteristic class by defining

$c(\nu) := \nu^*(c)$; we call $\langle\, c(\nu), [M]\,\rangle \in \mathbb{Z}$ the **characteristic number** associated to this characteristic class.

The map $M \mapsto \langle\, c(\nu), [M]\,\rangle$ is well-defined on U-bordism classes. Indeed, suppose $M = \partial W$ as U-manifolds, so $\nu = \nu_W i$ stably, where i is the inclusion $M \hookrightarrow W$. Then, under the boundary map $\partial : H_{n+1}(W, M) \to H_n(M)$, the orientation class $[W]$ is sent to $[M]$. So

$$\langle\, c(\nu), [M]\,\rangle = \langle\, \nu^*(c), \partial[W]\,\rangle = \langle\, \delta\nu^*(c), [W]\,\rangle.$$

But $\delta\nu^*(c) = \delta i^* \nu^*(c) = 0$, and thus we get a homomorphism

$$\Omega_n^U \longrightarrow \mathbb{Z}$$

for each c. Moreover, the natural map $\Omega_n^U \to \mathrm{Hom}(H^*(BU), \mathbb{Z})$ is simply the Hurewicz map:

$$\begin{array}{ccccc} \Omega_n^U & \longrightarrow & \mathrm{Hom}(H^*(BU), \mathbb{Z}) & & \\ & & & \nwarrow^{\cong} & \\ \downarrow{\cong} & & & & H_n(BU) \\ & & & \nearrow_{\cong} & \\ \pi_n(MU) & \xrightarrow[h]{} & H_n(MU) & . & \end{array}$$

Exercise 8.1. Check that this diagram commutes.

We now calculate the image of the U-bordism class of $\mathbb{C}\mathrm{P}^n$ under the Hurewicz map.[5]

Recall that $H^*(\mathbb{C}\mathrm{P}^\infty) = \mathbb{Z}[x]$, where $x = e(\lambda)$ is the Euler class of the canonical bundle. We saw in Section 2 that $\langle\, x, [\mathbb{C}\mathrm{P}^1]\,\rangle = -1$. It is more generally true that $\langle\, x^n, [\mathbb{C}\mathrm{P}^n]\,\rangle = (-1)^n$. Now $H_*(\mathbb{C}\mathrm{P}^\infty)$ is the free abelian group on generators β_n dual to x^n, and thus $\beta_n = (-1)^n[\mathbb{C}\mathrm{P}^n]$.

We need to study the homology Thom isomorphism for λ. We had the notation $MU(1)$ for the Thom space of λ, and we point out here that the zero section $\mathbb{C}\mathrm{P}^\infty \to MU(1)$ is in fact a homotopy equivalence.[6] We denote by β_n

[5] Any almost complex manifold has a natural U-manifold structure; when we write $\mathbb{C}\mathrm{P}^n$ we implicitly mean $\mathbb{C}\mathrm{P}^n$ with this natural structure.

[6] It is not hard to see by geometric arguments as in Section 1.7 that $MU(1)$ is homeomorphic to $\mathbb{C}\mathrm{P}^\infty$; one might say that $\mathbb{C}\mathrm{P}^\infty$ is $P(\mathbb{C}^\infty)$ while $MU(1)$ is $P(\mathbb{C} \oplus \mathbb{C}^\infty)$, where $P(V)$ denotes the space of one-dimensional subspaces of V. However, the zero section is the

the image of $\beta_n \in H_*(\mathbb{C}\mathrm{P}^\infty)$ under the zero section. The Thom isomorphism comes from the diagonal embedding of λ in $\mathbf{0} \times \lambda$. To study this we consider the diagram of bundle maps

$$\begin{array}{ccc} \mathbf{0} & \longrightarrow & \mathbf{0} \times \mathbf{0} \\ \downarrow & & \downarrow \\ \lambda & \longrightarrow & \mathbf{0} \times \lambda \end{array} ,$$

where the left-hand bundles are over $\mathbb{C}\mathrm{P}^\infty$, the right-hand bundles are over $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty$, the vertical maps cover the identity, and the horizontal maps cover the diagonal $\Delta : \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty$. By the functoriality of the Thom construction, we get a diagram

$$\begin{array}{ccccc} \mathbb{C}\mathrm{P}^\infty_+ & \xrightarrow{\Delta_+} & (\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty)_+ & = & \mathbb{C}\mathrm{P}^\infty_+ \wedge \mathbb{C}\mathrm{P}^\infty_+ \\ {\scriptstyle 0\text{-section}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{id} \otimes 0\text{-section}} \\ MU(1) & \longrightarrow & & & \mathbb{C}\mathrm{P}^\infty_+ \wedge MU(1) \end{array} .$$

Looking at this diagram in reduced homology, we find that $\beta_n \in \bar{H}_{2n}MU(1)$ gets sent to $\beta_{n-1} \otimes \beta_1 + \cdots + \beta_0 \otimes \beta_n$ in $H_*\mathbb{C}\mathrm{P}^\infty \otimes \bar{H}_*MU(1)$. Here we have used the fact that $(H_*(\mathbb{C}\mathrm{P}^\infty), \Delta_*)$ is a coalgebra dual to the polynomial algebra $\mathbb{Z}[x]$. Now the Thom isomorphism $\bar{H}_*MU(1) \to H_{*-2}\mathbb{C}\mathrm{P}^\infty$ is obtained by composing the map $\bar{H}_*MU(1) \to H_*\mathbb{C}\mathrm{P}^\infty \otimes \bar{H}_*MU(1)$ with evaluation on the Thom class $u \in H^2MU(1)$. Since the Thom class u pulls back to the Euler class x under the zero section, u is dual to β_1. Therefore, the Thom isomorphism is given by

$$\begin{array}{ccc} \bar{H}_{2n}MU(1) & \longrightarrow & H_{2n-2}\mathbb{C}\mathrm{P}^\infty \\ \beta_n & \longmapsto & \beta_{n-1}. \end{array}$$

The inclusion $\mathbb{C}\mathrm{P}^\infty \to BU$ defines classes β_i in H_*BU, and we've mentioned that H_*BU is polynomial on these classes with $\beta_0 = 1$. (The ring structure

map induced by the inclusion of $\mathbb{C}^\infty$ into the right summand of $\mathbb{C} \oplus \mathbb{C}^\infty$ and thus doesn't induce a homeomorphism. But all one has to do is show that on π_2, the only non-trivial homotopy group, the induced map is an isomorphism, and this isn't difficult. (There are other ways to do this.)

on H_*BU comes from the H-space structure on BU.) Just as the usual Thom construction is functorial, so is the stable version. Thus we get a map

$$\Sigma^{-2}MU(1) \longrightarrow MU$$

and a commutative square

$$\begin{array}{ccc} H_*(\Sigma^{-2}MU(1)) & \longrightarrow & H_*MU \\ \downarrow{\cong} & & \downarrow{\cong} \\ H_*\mathbb{C}\mathrm{P}^\infty & \longrightarrow & H_*BU \end{array}.$$

From this we see that $b_n \in H_*MU$, which was defined in the last section to be the image of $\beta_{n+1} \in H_*(\Sigma^{-2}MU(1))$, gets sent to $\beta_n \in H_*BU$ by the Thom isomorphism. If we write $\beta(T) = \beta_0 + \beta_1 T + \cdots$ in $(H_*BU)[T]$, and $b(T) = T + b_1T^2 + \cdots$ in $(H_*MU)[T]$, then $\beta(T)$ corresponds to $b(T)/T$ under the Thom isomorphism.

As in Section 7, the tangent bundle τ of $\mathbb{C}\mathrm{P}^n$ satisfies $\tau + \mathbf{1} = (n+1)\lambda^*$. Since the Whitney sum of the tangent bundle and the normal bundle of an embedding is a trivial bundle, we have that the stable normal bundle ν may be written as

$$\nu = (n+1)(1 - \lambda^*).$$

The map representing the stable bundle $\lambda - 1$ over $\mathbb{C}\mathrm{P}^\infty$ is the inclusion $\mathbb{C}\mathrm{P}^\infty \to BU$ we've discussed above. By definition, its effect in homology is to send β_n to β_n. Therefore, the map representing $\lambda^* - 1$ sends β_n to $(-1)^n\beta_n$; more concisely, it sends $\beta(T)$ to $\beta(-T)$. Since the H-space structure on BU corresponds both to the Whitney sum of bundles and to the ring structure on H_*BU, we have that the map representing $k(\lambda^* - 1)$ sends $\beta(T)$ to $\beta(-T)^k$. Thus the map representing $(n+1)(1-\lambda^*)$ sends $\beta(T)$ to $\beta(-T)^{-(n+1)}$. As ν is the restriction of this bundle to $\mathbb{C}\mathrm{P}^n$, we see that β_n is sent by the map representing ν to the coefficient of T^n in $\beta(-T)^{-(n+1)}$, which we denote $(\beta(-T)^{-(n+1)})_n$.

The Pontrjagin-Thom construction gives us a stable map

$$S^{2n} \longrightarrow \mathrm{Th}\,\nu \longrightarrow MU$$

representing the element of $\pi_{2n}(MU)$ corresponding to the U-bordism class of $\mathbb{C}\mathrm{P}^n$. To find the Hurewicz image of this class, we apply homology and look at the image of the fundamental class $[S^{2n}]$. The following diagram will allow us to calculate this image:

$$\begin{array}{ccccc} \bar{H}_{2n}S^{2n} & \longrightarrow & \bar{H}_{2n}\,\mathrm{Th}\,\nu & \longrightarrow & \bar{H}_{2n}MU \\ & & \big\downarrow \cong & & \big\downarrow \cong \\ & & H_{2n}\mathbb{C}\mathrm{P}^n & \longrightarrow & H_{2n}BU \end{array} .$$

We ask the reader to verify that the fundamental class of the $2n$-sphere gets mapped to the fundamental class of $\mathbb{C}\mathrm{P}^n$ in $H_{2n}\mathbb{C}\mathrm{P}^n$. Our previous work shows that this then gets sent to $((b(T)/T)^{-(n+1)})_n$ in $H_{2n}MU$. This can be written $(b(T)^{-(n+1)})_{-1}$, but it can be put more nicely. If we write $S = b(T)$, then $T = m(S) = S + m_1S^2 + \cdots$ for some power series m. Therefore,

$$\begin{aligned} (b(T)^{-(n+1)})_{-1} &= \operatorname{res} b(T)^{-(n+1)}dT \\ &= \operatorname{res} S^{-(n+1)}(1 + 2m_1S + 3m_2S^2 + \cdots)dS \\ &= (n+1)m_n. \end{aligned}$$

With $b(T) = \exp(T)$ and $m(S) = \log(S)$, we have a formula due to Miscenko:

$$\log(T) = \sum \frac{h[\mathbb{C}\mathrm{P}^n]}{n+1}T^{n+1}.$$

Now we do a calculation. Let E be a complex oriented ring spectrum with Euler class x. Then $E \wedge MU$ has two orientations defined in the following way. Let η_L be the composite $MU \to E \to E \wedge MU$ and let η_R be the obvious map $MU \to E \wedge MU$; these give orientations $x_L = \eta_{L*}x$ and $x_R = \eta_{R*}x$. We showed earlier that x_R must be expressible as a power series in x_L. The question is, what is this power series?

To answer this question, we make use of the **Boardman homomorphism** $b : [X, Y] \to [X, E \wedge Y]$ which sends a map $X \to Y$ to the composite $X \to Y \to E \wedge Y$. This is a generalization of the Hurewicz map as one sees by taking X to be the sphere spectrum and E to be the integral Eilenberg-Mac Lane spectrum. One may easily check that the diagram

$$\begin{array}{ccc} [X, Y] & \xrightarrow{\quad\quad b \quad\quad} & [X, E \wedge Y] \\ & {}_{E_*(-)}\searrow \quad \swarrow_{\alpha} & \\ & \mathrm{Hom}_{E_*}(E_*X, E_*Y) & \end{array}$$

commutes, where α sends a map $X \to E \wedge Y$ to the composite

$$E_*X \longrightarrow E_*(E \wedge Y) \xrightarrow{\mu \wedge 1} E_*Y.$$

Quite often, α is an isomorphism. In particular, if $X = \mathbb{C}\mathrm{P}^\infty$ and $Y = MU$, then this is true. (This follows from arguments similar to those we made in section 7. See also [**2**, Part II, Lemma 4.2].) Specializing to this case, we get

$$\begin{array}{ccc} MU^*\mathbb{C}\mathrm{P}^\infty & \longrightarrow & (E \wedge MU)^*\mathbb{C}\mathrm{P}^\infty \\ & \searrow \quad \cong \swarrow & \\ & \mathrm{Hom}_{E_*}(E_*\mathbb{C}\mathrm{P}^\infty, E_*MU) & . \end{array}$$

To compare x_L and x_R we find the relation between their images under α. Since $x \in MU^*\mathbb{C}\mathrm{P}^\infty$ is sent to x_R by the Boardman homomorphism, $\alpha(x_R)$ sends $\beta_i \in E_*\mathbb{C}\mathrm{P}^\infty$ to $b_{i-1} \in E_*MU$. (This is how b_i was defined.) Now $\alpha(x_L^n)$ is the composite along the top row and right edge of the diagram

$$\begin{array}{ccccc} E_*\mathbb{C}\mathrm{P}^\infty & \xrightarrow{{x^n}_*} & E_*E & \xrightarrow{1\wedge 1\wedge \eta} & E_*(E \wedge MU) \\ & {\scriptstyle\langle x^n, - \rangle} \searrow & \downarrow \mu & & \downarrow \mu\wedge 1 \\ & & E_*S & \xrightarrow{\eta} & E_*MU \quad , \end{array}$$

which is readily checked to commute. Under the pairing with cohomology, β_i goes to $\delta_{i,n}$, and so $\alpha(x_L^n)$ is the map sending β_i to $\delta_{i,n}$. Thus we obtain the following formula:

$$\alpha(x_R) = \sum_{n=1}^{\infty} b_{n-1}\alpha(x_L^n).$$

Finally, since α is an isomorphism, this gives the answer to our question:

$$x_R = \sum_{n=1}^{\infty} b_{n-1}x_L^n = \exp(x_L).$$

Note that this is a power series with coefficients lying in the homotopy of $E \wedge MU$, *i.e.,* in $E_*MU = E_*[b_1, b_2, b_3, \dots]$.

EXAMPLE 8.2. We can take E to be $H\mathbb{Z}$. Then $H\mathbb{Z}\wedge MU$ has two complex orientations: $x_L : \Sigma^{-2}MU(1) \longrightarrow MU \xrightarrow{u} H\mathbb{Z} \longrightarrow H\mathbb{Z} \wedge MU$ gives the additive formal group law $MU^* \longrightarrow H\mathbb{Z} \wedge MU^* = \mathbb{Z}[b_1, b_2, \cdots]$ and similarly x_R corresponds to the MU formal group law. We see that our notation was chosen well, since now $G = G_a^m \to G_a$ and $m(T) = b^{-1}(T)$ gives a logarithm. Our investigation leads to the following result.

PROPOSITION 8.3. *The Hurewicz map* $L \cong \pi_* MU \longrightarrow H_* MU$ *is an embedding. Therefore, integral cohomology characteristic numbers determine* U*-manifolds up to bordism.*

As a second example we choose E to be MU. Then the object MU_*MU carries the universal strict isomorphism of formal group laws. It comes to us with left and right units

$$MU_* \xrightarrow{\eta_L} MU_*MU \xleftarrow{\eta_R} MU_*$$

which turn MU_*MU into a two-sided module. Here η_L is induced by the inclusion of the unit into the right factor and is used to make MU_*MU into a right MU_*-module. Of course, η_R does things the other way. The switching map

$$c : MU_*MU \xrightarrow{\cong} MU_*MU$$

sends the source of an isomorphism to its target and vice versa. Therefore, MU_*MU is also free as a right MU_*-module. From this we see that the coaction map

$$\psi : MU_*X = MU_*(S^0 \wedge X) \longrightarrow MU_*(MU \wedge X) \xleftarrow{\cong} MU_*MU \otimes_{MU_*} MU_*X$$

is well-defined, since the last map[7] is an isomorphism[8] of left MU_*-modules. An associative ring spectrum E is called **flat** if E_*E is flat as a right E_*-module. It is under this assumption that we can define a coaction map.

PROPOSITION 8.4. *If E is a flat, commutative, associative ring spectrum then* $(E_*, E_*E, \eta_L, \eta_R, \psi, \varepsilon, c)$ *is a Hopf algebroid and* $E(-)$ *is a functor from the stable category to the category of* E_*E*-comodules.*

The details are tedious but easily verified.

[7] By our rule of thumb, this map should multiply two adjacent factors. To produce a map well-defined on the tensor product, it must combine the second and third factors.

[8] $MU_*MU \otimes_{MU_*} MU_* -$ and $MU_*(MU \wedge -)$ are homology theories (for the first, use that MU_*MU is free and hence flat as a right MU_*-module) and we have a natural transformation from the first to the second which is an isomorphism on the coefficients.

9. The Brown-Peterson Spectrum

We have seen that MU is universal for complex oriented ring theories. That is, if E is another complex oriented ring spectrum with orientation x_E and formal group law F, there is a unique ring spectrum map $MU \longrightarrow E$ sending x_{MU} to x_E and classifying F over the coefficients.

Let us write $MU_{(p)}$ for the $\mathbb{Z}_{(p)}$ localized spectrum $MU \wedge S(\mathbb{Z}_{(p)})$. The p-typicalization $\xi_G(x) \in MU^*_{(p)}(\mathbb{C}\mathrm{P}^\infty)$ defines another orientation for $MU_{(p)}$. Therefore, we obtain a ring spectrum map

$$MU \longrightarrow MU_{(p)} \xrightarrow{e} MU_{(p)}$$

such that on $\mathbb{C}\mathrm{P}^\infty$, $x \mapsto \xi(x)$, and on a point, $G \mapsto {}^{\xi}G = e_* G$. The map e is idempotent up to homotopy, since on the coefficients we have

$$\begin{array}{ccccc} MU^*_{(p)} & \xrightarrow{e_*} & MU^*_{(p)} & \xrightarrow{e_*} & MU^*_{(p)} \\ G & \rightsquigarrow & {}^{\xi}G & \rightsquigarrow & e_*^{\xi}G = {}^{\xi}G \ . \end{array}$$

We would like to look at the image of e, but unfortunately e is only idempotent up to homotopy. Therefore, let $e^{-1}MU_{(p)}$ be the telescope of

$$MU_{(p)} \xrightarrow{e} MU_{(p)} \xrightarrow{e} MU_{(p)} \longrightarrow \cdots$$

and similarly define $(1-e)^{-1}MU_{(p)}$. We obtain a splitting

$$MU_{(p)} \cong e^{-1}MU_{(p)} \vee (1-e)^{-1}MU_{(p)}.$$

The coefficient ring

$$\pi_*(e^{-1}MU_{(p)}) = \pi_*(\mathrm{tel}(e)) = \varinjlim(\pi_*(MU_{(p)}) \xrightarrow{e_*} \pi_* MU_{(p)} \xrightarrow{e_*} \cdots) = e{MU_{(p)}}_*$$

is the p-typical Lazard ring L^p.

Definition 9.1. *For each prime p, the* **Brown-Peterson spectrum** BP *is defined to be $e^{-1}MU_{(p)}$. (Honouring tradition, we omit the prime p from the notation.)*

BP is a complex oriented ring theory and carries the universal p-typical formal group law. It comes with a ring spectrum map

$$BP \longrightarrow BP[\, x_{2i}, i \neq (p^j - 1)] \cong MU_{(p)}.$$

BP_*BP supports the universal strict isomorphism between p-typical formal group laws. BP is also flat and thus defines a functor into the category of BP_*BP-comodules.

10. The Adams Spectral Sequence

We are now able to develop an extremely useful tool for computations in algebraic topology. Let E be an associative ring spectrum.

Definition 10.1. *A map $f : X \to Y$ is an **E-monomorphism** if the natural map $X \to E \wedge X$ factors through f. A spectrum I is said to be **E-injective** if it is a retract of $E \wedge Z$ for some Z.*

Proposition 10.2. *(i) $f : X \to Y$ is E-monic iff each map from X to an E-injective I can be factored through f:*

$$\begin{array}{ccc} X & \longrightarrow & I \\ {\scriptstyle f}\big\downarrow & \nearrow & \\ Y & . & \end{array}$$

(ii) I is E-injective iff for each E-monomorphism $X \to Y$ and each map $X \to I$ there is extension making

$$\begin{array}{ccc} X & \longrightarrow & I \\ \big\downarrow & \nearrow & \\ Y & & \end{array}$$

commutative.

That is, each class determines the other. The proof of the proposition is left as an exercise, but we mention that it only uses the fact that $E \wedge -$ is a triple (a.k.a. a monad).

Using the first part of the proposition, or working directly, one can show that if $X \to Y$ is E-monic then $E \wedge X \to E \wedge Y$ is split monic, and moreover that the splitting map $E \wedge Y \to E \wedge X$ can always be chosen to be a map of (left) E-module spectra.

Next we observe that there are **enough injectives**: for each X there is an E-injective I and an E-monomorphism $X \to I$. We can simply take I to be $E \wedge X$. When a class of maps and a class of objects in a category determine each other as in the proposition, and when there are enough injectives, we say that they form an **injective class**.

DEFINITION 10.3. *An* ***E*-Adams resolution** *of X is a diagram of cofibre sequences*

$$\begin{array}{ccccccccc} X \cong X^0 & \longleftarrow & X^1 & \longleftarrow & X^2 & \longleftarrow & \cdots & & \\ & \searrow^{i_0} & & \searrow^{i_1} & & \searrow^{i_2} & & & \\ & & I^0 & & I^1 & & I^2 & & \cdots\ , \end{array}$$

where the i's are E-monic and the I's are E-injective.

The **standard Adams resolution** is

$$\begin{array}{ccccccc} X = X^0 & \longleftarrow & X^1 & \longleftarrow & \cdots & & \\ & \searrow^{i_0} & & \searrow^{i_1} & & & \\ & & E \wedge X^0 & & E \wedge X^1 & & \cdots\ , \end{array}$$

where X^{k+1} is the desuspension of C_{i_k}, the mapping cone of i_k. One can show that $X^{k+1} = \bar{E}^{\wedge k+1} \wedge X$, where $\bar{E}$ is the desuspension of the mapping cone of the unit map $S \to E$.

When we now apply a homology functor F_* an exact couple is born, giving a spectral sequence with E_1 term

$$F_*(I^0) \longrightarrow F_*(I^1) \longrightarrow F_*(I^2) \longrightarrow \cdots .$$

This is called the **Adams spectral sequence**[9]. We want it to converge to F_*X, but this will not happen in general. However, if $E = BP$ and π_*X is a $\mathbb{Z}_{(p)}$-module which is bounded below then the spectral sequence converges when $F_* = \pi_*$. (See [**2**] for a proof.)

EXERCISE 10.4. Relate the notion of an Adams resolution to the notion of an injective resolution. Then formulate and prove a uniqueness up to chain homotopy result for Adams resolutions which implies that the spectral sequence is independent of the choice of Adams resolution from the E_2 term onwards.

We're particularly interested in the Adams spectral sequence when the exact functor being applied is π_*. In order to compute the E_2 term in this

[9]The construction makes sense given an injective class in any triangulated category.

case we assume that E is flat and commutative, so that E_*E is a Hopf algebroid and E_*X is an E_*E-comodule (see Prop. 8.4).

Now we notice that if we apply E_* to an E-Adams resolution of X we get sequences

$$E_*X^s \longrightarrow E_*I^s \longrightarrow E_*\Sigma X^{s+1}$$

which are short exact since the first map is split monic as a left E_*-module map. These short exact sequences can be spliced together to give a long exact sequence

$$0 \longrightarrow E_*X \longrightarrow E_*I^0 \longrightarrow E_*\Sigma I^1 \longrightarrow E_*\Sigma^2 I^2 \longrightarrow \cdots .$$

This leads to the development of some homological algebra. What we do below is a special case of what is called *relative homological algebra*. Chapter IX of Mac Lane's book [**10**] covers the general setting, but we will keep things simple.

Let W be a Hopf algebroid over a commutative ring L, and work in the category of left comodules over W, *i.e.,* left L-modules with a coaction $M \to W \otimes_L M$ which is coassociative and counital. We are simplifying things here by ignoring the fact that in our application everything is graded and graded-commutative.

DEFINITION 10.5. *A comodule map $M \to N$ is called a* **relative monomorphism** *if it is split monic as a map of left L-modules. A comodule I is called a* **relative injective** *if it is a (comodule) retract of $W \otimes_L N$ for some L-module N.*

EXERCISE 10.6. Show that relative monomorphisms and relative injectives form an injective class in the category of comodules. (Hint: The only fact you need is that $W \otimes_L -$ is right adjoint to the forgetful functor from W-comodules to L-modules.)

Note that an E-monomorphism $X \to Y$ of spectra gives a relative monomorphism $E_*X \to E_*Y$ of E_*E-comodules, and that an E-injective spectrum I produces a relative injective E_*E-comodule E_*I.

DEFINITION 10.7. *A short exact sequence $M \to N \to P$ of comodules is* **relative short exact** *if $M \to N$ is a relative monomorphism. A long exact* *sequence is a* **relative long exact sequence** *or a* **relative resolution** *if it is formed by splicing together relative short exact sequences. A relative resolution of relative injectives is called a* **relative injective resolution**.

With this terminology we can say that the long exact sequence obtained by applying E_* to an E-Adams resolution is a relative injective resolution of E_*E-comodules.

NOTE 10.8. A formal argument shows that relative (long or short) exact sequences are those that go to exact sequences of abelian groups under $\mathrm{Hom}_W(-, I)$ for all injectives I.

DEFINITION 10.9. *For comodules M and N we define $\mathrm{Ext}^*_W(M, N)$ in the usual way by replacing N by a relative injective resolution, applying $\mathrm{Hom}_W(M, -)$, and taking the homology of the resulting complex.*

The usual arguments show that the answer is independent of the choice of resolution.

It turns out that $\mathrm{Hom}_W(L, M)$ has a natural interpretation which makes it an interesting object to study. A element m of a comodule M is **primitive** if $\psi m = 1 \otimes m$ in $W \otimes_L M$. The following proposition is easy to check.

PROPOSITION 10.10. *Under the natural isomorphism $\mathrm{Hom}_L(L, M) \cong M$, the subgroup $\mathrm{Hom}_W(L, M)$ goes to the group of primitive elements in M.*

EXAMPLE 10.11. As an example we can take M to be L. The comodule structure on L is given by

$$\psi : L \xrightarrow{\eta_L} W \xleftarrow{\cong} W \otimes_L L,$$

so the primitive elements are those that satisfy $\eta_L(a) = \eta_R(a)$. This is not an L-submodule of L unless $\eta_L = \eta_R$.

In the application we have $L = E_*$ and $W = E_*E$. In this case, $\mathrm{Hom}_{E_*E}(E_*, E_*X)$ is a natural thing to study because the Hurewicz map

$$\pi_*X \xrightarrow{h} E_*X = \mathrm{Hom}_{E_*}(E_*, E_*X)$$

factors through $\mathrm{Hom}_{E_*E}(E_*, E_*X)$. We will now see that in some cases we can identify π_*X with the primitives.

For the following lemma, we need the notion of a split fork in a category. A diagram of the form

$$A \xrightarrow{i} B \underset{t}{\overset{s}{\rightrightarrows}} C$$

is called a **fork** if $si = ti$. If in addition there are maps

$$A \xleftarrow{r} B \xleftarrow{h} C$$

such that $ri = 1$, $hs = 1$ and $ht = ir$, then the diagram is called a **split fork**. In a split fork the map i is automatically an equalizer of s and t. The usefulness of split forks is that they are preserved under functors, and so $F(i)$ is an equalizer of $F(s)$ and $F(t)$ for any functor F.

LEMMA 10.12. *For an E-injective I, the natural map*

$$\pi_* I \longrightarrow \mathrm{Hom}_{E_*E}(E_*, E_*I)$$

is an isomorphism.

PROOF. The map $i : I \to E \wedge I$ obtained by including the unit is an E-monomorphism, so the E-injectivity of I implies that it has a retraction r. Consider the diagram

$$I \xrightarrow{i} E \wedge I \rightrightarrows E \wedge E \wedge I,$$

where the parallel arrows are the two insertions of the unit map. This fork is split by the maps r and $1 \wedge r$. Therefore, applying π_* produces an equalizer diagram

$$\pi_* I \longrightarrow E_*I \rightrightarrows E_*E \otimes_{E_*} E_*I$$

which identifies $\pi_* I$ with the primitives of E_*I. □

This allows us to describe the E_2 term of the Adams spectral sequence.

PROPOSITION 10.13. *If E is flat and commutative, then the E_2 term of the E-Adams spectral sequence for π_* is given by $\mathrm{Ext}^s_{E_*E}(E_*, E_*X)$.*

The **bar construction** produces for a comodule M a standard complex $\Omega^*(M)$ whose homology is $\mathrm{Ext}^*_W(L, M)$. We construct $\Omega^*(M)$ by first constructing a relative injective resolution of M. To do this, we need to be able to embed M in a relative injective via a relative monomorphism and compute the cokernel. Since $\epsilon \otimes 1$ splits $\psi : M \to W \otimes M$ as a module map, ψ is an appropriate embedding. Consider

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\psi} & W \otimes M & \longrightarrow & Q & \longrightarrow & 0 \\ & & & & \uparrow & \nearrow & & & \\ & & & & I \otimes M & & & & \end{array} ,$$

where the comodule Q is defined to be the cokernel of ψ and $I = \ker(\epsilon : W \to L)$ is the augmentation ideal of W, an L-bimodule. The map $I \otimes M \to W \otimes M$ is an inclusion since $0 \to I \to W \to L \to 0$ is split as right modules. It isn't hard to see that the dashed map is an isomorphism of left L-modules. Pulling back the comodule structure on Q we find that the coaction on $I \otimes M$ is

$$I \otimes M \xrightarrow{\psi \otimes 1 - 1 \otimes \psi} W \otimes I \otimes M.$$

By this we really mean the composite

$$I \otimes M \longrightarrow W \otimes M \xrightarrow{\psi \otimes 1 - 1 \otimes \psi} W \otimes W \otimes M$$

whose image actually lies in $W \otimes I \otimes M$. Thus for any comodule M we have a relative short exact sequence of comodules

$$0 \longrightarrow M \xrightarrow{\psi} W \otimes M \xrightarrow{\sigma} I \otimes M \longrightarrow 0,$$

where the coaction on $I \otimes M$ is that given above, and the map σ sends $w \otimes m$ to $w \otimes m - \epsilon(w)\psi(m) \in I \otimes M$. One can check directly that $I \otimes M$ is actually a comodule and that σ is a comodule map.

Piecing together such short exact sequences we obtain a relative injective resolution of M:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M & \xrightarrow{\psi} & W \otimes M & & \dashrightarrow & & W \otimes I \otimes M & & \cdots \\ & & & & & \searrow^{\sigma} & & \nearrow_{\psi} & & \searrow^{\sigma} & \\ & & & & & & I \otimes M & & & & I \otimes I \otimes M \quad \cdots \\ & & & & & \nearrow & & \searrow & & & \searrow \\ & & & & 0 & & & & 0 & & \quad 0 \end{array} .$$

The map $W \otimes M \to W \otimes I \otimes M$ is just $\psi \otimes 1 - 1 \otimes \psi$ since σ is $1 \otimes 1 - \psi(\epsilon \otimes 1)$ and $(\psi \otimes 1 - 1 \otimes \psi)\psi = 0$. Similarly, the map $W \otimes I \otimes M \to W \otimes I \otimes I \otimes M$ is a three term alternating sum, and so on.

It should be noted that if $M = E_*X$, $L = E_*$ and $W = E_*E$, then $I = E_*\bar{E}$ and this resolution is the one obtained from the standard E-Adams resolution of X by applying E-homology.

Now we must drop the initial M and apply $\mathrm{Hom}_W(L, -)$. To see what happens, note that $W \otimes -$ is right adjoint to the forgetful functor from W-comodules to left L-modules. That is,

$$\mathrm{Hom}_L(M, N) \cong \mathrm{Hom}_W(M, W \otimes N),$$

where M is a comodule, N is a module, and $\psi \otimes 1 : W \otimes N \to W \otimes W \otimes N$ is the coaction on $W \otimes N$. The isomorphism sends a module map $f : M \to N$ to $M \xrightarrow{\psi} W \otimes M \xrightarrow{1 \otimes f} W \otimes N$ and a comodule map $g : M \to W \otimes N$ to $M \xrightarrow{g} W \otimes N \xrightarrow{\epsilon \otimes 1} L \otimes N \cong N$. It is easy to check that this makes sense and that these maps are inverses.

From the adjointness we find[10] that

$$\mathrm{Hom}_W(L, W \otimes I^{\otimes k} \otimes M) \cong \mathrm{Hom}_L(L, I^{\otimes k} \otimes M) \cong I^{\otimes k} \otimes M,$$

and therefore that $\mathrm{Ext}_W(L, M)$ is the homology of the complex

$$0 \longrightarrow M \longrightarrow I \otimes M \longrightarrow I \otimes I \otimes M \longrightarrow \cdots .$$

This complex is denoted $\Omega^*(M)$. Some consideration reveals that the map $I^{\otimes k} \otimes M \to I^{\otimes (k+1)} \otimes M$ sends $[a_1|\cdots|a_k]m$ to $[1|a_1|\cdots|a_k]m - [a_1'|a_1''|a_2|\cdots|a_k]m + \cdots + (-1)^{k+1}[a_1|\cdots|a_k|m']m''$. Here we must explain that the notation $[a_1|\cdots|a_k]m$ refers to an element of $W^{\otimes k} \otimes M$ and that we write ψa as $\Sigma a' \otimes a''$ and then suppress the summation symbol. Note that Ω^* is a functor from W-comodules to chain complexes of abelian groups and is **exact**: relative short exact sequences are sent to short exact sequences.

EXERCISE 10.14. Compute $\mathrm{Ext}^0_{MU_*MU}(MU_*, MU_*)$.

Some of April 12, 1994 to be done by Dan.

$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$ acts on $H^s BP_*X = \mathrm{Ext}^s_{BP_*BP}(BP_*, BP_*X)$. In the following we want to invert the action of v_i.

[10] Another way to see this is to show that the primitives in $W \otimes M$ are precisely the elements of the form $1 \otimes m$ for m in M.

LEMMA 10.15. *Suppose M is a comodule over BP_* and $I_n^{k+1}M = 0$. Then*

$$v_n^{p^k} : M \longrightarrow \Sigma^{-|v_n|p^k} M$$

is a comodule map.

PROOF. Ravenel's formula gives

$$\begin{aligned} \eta_R(v_n) &= v_n + r \qquad (r \in I_n) \\ \eta_R(v_n^p) &= v_n^p + v_n^{p-1}pr + \cdots + r^p \end{aligned}$$

and inductively $\eta_R(v_n)^{p^k} \equiv v_n^{p^k}$ modulo I_n^{k+1}. This says that

$$v_n^{p^n} : BP_*/I_n^{k+1} \longrightarrow \Sigma^{-|\, v_n|p^k} BP_*/I_n^{k+1}$$

is a comodule map and the assertion follows after tensoring with M and using $BP_*/I_n^{k+1} \otimes_{BP_*} M = M$. □

In this case we write

$$v_n^{-1}M = \varinjlim (M \xrightarrow{v_n^{p^k}} M \xrightarrow{v_n^{p^k}} M \xrightarrow{v_n^{p^k}} \cdots)$$

and obtain a unique comodule structure on $v_n^{-1}M$ such that $M \to v_n^{-1}M$ is a comodule map.

EXAMPLE 10.16. We may take $M = BP_*$ to get an exact triangle

$$\begin{array}{ccc} BP_* & & BP_*/p^\infty \\ & \searrow \quad \nearrow & \\ & p^{-1}BP_* & \end{array} .$$

Here the situation is analogous to $\mathbb{Z}_{(p)} \to \mathbb{Q} \to \mathbb{Z}_{p^\infty}$.

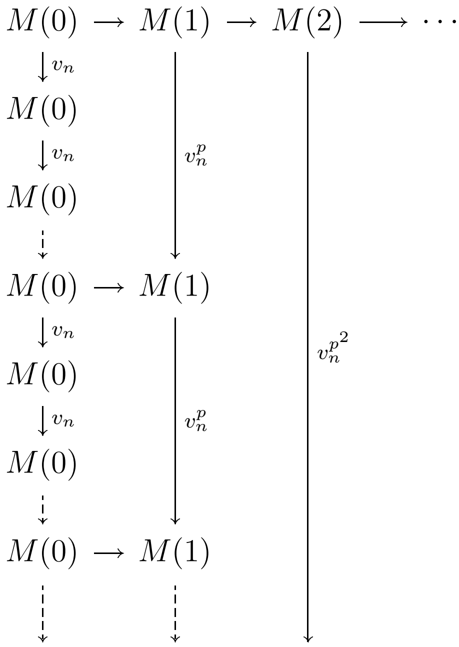
Unfortunately the above lemma does not apply to its extension

$$\begin{array}{ccc} BP_*/p^\infty & & BP_*/(p^\infty, v_1^\infty) \\ & \searrow \quad \nearrow & \\ & v_1^{-1}BP_*/p^\infty & \end{array} .$$

We need a little lemma.

LEMMA 10.17. *Let M be the direct limit of comodules $M(k)$ such that $I_n^{k+1}M(k) = 0$ for all k. Then there is a unique comodule structure on $v_n^{-1}M$ such that $M \to v_n^{-1}M$ is a comodule map.*

Proof. The previous lemma shows that all maps in the following diagram



are comodule maps. □

We want to take $M(k)$ to be $BP_*/p^k, BP_*/p^\infty, v_1^k, \ldots$. Then we can splice the short exact sequences together into a complex

$$0 \to p^{-1}BP_* \to v_1^{-1}BP_*/p^\infty \to v_2^{-1}BP_*/(p^\infty, v_1^\infty) \to \cdots$$

which is called the **chromatic resolution**. (Observe that this sequence is exact, except for the very first stage). The application of the exact functor Ω^* gives rise to a double complex

$$0 \to \Omega^* p^{-1}BP_* \to \Omega^* v_1^{-1}BP_*/p^\infty \to \Omega^* v_2^{-1}BP_*/(p^\infty, v_1^\infty) \to \cdots .$$

We obtain two spectral sequences ${}^{I}E, {}^{II}E$ coming from the vertical and horizontal filtration of the total complex respectively. According to our observation before, we have that ${}^{I}E = \Omega^* BP_*$ and ${}^{I}E_\infty = {}^{I}E_2 = H^*BP_*$. Hence the spectral sequences converge to the E_2-term of the Adams spectral sequence. The other spectral sequence ${}^{II}E$ is the **chromatic spectral sequence** with E_2 term $H^*(v_s^{-1}BP_*/(p^\infty, v_1^\infty, \cdots, v_{s-1}^\infty))$ and internal grading coming from H^*BP. The chromatic spectral sequence is a first quadrant spectral sequence.

$H^*(p^{-1}BP_*)$	$H^*(v_1^{-1}BP_*/p^\infty)$	$H^*(v_2^{-1}BP_*/(p^\infty, v_1^\infty)))$	$\cdots$

The first column is given by

$$H^*(p^{-1}BP_*) = \mathrm{Ext}_{BP_*BP}(BP_*, p^{-1}BP_*) = \mathrm{Ext}_{p^{-1}BP_*BP}(p^{-1}BP_*, p^{-1}BP_*).$$

But the pair $(p^{-1}BP_*, p^{-1}BP_*BP)$ corepresents the functor which takes a $\mathbb{Q}$-algebra into the set of strict isomorphisms between p-typical formal group laws. Obviously the cobar complex of $p^{-1}BP_*$ retracts to $\mathbb{Q} = p^{-1}\mathbb{Z}_{(p)}$

$$\begin{array}{ccccccccc} p^{-1}BP_* & \to & p^{-1}BP_*BP & \longrightarrow & p^{-1}BP_*BP \otimes_{p^{-1}BP_*} p^{-1}BP_*BP & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} & \longrightarrow & \cdots \end{array}$$

by classifying the additive formal group law. Hence, universality provides us with a ring homomorphism $p^{-1}BP_*BP \to p^{-1}BP_*$ when we write down the logarithm of the p-localized universal p-typical formal group. This provides a chain homotopy between the two complexes. That is, we have computed the first column of the E_2-term of the chromatic spectral sequence.

PROPOSITION 10.18. $H^k(p^{-1}BP_*) = \begin{cases} 0 & k \neq 0 \\ \mathbb{Q} & k = 0 \end{cases}$

We turn now to a technique for computing the other columns. Notice that we have short exact sequences of comodule maps

$$v_s^{-1}BP_*/(p, v_1^\infty, \ldots, v_{s-1}^\infty) \rightarrowtail v_s^{-1}BP_*/(p^\infty, v_1^\infty, \ldots, v_{s-1}^\infty) \xrightarrow{p} \!\!\!\!\rightarrow v_s^{-1}BP_*/(p^\infty, v_1^\infty, \ldots, v_{s-1}^\infty)$$

$$v_s^{-1}BP_*/(p, v_1, v_2^\infty, \ldots, v_{s-1}^\infty) \rightarrowtail v_s^{-1}BP_*/(p, v_1^\infty, \ldots, v_{s-1}^\infty) \xrightarrow{v_1} \!\!\!\!\rightarrow v_s^{-1}BP_*/(p, v_1^\infty, \ldots, v_{s-1}^\infty)$$

$$\cdots$$

$$v_s^{-1}BP_*/I_s \rightarrowtail v_s^{-1}BP_*/(p, v_1, \ldots, v_{s-2}, v_{s-1}^\infty) \xrightarrow{v_{s-1}} \!\!\!\!\rightarrow v_s^{-1}BP_*/(p, v_1, \ldots, v_{s-2}, v_{s-1})$$

Each of these give rise to a long exact sequence in cohomology and to an exact couple. The computation of $H^*(v_s^{-1}BP_*/(p^\infty, \cdots, v_{s-1}^\infty))$ is a long and complicated process, controlled by "Bockstein spectral sequences". It tells you what lies between the column of the chromatic spectral sequence and $H^*(v_n^{-1}BP_*/I_n)$, which is our next object of study.

We use the ring map

$$\begin{array}{ccc} BP_* & \longrightarrow & K(n)_* := \mathbb{F}_p(v_n^{\pm}) \\ v_i & \mapsto & \begin{cases} v_n & i = n,\ n > 0 \\ 0 & i = 0 \end{cases} \end{array}$$

and focus on the Honda formal group law over $K(n)_*$. Now

$$\Sigma(n) := K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_*$$

is a commutative Hopf algebra over the graded field $K(n)_*$, since $\eta_R(v_n) \equiv \eta_L(v_n) \bmod I_n$. We have that

$$\Sigma(n) = K(n)_*[t_1, t_2, \cdots]/\eta_R(v_{n+1}), \eta_R(v_{n+2}), \ldots .$$

We can even be more precise. By using Ravenel's formula we have $v_n t_j^{p^n} = t_i v_n^{p^j}$ in $\Sigma(n)$ and these are the only relations.

Proposition 10.19. *$\Sigma(n) = K(n)_*[t_1, t_2, \cdots, t_n]/(t_j^{p^n} = v_n^{p^{j-1}} t_j)$.*

Let us return to our map $v_n^{-1}BP_*/I_n \to K(n)_*$ which induces a map $H^*(v_n^{-1}BP_*/I_n) \to \mathrm{Ext}_{\Sigma(n)}(K(n), K(n))$. We mention without proof (we don't use this fact later) that this map is in fact an isomorphism. We also state

Theorem 10.20 (Morava). *$H^t(v_n^{-1}BP_*/I_n) = 0$ for $t > n^2$ provided $p-1$ does not divide n.*

Hence in the Bockstein spectral sequences we have

$$E_1^{s,t} = 0 \quad \text{if} \quad (p-1) \nmid s,\ t > s^2.$$

Let us try to calculate the second column of the chromatic spectral sequence. The short exact sequence

$$\begin{array}{ccccc} v_1^{-1}BP_*/p & \rightarrowtail & v_1^{-1}BP_*/p^\infty & \overset{p}{\twoheadrightarrow} & v_1^{-1}BP_*/p^\infty \\ w & \longrightarrow & w/p & \longrightarrow & 0 \end{array}$$

gives rise to an long exact sequence in cohomology:

$$H^0(v_1^{-1}BP_*/p) \to H^0(v_1^{-1}BP_*/p^\infty) \xrightarrow{p} H^0(v_1^{-1}BP_*/p^\infty) \xrightarrow{\delta} H^1(v_1^{-1}BP_*/p) \to \cdots .$$

Hence, we are interested in the primitives in $v_1^{-1}BP_*/p^\infty$. Using the BP formula $\eta_R(v_1) = v_1 + pt_1$ we see

$$\eta_R(v_1^{p^k s}) = v_1^{p^k s} + p^{k+1} s t_1 v_1^{p^k s - 1} \quad \bmod \quad (p^{k+2})$$

for $p > 2$. The prime 2 becomes a different story which we want to investigate later. It follows that $v_1^{p^k s}/p^{k+1}$ is a cocycle. The homomorphism δ maps $v_1^{p^k s}/p^{k+1}$ to $s v_1^{p^k s - 1} t_1$ since the first differential in the cobar resolution of $v_1^{-1}BP_*/p^\infty$ can be identified with $\eta_R - \eta_L$.

Now the first arrow in

$$\mathbb{F}_p[\, v_1^{\pm}] \, \langle t_1 \rangle \longrightarrow H^1(v_1^{-1}BP_*/p) \longrightarrow H^1(\Sigma(1))$$

has to be a monomorphism, since there is no relation in the composite. Hence we obtain a diagram

$$\begin{array}{ccccccc}
H^0(v_1^{-1}BP_*/p) & \rightarrowtail & H^0(v_1^{-1}BP_*/p^\infty) & \xrightarrow{p} & H^0(v_1^{-1}BP_*/p^\infty) & \longrightarrow & H^1(v_1^{-1}BP_*/p)\cdots \\
\cong \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{F}_p[\, v_1^{\pm}] & \rightarrowtail & \left\langle \frac{v_1^{p^k s}}{p^{k+1}} : p \not| s \in \mathbb{Z}, k \geq 0 \right\rangle \oplus \mathbb{Z}_{p^\infty} & \xrightarrow{p} & \langle same \rangle \oplus \mathbb{Z}_{p^\infty} & \longrightarrow & \mathbb{F}_p[\, v_1^{\pm}] \langle t_1 \rangle \cdots
\end{array} .$$

The following little argument reveals that we have calculated all of $H^0(v_1^{-1}BP_*/p^\infty)$.

Lemma 10.21. *Let $p : A \to A$ and $p : B \to B$ be self maps of abelian groups such that*

$$A = \bigcup \ker(p^i), \quad B = \bigcup \ker(p^i).$$

Assume $f : A \to B$ commutes with p and that

$$\begin{array}{rcll}
\ker(p) & \longrightarrow & \ker(p) & \textit{is an isomorphism} \\
\operatorname{coker}(p) & \longrightarrow & \operatorname{coker}(p) & \textit{is a monomorphism.}
\end{array}$$

Then f is an isomorphism.

PROOF. A diagram chase in

$$\begin{array}{ccccccccc}
\ker(p) & \rightarrowtail & \ker(p^2) & \xrightarrow{p} & \ker(p) & \longrightarrow & \ker(p)/p\ker(p^2) & \hookrightarrow & \operatorname{coker}(p) \\
\downarrow\cong & & \downarrow f & & \downarrow\cong & & \downarrow f & & \downarrow f \\
\ker(p) & \rightarrowtail & \ker(p^2) & \xrightarrow{p} & \ker(p) & \longrightarrow & \ker(p)/p\ker(p^2) & \hookrightarrow & \operatorname{coker}(p)
\end{array}$$

and

$$\begin{array}{ccccccc}
\operatorname{coker}(p) & \longleftarrow & \operatorname{coker}(p^2) & \xleftarrow{p} & \operatorname{coker}(p) & \longleftarrow & \ker(p)/p\ker A \\
\downarrow f & & \downarrow f & & \downarrow f & & \downarrow f \\
\operatorname{coker}(p) & \longleftarrow & \operatorname{coker}(p^2) & \xleftarrow{p} & \operatorname{coker}(p) & \longleftarrow & \ker(p)/pB
\end{array}$$

gives the inductive step. □

The differential of the chromatic spectral sequence kills the elements with v_1 in the denominator

$$\frac{v_1^{p^k s}}{p^{k+1}} \stackrel{d}{\longmapsto} \begin{cases} 0 & \text{if} \quad s > 0 \\ \frac{1}{p^{k+1} v_1^{p^k(-s)}} & \text{if} \quad s < 0 \end{cases}.$$

As a consequence we obtain

COROLLARY 10.22. *For* $p > 2$

$$H^{1,l}(BP_*) = \begin{cases} \mathbb{Z}_{p^{k+1}} & l = qp^k s,\ p \not| s > 0, q = |v_1| = 2(p-1) \\ 0 & \textit{otherwise} \end{cases}$$

generated by

$$\alpha_{p^k s/k+1} = \delta(\frac{v_1^{p^k s}}{p^{k+1}})$$

where δ *is the connecting homomorphism associated to*

$$0 \longrightarrow BP_* \longrightarrow p^{-1}BP_* \longrightarrow BP_*/p^\infty \longrightarrow 0$$

The case $p = 2$ is left as an exercise. Observe that the elements $v_1^{2^k s}/2^{k+1}$ are not primitive, but $x = v_1^2 - 4v_1^{-1}v_2, x^2, x^3, \ldots$ are. Let us reward ourselves with a geometrical application.

11. The BP-Hopf Invariant

Let $f : S^n \to S^0$, $n > 0$ be a stable map. The induced homomorphism BP_*f is zero, since there are no primitives of positive dimension in BP_*. Thus we have a short exact sequence of comodules

$$0 \to BP_*S^0 \to BP_*C_f \to BP_*S^{n+1} \to 0$$

which splits as a sequence of BP_*-modules. We can view it as an element in $\mathrm{Ext}^{1,n+1}_{BP_*BP}(BP_*, BP_*) = H^{1,n+1}BP_*$ and obtain a homomorphism

$$h : \pi_n^S(S^0) \longrightarrow H^{1,n+1}BP_*$$

for $n > 0$ which is the ***BP*-Hopf invariant**. Use the embedding of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & BP_* & \longrightarrow & p^{-1}BP_* & \longrightarrow & BP_*/p^{\infty} & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle p^{-(k+1)}} & & \uparrow & & \\ 0 & \longrightarrow & BP_* & \xrightarrow{p^{k+1}} & BP_* & \longrightarrow & BP_*/p^{k+1} & \longrightarrow & 0 \end{array}$$

to compute the boundary of

$$v_1^{p^k s} \in H^0(BP_*/p^{k+1}) \longleftarrow H^0(BP_*/p^{k+1}).$$

The last is induced by the cofibre sequence

$$S^0 \xrightarrow{p^{k+1}} S^0 \longrightarrow S^0/p^{k+1} \xrightarrow{\partial} S^1 \longrightarrow \cdots .$$

THEOREM 11.1 (J.F. Adams). *There is a map between the Moore spaces*

$$\cdots \longrightarrow S^{2qp^k}/p^{k+1} \xrightarrow{\Sigma^{qp^k}\Phi_k} S^{qp^k}/p^{k+1} \xrightarrow{\Phi_k} S^0/p^{k+1}$$

inducing multiplication by $v_1^{p^k}$ in BP_*-homology.

The composite

$$S^{qp^k s} \to S^{qp^k s}/p^{k+1} \xrightarrow{\Phi_k^s} S^0/p^{k+1} \xrightarrow{\partial} S^1$$

gives an exciting homotopy class of spheres.

LEMMA 11.2 (Ravenel 2.3.4). *Let $X \to Y \to Z$ be a cofibre sequence which is short exact in E_* − homology. Then there is a map of spectral sequences*

$$E_r^s(Z, E) \to E_r^{s+1}(\Sigma X, E)$$

which represents $\partial : Z \to \Sigma X$ and is given at E_2 by the obvious algebraic δ.

COROLLARY 11.3. *There exist summands $\mathbb{Z}/p^{k+1}$ in $\pi_{qp^k s-1}$ if $p \not| s > 0$.*

These classes are usually constructed using the J-homomorphism which is discussed in [**1**]. It enters here in the construction of Φ_k. The cofibres are usually denoted by $V(k)$. For example $S \xrightarrow{p} S \longrightarrow V(0)$ induces multiplication by p^n in homotopy. Furthermore we saw for $p > 2$ that

$$\Sigma^{2(p-1)}V(0) \xrightarrow{\Phi(0)} V(0) \longrightarrow V(1)$$

give rise to classes α_n. We mention the construction of other families of Greek letters: For $p > 2$ there are β_n with chromatic name $\frac{v_2^n}{pv_1}$ coming from $\Sigma^{2(p^2-1)}V(1) \longrightarrow V(2)$ and γ_n from $\Sigma^{2(p^3-1)}V(2) \longrightarrow V(2) \longrightarrow V(3)$. It is still unknown which classes survive and represent a homotopy class of spheres.

Part of April 21, 1994 to be done.

12. The MU-Cohomology of a Finite Complex

If X is a finite complex, is $MU_*(X)$ finitely generated as a module over the coefficient ring MU_*? This is the question that motivates the present section, which is based on work of L. Smith and P. Conner.

Let R be a commutative ring.

DEFINITION 12.1. *An R-module is* **Noetherian** *if every submodule is finitely generated as an R-module. The ring R is said to be* **Noetherian** *if it is Noetherian as a module over itself.*

NOTE 12.2. If E is a cohomology theory and E_* is a Noetherian ring, then E_*X is a finitely generated E_*-module for finite X. Unfortunately, MU_* is not Noetherian.

We will make use of the following standard theorem.

THEOREM 12.3 (Hilbert). *If R is a Noetherian ring, then so is $R[x]$.*

DEFINITION 12.4. *An R-module M is* **finitely presented** *if there exists an exact sequence $F_1 \to F_0 \to M \to 0$ with F_0 and F_1 finitely generated free R-modules. We say that M is* **coherent** *if M is finitely generated and every finitely generated submodule is finitely presented. The ring R is* **coherent** *if it is coherent as a module over itself.*

It isn't hard to see that the direct sum of two Noetherian modules is Noetherian. This implies that every finitely generated module over a Noetherian ring is coherent, and in particular that the Noetherian ring is itself coherent.

LEMMA 12.5. *If every finite subset of R is contained in a Noetherian subring S such that R is flat over S, then R is coherent.*

PROOF. Let I be a finitely generated submodule of R, that is, a finitely generated ideal in R. We must show that I is finitely presented. The finite set of generators lie in some Noetherian subring S such that R is flat over S; let J be the ideal they generate in S. Since S is Noetherian, it is coherent and there exists an exact sequence

$$F_1 \longrightarrow F_0 \longrightarrow J \longrightarrow 0,$$

where F_1 and F_0 are finitely generated free S-modules. That R is S-flat means that the functor $R \otimes_S -$ is exact, so

$$R \otimes_S F_1 \longrightarrow R \otimes_S F_0 \longrightarrow R \otimes_S J \longrightarrow 0$$

is an exact sequence. But the first two modules are finitely generated free R-modules and the third is I, so this shows that I is finitely presented. $\square$

EXAMPLE 12.6. If R is Noetherian, then any polynomial algebra over R is coherent (even if uncountably generated). So, by the theorem of Quillen, MU_* is coherent. This uses Theorem 12.3.

NOTE 12.7. Every finitely generated submodule of a coherent module is coherent.

Now we prove three lemmas regarding the short exact sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

of R-modules, and deduce several corollaries. In the following, F, F', etc. will denote free modules.

LEMMA 12.8. *If N is finitely generated and N'' is finitely presented, then N' is finitely generated.*

PROOF. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle f} & & \| & & \\ 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & N'' & \longrightarrow & 0 \end{array} .$$

The bottom row exists because N'' is finitely presented, the map f exists because F is free, and the map g is the map induced by f on the kernels. (Saying that N'' is finitely presented is equivalent to saying that there exists an exact sequence like the bottom row with F finitely generated and free, and K finitely generated.) Now a diagram chase (or the serpent lemma) tells us that

$$K \longrightarrow N' \longrightarrow N/\operatorname{im} f \longrightarrow 0$$

is exact. Both K and $N/\operatorname{im} f$ are finitely generated, and taking the image and preimage of respective generating sets gives a finite set of generators for N'. □

COROLLARY 12.9. *If N is coherent and N'' finitely presented, then N' is coherent.*

PROOF. Use the lemma and the note preceding it. □

LEMMA 12.10. *If N' and N'' are finitely presented, then N is finitely presented.*

Proof. Consider the following diagram.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow{\scriptstyle 3} & & \uparrow & & \\
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle 2} & & \uparrow & & \\
0 & \overset{1}{\dashrightarrow} & F' & \overset{1}{\dashrightarrow} & F' \oplus F'' & \overset{1}{\dashrightarrow} & F'' & \overset{1}{\dashrightarrow} & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle 4} & & \uparrow & & \\
0 & \overset{5}{\dashrightarrow} & K' & \overset{5}{\dashrightarrow} & K & \overset{5}{\dashrightarrow} & K'' & \overset{5}{\dashrightarrow} & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle 4} & & \uparrow & & \\
 & & 0 & & 0 & & 0\ . & &
\end{array}
$$

The left and right columns are those showing N' and N'' finitely presented. The dashed arrows (and the necessary objects) are added in the order indicated. All of the squares commute and all horizontal or vertical composable pairs are exact. Thus K is finitely generated, and so N is finitely presented. □

Corollary 12.11. *If N' and N'' are coherent, then so is N.*

Proof. Let M be a finitely generated submodule of N, let M'' be the image of M in N'', and let M' be the kernel of the surjection $M \to M''$. Then we have the following situation:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0\ .
\end{array}
$$

Since M is finitely generated, so is M'', and since N'' is coherent, M'' is finitely presented. By the first lemma of the series, M' is finitely generated and hence finitely presented. Thus by the second lemma, M is finitely presented. □

Corollary 12.12. *If N' and N'' are coherent, then so is $N' \oplus N''$.*

LEMMA 12.13. *If N' is finitely generated and N is finitely presented, then N'' is finitely presented.*

In particular, a retract of a finitely presented module is finitely presented.

PROOF. A diagram chase shows that the right column of the following diagram, in which all modules and maps are the obvious ones, is exact:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & F' & \longrightarrow & F & = & F & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & K & & K \oplus F' & & \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & . & &
\end{array}$$

Since $K \oplus F'$ is finitely generated, it is the target of a surjection from a finitely generated free module, and we are done. □

COROLLARY 12.14. *If N' is finitely generated and N is coherent then N'' is coherent.*

PROOF. Let M'' be a finitely generated submodule of N'', let M be the preimage of M'' in N, and let M' be the kernel of the surjection $M \to M''$, so we have the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 .
\end{array}$$

A diagram chase shows that the induced map $M' \to N'$ is an isomorphism, and thus M' is finitely generated. Therefore M is finitely generated, and hence finitely presented. Now the lemma implies that M'' is finitely presented. □

Corollary 12.15. *Coherent modules form an abelian category.*

Now let's focus our attention on the case when R is coherent. Then any finitely generated free R-module is coherent by an earlier corollary. So if M is finitely presented over R, then M is coherent.

Corollary 12.16. *Over a coherent ring, a module M is coherent if and only if it is finitely presented.*

Now we can answer the question that introduced this section. It isn't hard to see that if we have an exact sequence

$$A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E$$

of modules over a ring R with A, B, D and E coherent, then C is coherent. If X is a finite CW-complex with skeleta X_i, then we get for each i an exact sequence

$$MU^*(X_{i-1}) \to MU^*(X_i/X_{i-1}) \to MU^*(X_i) \to MU^*(X_{i-1}) \to MU^*(X_i/X_{i-1}).$$

Since X_i/X_{i-1} is a finite wedge of spheres and $X_i = X$ for large i, we can work inductively and conclude that $MU^*(X)$ is coherent. In particular, it is finitely generated.

We also have the following result of Quillen, for which we refer the reader to [**18**].

Theorem 12.17. *If X is a finite CW-complex, then $MU^*(X)$ is generated as an MU^*-module by elements of non-negative dimension.*

13. The Landweber Filtration Theorem

The algebraic situation that we will investigate is the following. We have a base ring R, over which we have a connected graded commutative Hopf algebra S which is free as an R-module. In addition, there is an S-comodule A which is also a connected graded commutative R-algebra such that the structure maps

$$R \longrightarrow A \longleftarrow A \otimes A$$

are S-comodule maps. That is, A is an S-comodule algebra.

We would like to determine something about the structure of an object M which is an A-module and an S-comodule such that

$$A \otimes M \longrightarrow M$$

is an S-comodule map.

EXAMPLE 13.1. Take R to be $\mathbb{F}_2$ and S to be the dual Steenrod algebra $\mathcal{A}_*$. If $X \to B$ is a map of finite complexes, then $A = H^*B$ is an $\mathcal{A}_*$-comodule algebra and $M = H^*X$ is an $\mathcal{A}_*$-comodule and an H^*B-module. (The $\mathcal{A}_*$-comodule structures come from the fact that $H^*Y = H_*(DY)$, where DY is the Spanier-Whitehead dual of Y.

A more interesting example, in which A is MU_* and S is the Landweber-Novikov algebra, will be discussed shortly.

If $x \in M$ is primitive ($\psi x = x \otimes 1$) then the ideal $\operatorname{Ann} x := \{a \in A : ax = 0\}$ is an invariant ideal of A, *i.e.,* a subcomodule of A. This is because the map $A \to \Sigma^{-|x|}M$ sending a to ax is a comodule map, and $\operatorname{Ann} x$ is the kernel. Thus we have the following diagram of A-module S-comodules:

$$\begin{array}{ccc} \Sigma^{|x|}A/\operatorname{Ann} x & \rightarrowtail & M \\ & \searrow^{\cong} \quad \nearrow & \\ & Ax & \end{array} .$$

Now assume that M is a coherent A-module. As Ax is finitely generated, it is also coherent. So M/Ax is also coherent. The trick now is to find a primitive x so that $\operatorname{Ann} x$ is a prime ideal.

Let I be maximal among annihilators of non-zero elements of M. Then I is prime. For suppose that $I = \operatorname{Ann} x$ and rs kills x but s doesn't. Then r is in $\operatorname{Ann} sx$. But $\operatorname{Ann} sx$ contains and hence is equal to $\operatorname{Ann} x = I$. This argument shows that if y is killed by I then $I = \operatorname{Ann} y$.

LEMMA 13.2. *If I is an invariant ideal, then $N := \{x \in M : Ix = 0\}$ is a subcomodule.*

PROOF. Exercise. □

The following is a difficult result of Landweber.

Theorem 13.3. *Assume that A and S are polynomial over R as algebras and that M is finitely presented over A. Then all annihilator ideals of M are invariant.*

A bit of April 26, 1994 to be done by Dan.
April 28, 1994 to be done by Dan.
May 3, 1994 to be done by Dan.

CHAPTER 3

The Nilpotence Theorem

1. Statement of Nilpotence Theorems

The starting point for nilpotence theorems was Nishida's theorem in 1973 asserting that any element in the ring π_*^S of positive dimension is nilpotent. There are two faces to this theorem:

(i) For each $\alpha : S^q \to S^0$, $\alpha^{\wedge n} : S^{qn} \to S^0$ is null for n large enough.

(ii) Each $\alpha : S^q \to S^0$ gives the null class after iterating

$$S^{qn} \to S^{q(n-1)} \to S^{q(n-2)} \to \cdots \to S^0,$$

for n large enough.

Around 1976 Ravenel conjectured that the non-nilpotent self-maps of finite spectra are detected by complex cobordism. This *nilpotence conjecture* will be a consequence of the following result.

THEOREM 1.1 (Devinatz, Hopkins, Smith). *Let R be an associative ring spectrum of finite type (*i.e., *connective and with finitely generated homotopy). Then the kernel of the Hurewicz map $\pi_* R \to MU_* R$ is a nilideal. That is, every element is nilpotent.*

COROLLARY 1.2. *Let F be finite and let X be an arbitrary spectrum. In addition, let $f : F \to X$ be such that the composition*

$$F \xrightarrow{f} X \xrightarrow{\eta \wedge 1} MU \wedge X$$

is nullhomotopic. Then $f^{\wedge n} : F^{\wedge n} \to X^{\wedge n}$ is null for n large enough.

PROOF. First we observe that we can assume X to be finite. Since also F is finite, f induces

$$\hat{f} : S^0 \longrightarrow F \wedge DF \xrightarrow{f \wedge 1} X \wedge DF$$

where DF is the Spanier-Whitehead dual of F. Both f and $\hat{f}$ are $\wedge$-nilpotent. Moreover the null map $F \to X \to MU \wedge X$ gives the contractibility of $S^0 \to X \wedge DF \to MU \wedge X \wedge DF$ and vice versa. All in all we may assume F to be S^0 or better F to be S^n and X 0-connected.

Now form the ring spectrum $R = \bigvee_{n=0}^{\infty} X^{\wedge n}$, where $X^{\wedge 0} = S^0$, analogous to the tensor algebra. Since $X^{\wedge n}$ is highly connected, R is of finite type. $f \in \pi_n(X) \hookrightarrow \pi_n(R)$ is $\wedge$-nilpotent iff its image in $\pi_n(R)$ is so. But we know that the Hurewicz image of f in MU_*R is zero, which places us in the situation of our theorem. □

Corollary 1.3. *Let R be a ring spectrum with unit (not necessarily associative). Then for every $\alpha \in \ker(\pi_* R \to MU_* R)$ there exists an n such that any bracket of $\alpha^{\wedge n}$ is zero.*

Proof. Apply Corollary 1 to the diagram

$$\begin{array}{ccc} S^{nq} & \xrightarrow{\alpha^{\wedge n}} & R^{\wedge n} \\ & {\scriptstyle \alpha^{\wedge n}} \searrow & \downarrow{\scriptstyle \mu} \\ & & R \end{array} .$$

□

Also, the following theorem is a consequence of the preceding one.

Theorem 1.4. *Let*

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots$$

be a direct system of spectra. Assume there are integers m and b such that each X_n is at least $(mn+b)$-connected. If $MU_* f_n = 0$ for all n, then $\mathrm{tel}(f_*) \simeq *$.

Example 1.5. Let $f : F \to \Sigma^{-q} F$ be a self-map of a finite spectrum and suppose that $MU_* f = 0$. Then the theorem gives

$$* \cong [\, F, \mathrm{tel}(\Sigma^{-nq} F)\,] \cong \varprojlim [\, F, \Sigma^{-nq} F\,] \ni [\, 1 : F \to F\,],$$

so f is composition nilpotent.

Before turning to the proofs of the two theorems we state another application which guarantees the existence of v_n-self-maps. A proof can be found in [**8**].

THEOREM 1.6 (Hopkins, Smith). *Let X be finite p-torsion spectrum with $K(n-1)_*X = 0$. Then there exists $v : X \to \Sigma^{-q}X$ such that*

$$K(m)_*v = \begin{cases} 0 & m \neq n \\ \textit{isomorphism} & m = n \end{cases}.$$

(So if $K(n)_*X \neq 0$ then v is not nilpotent and we refer to it as a v_n-self-map.) Moreover, let Y be another finite $K(n-1)$-acyclic spectrum and $w : Y \to \Sigma^{-r}Y$ a v_n-self-map. Then for every $f : X \to Y$ there exists $i, j \geq 0$ with $qi = rj$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle v^i}\downarrow & & \downarrow{\scriptstyle w^j} \\ \Sigma^{-qi}X & \xrightarrow{f} & \Sigma^{-ij}Y \end{array}$$

commute. In particular, there is a v_n-self-map in the center of $\mathrm{End}(X)$*. Finally if $K(n)_*X \neq 0$ it can be shown that*

$$\text{center } \mathrm{End}(X)/\text{ nilpotents} \cong \mathbb{F}_p[\,v\,].$$

2. An Outline of the Proof

In order to prove the main theorem we approximate the complex cobordism spectrum MU by a sequence of ring spectra. Let $X(n)$ be the Thom spectrum of

$$\Omega SU(n) \longrightarrow \Omega SU \overset{\text{Bott}}{\cong} BU.$$

For example, $X(1) = S^0$. The adjoint to

$$\begin{array}{ccc} \Sigma\,\mathbb{C}\mathrm{P}^{n-1} & \longrightarrow & SU(n) \\ \uparrow & \nearrow & \\ S^1 \times \mathbb{C}\mathrm{P}^{n-1} & & \end{array} \qquad \begin{pmatrix} z_1^{-1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & 1 \end{pmatrix} (z\pi_L + (1-\pi_L))$$

$$(z, L) \nearrow$$

defines the left vertical arrow in the commutative diagram

$$\begin{array}{ccc} \mathbb{C}\mathrm{P}^{n-1} & \longrightarrow & \mathbb{C}\mathrm{P}^{\infty} \\ \downarrow & & \downarrow \\ \Omega SU(n) & \longrightarrow & \Omega SU = BU \end{array} .$$

It is not hard to see (for example using the Leray-Serre spectral sequence) that

$$H_*\Omega SU(n) = \mathbb{Z}[\,\beta_0, \beta_1, \cdots, \beta_{n-1}]/(\beta_0 = 1),$$

where $H_*\,\mathbb{C}\mathrm{P}^{n-1} = \langle \beta_0, \beta_1, \cdots \rangle$. From the Thom isomorphism we can conclude that

$$H_*(X(n)) = \mathbb{Z}[\,b_1, \cdots, b_{n-1}] \subset \mathbb{Z}[\,b_1, b_2, \cdots] \subset H_*MU.$$

Since now the inclusion $X(n) \to MU$ is $2(n-1)$-connected we see that the Hurewicz map

$$\begin{array}{ccc} \pi_*R & \longrightarrow & MU_*R \\ {\scriptstyle\varphi}\downarrow & \nearrow & \\ X(n)_*R & & \end{array}$$

sends every element $\alpha \in \ker(\pi_*R \to MU_*R)$ to the zero element in $X(n)_*R$ for $2n > |\alpha|$. Thus it remains to show that *if* $\varphi_{n+1}\alpha$ *is nilpotent, then so is* $\varphi_n\alpha$.

LEMMA 2.1. *Let R and E be commutative, associative ring spectra and $\alpha \in \pi_n R$. Then the Hurewicz image $\varphi(\alpha) \in E_*R$ is nilpotent iff the telescope*

$$\alpha^{-1}R = \mathrm{tel}(R \xrightarrow{\bar{\alpha}} \Sigma^{-n}R \xrightarrow{\Sigma^{-n}\bar{\alpha}} \Sigma^{-2n}R \longrightarrow \cdots)$$

$$\bar{\alpha} : \Sigma^n R \cong R \wedge S^n \xrightarrow{1\wedge\alpha} R \wedge R \xrightarrow{\mu} R$$

is E_*-acyclic, i.e. $E \wedge \alpha^{-1}R$ is contractible.

Proof. We leave the proof as an exercise. The reader may use the diagram

$$
\begin{array}{ccccc}
 & & E \wedge R \wedge R & & \\
 & \nearrow^{E\wedge\alpha^m\wedge 1} & \uparrow{\scriptstyle \eta\wedge\mu^m\wedge 1} & \searrow^{1\wedge\mu} & \\
\Sigma^{nm} R \cong S^{nm} \wedge R & \xrightarrow{\alpha^{\wedge m}\wedge 1} & R^m \wedge R \cong S^0 \wedge R^m \wedge R & \xrightarrow{\eta\wedge\mu^{m+1}} & E \wedge R \\
 & \searrow_{\bar{\alpha}^m} & \downarrow{\scriptstyle 1\wedge\mu^{m+1}} & \nearrow_{\eta\wedge 1} & \\
 & & R \cong S^0 \wedge R & &
\end{array}
.
$$

□

Note that X is contractible iff each localized spectrum $X_{(p)}$ is contractible for all primes p. So we may assume that R is p-local (and of finite type). However, we only deal with the case $p = 2$ for simplicity's sake.

For the key step from $X(n{+}1)$ to $X(n)$ we define spectra F_k by the pullback diagram of fibrations

$$
\begin{array}{ccc}
\Omega SU(n) & \longrightarrow & \Omega SU(n) \\
\downarrow & & \downarrow \\
F_k & \longrightarrow & \Omega SU(n+1) \\
\downarrow & & \downarrow \\
J_{p^k-1} S^{2n} & \hookrightarrow & \Omega S^{2n+1}
\end{array}
.
$$

Here J_{p^k-1} denotes the $2^n(p^k-1)$-skeleton of ΩS^{n+1}.

We are able to compute the singular homology of this diagram. Recall that $H_*\Omega S^{2n+1} = \mathbb{Z}[\beta_n]$, b_n is of degree $2n$. $H_* J_l S^{2n}$ is the subgroup generated by $1, b_n, \cdots, b_n^k$ (see Whitehead's book for details.) The Leray-Serre spectral sequence reads

$$H_* F_k = H_*(\Omega SU(n)) \left\langle 1, \beta_n, \cdots, \beta_n^{p^k-1} \right\rangle .$$

Let G_k be the Thom spectrum of $F_k \to BU$. Then again the Thom isomorphism gives

$$H_* G_k = H_*(X(n)) \left\langle 1, b_n, \cdots, b_n^{p^k-1} \right\rangle .$$

We now outline our program for the proof of the key step. The last lemma said that the nilpotence of $\varphi_{n+1}(\alpha)$ implies the contractibility of $X(n+1) \wedge \alpha^{-1}R$. A *vanishing line argument* will show that $G_N \wedge \alpha^{-1}R$ is also contractible for large enough N. The *nilpotence cofibration lemma* will let us conclude that

$$G_{k+1} \wedge \alpha^{-1}R \simeq * \Rightarrow G_k \wedge \alpha^{-1}R \simeq *.$$

Finally the lemma applied to $* \simeq G_0 \wedge \alpha^{-1}R = X(n) \wedge \alpha^{-1}R$ furnishes the nilpotence of $\varphi_n \alpha$.

3. The Vanishing Line Lemma

We will study the $X(n+1)$-based Adams spectral sequence

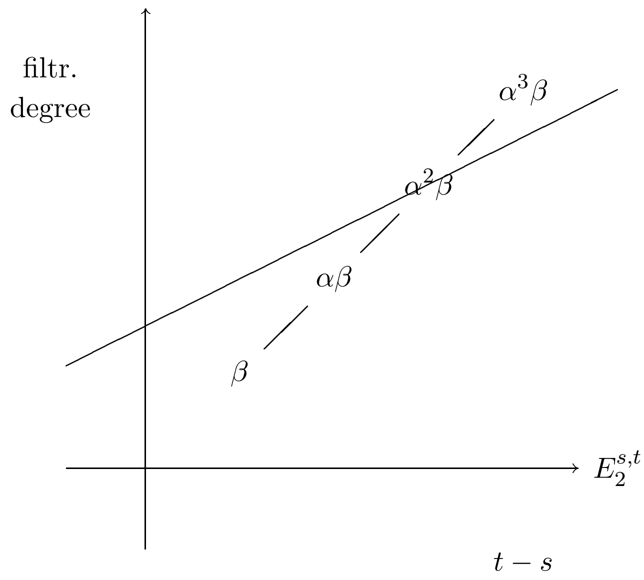
$$E_2(G_k \wedge R, X(n+1)) \Rightarrow \pi_*(G_k \wedge R).$$

By taking a power of α, assume that $\varphi_{n+1}(\alpha) = 0$, i.e. α is in positive filtration, say in $\mathrm{Ext}^{s,t}_{X(n+1)_*X(n+1)}(X(n+1)_*, X(n+1)_*R)$, $s > 0$. Furthermore, $E_r(R, X(n+1))$ acts on $E_r(G_k \wedge R, X(n+1))$ and multiplication by α increases the filtration in $E_*(G_k \wedge R)$.

LEMMA 3.1. *$E_2(G_k \wedge R, X(n+1))$ has a vanishing line of slope tending to 0 as $k \to \infty$.*

We postpone the proof for a while and show how the contractibility of $G_k \wedge \alpha^{-1}R$ for large k follows.

Let k be such that the slope of the vanishing line is less than $s(t-1)^{-1}$. Therefore, $\beta \in \pi_*(g_k \wedge R)$ is transported into the vanishing area by a high power of α.



This implies $\alpha^m\beta = 0$ and what we wanted: $G_k \wedge \alpha^{-1}R \simeq *$.

Proof. (*Sketch*) Using the diagram of "orientations"

$$\begin{array}{ccc} \mathbb{C}\mathrm{P}^n & \longrightarrow & X(n+1) \\ \big\downarrow & & \big\downarrow \\ \mathbb{C}\mathrm{P}^\infty & \longrightarrow & MU \end{array}$$

together with routine A-H spectral sequence arguments we can see that

$$\begin{aligned} X(n+1)^*\mathbb{C}\mathrm{P}^n &= X(n+1)^*[\,x\,]/(x^{n+1}) \\ X(n+1)_*\mathbb{C}\mathrm{P}^n &= X(n+1)_*\,\langle\beta_1,\beta_2,\cdots,\beta_n\rangle \\ X(n+1)_*\Omega SU(n+1) &= X(n+1)_*[\,\beta_1,\beta_2,\cdots,\beta_n] \\ X(n+1)_*X(n+1) &= X(n+1)_*[\,b_1,b_2,\cdots,b_n]. \end{aligned}$$

The Hopf algebroid $X(n+1)_*X(n+1)$ agrees with MU_*MU in the range where the b_i are defined. The reader may verify that $X(n+1)_*F_k$ is a submodule of $X(n+1)_*X(n+1)$ (e.g. by staring first at singular homology). It is the free module over $X(n+1)_*X(n+1)$ with basis $\{1, b_n, \cdots, b_n^k\}$. The vanishing line follows now with standard means of homological algebra. See [**4**]. □

4. The Nilpotent Cofibration Lemma

We have seen that

$$H_*G_k = H_*X(n)\left\langle 1, b_n, \cdots, b_n^{p^k-1}\right\rangle.$$

Therefore, we can think of $X(n+1)$ as cellular complex with cells $X(n)$ and skeleta G_k. We will show:

THEOREM 4.1. *There is a map h and a cofibre sequence*

$$\Sigma^q G_k \xrightarrow{h} G_k \longrightarrow G_{k+1}, \quad q = |\, b_n^{2^k} \,| - 1$$

such that h has contractible telescope.

Let us first see how this implies what we wanted, i.e. for any spectrum Z

$$G_{k+1} \wedge Z \simeq * \Rightarrow G_k \wedge Z \simeq *.$$

The following lemma will certainly be enough.

LEMMA 4.2. *If $\Sigma^q X \xrightarrow{f} X \longrightarrow C_f$ is an cofibre sequence, then $X \wedge Z$ is contractible iff both $\mathrm{tel}(f) \wedge Z$ and $C_f \wedge Z$ are so.*

PROOF. Let $\mathrm{tel}(f) \wedge Z$ and $C_f \wedge Z$ be contractible and $\alpha : S^n \to X \wedge Z$ a map. The nullhomotopy of α in $\mathrm{tel}(f) \wedge Z$ compresses through some $\Sigma^{-qn} X \wedge Z$.

$$\begin{array}{ccccc}
S^n & \xrightarrow{\alpha} & X \wedge Z & \longrightarrow & \mathrm{tel}(f) \wedge Z \\
\downarrow & \searrow & & & \uparrow \\
\Sigma^{-q(n-1)-1} C_f \wedge Z & \longrightarrow & \Sigma^{-q(n-1)} X \wedge Z & \longrightarrow & \Sigma^{-qn} X \wedge Z
\end{array}.$$

Hence α is nullhomotopic in $\Sigma^{-qn} X \wedge Z$ and factors through a map to $\Sigma^{-q(n-1)-1} C_f \wedge Z \simeq *$. Now α is already nullhomotopic in $\Sigma^{-q(n-1)} X \wedge Z$ and an induction completes the proof. $\square$

In order to prove the theorem we first give an alternative description of F_k which reveals more structure.

The Hopf map induces a fibration

$$S^{2n} \longrightarrow \Omega S^{2n+1} \xrightarrow{h} \Omega S^{4n+1}.$$

More generally there is a fibration

$$J_{2^k}S^{2n} \longrightarrow \Omega S^{2n+1} \xrightarrow{h^k} \Omega S^{2^{k+1}n+1}.$$

The reader may check that

$$F_k := \text{pullback}(J_{2^k-1}S^{2n} \longrightarrow \Omega S^{2n+1} \longleftarrow \pi\Omega SU(n+1))$$

can serve as the homotopy fibre of $\Omega SU(n+1) \xrightarrow{h^k\pi} \Omega S^{2^{k+1}n+1}$. We are led to the following diagram

$$\begin{array}{ccccc} F_k & \longrightarrow & F_k & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ F_{k+1} & \longrightarrow & \Omega SU(n+1) & \xrightarrow{h^{k+1}\pi} & \Omega S^{2^{k+2}n+1} \\ \downarrow & & \downarrow & & \| \\ S^{2^{k+1}n} & \longrightarrow & \Omega S^{2^{k+1}n+1} & \xrightarrow{h} & \Omega S^{2^{k+2}n+1} \end{array}.$$

Let $E \xrightarrow{\pi} S^{2m}$ be a fibration with fibre F and let $E \xrightarrow{\xi} BU$ be continuous. Then E is homotopy equivalent over S^{2m} to

$$F' \longrightarrow E' = \{(w,e)|\, w : I \to S^{2m}, w(1) = \pi e\} \xrightarrow{ev_0} S^{2m}$$

and thus admits an obvious action

$$\begin{array}{ccc} \Omega S^{2m} \times F' & \longrightarrow & F' \\ \downarrow & & \downarrow \\ PS^{2m} \times F' & \longrightarrow & E' \end{array}.$$

Passing to Thom spectra it yields an action $(\Omega S^{2m})_+ \wedge F^\xi \to F^\xi$, because $PS^{2m} \simeq *$. That is, we have expressed F^ξ as module spectrum over the ring spectrum $(\Omega S^{2n})_+$. It is well known that

$$\Sigma^\infty(\Omega S^{2m})_+ \simeq \bigvee_{j=0}^{\infty} S^{(2m-1)j}$$

as ring spectra. We obtain operations

$$\beta_j : S^{(2m-1)j} \wedge F^\xi \longrightarrow \Omega(S^{2m})_+ \wedge f^\xi \longrightarrow F^\xi.$$

It can be shown using the associativity of the action that $\beta_j = \beta_1^j$ and $C_{\beta_1} \simeq E^\xi$.

In our case we have an extension of the action of ΩS^{2n} on F to an action of $\Omega^2 S^{n+1}$.

$$\begin{array}{ccccc} F & \longrightarrow & F & & \\ \downarrow & & \downarrow & & \\ E & \longrightarrow & E' & \longrightarrow & BU \\ \downarrow & & \downarrow & & \\ S^{2m} & \longrightarrow & \Omega S^{2m+1} & & \end{array}$$

Recall $H_*\Omega^2 S^{2m+1} = \mathbb{F}_p[\,x_0, x_1, \cdots],\ |\,x_i| = 2^{j+1}m - 1$.

THEOREM 4.3 (Snaith). *Stably,* $\Omega^2 S^{2m+1}$ *splits as* $\bigvee_{k=0}^{\infty} \Sigma^{(2m-1)k} D_k$ *where*

$$H_*\Sigma^{(2m-1)k} D_k = \textit{weight}\, k \textit{ summand in}\, H_*\Omega^2 S^{2m+1}.$$

$H^* D_k$ *is free over* $\mathcal{A}$ *through dim*$[\frac{k}{2}]$.

Now we define h in the cofibre sequence

$$\Sigma^q G_k \xrightarrow{h} G_k \longrightarrow G_{k+1}$$

to be the composite

$$\Sigma^{(2m-1)j} F^{\xi} \longrightarrow \Sigma^{(2m-1)j} D_j \wedge F \longrightarrow (\Omega^2 S^{2m+1})_+ \wedge F^{\xi} \longrightarrow F^{\xi}.$$

It remains to show that the telescope of h is contractible. But we know that h^j induces zero in $H_*(-, \mathbb{F}_p)$. Take $\alpha : S^q \longrightarrow F^{\xi}$. Then for j such that $|j/2| > q$ we factored $\beta_1^j \alpha$ through the Eilenberg–Mac Lane spectrum. Hence it is null.

APPENDIX A

A Construction of the Steenrod Squares

The following construction is important in that it easily generalizes in several ways. See Steenrod-Epstein [**22**] for more information; in particular, the first section of Chapter VII gives a good overview of the construction. The details of this construction were taken from [**11**].

1. The Definition

Let X be a pointed space. This gives us a filtration $F_k = F_k X^n$ of X^n, where $F_k = \{(x_1, \ldots, x_n) \ : \text{ at least } n-k \text{ points are the basepoint } \}$. For example, $F_0 = \{(*, \ldots, *)\}$, $F_1 = X^{\vee n}$, and $F_n = X^n$. Consider a subgroup π of the symmetric group Σ_n on n letters acting on X^n (on the left) by permuting the factors. To the group π corresponds a universal π-bundle

$$\begin{array}{ccc} \pi & \longrightarrow & E\pi \\ & & \downarrow \\ & & B\pi \ . \end{array}$$

Here $E\pi$ is a contractible CW-complex with a free (right) π-action and $B\pi \cong E\pi/\pi$. Since π is discrete, the bundle is in fact a covering space and $B\pi$ is a $K(\pi, 1)$. Most important for us will be the case $n = 2$, $\pi = \Sigma_2 = \mathbb{Z}/2$; here we can take $E\pi = S^\infty$ with the antipodal action, and $B\pi = \mathbb{R}\mathrm{P}^\infty$.

The Borel construction gives us a bundle

$$\begin{array}{ccc} X^n & \longrightarrow & E\pi \times_\pi X^n \\ & & \downarrow \\ & & B\pi \ , \end{array}$$

where $E\pi \times_\pi X^n = (E\pi \times X^n)/\sim$ and $(eg, x) \sim (e, gx)$. Because the action of π on X^n preserves the filtration, we have a subbundle with total space

$E\pi \times_\pi F_{n-1}$. We define $D_\pi X$ to be the quotient $\frac{E\pi\times_\pi X^n}{E\pi\times_\pi F_{n-1}}$. It is easy to check that we have the following homeomorphisms

$$\frac{E\pi \times_\pi X^n}{E\pi \times_\pi F_{n-1}} \cong \frac{E\pi \times_\pi X^{\wedge n}}{E\pi \times_\pi \{*\}} \cong E\pi_+ \wedge_\pi X^{\wedge n},$$

where we've used that $X^{\wedge n} \cong X^n/F_{n-1}$. Thus $D_\pi X$ is the pointed *homotopy* quotient of $X^{\wedge n}$ by π.

The following lemma is crucial for the construction.

LEMMA A.1. *Suppose $\bar{H}^i X = 0$ for $i < q$ and that $\bar{H}^q X$ is finite-dimensional, where we are taking coefficients in a field. Then*

$$\bar{H}^i(D_\pi X) = 0 \text{ for } i < nq$$

and

$$\bar{H}^{nq}(D_\pi X) = ((H^q X)^{\otimes n})^\pi.$$

The last expression denotes the π-invariant elements of the tensor product, where π acts by permuting the factors with the usual convention that interchanging two items of odd degree introduces a sign.

PROOF. Associated to the bundle $E\pi \times_\pi X^n$ and subbundle $E\pi \times_\pi F_{n-1}$ is a relative Serre spectral sequence converging to $H^*(E\pi \times_\pi X^n, E\pi \times_\pi F_{n-1}) = \bar{H}^*(D_\pi X)$ with E_2 term

$$H^*(B\pi; \{H^*(X^n, F_{n-1})\}).$$

Coefficients are taken in a local system as $B\pi$ is not simply connected. Now $H^i(X^n, F_{n-1}) \cong \bar{H}^i(X^{\wedge n}) \cong ((\bar{H}^* X)^{\otimes n})^i$, so the coefficients are zero for $i < nq$. Thus the E_2 term is zero below the line at nq and so $\bar{H}(D_\pi X) = 0$ for $i < nq$. Moreover,

$$\bar{H}^{nq}(D_\pi X) \cong H^0(B\pi, \{H^{nq}(X^n, F_{n-1})\}) \cong (H^{nq}(X^n, F_{n-1}))^\pi;$$

for the last isomorphism, see G.W. Whitehead [**25**, p. 275]. Now π acts on $H^{nq}(X^n, F_{n-1})$ in two ways. The first is through its identification with the fundamental group of the base; this is the action that gives us the structure of a local system, and is the action that is meant in the last displayed expression above. But π also acts on this cohomology group because it acts on the pair

(X^n, F_{n-1}). We claim that these actions are one and the same, at least up to an inverse.

How does the fundamental group of the base act on $H^*(X^n, F_{n-1})$? Let $\gamma : (I, \{0,1\}) \to (B\pi, b)$ be a loop at b. Since the pullback of $E\pi \times_\pi X^n$ along γ is trivial, γ is covered by a bundle map from the trivial bundle $I \times X^n$ to $E\pi \times_\pi X^n$ which is a homeomorphism on each fibre. Since $\{0\} \times X^n$ is canonically homeomorphic to $\{1\} \times X^n$, we get a well-defined map from the fibre above b to itself. The effect of this map in cohomology is independent of the choice of representative of the homotopy class and of the choice of bundle map.

Now we'll prove that the actions are the same. Choose a point $e \in E\pi$ in the fibre above b, and let γ be as above. Since $E\pi$ is a covering space, there exists a unique lifting $\bar{\gamma} : I \to E\pi$ of γ such that $\bar{\gamma}(0) = e$. This lifting has the property that $\bar{\gamma}(1) = eg$ for some $g \in \pi$; in fact, this is how the isomorphism between π and $\pi_1(B\pi)$ is defined. We judiciously define a bundle map $I \times X^n \to E\pi \times_\pi X^n$ by $(t, x) \mapsto [\bar{\gamma}(t), x]$. Then we have that $(0, x) \mapsto [\bar{\gamma}(0), x] = [e, x]$ and that $(1, x) \mapsto [\bar{\gamma}(1), x] = [eg, x] = [e, gx]$. Thus under the correspondence described above, $x \mapsto gx$. (We have been sloppy about inverses, but of course this does not matter for the situation at hand, as the invariant elements are the same in either case.)

So we now know that $\bar{H}^{nq}(D_\pi X) \cong (H^{nq}(X^n, F_{n-1}))^\pi$, where π acts through its embedding in Σ_n. Now

$$H^{nq}(X^n, F_{n-1}) \cong \bar{H}^{nq}(X^{\wedge n}) \cong (\bar{H}^q X)^{\otimes n}$$

and these isomorphisms are equivariant with respect to the Σ_n action. Hence,

$$\bar{H}^{nq}(D_\pi X) \cong ((\bar{H}^{nq} X)^{\otimes n})^\pi$$

as required. □

As an application of the lemma, take $X = K(\mathbb{F}_2, q) = K$. The Hurewicz theorem implies that $\bar{H}^i K = 0$ for $i < q$ and $\bar{H}^q K = \mathbb{F}_2$. Therefore, the lemma says that $\bar{H}^{nq}(D_\pi X)$ is $\mathbb{F}_2$ as well, since the action on $\mathbb{F}_2$ is trivial.

The inclusion of the fibre

$$K^{\wedge n} \xhookrightarrow{i} E\pi \times_\pi K^{\wedge n} = D_\pi K$$

induces the edge homomorphism in the spectral sequence, and so

$$\bar{H}^{nq}K^{\wedge n} \xleftarrow{i^*} \bar{H}^{nq}(D_\pi K) = \mathbb{F}_2$$

is an isomorphism. The non-zero element of $\bar{H}^{nq}K^{\wedge n}$ is $\iota_q^{\wedge n}$, where $\iota_q \in \bar{H}^q K$ is the fundamental class.

Corollary A.2. *There exists a unique class $P_\pi \iota_q \in \bar{H}^{nq}(D_\pi K)$ such that $i^* P_\pi \iota_q = \iota_q^{\wedge n}$.*

Now consider an arbitrary class $u \in \bar{H}^q X$. Represent u by a map $X \xrightarrow{u} K$. Then

$$\begin{array}{ccccc} X^{\wedge n} & \xrightarrow{u^{\wedge n}} & K^{\wedge n} & \xrightarrow{\iota_q^{\wedge n}} & K_{nq} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & \nearrow{\scriptstyle P_\pi \iota_q} & \\ D_\pi X & \xrightarrow[D_\pi u]{} & D_\pi K & & \end{array}$$

commutes. Here $D_\pi u$ is defined by

$$\begin{array}{ccc} D_\pi X & \xrightarrow{D_\pi u} & D_\pi K \\ \| & & \| \\ E\pi_+ \wedge_\pi X^{\wedge n} & \xrightarrow[1 \wedge u^{\wedge n}]{} & E\pi_+ \wedge_\pi K^{\wedge n} \end{array} ,$$

which makes sense since both 1 and $u^{\wedge n}$ are π-equivariant. Note that the top row of the larger diagram represents the class $u^{\wedge n}$ since $(u^{\wedge n})^*(\iota_q^{\wedge n}) = (u^*(\iota_q))^{\wedge n} = u^{\wedge n}$, so there exists a class $P_\pi u \in \bar{H}^{nq}(D_\pi X)$ such that $i^* P_\pi u = u^{\wedge n}$, namely $(D_\pi u)^* P_\pi \iota_q$, which is represented by the composite $P_\pi \iota_q \circ D_\pi u$. It is easy to check that P_π is a natural transformation from $\bar{H}^q(-)$ to the composite functor $\bar{H}^{nq} D_\pi(-)$ and that it is the unique such transformation satisfying $i^* P_\pi u = u^{\wedge n}$.

The diagonal map $X \xrightarrow{\Delta} X^{\wedge n}$ is equivariant when we consider X as a trivial π-space, so we have a map j defined by

$$\begin{array}{ccc} B\pi_+ \wedge X & \xrightarrow{j} & D_\pi X \\ \| & & \| \\ E\pi_+ \wedge_\pi X & \xrightarrow[1 \wedge \Delta]{} & E\pi_+ \wedge_\pi X^{\wedge n} \end{array} .$$

Now we fix $n = 2$ and $\pi = \Sigma_2 = \mathbb{Z}/2$, and write P for P_π. Then $B\pi = \mathbb{R}\mathrm{P}^\infty$ and $\bar{H}^*(B\pi_+) = H^*B\pi = \mathbb{F}_2[x]$ with $|x| = 1$. Since $\bar{H}^*(B\pi_+ \wedge X) \cong \mathbb{F}_2[x] \otimes \bar{H}^*X$, we may write

$$j^*Pu = \sum_{i=-q}^{q} x^{q-i} \otimes \mathrm{Sq}^i\, u \qquad (|u| = q)$$

for unique classes $\mathrm{Sq}^i\, u \in \bar{H}^{q+i}X$, $-q \leq i \leq q$. These are the **Steenrod Squares**. We will find that for $i < 0$ the squares are zero.

2. Properties of the Squares

PROPOSITION A.3 (Naturality). Sq^i *is a natural transformation from* $\bar{H}^q(-)$ *to* $\bar{H}^{q+i}(-)$.

PROOF. We clearly have that

$$\begin{array}{ccc} B\pi_+ \wedge X & \xrightarrow{1\wedge f} & B\pi_+ \wedge Y \\ {\scriptstyle j=1\wedge\Delta}\downarrow & & \downarrow{\scriptstyle 1\wedge\Delta=j} \\ E\pi_+ \wedge_\pi X^{\wedge 2} & \xrightarrow[1\wedge f^{\wedge 2}]{} & E\pi_+ \wedge_\pi Y^{\wedge 2} \end{array}$$

commutes, and so $j\,(1 \wedge f) = (1 \wedge f^{\wedge 2})\, j = (D_\pi f)\, j$. Therefore, for $u \in H^qY$,

$$\begin{aligned} \sum x^{q-i} \otimes f^*\,\mathrm{Sq}^i\, u &= (1 \wedge f)^* \sum x^{q-i} \otimes \mathrm{Sq}^i\, u \\ &= (1 \wedge f)^* j^* Pu \\ &= j^*(D_\pi f)^* Pu \\ &= j^* P(f^* u) \qquad \text{(by naturality of } P\text{)} \\ &= \sum x^{q-i} \otimes \mathrm{Sq}^i\, f^* u\,. \end{aligned}$$

□

PROPOSITION A.4. Sq^i *is zero for* $i < 0$.

PROOF. Use naturality and the fact that $\bar{H}^i(K_q) = 0$ for $i < q$. □

PROPOSITION A.5. Sq^q *is the cup square on classes of dimension* q.

Proof. Consider the commutative square

$$\begin{array}{ccc} B\pi_+ \wedge X & \xrightarrow{j} & D_\pi X \\ {\scriptstyle k}\uparrow & & \uparrow{\scriptstyle i} \\ X & \xrightarrow{\Delta} & X^{\wedge 2} \end{array},$$

where k is the inclusion of X as $b \wedge X$ with b a point in $B\pi_+$ other than the basepoint. Put another way, k is given by smashing a map $S^0 \to B\pi_+$ with the identity map. (The maps k and i can be chosen so that the diagram commutes on the nose.) Examining the effect of induced maps on $Pu \in \bar{H}^{2q}(D_\pi X)$ and noting that $k = l1$, where $l : S^0 \to B\pi_+$ sends x to 0 and 1 to 1, we find that $\mathrm{Sq}^q u = u \smile u$. □

Consider the map δ defined by

$$D_\pi(X \wedge Y) =$$

$$[e \wedge ((x$$

*endeqnarray*Itisstraightforwardtocheckthat*

$$\begin{array}{ccccc} (X \wedge Y)^{\wedge 2} & \xrightarrow{i} & D_\pi(X \wedge Y) & \xleftarrow{j} & \\ \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \delta} & & \\ X^{\wedge 2} \wedge Y^{\wedge 2} & \xrightarrow{i \wedge i} & D_\pi X \wedge D_\pi Y & \xleftarrow{j \wedge j} & B\pi \end{array}$$

commutes, where T and $\tilde{\Delta}$ are the obvious maps. We will use the left hand square in the proof of the following lemma, and the right hand square in the proof of the corollary.

Lemma A.6. *$\delta^*(Pu \wedge Pv) = P(u \wedge v)$ for $u \in \bar{H}^p X$ and $v \in \bar{H}^q Y$.*

The philosophy here and in many proofs in this section is to use special properties of the universal example to prove the result in that case, and then use naturality to deduce the result in the general case.

Proof. Consider first the case $X = K_p$, $Y = K_q$, $u = \iota_p$ and $v = \iota_q$. Then using the left-hand square of the above diagram we find that

$$\begin{aligned} i^*\delta^*(P\iota_p \wedge P\iota_q) &= T(i \wedge i)^*(P\iota_p \wedge P\iota_q) \\ &= T(i^*P\iota_p \wedge i^*P\iota_q) \\ &= T(\iota_p^{\wedge 2} \wedge \iota_q^{\wedge 2}) \\ &= (\iota_p \wedge \iota_q)^{\wedge 2}. \end{aligned}$$

That i^* is an isomorphism for this universal example gives the desired result in this case.

The general case follows by naturality:

$$\begin{aligned} \delta^*(Pu \wedge Pv) &= \delta^*((D_\pi u)^*P\iota_p \wedge (D_\pi v)^*P\iota_q) \\ &= \delta^*(D_\pi u \wedge D_\pi v)^*(P\iota_p \wedge P\iota_q) \\ &= D_\pi(u \wedge v)^*\delta^*(P\iota_p \wedge P\iota_q) \\ &= D_\pi(u \wedge v)^*P(\iota_p \wedge \iota_q) \\ &= P(u \wedge v). \end{aligned}$$

□

Corollary A.7 (Cartan Formula). $\mathrm{Sq}^k(u \wedge v) = \sum_{i+j=k} \mathrm{Sq}^i u \wedge \mathrm{Sq}^j v.$

Proof. We compute

$$\begin{aligned} \sum_k x^{p+q-k} \otimes \mathrm{Sq}^k(u \wedge v) &= j^*P(u \wedge v) \\ &= j^*\delta^*(Pu \wedge Pv) \qquad \text{(by the lemma)} \\ &= \tilde{\Delta}^*(j \wedge j)^*(Pu \wedge Pv) \\ &= \tilde{\Delta}^*[(\sum x^{p-i} \otimes \mathrm{Sq}^i u) \wedge (\sum x^{q-j} \otimes \mathrm{Sq}^j v)] \\ &= \sum_{i,j} x^{p+q-(i+j)} \otimes \mathrm{Sq}^i u \wedge \mathrm{Sq}^j v, \end{aligned}$$

and compare coefficients. The third equality uses the right hand square in the commutative diagram before the lemma. □

The cup product form of the Cartan formula follows by pullback along the diagonal map.

EXERCISE A.8. Prove that $\mathrm{Sq}^0 \sigma = \sigma$, for σ the generator of $\bar{H}^1(S^1)$.

The exercise can be proved directly from the definition, but the proof is not just a formality. Indeed, one expects it to be a little trickier than the previous proofs in this section, since it is a property that does *not* hold for variations of this construction in other settings.

COROLLARY A.9 (Stability). Sq^i *commutes with the suspension homomorphism* $\bar{H}^q(X) \to \bar{H}^{q+1}(\Sigma X)$.

PROOF. This follows from the exercise and the Cartan formula. See Section 6. □

COROLLARY A.10. $\mathrm{Sq}^0 : \bar{H}^q(X) \to \bar{H}^q(X)$ *is the identity.*

PROOF. $\mathrm{Sq}^0 \sigma = \sigma \neq 0$, so $\mathrm{Sq}^0 \iota_1$ is non-zero by naturality. But $\bar{H}^1(K_1) \cong \mathbb{F}_2$, so $\mathrm{Sq}^0 \iota_1 = \iota_1$. Thus, using naturality again, we see that the claim is true for $q = 1$. The Cartan formula implies that $\mathrm{Sq}^0(\sigma^{\wedge q}) = \sigma^{\wedge q} \neq 0$. So in the same way as above we see that the claim holds for arbitrary q. □

PROPOSITION A.11. Sq^i *is a homomorphism.*

PROOF. We saw in Section 9 of Chapter 1 that stable operations are additive. □

APPENDIX B

References on Complex Cobordism

The original computation of the complex bordism ring was:

J.W. Milnor, *On the cobordism ring Ω^* and a complex analogue*, Amer. J. Math. 82 (1960) 505–521.

Quillen's introduction of formal group laws occurs in

D.G. Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. 75 (1969) 1293–1298.

In this paper he outlines a computation of the complex cobordism ring, using Steenrod operations in cobordism theory. He also shows that unoriented cobordism is mod 2 cohomology tensored with a certain polynomial algebra *as an algebra*, by the same methods. Moreover he shows there how to split MU localized at a prime. More complete accounts occur in:

D.G. Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Adv. in Math. 7 (1971) 29–56.

M. Karoubi,

Some of these results were also obtained by tom Dieck:

D. tom Dieck, *Steenrod Operationen in Kobordismen-Theorien*, Math. Zeit. 107 (1968) 380–401.

The basic structural implications of the formal group machinery are described in

P.S. Landweber, *$BP_*(BP)$ and typical formal group laws*, Osaka J. Math. 12 (1975) 357–363

as well as in "Part II: Quillen's work on formal groups and complex cobordism," in

J.F. Adams, *Stable Homotopy and Generalized Homology*, Chicago Lectures in Mathematics, 1974.

Miscenko's theorem is in

Miscenko,

The original construction of BP and the splitting of MU is in

E.H. Brown and F.P. Peterson, *A spectrum whose $\mathbb{Z}_p$ cohomology is the algebra of reduced p^{th} powers*, Topology 5 (1966) 149–154.

The first serious investigation of the MU-Adams spectral sequence occurs in

S.P. Novikov, *The methods of algebraic topology from the viewpoint of cobordism theories*, Math. USSR Izvestija 1 (1967) 827–913.

This article contains a huge variety of other deep ideas as well.

A good basic reference for formal groups is

A. Frölich, *Formal Groups*, Springer Lecture Notes in Math. 74, 1968.

The notion of p-typicality was introduced in

P. Cartier, *Modules associés à un groupe formel commutatif: Courbes typiques*, C. R. Acad. Sci. Paris, Sér. A 265 (1967) 129–132.

Other references on this material:

S. Araki, *Typical Formal Groups in Complex Cobordism and K-theory*, Lecture Notes in Math., Kyoto Univ., Kinokuniya Book-Store Co., n.d.

M. Lazard, *Commutative Formal Groups*, Springer Lecture Notes in Math. 443, 1975.

The book

D.C. Ravenel, *Complex Cobordism and Stable Homotopy Groups of Spheres*, Academic Press, 1986.

contains a useful appendix on formal groups, as well as a systematic development of Adams spectral sequences. It also contains essentially verbatim accounts of a number of basic papers on the chromatic approach to the BP spectral sequence.

The first paper to use the computational power of the explicit generators for BP_* was

H.R. Miller, D.C. Ravenel, and W.S. Wilson, *Periodic phenomena in the Adams-Novikov spectral sequence*, Annals of Math. 106 (1977) 469–516.

The chromatic spectral sequence is developed there, and used for example to compute the 2-line of the BP E_2-term. An interesting interpretation of the chromatic resolution is described in

D.C. Johnson, P.S. Landweber, and Z. Yosimura, *Injective BP_*BP-comodules and localizations of Brown-Peterson homology*, ???.

The localization theorem is proved in

H.R. Miller and D.C. Ravenel, *Morava stabilizer algebras and the localizaton of Novikov's E_2-term*, Duke Math. J. 44 (1977) 433–447.

Jack Morava has another approach which gives a slightly different theorem, explained along with much further material in

J. Morava, *Noetherian localizations of categories of cobordism comodules*, Ann. of Math. 121 (1985) 1–39.

and again in

E. Devinatz,

The coherence of the complex bordism ring was explored in

L. Smith, *On the finite generation of* $\Omega^U_*(X)$, J. Math. Mech. 18 (1969) 1017–1024.

The invariant prime ideal theorem is in

P.S. Landwber, *Annihilator ideals and primitive elements in complex bordism*, Ill. J. Math. 17 (1973) 273–284.

The Landweber filtration is constructed in

P.S. Landweber, *Associated prime ideals and Hopf algebras*, J. Pure and Appl. Math. 3 (1973) 43–58

and the exactness theorem is proved in

P.S. Landweber, *Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$*, Amer. J. Math. 98 (1976) 591–610.

The theory $B(n)$ with coefficient ring $B(n)_* = \mathbb{F}_p[v_n^{\pm 1}, v_{n+1}, \ldots]$ is introduced and shown to be free over $K(n)$ in

D.C. Johnson and W.S. Wilson, *B operations and Morava's extraordinary K-theories*, Math. Zeit 144 (1975) 55–75.

This is exploited to prove that the dimensions of $K(n)_*(X)$ form a non-decreasing sequence if X is a finite complex in

D.C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. 106 (1984) 351–414.

This paper contains a wealth of other ideas. It introduced the most important examples of Landweber localization functors, and enunciated the "nilpotence" and "telescope" conjectures.

Useful further information is derived in

D.C. Johnson and Z. Yosimura, *Torsion in Brown-Peterson homology and Hurewicz homomorphisms*, Osaka J. Math. 17 (1980) 117–136.

P.S. Landweber, *New applications of commutative algebra to Brown-Peterson homology*, Algebraic Topology, Waterloo 1978, Springer Lect. Notes in Math. 741 (1979) 449–460.

The Nilpotence Theorem is proven in

E. Devinatz, M.J. Hopkins, and J. Smith, *Nilpotence and stable homotopy theory*, Annals of Math. 128 (1988) 207–242.

The Thick Subcategory Theorem and its first consequences are described in

M.J. Hopkins and J. Smith, *Nilpotence and stable homotopy theory II*,

A beautiful summary of this development is given in

M.J. Hopkins, *Global methods in homotopy theory*, Homotopy Theory, Proceedings of the Durham Symposium 1985, Lon. Math. Soc. Lect. Note Series 117 (1987) 73–96.

Ravenel has also written an account of this work in

D.C. Ravenel, *Nilpotence and Periodicity in Stable Homotopy Theory*, Ann. of Math. Studies 128, 1992.

Bibliography

[1] J.F. Adams, *On the Groups $J(X)$*, Topology 5 (1966) 21–71.

[2] J.F. Adams, *Stable Homotopy and Generalised Homology*, Univ. of Chicago Press, 1974.

[3] M.F. Atiyah, *Thom Complexes*, Proc. Lond. Math. Soc. 11 (1961) 291–310; appears in Collected Works, Oxford Univ. Press (1988) V. II, 229–310.

[4] E. Devinatz, M.J. Hopkins, and J. Smith, *Nilpotence and stable homotopy theory*, Annals of Math. 128 (1988) 207–242.

[5] M.W. Hirsch, *Differential Topology*, Graduate Texts in Math. 33, Springer, 1976.

[6] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Grundlerhen der math. Wissenschaften 131, Springer, 1966.

[7] F. Hirzebruch, T. Berger, and R. Jung, *Manifolds and Modular Forms*, Aspects of Mathematics E20, Vieweg, 1992.

[8] M.J. Hopkins, *Global methods in homotopy theory*, Homotopy Theory, Proceedings of the Durham Symposium 1985, Lon. Math. Soc. Lect. Note Series 117 (1987) 73–96.

[9] S. Lang, *Introduction to Differentiable Manifolds*, Interscience, 1966.

[10] S. Mac Lane, *Homology*, Springer-Verlag, 1975.

[11] H.R. Miller, Unpublished lecture notes, spring 1991. Notes written by Matthew Ando.

[12] J.W. Milnor, *Construction of Universal Bundles, II*, Annals of Math. (2) 63 (1956) 430–436.

[13] J.W. Milnor, *Topology from the Differentiable Viewpoint*, University Press of Virginia, 1965.

[14] J.W. Milnor and J.C. Moore, *On the Structure of Hopf Algebras*, Annals of Math. (2) 81 (1965) 211–264.

[15] J.W. Milnor and J.D. Stasheff, *Characteristic Classes*, Annals of Math. Studies 76, Princeton Univ. Press, 1974.

[16] R.E. Mosher and M.C. Tangora, *Cohomology Operations and Applications in Homotopy Theory*, Harper and Row, 1968.

[17] S. Priddy.

[18] Quillen, Advances in Math., ca. 1972.

[19] D.C. Ravenel, *Complex Cobordism and Stable Homotopy Groups of Spheres*, Academic Press, 1986.

[20] D.C. Ravenel, *Nilpotence and Periodicity in Stable Homotopy Theory*, Annals of Math. Studies 128, Princeton Univ. Press, 1992.

[21] G. Segal, *Classifying Spaces and Spectral Sequences*, I.H.E.S. 34 (1968) 105–112.

[22] N.E. Steenrod and D.B.A. Epstein, *Cohomology Operations*, Annals of Math. Studies, Number 50, Princeton Univ. Press, 1962.

[23] R.E. Stong, *Notes on Cobordism Theory*, Math. Notes, Princeton Univ. Press, 1968.

[24] R. Thom, *Quelque propriétés globales des variétiés différentiables*, Comment. Math. Helv. 28 (1954) 17–86.

[25] G.W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics, Springer-Verlag.

[26] W.S. Wilson, *Brown-Peterson Homology, an Introduction and Sampler*, Regional Conf. Series in Math. 48, Amer. Math. Soc., 1980.