Splittings of bicommutative Hopf algebras

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Abstract

We use the theory of Dieudonné modules to show that certain types of short exact sequences of Hopf algebras split. Several examples occur naturally with Morava K-theory.
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MSC: 55Q51; 55N20; 57T05

1. Introduction

Morava K-theory, $K(n)_*(-)$, is a generalized homology theory with coefficient ring $K(n)_* \simeq \mathbb{F}_p[v_n^{\pm 1}]$ where p is a prime and the degree of v_n is $2(p^n - 1)$. The first two authors were led by their study [6] to an interest in the fibration $K(\mathbb{Z}, 3) \to BO\langle 8 \rangle \to BSpin$. The Morava K-theory for $p = 2$ of this was analyzed in [7]. In particular, for $n = 2$, although the first map does not inject, its image is an algebraic copy of $K(2)_*(K(\mathbb{Z}, 3))$ and there is a short exact sequence of Hopf algebras

$$K(2)_*(K(\mathbb{Z}, 3)) \to K(2)_*(BO\langle 8 \rangle) \to K(2)_*(BSpin).$$

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Independently, this short exact sequence was found and studied (in a very different form) in a preliminary version of [1]. The middle term is the one of interest. The right hand term is polynomial and the left term is cofree as a coalgebra. The question of whether this splits as Hopf algebras or not is what led to this paper. It splits. In this special case we investigate the number of splittings which turns out to be uncountable. Furthermore, in [7], we show that there is an algebraic copy of $K(n)_*(K(\mathbb{Z}, 3))$ sitting inside of $K(n)_*(BO\langle 8 \rangle)$ and our main theorem shows that this too splits off as Hopf algebras.

Our main result:

Theorem 1.1. *Let*

$$A \to B \to C$$

be a short exact sequence in the abelian category of bicommutative, biassociative Hopf algebras with an exhaustive primitive filtration. If A is cofree as a coalgebra and C is polynomial (free) as an algebra, then the sequence splits as Hopf algebras.

There are fairly simple examples of such short exact sequences which split both as algebras and coalgebras but not as Hopf algebras. This can be done with all three Hopf algebras polynomial (free) or all three cofree. Such examples, of course, give rise to pairs of Hopf algebras which are isomorphic both as coalgebras and as algebras but not as Hopf algebras. C being polynomial (free) guarantees that B splits as algebras, ([2, Theorem B.9]), and A being cofree guarantees that B splits as coalgebras. As examples show, a Hopf algebra is more than just its coalgebra and algebra structures and so the above splitting as Hopf algebras is just a bit of a surprise.

Our theorem takes place in several categories at once. For the standard graded case, connected Hopf algebras have exhaustive primitive filtrations. Our theorem is also true in the ungraded case. However, all of our examples are in the periodically graded case of Morava homology Hopf algebras studied in [3] and [11]. These Hopf algebras come up naturally from the Morava K-theory of homotopy commutative connected H-spaces. Odd degree generators are exterior and so split off and we tend to consider only the even degree Hopf algebras. By setting $v_n = 1$ we get Hopf algebras over $\mathbb{F}_p$ which are (evenly) graded over $\mathbb{Z}/(p^n - 1)$.

We only prove the Morava homology Hopf algebra version of the theorem. This contains (degree 0) the ungraded case. The proof works for the standard graded case but some of the limit arguments are unnecessary there.

All of our examples are for Morava K-theory. There are several such short exact sequences in the literature and their splitting had remained unresolved until now. One of the most interesting to us is the following. Let $g(q) = 2(p^q + p^{q-1} + \cdots + p + 1)$. In [10, Proposition 1.16, p. 153] it is shown that $K(n)_*(K(\mathbb{Z}, q+2))$ injects (from a map) into $K(n)_*(\underline{BP\langle q \rangle}_{g(q)})$ and that the cokernel is polynomial. The first term is cofree from [9] so our theorem applies and we see that:

Corollary 1.2. *As Hopf algebras, $K(n)_*(K(\mathbb{Z}, q+2))$ splits off of $K(n)_*(\underline{BP\langle q \rangle}_{g(q)})$.*

This includes the $p = 2$ case of $K(Z, 3) \rightarrow BU\langle 6 \rangle$. In [5] it is shown that if you deloop the map in the corollary then the Hopf algebra splits off but they were unable to show this particular case which is essential for the computation of $BP^*(K(\mathbb{Z}, q+2))$ in [10].

In [4], Kashiwabara showed that $K(n)_*(QS^{2q})$ is cofree and injects into $K(n)_*(\underline{BP}_{2q})$ with cokernel polynomial. Thus:

Corollary 1.3. $K(n)_*(QS^{2q})$ *splits off of* $K(n)_*(\underline{BP}_{2q})$ *as Hopf algebras.*

Actually, Kashiwabara does the same for the spectra $T(j)$ between S^0 and BP.

In [5] the Morava K-theories of the spaces in the Omega spectrum for $BP\langle q \rangle$ were computed. There are several descriptions of the results. In one version a fibration is found

$$X_{2r} \rightarrow \underline{BP\langle q \rangle}_{2r} \rightarrow Y_{2r}$$

which gives a short exact sequence in Morava K-theory and where the Morava K-theory of the left term is cofree and the right term is polynomial. Our theorem applies:

Corollary 1.4. *The short exact sequence of* [5, Theorem 2.2 (iii), p. 159] *is split as Hopf algebras.*

Just a few more examples:

Corollary 1.5. *Let* $p = 2$, *the short exact sequences in Morava K-theories coming from the fibrations* $K(\mathbb{Z}, 3) \rightarrow \underline{BO}_4 \rightarrow BSp$, [7, Theorem 2.26(vi)], $Sp/U \rightarrow BU \rightarrow BSp$, [10, p.162], *and* $Sp/SU \rightarrow BSU \rightarrow BSp$, [7, Theorem 2.36], *all split as Hopf algebras.*

2. Morava Dieudonné modules

In the following we let R be the ring $\mathbb{Z}[V, F]/(VF - p)$. Define the category $\mathscr{D}$, of Dieudonné modules to consist of $\mathbb{Z}/(p^n - 1)$-graded groups M_* with an R-module structure where $F : M_t \rightarrow M_{pt}$ (the Frobenius) and $V : M_t \rightarrow M_{t/p}$ (the Verschiebung). The morphisms in $\mathscr{D}$ are graded R-module maps.

Let $\mathscr{MD}$ be the category of Morava Dieudonné modules given by the full subcategory of $\mathscr{D}$ consisting of objects where every element is V-torsion.

We also consider the category $\mathscr{EC}$ of evenly graded Hopf algebras known as Morava Hopf algebras, see above.

Let W be the polynomial ring $\mathbb{Z}_{(p)}[x_1, x_2, x_3, \dots]$. Then W admits the structure of a bicommutative Hopf algebra such that the Witt vectors

$$w_n = x_0^{p^n} + p x_1^{p^{n-1}} + \cdots + p^n x_n$$

are primitive (see [11].) For $s \geqslant 0$, and $t \in \mathbb{Z}/(p^n - 1)$ define the graded sub Hopf algebra of $W \otimes \mathbb{F}_p$

$$W(s,t) = \mathbb{F}_p[x_0, x_1, \ldots, x_s]$$

by setting $|x_i| = p^i 2t$. Notice that $W(s,t)$ belongs to $\mathscr{EC}$. For each algebra A in $\mathscr{EC}$ we define a Morava Dieudonné-module by

$$m_t(A) = \lim_{\rightarrow} \mathrm{Hom}_{\mathscr{EC}}(W(s, t/p^s), A),$$

where the maps are induced by the sequence

$$W(0,t) \leftarrow W(1, t/p) \leftarrow \cdots \leftarrow W(s, t/p^s) \leftarrow \cdots,$$

where the map $W(s, t/p) \rightarrow W(s-1, t)$ takes the primitive generator in degree $2t/p$ to zero. The action of V and F are as described in [11].

Theorem 2.1 (Sadofsky and Wilson [11]). *The functor*

$$m_* : \mathscr{EC} \rightarrow \mathscr{MD}$$

induces an equivalence of categories.

The Morava Dieudonné module for a Morava Hopf algebra may be represented as a picture. The picture below shows the Morava Dieudonné module of $V_t = W(0,t)$. The Frobenius acts injectively as shown, and the Verschiebung is trivial. Each circle represents a copy of $\mathbb{F}_p$.

$$\circ \xrightarrow[V=0]{F} \circ \xrightarrow[V=0]{F} \circ \xrightarrow[V=0]{F} \circ \xrightarrow[V=0]{F} \circ \quad \ldots$$

Now assume that X is a connected, homotopy commutative H-space such that $K(n)_*(X)$ is evenly graded. Then setting $v_n = 1$, we get an object of $\mathscr{EC}$. We shall identify $K(n)_*(X)$ with the corresponding Morava Hopf algebra. Examples of spaces X as described above are given by Eilenberg–MacLane spaces [11]. The Morava Dieudonné modules for some Eilenberg–MacLane spaces are:
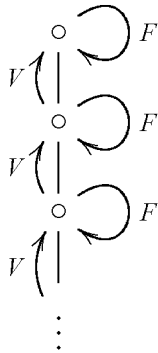
$$m_*(K(n)_* K(\mathbb{Z}/p, n)) = \mathbb{Z}/p$$

is concentrated in degree $(p^n - 1)/(p - 1)$, where V operates trivially and $F(1) = (-1)^{n-1}$. Moreover,

$$m_*(K(n)_* K(\mathbb{Z}, n+1)) = \mathbb{Z}_{p^\infty}$$

concentrated in degree $(p^n - 1)/(p - 1)$, where V operates by multiplication by $(-1)^{n-1} p$ and F by multiplication by $(-1)^{n-1}$. This module may be pictured
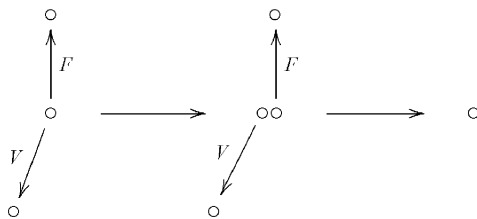
as follows:

A circle represents a copy of $\mathbb{F}_p$ and a vertical line is a non-trivial extension. Hence in the picture the Verschiebung coincides with the multiplication by p map (up to an isomorphism) whereas the Frobenius is an isomorphism.
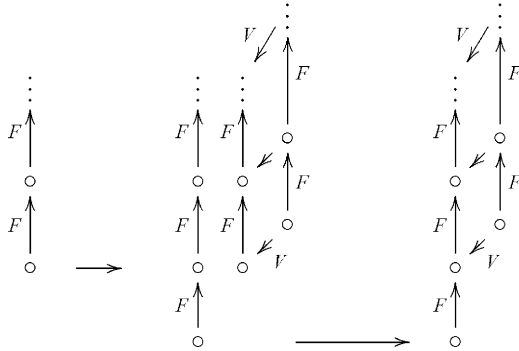
We now give some examples of short exact sequences of Hopf algebras which split both as algebras and as coalgebras but do not split as Hopf algebras. There are three reasons for doing this: (1) to show that we are actually proving something, (2) to show our result is not readily generalizable, and (3) because these types of examples are not much found in the literature and are very easy to lose track of. We will do our examples in the standard graded category. The same examples will do for the other two categories either by forgetting the grading or making it periodic.

Example 2.2 (A finite example). We construct our example as a Dieudonné module. We want a short exact sequence $A \to B \to C$. Let A be three copies of $\mathbb{Z}/(p)$ on generators a, b, and c, of degrees $2k$, $2pk$, and $2p^2k$, respectively, with $V(b) = a$, $V(a) = V(c) = 0$, $F(b) = c$, and $F(c) = F(a) = 0$. Let B be four copies of $\mathbb{Z}/(p)$ on generators d, e, f, and g of degrees $2k$, $2pk$, $2pk$ and $2p^2k$ respectively with $V(e) = d$, $F(f) = g$ and F and V zero on all the other generators. The map $A \to B$ is given by $a \to d$, $b \to e + f$, and $c \to g$. The cokernel is just a $\mathbb{Z}/(p)$ in degree $2pk$ with trivial F and V. It is easy to see that the algebra structure, given by F, splits, and that the coalgebra structure, given by V, splits but that the Hopf algebra structure doesn't split. Pictorially this is

Of course this example can be dualized by changing the direction of all the arrows and interchanging F and V.

Example 2.3 (A polynomial example). We thank A.K. Bousfield for this example. This example is easy to describe in terms of Hopf algebras. We let A be $W(0, pt)$ and B be $W(0, t) \otimes W(1, pt)$. C is the cokernel of the map which takes x_0 to $x_0^p \otimes 1 - 1 \otimes x_0$ (where the three x_0 are in $W(0, pt)$, $W(0, t)$, and $W(1, pt)$, respectively). This splits as coalgebras and algebras but not as Hopf algebras. All of the Hopf algebras are polynomial. This may be easier to see with the Dieudonné modules

where each circle represents a $\mathbb{Z}/(p)$ and the compositions $VF = FV$ give extensions with multiplication by p.

Example 2.4 (A cofree example). To get an example where each Hopf algebra is cofree just dualize the preceding polynomial example by reversing all the arrows and interchanging F and V.

Example 2.5 (Another type of example). All of our examples lead to another kind of example as well. We can take the middle terms of all of the above non-split short exact sequences and compare them to the middle terms of the split short exact sequences with the same ends. This gives examples of Hopf algebras which are isomorphic as coalgebras and as algebras but not as Hopf algebras. We get two examples for finite Hopf algebras, an example where both Hopf algebras are polynomial and an example where both are cofree. A previous theorem with a similar flavor along this line is the result of [8] which states that the Hopf algebra structure of graded bicommutative Hopf algebras over $\mathbb{Z}_{(p)}$ which are polynomial and cofree is uniquely determined.

3. Extensions of Morava Dieudonné modules

Notice that $\mathscr{D}$ is an abelian category, and $\mathscr{MD}$ is an abelian subcategory of $\mathscr{D}$. Hence equivalence classes of extension of length n form a group $\mathrm{Ext}^n_{\mathscr{MD}}(A, B)$ for Morava Dieudonné modules A, B. In $\mathscr{D}$ the extension groups can be computed by projective resolutions of the range or injective resolutions of the target. In $\mathscr{MD}$ projectives are hard to come by but there is a canonical isomorphism

$$\mathrm{Ext}^1_{\mathscr{MD}}(A, B) \cong \mathrm{Ext}^1_{\mathscr{D}}(A, B) \tag{3.1}$$

since all extensions of Morava Dieudonné modules in $\mathscr{D}$ are Morava Dieudonné modules.

Definition 3.1. A Morava Dieudonné module is called cofree if V is surjective. This corresponds to Hopf algebras which are cofree as coalgebras.

Proposition 3.2. *Let V_t denote the polynomial algebra with one primitive generator in degree t and let C be a Morava Dieudonné module. Then we have*

$$\mathrm{Ext}^i_{\mathscr{M}\mathscr{D}}(m(V_t), C) = \begin{cases} C^V_t = ker(V) & for\ i = 0, \\ C_{t/p}/V = coker(V) & for\ i = 1, \end{cases}$$

where C_t is the homogeneous part of C in degree t. In particular, $\mathrm{Ext}^1_{\mathscr{M}\mathscr{D}}(m(V_t), C)$ vanishes for all cofree C.

Proof. As explained in (3.1), it is sufficient to prove the proposition in the category $\mathscr{D}$. Consider the following projective resolution of $m(V_t)$ in $\mathscr{D}$

$$0 \to R\langle x_{t/p}\rangle \xrightarrow{V} R\langle x_t\rangle \to m(V_t) \to 0,$$

where $R\langle x_t\rangle$ denotes the free R-module generated by one element x_t in degree t. The name V is intended to be illustrative, but could be confusing. The *map V* carries the element $x_{t/p}$ to the element $V(x_t)$ where this second V is the Verschiebung. It is easy to see that $R\langle x_t\rangle$ is projective in $\mathscr{D}$. Notice that $\mathrm{Hom}_{\mathscr{D}}(R\langle x_t\rangle, C) = C_t$, where a morphism φ is identified with $\varphi(x_t)$. Therefore, on mapping out of the above resolution into C, we get the cochain complex that computes the required Ext groups

$$0 \leftarrow C_{t/p} \xleftarrow{V} C_t \leftarrow 0. \qquad \square$$

Theorem 3.3. *Let C be cofree in $\mathscr{M}\mathscr{D}$, and let D be a polynomial algebra in $\mathscr{E}\mathscr{C}$. Then the group $\mathrm{Ext}^1_{\mathscr{M}\mathscr{D}}(m(D), C)$ is trivial.*

Proof. As before, it is sufficient to prove this in the category $\mathscr{D}$. Consider the following exhaustive filtration of $m(D)$ given by the Morava Dieudonné submodules

$$F^i = ker(V^i : m(D) \to m(D)).$$

We claim that the associated quotients Q^{i+1} are sums of Morava Dieudonné modules of type $m(V_t)$. To see this, observe that Q^{i+1} is isomorphic to the image of the map

$$V^i : F^{i+1} \to m(D)$$

and as such is isomorphic to a submodule of the kernel of V on $m(D)$. Since D is polynomial, this image corresponds to $m(U)$, where U is polynomial [2, Theorem B.7]. Since V acts trivially on $m(U)$, it follows that this is a sum of copies of $m(V_t)$. By induction and Proposition 3.2, we see with that

$$\mathrm{Ext}^1_{\mathscr{D}}(F^i, C) = 0.$$

Now consider the short exact sequence

$$0 \longrightarrow \bigoplus_i F^i \xrightarrow{\ 1-\sigma\ } \bigoplus_i F^i \to m(D) \to 0,$$

where σ is the inclusion map $\sigma : F^i \subseteq F^{i+1}$. This yields a long exact sequence for Ext groups, which splits, by the definitions of *limit* and $\lim^1$, into short exact sequences:

$$0 \to \lim{}^1 \operatorname{Ext}_{\mathscr{D}}^{n-1}(F^i, C) \to \operatorname{Ext}_{\mathscr{D}}^n(m(D), C) \to \lim \operatorname{Ext}_{\mathscr{D}}^n(F^i, C) \to 0.$$

For $n = 1$, the term on the right vanishes by what we have shown above. Hence, to prove the theorem, we need to show that the $\lim^1 \operatorname{Hom}_{\mathscr{D}}(F^i, C) = 0$. This follows if we can verify the Mittag–Leffler condition that requires the natural map

$$\operatorname{Hom}_{\mathscr{D}}(F^{i+1}, C) \to \operatorname{Hom}_{\mathscr{D}}(F^i, C)$$

to be surjective. The cokernel of the above map is $\operatorname{Ext}_{\mathscr{D}}^1(Q^{i+1}, C)$, which we know to be trivial, and therefore the proof is complete. $\quad\square$

4. The torsor of splittings

Notice that the space of splittings of a split exact sequence in any abelian category:

$$A \to B \to C$$

is a torsor over the group $\operatorname{Hom}(C, A)$. In this section, we give a concrete description of this group in the case when $A = m_*(K(n)_* K(\mathbb{Z}, n+1))$ in the category $\mathscr{MD}$.

Let C, D be any objects of $\mathscr{MD}$. Recall that we have an isomorphism of groups

$$\operatorname{Hom}_{\mathscr{MD}}(C, D) = \operatorname{Hom}_{\mathscr{D}}(C, D),$$

where $\mathscr{D}$ is the category of $\mathbb{Z}/(p^n - 1)$-graded modules M_*, over the ring $R = \mathbb{Z}[V, F]/(VF - p)$. The action of R changes the grading by $F : M_t \to M_{pt}$ (the Frobenius) and $V : M_t \to M_{t/p}$ (the Verschiebung). The morphisms in $\mathscr{D}$ are graded R-module maps.

Theorem 4.1. *Let K denote the object $m_*(K(n)_* K(\mathbb{Z}, n+1))$, and let C be any object of $\mathscr{MD}$. Then there is an isomorphism of groups*

$$\operatorname{Hom}_{\mathscr{MD}}(C, K) = \operatorname{Hom}_{\mathbb{Z}}(\bar{C}_t, \mathbb{Z}_{p^\infty}),$$

where t denotes the degree $(p^n - 1)/(p - 1)$, and $\bar{C}_t = C_t \otimes_R \mathbb{Z}$. Here $\mathbb{Z}$ is seen as an ungraded R-module, with F acting as $(-1)^{n-1}$ and V acting as $(-1)^{n-1} p$.

Proof. We begin by recalling that

$$K = m_*(K(n)_* K(\mathbb{Z}, n+1)) = \mathbb{Z}_{p^\infty}$$

concentrated in degree $t = (p^n - 1)/(p - 1)$, where V operates by multiplication by $(-1)^{n-1} p$ and F by multiplication by $(-1)^{n-1}$. We may therefore change base to get

the required isomorphism:

$$\text{Hom}_{\mathcal{MD}}(C, K) = \text{Hom}_R(C, K) = \text{Hom}_{\mathbb{Z}}(\bar{C}_t, \mathbb{Z}_{p^\infty}). \qquad \square$$

Example 4.2. Recall the Morava Hopf algebra $W(\infty, s)$:

$$W(\infty, s) = \mathbb{F}_p[x_0, x_1, x_2, \ldots], \qquad |x_i| = p^i 2s.$$

Let $M(s) = m_*(W(\infty, s))$. It is easy to see that

$$\bar{M}(s)_t = \begin{cases} \mathbb{Z}_{p^\infty}, & s = t, \\ 0, & s \neq t, \end{cases}$$

where $t = (p^n - 1)/(p - 1)$. It follows from the above theorem that

$$\text{Hom}_{\mathcal{MD}}(M(s), K) = \begin{cases} \mathbb{Z}_p, & s = t, \\ 0, & s \neq t. \end{cases}$$

Example 4.3. Now it is well known that we have an isomorphism of Morava Hopf algebras:

$$K(n)_*(BU) = \bigotimes_{(s,p)=1} \mathbb{F}_p[v_n^{\pm 1}] \otimes_{\mathbb{F}_p} W(\infty, s).$$

It follows from the previous example that

$$\text{Hom}_{\mathcal{MD}}(m_*(K(n)_*(BU)), K) = \prod_{i=1}^{\infty} \mathbb{Z}_p.$$

The above example may be used to compute yet another interesting case. Let $\mathbb{Z}/2$ act on the Morava Hopf algebra $K(n)_*(BU)$ via the induced action of $\mathbb{Z}/2$ on BU by complex conjugation. Let Q denote the coinvariants of this action in the category of Morava Hopf algebras. It follows that:

$$\text{Hom}_{\mathcal{MD}}(m_*(Q), K) = \text{Hom}_{\mathcal{MD}}(m_*(K(n)_*(BU)), K)^{\mathbb{Z}/2} = \left(\prod_{i=1}^{\infty} \mathbb{Z}_p\right)^{\mathbb{Z}/2}. \qquad (4.1)$$

If the prime p under consideration is $p = 2$, then the algebra structure of the Morava Hopf algebra Q was calculated in [6] (Propositions 2.5, 2.7). In particular, it was shown that Q was a free algebra on generators a_i in degree $4i$.

Claim 4.4. *For $p=2$, the group $\text{Hom}_{\mathcal{MD}}(m_*(Q), K)$ is a nontrivial 2-complete, torsion free $\mathbb{Z}_2$-module.*

Proof. Using the usual technique, consider the following exhaustive filtration of $m(Q)$ given by the Morava Dieudonné submodules

$$F^i = ker(V^i : m(Q) \to m(Q)).$$

As before, we may observe that the associated quotients are sums of Morava Dieudonné modules of type $m(V_s)$, where V_s is a primitively generated polynomial algebra on an element in degree s. We may now use Theorem 3.3 to show that

$$\mathrm{Hom}_{\mathscr{MD}}(m_*(Q), K) = \lim \mathrm{Hom}_{\mathscr{MD}}(F^i, K)$$

and that each map in the inverse system is an epimorphism. Since the degrees of the polynomial generators of Q are multiples of 4, it follows that some $\mathrm{Hom}_{\mathscr{MD}}(F^i, K)$ is nontrivial. The claim now follows from Eq. (4.1). □

Recall that in [6] it was also shown that $K(2)_* BSpin$ is isomorphic to Q. Hence we have

Corollary 4.5. *The group* $\mathrm{Hom}_{\mathscr{MD}}(m_*(K(2)_* BSpin), K)$ *is a nontrivial* 2-*complete, torsion free* $\mathbb{Z}_2$-*module.*

This shows in particular that the space of splittings for the following short exact sequence is uncountable:

$$K(2)_*(K(\mathbb{Z}, 3)) \to K(2)_*(BO\langle 8 \rangle) \to K(2)_*(BSpin).$$

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