

THE L-HOMOLOGY FUNDAMENTAL CLASS FOR IP-SPACES AND THE STRATIFIED NOVIKOV CONJECTURE

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ABSTRACT. An IP-space is a pseudomanifold whose defining local properties imply that its middle perversity global intersection homology groups satisfy Poincaré duality integrally. We show that the symmetric signature induces a map of Quinn spectra from IP bordism to the symmetric L -spectrum of \mathbb{Z} , which is, up to weak equivalence, an E_∞ ring map. Using this map, we construct a fundamental L -homology class for IP-spaces, and as a consequence we prove the stratified Novikov conjecture for IP-spaces.

1. INTRODUCTION

An intersection homology Poincaré space, or IP-space, is a piecewise linear pseudomanifold such that the middle dimensional, lower middle perversity integral intersection homology of even-dimensional links vanishes and the lower middle dimensional, lower middle perversity intersection homology of odd-dimensional links is torsion free. This class of spaces was introduced by Goresky and Siegel in [GS83] as a natural solution, assuming the IP-space to be compact and oriented, to the question: For which class of spaces does intersection homology (with middle perversity) satisfy Poincaré duality over the integers?

If X is a compact oriented IP-space whose dimension n is a multiple of 4, then the signature $\sigma(X)$ of X is the signature of the intersection form

$$IH_{n/2}(X; \mathbb{Z}) / \text{Tors} \times IH_{n/2}(X; \mathbb{Z}) / \text{Tors} \longrightarrow \mathbb{Z},$$

where IH_* denotes intersection homology with the lower middle perversity, [GM80], [GM83]. This signature is a bordism invariant for bordisms of IP-spaces. The IP-space bordism groups have been investigated by Pardon in [Par90], where it is shown that the signature (when $n = 4k$) together with

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the de Rham invariant (when $n = 4k + 1$) form a complete system of invariants.

Next we recall the theory of the L -homology fundamental class for manifolds. Let M^n be a closed oriented n -dimensional manifold. The symmetric signature $\sigma^*(M)$ of [M71],[Ran92] is an element of the symmetric L -group $L^n(\mathbb{Z}[G])$, where $\mathbb{Z}[G]$ is the integral group ring of the fundamental group $G = \pi_1(M)$ of M . It is a non-simply-connected generalization of the signature $\sigma(M)$, since for $n = 4k$ the canonical homomorphism $L^n(\mathbb{Z}[G]) \rightarrow L^n(\mathbb{Z}) = \mathbb{Z}$ maps $\sigma^*(M)$ to $\sigma(M)$. Moreover, σ^* is homotopy invariant and bordism invariant for bordisms over the classifying space BG . Let $\mathbb{L}^\bullet = \mathbb{L}^\bullet\langle 0 \rangle(\mathbb{Z})$ denote the symmetric L -spectrum with homotopy groups $\pi_n(\mathbb{L}^\bullet) = L^n(\mathbb{Z})$ and let $L_n^\bullet(-)$ denote the homology theory determined by \mathbb{L}^\bullet . For an n -dimensional Poincaré space M which is either a topological manifold or a combinatorial homology manifold (i.e. a polyhedron whose links of simplices are homology spheres), Ranicki defines a canonical L^\bullet -homology fundamental class $[M]_{\mathbb{L}} \in L_n^\bullet(M)$, see [Ran92]. Its image under the assembly map

$$L_n^\bullet(M) \xrightarrow{\alpha} L^n(\mathbb{Z}[G])$$

is the symmetric signature $\sigma^*(M)$. The class $[M]_{\mathbb{L}}$ is a topological invariant, but, unlike the symmetric signature, not a homotopy invariant in general. The geometric meaning of the L^\bullet -homology fundamental class is that its existence for a geometric Poincaré complex X^n , $n \geq 5$, assembling to the symmetric signature (which in fact any Poincaré complex possesses), implies up to 2-torsion that X is homotopy equivalent to a compact topological manifold. (More precisely, X is homotopy equivalent to a compact manifold if it has an L^\bullet -homology fundamental class, which assembles to the so-called visible symmetric signature of X .) Smooth manifolds M possess a Hirzebruch L -class in $H^*(M; \mathbb{Q})$, whose Poincaré dual we denote by $L(M) \in H_*(M; \mathbb{Q})$. Rationally, $[M]_{\mathbb{L}}$ is then given by $L(M)$,

$$[M]_{\mathbb{L}} \otimes 1 = L(M) \in L_n^\bullet(M) \otimes \mathbb{Q} \cong \bigoplus_{j \geq 0} H_{n-4j}(M; \mathbb{Q}).$$

Thus, we may view $[M]_{\mathbb{L}}$ as an integral refinement of the L -class of M . The identity $\alpha[M]_{\mathbb{L}} = \sigma^*(M)$ may then be interpreted as a non-simply connected generalization of the Hirzebruch signature formula. These facts show that the L^\bullet -homology fundamental class is much more powerful than $\sigma^*(M)$. For example, there exist infinitely many manifolds $M_i, i = 1, 2, \dots$, in the homotopy type of $S^2 \times S^4$, distinguished by the first Pontrjagin class of their tangent bundle $p_1(TM_i) \in H^4(S^2 \times S^4) \cong \mathbb{Z}$, namely $p_1(TM_i) = Ki$, K a fixed nonzero integer. On the other hand, $\sigma^*(M_i) = \sigma^*(S^2 \times S^4) = 0 \in L^6(\mathbb{Z}[\pi_1(S^2 \times S^4)]) = L^6(\mathbb{Z}) = 0$.

We return to singular spaces. A Witt space is a piecewise linear pseudo-manifold such that the middle dimensional, lower middle perversity rational

intersection homology of even-dimensional links vanishes, [Sie83]. The symmetric signature $\sigma^*(X) \in L^n(\mathbb{Q}[G])$ and the L^\bullet -homology fundamental class $[X]_{\mathbb{L}} \in (L^\bullet(\mathbb{Q}))_n(X)$ of an oriented Witt space X^n appeared first in the work of Cappell, Shaneson and Weinberger, see [CSW91] and [W94], though a detailed construction is not provided there.

In [Ban11], the first author outlined a construction of $[X]_{\mathbb{L}}$ for IP-spaces X based on ideas of Eppelmann [Epp07], and pointed out that the existence of this class implies in particular a definition of a symmetric signature $\sigma^*(X)$ as the image of $[X]_{\mathbb{L}}$ under assembly. In [ALMP12], it is shown that this symmetric signature, adapted to Witt spaces and pushed into $K_*(C_r^*G)$ via

$$L^*(\mathbb{Q}[G]) \longrightarrow L^*(C_r^*G) \longrightarrow K_*(C_r^*G),$$

agrees rationally with the Albin-Leichtnam-Mazzeo-Piazza signature index class. The first fully detailed construction of $\sigma^*(X)$ for Witt spaces X has been provided in [FM13b]. That construction is closely parallel to the original construction of Miščenko, but using singular intersection chains on the universal cover instead of ordinary chains. The methods of [FM13b] carry over to IP-spaces and yield a symmetric signature over \mathbb{Z} for such spaces, as we show in Section 7.

In the present paper, we give the first detailed construction of an L^\bullet -homology fundamental class $[X]_{\mathbb{L}} \in L_n^\bullet(X)$ for IP-spaces X . While Eppelmann used complexes of sheaves, we are able to use the, for our purposes, more precise and geometric methods of [FM13b]. The main issue is to construct a map (at least in the derived category) on the spectrum level from IP bordism to L^\bullet , for then $[X]_{\mathbb{L}}$ can readily be defined as the image of the identity map $[\text{id}_X] \in (\Omega_{\text{IP}})_n(X)$ under $(\Omega_{\text{IP}})_n(X) \rightarrow L_n^\bullet(X)$, see Definition 8.5. To obtain this map of spectra, we rely heavily on the technology of ad theories and their associated Quinn spectra as developed by the second and third author in [LM13], [LM]. Roughly, we construct first an ad theory of IP spaces, which automatically gives an associated Quinn spectrum \mathbf{Q}_{IP} , whose homotopy groups are Pardon's IP bordism groups. Using the symmetric signature, we define a morphism of ad theories from the IP ad theory to the ad theory of symmetric algebraic Poincaré complexes over \mathbb{Z} . The spectrum of the latter ad theory is the symmetric L -spectrum L^\bullet . The morphism of ad theories then induces the desired map of spectra. We prove that our L^\bullet -homology fundamental class has all the expected properties (Theorem 8.2): It is an oriented PL homeomorphism invariant, its image under assembly is the symmetric signature and it agrees with Ranicki's L^\bullet -homology fundamental class when X is a PL manifold.

As an application of our L^\bullet -homology fundamental class, we discuss the stratified homotopy invariance of the higher signatures of IP-spaces. Let X be an n -dimensional compact oriented IP-space, whose fundamental group

$G = \pi_1(X)$ satisfies the strong Novikov conjecture, that is, the assembly map

$$L_n^\bullet(BG) \longrightarrow L^n(\mathbb{Z}[G])$$

is rationally injective. Then we prove that the stratified Novikov conjecture holds for X , i.e. the higher signatures

$$\langle a, r_*L(X) \rangle, \quad a \in H^*(BG; \mathbb{Q}),$$

where $r : X \rightarrow BG$ is a classifying map for the universal cover of X and $L(X) \in H_*(X; \mathbb{Q})$ is the Goresky-MacPherson L -class of X , are stratified homotopy invariants, see Theorem 9.2. The stratified Novikov conjecture has been treated from the analytic viewpoint in [ALMP13].

Here is an outline of the paper. Sections 2 and 3 review the basic facts about IP-spaces and ad theories. Section 4 constructs an ad theory associated to IP-spaces. Section 5 reviews two (equivalent) ad theories associated to symmetric Poincaré complexes. Section 6 uses the symmetric signature (ignoring the fundamental group) to construct a map Sig of Quinn spectra from IP bordism to the symmetric L -spectrum of \mathbb{Z} . Section 7 constructs the symmetric signature of an IP-space X as an element of $L^n(\mathbb{Z}[\pi_1 X])$. Section 8 constructs the L^\bullet -theory fundamental class of an IP-space and shows that it assembles to the symmetric signature constructed in Section 7. Section 9 uses the results of Section 8 to prove the stratified Novikov conjecture for IP-spaces. Section 10 shows that the map Sig constructed in Section 6 is, up to weak equivalence, an E_∞ ring map; this is applied in Section 11 to prove that $[X \times Y]_{\mathbb{L}} = [X]_{\mathbb{L}} \times [Y]_{\mathbb{L}}$. Section 12 proves a result needed for Sections 8–11, namely the fact that the assembly map for IP bordism is a weak equivalence. There are four appendices. Appendix A reviews the basic facts about the intrinsic filtration of a PL space, and Appendix B gives generalizations of some technical results from [FM13b] which are needed in Section 7. Appendix C proves a multiplicative property of the assembly map which may be of independent interest. Appendix D corrects some signs in [LM13].

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2. REVIEW OF IP BORDISM

We use the term *polyhedron* as defined in [RS72, Definition 1.1].

Definition 2.1. An n -dimensional PL pseudomanifold is a polyhedron X for which some (and hence every) triangulation has the following properties.

- (a) Every simplex is contained in an n -simplex.
- (b) Every $(n - 1)$ -simplex is a face of exactly two n -simplices.

Definition 2.2. An n -dimensional PL ∂ -pseudomanifold is a polyhedron X with the property that some (and hence every) triangulation has the following properties.

- (a) Every simplex is contained in an n -simplex.
- (b) Every $(n-1)$ -simplex is a face of either one or two n -simplices; the union of the $(n-1)$ -simplices which are faces of one n -simplex is called the *boundary* of X and denoted ∂X .
- (c) The boundary ∂X is an $(n-1)$ -dimensional pseudomanifold.
- (d) The boundary is collared, that is, there is a PL embedding $\partial X \times [0, 1) \rightarrow X$ with open image which is the identity on ∂X .

Remark 2.3. (i) The subspace ∂X is independent of the triangulation.
(ii) The collaring condition is needed in order for Lefschetz duality to hold in intersection homology (see [FM13a, Section 7.3]).

Definition 2.4. An *orientation* of an n -dimensional PL pseudomanifold or PL ∂ -pseudomanifold is a set of orientations of the n -simplices of some triangulation such that the sum of the n -simplices with these orientations is a cycle (a relative cycle in the case of a ∂ -pseudomanifold).

For some purposes we need a stratification. For a polyhedron Y , let $c^\circ Y$ denote the open cone $([0, 1) \times Y)/(0 \times Y)$. We recall the inductive definition of stratified pseudomanifold:

Definition 2.5. A 0 -dimensional stratified PL pseudomanifold X is a discrete set of points with the trivial filtration $X = X^0 \supseteq X^{-1} = \emptyset$. An n -dimensional stratified PL pseudomanifold X is a polyhedron together with a filtration by closed polyhedra

$$X = X^n \supseteq X^{n-1} = X^{n-2} \supseteq \dots \supseteq X^0 \supseteq X^{-1} = \emptyset$$

such that

- (a) $X - X^{n-1}$ is dense in X , and
- (b) for each point $x \in X^i - X^{i-1}$, there exists a neighborhood U of x for which there is a compact $n-i-1$ dimensional stratified PL pseudomanifold L and a PL homeomorphism

$$\phi : \mathbb{R}^i \times c^\circ L \rightarrow U$$

that takes $\mathbb{R}^i \times c^\circ(L^{j-1})$ onto $X^{i+j} \cap U$.

The space L in part (b) is determined up to PL homeomorphism by x and the stratification ([F, Lemma 2.56]); it is called the *link* of X at x and denoted L_x . A PL pseudomanifold always possesses a stratification in the sense of Definition 2.5, by Proposition A.1(iv).

Definition 2.6. An n -dimensional stratified PL ∂ -pseudomanifold is a PL ∂ -pseudomanifold X together with a filtration by closed polyhedra such that

- (a) $X - \partial X$, with the induced filtration, is an n -dimensional stratified PL pseudomanifold,

(b) ∂X , with the induced filtration, is an $n - 1$ dimensional stratified PL pseudomanifold, and

(c) there is a neighborhood N of ∂X with a homeomorphism of filtered spaces $N \rightarrow \partial X \times [0, 1)$ (where $[0, 1)$ is given the trivial filtration) which is the identity on ∂X .

A PL ∂ -pseudomanifold always possesses a stratification in the sense of Definition 2.6, by Proposition A.2. Next recall the definition of intersection homology ([GM80], [GM83], [Bo84], [KW06], [Ban07]). We will denote the lower middle perversity, as usual, by \bar{m} .

Definition 2.7. ([GS83, Par90]) An n -dimensional *IP-space* is an n -dimensional PL pseudomanifold X for which some stratification has the following properties.

- (a) $IH_l^{\bar{m}}(L_x; \mathbb{Z}) = 0$ for all $x \in X^{n-2l-1} - X^{n-2l-2}$, and
- (b) $IH_{l-1}^{\bar{m}}(L_x; \mathbb{Z})$ is torsion free for all $x \in X^{n-2l} - X^{n-2l-1}$.

Remark 2.8. (i) IP stands for “intersection homology Poincaré”.

(ii) Note that the stratification is not considered as part of the structure of an IP-space.

(iii) If conditions (a) and (b) hold for some stratification then they hold for every stratification (by the Proposition in [GM83, Section 2.4]).

Definition 2.9. ([Par90]) An n -dimensional ∂ -*IP-space* is an n -dimensional PL ∂ -pseudomanifold X for which $X - \partial X$ is an IP-space.

Proposition 2.10. *If X is a ∂ -IP-space then ∂X is an IP-space.*

Proof. Give X the stratification of Proposition A.2. By Remark 2.8(iii), the restriction of this stratification to $X - \partial X$ has properties (a) and (b) of Definition 2.7. Give ∂X the stratification of Proposition A.1(iv). Part (c) of Definition 2.6 implies that the links of ∂X are also links of $X - \partial X$, so ∂X satisfies Definition 2.7. \square

Next we consider IP bordism groups. There are two ways to define them:

- (1) The objects and bordisms are the compact oriented IP-spaces and ∂ -IP-spaces.
- (2) An object is a compact oriented IP-space with a given stratification, and similarly for the bordisms.

Pardon [Par90] does not make it clear which definition he is using, but fortunately the two definitions give the same bordism groups by [Fa]. We will use the first definition.

3. REVIEW OF ad THEORIES

We recall some definitions from [LM13, Sections 2 and 3]. For a ball complex K (that is, a CW complex with a compatible PL structure [LM13, Definition 2.1]) and a subcomplex L we define $\mathcal{C}ell(K, L)$ to be the category in which the objects are the oriented closed cells of K which are not in L ,

together with an empty cell \emptyset_n for each dimension n , and the non-identity morphisms are given by inclusions of cells (with no requirement on the orientations). The category $\mathcal{C}ell(K, L)$ is \mathbb{Z} -graded ([LM13, Definition 3.3]), that is, it comes with an involution i (which reverses the orientation), a dimension functor d into the poset \mathbb{Z} and a section functor given by \emptyset_n .

Given a \mathbb{Z} -graded category \mathcal{A} , a *pre* (K, L) -*ad of degree* k is a functor $\mathcal{C}ell(K, L) \rightarrow \mathcal{A}$ which decreases dimensions by k . The set of these is denoted $\text{pre}^k(K, L)$. An *ad theory* with values in \mathcal{A} (called the *target category* of the ad theory) consists of a subset $\text{ad}^k(K, L) \subset \text{pre}^k(K, L)$ for each (K, L) and each k , satisfying certain axioms ([LM13, Definition 3.10]). One of the axioms says that an element of $\text{pre}^k(K, L)$ is in $\text{ad}^k(K, L)$ if and only if its image in $\text{pre}^k(K)$ is in $\text{ad}^k(K)$, so to describe an ad theory it suffices to specify the sets $\text{ad}^k(K)$.

An ad theory gives rise to bordism groups Ω_* , a cohomology theory T^* with $T^k(\text{pt}) = \Omega_{-k}$, a spectrum \mathbf{Q} ([Q95], [LM13, Section 15]), and to a weakly equivalent symmetric spectrum \mathbf{M} ([LM13, Section 17]) such that the cohomology theory represented by \mathbf{Q} is naturally isomorphic to T^* . A *morphism* of ad theories is a functor of target categories which takes ads to ads. A morphism $\text{ad}_1 \rightarrow \text{ad}_2$ of ad theories induces a map $\mathbf{Q}_1 \rightarrow \mathbf{Q}_2$ of associated Quinn spectra.

3.1. The ad theory of oriented topological manifolds. As motivation for the ad theory of IP-spaces, we briefly recall the ad theory ad_{STop} ([LM13, Example 3.5 and Section 6]; also see [LM, Section 2]). The target category $\mathcal{A}_{\text{STop}}$ has as objects the compact oriented topological manifolds with boundary. The morphisms between objects of the same dimension are the orientation-preserving homeomorphisms, and the other morphisms are the inclusions with image in the boundary.

To describe the set $\text{ad}_{\text{STop}}^k(K)$ we need to recall two definitions from [LM13, Section 5]. A \mathbb{Z} -graded category \mathcal{A} is called *balanced* if it comes with a natural involutive bijection

$$\eta : \mathcal{A}(A, B) \rightarrow \mathcal{A}(A, i(B))$$

which commutes with the involution i ; examples are $\mathcal{C}ell(K)$ and $\mathcal{A}_{\text{STop}}$. Functors between balanced categories are called *balanced* if they commute with η .

Let $\mathcal{C}ell^b(K)$ be the category whose objects are the (unoriented) cells of K and whose morphisms are the inclusions. Let $\mathcal{A}_{\text{STop}}^b$ be the category whose objects are compact orientable topological manifolds, whose morphisms between objects of the same dimension are homeomorphisms, and whose other morphisms are the inclusions with image in the boundary. A balanced functor

$$F : \mathcal{C}ell(K) \rightarrow \mathcal{A}_{\text{STop}}$$

induces a functor

$$F^b : \mathcal{C}ell^b(K) \longrightarrow \mathcal{A}_{\text{STop}}^b.$$

We define $\text{ad}_{\text{STop}}^k(K) \subset \text{pre}_{\text{STop}}^k(K)$ to be the set of functors F with the following properties.

- (a) F is balanced.
- (b) If (σ', o') and (σ, o) are oriented cells with $\dim \sigma' = \dim \sigma - 1$, and if the incidence number $[o, o']$ is equal to $(-1)^k$, then the map

$$F(\sigma', o') \rightarrow \partial F(\sigma, o)$$

is orientation preserving.

- (c) For each σ , $\partial F^b(\sigma)$ is the colimit in Top of $F^b|_{\text{Cell}^b(\partial\sigma)}$.

It is shown in [LM13, Appendix B] and [LM, Appendix A] that the spectrum \mathbf{Q}_{STop} (resp., the symmetric spectrum \mathbf{M}_{STop}) obtained from this ad theory is weakly equivalent to the usual Thom spectrum MSTop (resp., as a symmetric spectrum).

4. THE ad THEORY OF IP-SPACES

Recall Proposition A.2.

Definition 4.1. Let X and X' be PL ∂ -pseudomanifolds of dimensions n, n' . A *strong embedding* $f : X \rightarrow X'$ is a PL embedding for which $X[n - i] = f^{-1}((X')[n' - i])$ for $0 \leq i \leq n$.

Let \mathcal{A}_{IP} be the \mathbb{Z} -graded category whose objects are compact oriented ∂ -IP-spaces (with an empty space of dimension n for each n), whose morphisms between objects of the same dimension are the orientation-preserving PL homeomorphisms and whose other morphisms are the strong embeddings with image in the boundary. The involution i reverses the orientation. Then \mathcal{A}_{IP} is a balanced \mathbb{Z} -graded category. (The requirement that the morphisms between objects of different dimensions are strong embeddings will not actually be used until the proof of Lemma 6.5(ii)). Before defining $\text{ad}_{\text{IP}}^k(K)$, we need a fact about PL topology which will be proved at the end of this section.

Lemma 4.2. *Let \mathcal{P} be the category whose objects are compact polyhedra and whose morphisms are PL embeddings. Let K be a ball complex and $G : \text{Cell}^b(K) \rightarrow \mathcal{P}$ a covariant functor such that, for every σ , the map*

$$\text{colim}_{\tau \in \partial\sigma} G(\tau) \rightarrow G(\sigma)$$

is a monomorphism. Then for every subcomplex L of K

- (i) *the space $\text{colim}_{\sigma \in L} G(\sigma)$ has a PL structure for which the maps $G(\sigma) \rightarrow \text{colim}_{\sigma \in L} G(\sigma)$ for $\sigma \in L$ are PL embeddings, and*
- (ii) *the map*

$$\text{colim}_{\sigma \in L} G(\sigma) \rightarrow \text{colim}_{\sigma \in K} G(\sigma)$$

is a PL embedding.

Now let $\mathcal{A}_{\text{IP}}^{\flat}$ be the category whose objects are compact orientable IP-spaces with boundary, whose morphisms between objects of the same dimension are PL homeomorphisms, and whose other morphisms are the strong embeddings with image in the boundary. A balanced functor

$$F : \mathcal{C}ell(K) \longrightarrow \mathcal{A}_{\text{IP}}$$

induces a functor

$$F^{\flat} : \mathcal{C}ell^{\flat}(K) \longrightarrow \mathcal{A}_{\text{IP}}^{\flat}.$$

Definition 4.3. Let K be a ball complex. Define $\text{ad}_{\text{IP}}^k(K) \subset \text{pre}_{\text{IP}}^k(K)$ to be the set of functors F with the following properties:

- (a) F is balanced.
- (b) If (σ', o') and (σ, o) are oriented cells with $\dim \sigma' = \dim \sigma - 1$, and if the incidence number $[o, o']$ is equal to $(-1)^k$, then the map

$$F(\sigma', o') \longrightarrow \partial F(\sigma, o)$$

is orientation preserving.

- (c) For each $\sigma \in K$, the map

$$\text{colim}_{\tau \in \partial \sigma} F^{\flat}(\tau) \rightarrow \partial F^{\flat}(\sigma)$$

is a bijection.

Theorem 4.4. ad_{IP} is an ad theory.

Remark 4.5. The Cartesian product of ∂ -IP-spaces is a ∂ -IP-space, and the product of an element of $\text{ad}_{\text{IP}}^k(K)$ with an element of $\text{ad}_{\text{IP}}^l(L)$ is an element of $\text{ad}_{\text{IP}}^{k+l}(K \times L)$. Thus ad_{IP} is a multiplicative ad theory ([LM13, Definition 18.4]) and the associated symmetric spectrum \mathbf{M}_{IP} is a symmetric ring spectrum ([LM13, Theorem 18.5]).

Moreover, ad_{IP} is a commutative ad theory ([LM, Definition 3.3]), so by Theorem 1.1 of [LM] there is a commutative symmetric ring spectrum $\mathbf{M}_{\text{IP}}^{\text{comm}}$ which is weakly equivalent as a symmetric ring spectrum to \mathbf{M}_{IP} . Specifically, there is a symmetric ring spectrum \mathbf{A} and ring maps

$$\mathbf{M}_{\text{IP}} \leftarrow \mathbf{A} \rightarrow \mathbf{M}_{\text{IP}}^{\text{comm}}$$

which are weak equivalences.

Proof of Theorem 4.4. The only parts of [LM13, Definition 3.10] which are not obvious are (f) (the gluing axiom) and (g) (the cylinder axiom).

For part (g), let F be a K -ad; we need to define $J(F) : \mathcal{C}ell(K \times I) \rightarrow \mathcal{A}_{\text{IP}}$. First note that the statement of part (g) specifies what $J(F)$ has to be on the subcategories $\mathcal{C}ell(K \times 0)$ and $\mathcal{C}ell(K \times 1)$. The remaining objects have the form $(\sigma \times I, o \times o')$ and we define $J(F)$ for such an object to be $F(\sigma, o) \times (I, o')$, where (I, o') denotes the PL ∂ -manifold I with orientation o' . $F(\sigma, o) \times I$ is a ∂ -IP-space because the link at a point (x, t) is the link in $F(\sigma, o)$ at x . The inclusions of $F(\sigma, o) \times \{0\}$ and $F(\sigma, o) \times \{1\}$ in $F(\sigma, o) \times I$ are strong embeddings (see Definition 4.1) by the definition of the stratification in Proposition A.2.

For part (f), let K be a ball complex and K' a subdivision of K . Let F be a K' -ad. We need to show that there is a K -ad E which agrees with F on each residual subcomplex of K . We may assume inductively that $|K|$ is a PL n -ball, that K has exactly one n -cell, and that K' is a subdivision of K which agrees with K on the boundary of $|K|$. Let L be the subcomplex of K' consisting of cells in the boundary of $|K|$. The proof of [Wh78, II.6.2] shows that K' has the following properties (where “cell” means closed cell):

- (1) Every cell is contained in an n -cell.
- (2) Every $(n - 1)$ -cell in L is contained in exactly one n -cell of K' .
- (3) Every $(n - 1)$ -cell not in L is contained in exactly two n -cells.
- (4) For any two n -cells σ, σ' , there is a finite sequence $\sigma = \sigma_1, \dots, \sigma_m = \sigma'$ of n -cells such that each consecutive pair has an $(n - 1)$ -face in common.

Now let τ denote the n -cell of K , and choose an orientation o of τ . To construct the K -ad E , we only need to define $E(\tau, o)$. Let X denote $\operatorname{colim}_{\sigma \in K'} F^b(\sigma)$ and give X the PL structure provided by Lemma 4.2(i). We first claim that X is a PL ∂ -pseudomanifold. Part (a) of Definition 2.2 follows from property (1) of K' , and part (b) follows from properties (2) and (3); this also shows that $\partial X = \operatorname{colim}_{\sigma \in L} F^b(\sigma)$. Part (c) follows from [Wh78, II.6.2]. For part (d), we first observe that the proof of [LM13, Proposition 6.6] shows that ∂X is locally collared (in the sense of [RS72, page 24]); now [RS72, Theorem 2.25] shows that ∂X is collared.

Next give the n -cells σ of K' the orientations o_σ which agree with o . Then X has an orientation which agrees with the orientations of the $F(\sigma, o_\sigma)$, and we define $E(\tau, o)$ to be X with this orientation. To justify this choice, we need to show that $X - \partial X$ is an IP-space. We give $X - \partial X$ the intrinsic stratification (see Proposition A.1(iv)). Let $x \in X - \partial X$. There is a unique $\sigma \in K' - L$ for which x is in the interior of $F^b(\sigma)$; give $F^b(\sigma)$ the intrinsic stratification and let U be a distinguished neighborhood of x in $F^b(\sigma)$. The proof of [LM13, Proposition 6.6] shows that x has a neighborhood V in X such that there is a PL homeomorphism

$$f : V \rightarrow U \times E,$$

where E is a Euclidean space. The filtration of V inherited from X is the same as the intrinsic stratification of V by Proposition A.1(i), and (by Proposition A.1(ii) and (iii)) f takes this filtration to the Cartesian product of the intrinsic stratification of U with the trivial stratification of E . This implies that the link of x in X is the same as the link of x in $F^b(\sigma)$, and so conditions (a) and (b) of Definition 2.7 are satisfied.

It remains to show that $F^b(\sigma) \rightarrow X$ is a strong embedding when σ is an n -cell of K' . We denote the stratification on a PL pseudomanifold (resp., PL ∂ -pseudomanifold) Y provided by Proposition A.1(iv) (resp., Proposition A.2) by Y^* (resp., $Y[*]$). By its definition, the filtration $X[*]$ agrees (up to a dimension shift) with $(\partial X)^*$, so it suffices to show that $(\partial X)^*$ agrees with

$(F^b(\sigma))[*]$. Next we observe that (by the proof of [LM13, Proposition 6.6]) each point of $\partial F^b(\sigma)$ has a neighborhood U in $\partial F^b(\sigma)$ and a neighborhood V in ∂X with $V \approx U \times (-1, 1)$. Then $(\partial X)^*$ agrees with V^* by Proposition A.1(i), and (by Proposition A.1(ii) and (iii)) the latter agrees with $U^* \times (-1, 1)$ (where $(-1, 1)$ is given the trivial filtration). This implies that $(\partial X)^*$ agrees with the restriction of $(F^b(\sigma))[*]$ to $\partial F^b(\sigma)$. Moreover, $(\partial X)^*$ also agrees with the restriction of $(F^b(\sigma))[*]$ to $F^b(\sigma) - \partial F^b(\sigma)$ by Proposition A.1(i), so the two filtrations agree on all of $F^b(\sigma)$. \square

It remains to prove Lemma 4.2. The main ingredient is the following, which is Exercise 2.27(2) in [RS72].

Lemma 4.6. *Let P, Q and R be polyhedra and let $f : R \hookrightarrow P$, $g : R \hookrightarrow Q$ be PL embeddings. Then the pushout of*

$$P \hookleftarrow R \hookrightarrow Q,$$

formed in the category of topological spaces, has a PL structure for which the inclusions of P and Q are PL embeddings. \square

Proof of Lemma 4.2. Assume inductively that (i) and (ii) hold for any ball complex with at most k cells. Let K be a ball complex with $k + 1$ cells and let $\sigma \in K$ be a top-dimensional cell. Let $K_0 = K - \{\sigma\}$. Let $P = \operatorname{colim}_{\tau \in K_0} G(\tau)$ and $R = \operatorname{colim}_{\tau \subset \partial \sigma} G(\tau)$; the inductive hypothesis implies that P and R have PL structures for which all maps $G(\tau) \rightarrow P$ and $G(\tau) \rightarrow R$ are PL embeddings, and it also implies that $R \rightarrow P$ is a PL embedding. We are given that the map $R \rightarrow G(\sigma)$ is a monomorphism, and it is PL since its restriction to each $G(\tau)$ is PL. Now let S denote $\operatorname{colim}_{\tau \in K} G(\tau)$. Then S is the pushout of

$$P \hookleftarrow R \hookrightarrow G(\sigma),$$

so by Lemma 4.6 it has a PL structure for which $P \rightarrow S$ and $G(\sigma) \rightarrow S$ are PL maps; it follows that $G(\tau) \rightarrow S$ is a PL map for every τ . It remains to check that part (ii) of Lemma 4.2 holds, so let L be a subcomplex of K . The map

$$i : \operatorname{colim}_{\tau \in L} G(\tau) \rightarrow S$$

is PL, since its restriction to each $G(\tau)$ is PL, so we only need to check that i is a monomorphism. If $\sigma \notin L$ this follows from the inductive hypothesis and the fact that $P \rightarrow S$ is a monomorphism. If $\sigma \in L$ then $\operatorname{colim}_{\tau \in L} G(\tau)$ is the pushout of

$$\operatorname{colim}_{\tau \in L - \{\sigma\}} G(\tau) \hookleftarrow R \hookrightarrow G(\sigma),$$

and this pushout maps by a monomorphism to the pushout of

$$P \hookleftarrow R \hookrightarrow G(\sigma)$$

which is S . \square

5. ad THEORIES OF SYMMETRIC POINCARÉ COMPLEXES

The ad theory constructed in the previous section gives a spectrum $\mathbf{Q}_{\mathbb{I}\mathbb{P}}$ and a symmetric spectrum $\mathbf{M}_{\mathbb{I}\mathbb{P}}$. Our next goal is to use the symmetric signature to construct maps (in the derived category of spectra and the derived category of symmetric spectra) from $\mathbf{Q}_{\mathbb{I}\mathbb{P}}$ and $\mathbf{M}_{\mathbb{I}\mathbb{P}}$ to suitable versions of the symmetric L-theory spectrum of \mathbb{Z} . In order to do this we need an ad theory for symmetric Poincaré complexes over \mathbb{Z} . In [LM13, Section 9] we gave an ad theory (denoted $\text{ad}^{\mathbb{Z}}$) which was suggested by definitions from [WW89] and [Ran92]; in particular this leads to a spectrum $\mathbf{Q}^{\mathbb{Z}}$ which is identical to Ranicki's spectrum $\mathbb{L}^{\bullet}(\mathbb{Z})$. But this turns out not to be well-adapted to questions of commutativity (see the beginning of [LM, Section 11]) or to intersection homology (see the introduction to [FM13b]), so in [LM, Section 11] the second and third authors introduced a modification $\text{ad}_{\text{rel}}^{\mathbb{Z}}$ (rel stands for “relaxed”) which gives a spectrum $\mathbf{Q}_{\text{rel}}^{\mathbb{Z}}$ weakly equivalent to Ranicki's $\mathbb{L}^{\bullet}(\mathbb{Z})$. In this section we review this material; we should mention that everything extends from \mathbb{Z} to an arbitrary ring-with-involution R and that [LM13] and [LM] develop the theory in this generality.

5.1. The ad theory $\text{ad}^{\mathbb{Z}}$. As motivation we begin with the theory $\text{ad}^{\mathbb{Z}}$. A chain complex over \mathbb{Z} is called *finite* if it is free abelian and finitely generated in each degree and nonzero in only finitely many degrees; it is called *homotopy finite* if it is free abelian in each degree and chain homotopic to a finite complex. Let \mathcal{D} be the category of homotopy finite chain complexes. Let W be the standard free resolution of \mathbb{Z} by $\mathbb{Z}[\mathbb{Z}/2]$ modules. The n -dimensional objects of the target category $\mathcal{A}^{\mathbb{Z}}$ are pairs (C, φ) , where C is an object of \mathcal{D} and

$$\varphi : W \rightarrow C \otimes C$$

is a $\mathbb{Z}/2$ equivariant map which raises degrees by n . The morphisms $(C, \varphi) \rightarrow (C', \varphi')$ are the chain maps $f : C \rightarrow C'$, with the additional requirement that $(f \otimes f) \circ \varphi = \varphi'$ when the dimensions are equal. The involution reverses the sign of φ . Next, $(\text{ad}^{\mathbb{Z}})^k(K) \subset (\text{pre}^{\mathbb{Z}})^k(K)$ is defined to be the set of functors F with the following properties:

- (a) F is balanced. This allows us to write $F(\sigma, o)$ as $(C_{\sigma}, \varphi_{\sigma, o})$.
- (b) F is well-behaved, that is, each map $C_{\tau} \rightarrow C_{\sigma}$ is a split monomorphism in each dimension, and (writing $C_{\partial\sigma}$ for $\text{colim}_{\tau \subset \partial\sigma} C_{\tau}$) each map

$$C_{\partial\sigma} \rightarrow C_{\sigma}$$

is a split monomorphism in each dimension.

- (c) F is closed, that is, for each cell σ of K the map from the cellular chain complex $\text{cl}(\sigma)$ to $\text{Hom}(W, C_{\sigma} \otimes C_{\sigma})$ which takes (τ, o) to the composite

$$W \xrightarrow{\varphi(\tau, o)} C_{\tau} \otimes C_{\tau} \rightarrow C_{\sigma} \otimes C_{\sigma}$$

is a chain map. This implies that $\varphi_{\sigma,o}$ represents a class $[\varphi_{\sigma,o}]$ in $H_n(((C_\sigma/C_{\partial\sigma}) \otimes C_\sigma)^W)$; this group is isomorphic to $H_n((C_\sigma/C_{\partial\sigma}) \otimes C_\sigma)$, and we denote the image of $[\varphi_{\sigma,o}]$ in the latter group by \mathbf{c} .

(d) F is nondegenerate, that is, for each σ the slant product with \mathbf{c} gives an isomorphism

$$H^*(\mathrm{Hom}(C, \mathbb{Z})) \rightarrow H_{\dim \sigma - k - *}(C_\sigma/C_{\partial\sigma}).$$

Remark 5.1. In [LM13], the second and third authors give a construction of a symmetric signature map $\mathbf{Q}_{\mathrm{STOP}} \rightarrow \mathbf{Q}^{\mathbb{Z}}$ in the derived category of spectra, using ideas from [Ran92] (see [LM13, Section 10 and the end of Section 8]). The starting point for this construction is the observation that, if M is a compact oriented ∂ -manifold and $\xi \in S_n(M)$, where $S_*(-)$ denotes singular chains, represents the fundamental class of M then the composite

$$W \cong W \otimes \mathbb{Z} \xrightarrow{1 \otimes \xi} W \otimes S_*M \xrightarrow{EAW} S_*M \otimes S_*M$$

(where EAW is the extended Alexander-Whitney map, which can be constructed using acyclic models) is an object of $\mathcal{A}^{\mathbb{Z}}$. The Alexander-Whitney map (and *a fortiori* the extended Alexander-Whitney map) does not exist for intersection chains, which is one reason we need the modification of $\mathcal{A}^{\mathbb{Z}}$ given in the next subsection.

Remark 5.2. The ad theory $\mathrm{ad}^{\mathbb{Z}}$ is multiplicative ([LM13, Definitions 18.1 and 9.12]) but not commutative (see the beginning of Section 11 of [LM]).

5.2. The ad theory $\mathrm{ad}_{\mathrm{rel}}^{\mathbb{Z}}$. An object of the category $\mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}$ is a quadruple (C, D, β, φ) , where C is an object of \mathcal{D} , D is a chain complex with a $\mathbb{Z}/2$ action, β is a quasi-isomorphism $C \otimes C \rightarrow D$ which is also a $\mathbb{Z}/2$ equivariant map, and φ is an element of $D_n^{\mathbb{Z}/2}$. A morphism $(C, D, \beta, \varphi) \rightarrow (C', D', \beta', \varphi')$ is a pair $(f : C \rightarrow C', g : D \rightarrow D')$, where f and g are chain maps, g is $\mathbb{Z}/2$ equivariant, $g\beta = \beta'(f \otimes f)$, and (if the dimensions are equal) $g_*(\varphi) = \varphi'$.

Example 5.3. If (C, φ) is an object of $\mathcal{A}^{\mathbb{Z}}$ then the quadruple $(C, (C \otimes C)^W, \beta, \varphi)$ is a relaxed quasi-symmetric complex, where $\beta : C \otimes C \rightarrow (C \otimes C)^W$ is induced by the augmentation $W \rightarrow \mathbb{Z}$. This construction gives a functor $\mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}$.

Example 5.4. In the situation of Remark 5.1, we obtain an object of $\mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}$ by letting C be S_*M , D be $S_*(M \times M)$, β be the cross product, and φ be the image of $\xi \in S_n(M)$ under the diagonal map.

Now $(\mathrm{ad}_{\mathrm{rel}}^{\mathbb{Z}})^k(K) \subset (\mathrm{pre}_{\mathrm{rel}}^{\mathbb{Z}})^k(K)$ is defined to be the set of functors F with the following properties:

- (a) F is balanced. This allows us to write $F(\sigma, o)$ as $(C_\sigma, D_\sigma, \beta_\sigma, \varphi_{\sigma,o})$.
- (b) F is well-behaved, that is, all maps $C_\tau \rightarrow C_\sigma$, $D_\tau \rightarrow D_\sigma$, $C_{\partial\sigma} \rightarrow C_\sigma$ and $D_{\partial\sigma} \rightarrow D_\sigma$ are split monomorphisms in each dimension. This implies that the map $\beta_* : H_*(C_\sigma \otimes C_\sigma, (C \otimes C)_{\partial\sigma}) \rightarrow H_*(D_\sigma, D_{\partial\sigma})$ is an isomorphism ([LM, Lemma 11.6(ii)]).

(c) F is closed, that is, for each σ the map

$$\text{cl}(\sigma) \rightarrow D_\sigma$$

which takes $\langle \tau, o \rangle$ to $\varphi_{\tau, o}$ is a chain map. This implies that $\varphi_{\sigma, o}$ represents a class $[\varphi_{\sigma, o}]$ in $H_n(D_\sigma, D_{\partial\sigma})$.

(d) F is nondegenerate, that is, for each σ the slant product with $\beta_*^{-1}([\varphi_{\sigma, o}])$ is an isomorphism

$$H^*(\text{Hom}(C_\sigma, \mathbb{Z})) \rightarrow H_{\dim \sigma - k - *}(C_\sigma / C_{\partial\sigma}).$$

Remark 5.5. (i) The functor $\mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}_{\text{rel}}^{\mathbb{Z}}$ in Example 5.3 gives a map of spectra $\mathbf{Q}^{\mathbb{Z}} \rightarrow \mathbf{Q}_{\text{rel}}^{\mathbb{Z}}$ which is a weak equivalence ([LM, Section 12]).

(ii) The ad theory $\text{ad}_{\text{rel}}^{\mathbb{Z}}$ is commutative ([LM, Definition 3.3 and Remark 11.11]) so Theorem 1.1 of [LM] shows that there is a commutative symmetric ring spectrum $(\mathbf{M}_{\text{rel}}^{\mathbb{Z}})^{\text{comm}}$ which is weakly equivalent as a symmetric ring spectrum to $\mathbf{M}_{\text{rel}}^{\mathbb{Z}}$.

5.3. Connective versions. In the sequel, we will only deal with connective versions of L -theory rather than with periodic ones. There is a general procedure which takes an ad theory to another ad theory and which makes the associated Quinn spectrum connective: define the sub functor $\text{ad}_{\geq 0}$ of an ad theory by

$$\text{ad}_{\geq 0}^k(K, L) = \text{ad}^k(K, L \cup K^{(-k-1)}).$$

Here $K^{(n)}$ denotes the n -skeleton of K . We leave it to the reader to check the properties of an ad theory for $\text{ad}_{\geq 0}$. Clearly, we get a map of Quinn spectra $\mathbf{Q}_{\geq 0} \rightarrow \mathbf{Q}$ which does the right job on homotopy groups. Moreover, if we start with a multiplicative ad theory then the associated connective one is multiplicative as well.

Note that that the Quinn spectrum $\mathbf{Q}_{\geq 0}^{\mathbb{Z}}$ coincides with the usual connective L -theory spectrum up to a canonical weak equivalence: the restrictions of k -ads on their $(-k-1)$ skeleton vanish and hence consist of acyclic complexes.

6. THE SYMMETRIC SIGNATURE AS A MAP OF SPECTRA

In this section we construct symmetric signature maps

$$\text{Sig} : \mathbf{Q}_{\text{IP}} \rightarrow \mathbf{Q}_{\geq 0, \text{rel}}^{\mathbb{Z}}$$

(in the derived category of spectra) and

$$\text{Sig} : \mathbf{M}_{\text{IP}} \rightarrow \mathbf{M}_{\geq 0, \text{rel}}^{\mathbb{Z}}$$

(in the derived category of symmetric spectra). The first step is to give a variant of the ad theory ad_{IP} . As we have seen in Remark 5.1 and Example 5.4, in order for a compact oriented ∂ -manifold to give rise to an object of $\mathcal{A}_{\geq 0, \text{rel}}^{\mathbb{Z}}$ we must choose a chain representative for the fundamental class. The same is true for ∂ -IP-spaces, so in Subsection 6.1 we construct a suitable ad theory ad_{IPFun} and we show that the forgetful maps $\mathbf{Q}_{\text{IPFun}} \rightarrow \mathbf{Q}_{\text{IP}}$ and

$\mathbf{M}_{\text{IPFun}} \rightarrow \mathbf{M}_{\text{IP}}$ are weak equivalences. Next, in Subsection 6.3 we construct a morphism of ad theories

$$\text{sig} : \text{ad}_{\text{IPFun}} \rightarrow \text{ad}_{\geq 0, \text{rel}}^{\mathbb{Z}}.$$

These results allow us to make the following definition.

Definition 6.1. The symmetric signature map

$$\text{Sig} : \mathbf{Q}_{\text{IP}} \rightarrow \mathbf{Q}_{\geq 0, \text{rel}}^{\mathbb{Z}}$$

is the composite

$$\mathbf{Q}_{\text{IP}} \xleftarrow{\simeq} \mathbf{Q}_{\text{IPFun}} \xrightarrow{\text{sig}} \mathbf{Q}_{\geq 0, \text{rel}}^{\mathbb{Z}}.$$

The symmetric signature map

$$\text{Sig} : \mathbf{M}_{\text{IP}} \rightarrow \mathbf{M}_{\geq 0, \text{rel}}^{\mathbb{Z}}$$

is the composite

$$\mathbf{M}_{\text{IP}} \xleftarrow{\simeq} \mathbf{M}_{\text{IPFun}} \xrightarrow{\text{sig}} \mathbf{M}_{\geq 0, \text{rel}}^{\mathbb{Z}}.$$

6.1. The ad theory ad_{IPFun} . We denote singular intersection chains with perversity \bar{p} by $IS_{*}^{\bar{p}}$. By [FM13a, Proposition 7.7], an orientation of a compact n -dimensional ∂ -IP-space X determines a fundamental class $\Gamma_X \in IH_n^{\bar{0}}(X, \partial X; \mathbb{Z})$, where $\bar{0}$ denotes the 0 perversity.

We define a category $\mathcal{A}_{\text{IPFun}}$ as follows. The objects are pairs (X, ξ) , where X is a compact oriented ∂ -IP-space and $\xi \in IS_n^{\bar{0}}(X, \partial X; \mathbb{Z})$ is a chain representative for the fundamental class Γ_X ; there is also an empty object of dimension n for each n . The morphisms $(X, \xi) \rightarrow (X', \xi')$ between objects of the same dimension are PL homeomorphisms which take ξ to ξ' , and the other morphisms are strong embeddings with image in the boundary. There is a forgetful functor $\mathcal{A}_{\text{IPFun}} \rightarrow \mathcal{A}_{\text{IP}}$, and we define $\text{ad}_{\text{IPFun}}^k(K) \subset \text{pre}_{\text{IPFun}}^k(K)$ to be the set of functors F such that

- (a) F is balanced,
- (b) the composite of F with the forgetful functor is an element of $\text{ad}_{\text{IP}}^k(K)$, and
- (c) for each oriented cell (σ, o) of K , the equation

$$\partial \xi_{\sigma, o} = \sum \xi_{\sigma', o'}$$

holds, where σ' runs through the cells of $\partial \sigma$ and o' is the orientation for which the incidence number $[o, o']$ is $(-1)^k$.

Proposition 6.2. ad_{IPFun} is an ad theory.

Proof. We only need to check parts (f) and (g) of [LM13, Definition 3.10]. For the proof of (f) we use the gluing construction in the proof of Theorem 4.4 (and the notation there) and we define $\xi_{\tau, o}$ to be $\sum \xi_{\sigma, o_{\sigma}}$, where σ runs through the n -cells of K' . The proof of (g) is the same as the corresponding part of the proof of [LM13, Theorem 7.13]. \square

The forgetful functor $\mathcal{A}_{\text{IPFun}} \rightarrow \mathcal{A}_{\text{IP}}$ gives rise to a morphism $\text{ad}_{\text{IPFun}} \rightarrow \text{ad}_{\text{IP}}$ of ad theories.

Proposition 6.3. *The maps*

$$\mathbf{Q}_{\text{IPFun}} \rightarrow \mathbf{Q}_{\text{IP}}$$

and

$$\mathbf{M}_{\text{IPFun}} \rightarrow \mathbf{M}_{\text{IP}}$$

induced by the forgetful functor $\mathcal{A}_{\text{IPFun}} \rightarrow \mathcal{A}_{\text{IP}}$ are weak equivalences.

Proof. Recall the definition of the bordism groups of an ad theory ([LM13, Definitions 4.1 and 4.2]). By [LM13, Proposition 16.4(i), Remark 14.2(i), and Proposition 17.7], it suffices to show that the map of bordism groups

$$(\Omega_{\text{IPFun}})_* \rightarrow (\Omega_{\text{IP}})_*$$

is an isomorphism. This map is obviously onto, and it is a monomorphism by the proof of [LM13, Lemma 8.2]. \square

Remark 6.4. ad_{IPFun} is a commutative ad theory, so by Theorem 1.1 of [LM] there is a commutative symmetric ring spectrum $\mathbf{M}_{\text{IPFun}}^{\text{comm}}$ which is weakly equivalent as a symmetric ring spectrum to $\mathbf{M}_{\text{IPFun}}$. Moreover, the forgetful map $\text{ad}_{\text{IPFun}} \rightarrow \text{ad}_{\text{IP}}$ is strictly multiplicative, so the proof of [LM, Theorem 1.1] gives a commutative diagram

$$\begin{array}{ccccc} \mathbf{M}_{\text{IP}} & \xleftarrow{\simeq} & \mathbf{A} & \xrightarrow{\simeq} & \mathbf{M}_{\text{IP}}^{\text{comm}} \\ \uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq \\ \mathbf{M}_{\text{IPFun}} & \xleftarrow{\simeq} & \mathbf{B} & \xrightarrow{\simeq} & \mathbf{M}_{\text{IPFun}}^{\text{comm}} \end{array}$$

in which \mathbf{A} and \mathbf{B} are symmetric ring spectra and all arrows are ring maps.

6.2. Background. Before proceeding we need to recall some information about generalized perversities. For a stratified n -dimensional ∂ -pseudomanifold Y the components of $Y^i - Y^{i-1}$ are called *i -dimensional strata*; the n -dimensional strata are called *regular* and the others *singular*. Recall ([F, Definition 3.1]) that a *generalized perversity*¹ on Y is a function \bar{p} from the set of strata of Y to \mathbb{Z} which is 0 on the regular strata; an ordinary perversity \bar{q} can be thought of as a generalized perversity taking a stratum S to $\bar{q}(\text{codim}(S))$. We use the definition of intersection homology for general perversities given in [F, Definition 6.2]. Let \bar{n} be the upper middle perversity.

Let X be a ∂ -IP space, and give X the stratification of Proposition A.2. Give $X \times X$ the product stratification. Define a generalized perversity $Q_{\bar{n}, \bar{n}}$

¹These are simply called perversities in [F].

on $X \times X$ as follows.

$$Q_{\bar{n}, \bar{n}}(S_1 \times S_2) = \begin{cases} \bar{n}(S_1) + \bar{n}(S_2) + 2, & S_1, S_2 \text{ both singular strata,} \\ \bar{n}(S_1), & S_2 \text{ a regular stratum and } S_1 \text{ singular,} \\ \bar{n}(S_2), & S_1 \text{ a regular stratum and } S_2 \text{ singular,} \\ 0, & S_1, S_2 \text{ both regular strata.} \end{cases}$$

By [FM13a, Subsection 4.1], the diagonal map induces a map

$$(6.1) \quad d : IS_*^{\bar{0}}(X; \mathbb{Z}) \rightarrow IS_*^{Q_{\bar{n}, \bar{n}}}(X \times X; \mathbb{Z})$$

(this is the reason we need generalized perversities). By [F, Lemma 6.43 and Remark 6.46], the cross product induces an equivalence

$$IS_*^{\bar{n}}(X; \mathbb{Z}) \otimes IS_*^{\bar{n}}(X; \mathbb{Z}) \rightarrow IS_*^{Q_{\bar{n}, \bar{n}}}(X \times X; \mathbb{Z}).$$

6.3. The map $\text{sig} : \text{ad}_{\text{IPFun}} \rightarrow \text{ad}_{\geq 0, \text{rel}}^{\mathbb{Z}}$.

Lemma 6.5. (i) *Let (X, ξ) be an object of $\mathcal{A}_{\text{IPFun}}$. Give X the stratification of Proposition A.2 and give $X \times X$ the product stratification. Then (C, D, β, φ) is an object of $\mathcal{A}_{\text{rel}}^{\mathbb{Z}}$, where*

$$C = IS_*^{\bar{n}}(X; \mathbb{Z}),$$

$$D = IS_*^{Q_{\bar{n}, \bar{n}}}(X \times X; \mathbb{Z}),$$

β is the cross product, and

φ is the image of ξ under the diagonal map (6.1).

(ii) *Let $f : (X, \xi) \rightarrow (X', \xi')$ be a morphism in $\mathcal{A}_{\text{IPFun}}$ and let (C, D, β, φ) and $(C', D', \beta', \varphi')$ be the objects of $\mathcal{A}_{\text{rel}}^{\mathbb{Z}}$ corresponding to (X, ξ) and (X', ξ') . Then f induces a morphism $(C, D, \beta, \varphi) \rightarrow (C', D', \beta', \varphi')$.*

Proof. (i) We only need to show that $IS_*^{\bar{n}}(X; \mathbb{Z})$ is homotopy finite. The complex $IS_*^{\bar{n}}(X; \mathbb{Z})$ is free because it is a subcomplex of the singular chain complex $S_*(X; \mathbb{Z})$. Next let T be a triangulation of X which is compatible with the stratification of X . Let $IC_*^{T, \bar{n}}(X; \mathbb{Z})$ denote the complex of PL intersection chains which are simplicial with respect to T . This is free, finitely generated in each degree, and nonzero in only finitely many degrees. By [F, Corollary 5.49], the inclusion

$$IC_*^{T, \bar{n}}(X; \mathbb{Z}) \rightarrow IS_*^{\bar{n}}(X; \mathbb{Z})$$

is a quasi-isomorphism. Since the domain and range are free, it is a chain homotopy equivalence, and thus $IS_*^{\bar{n}}(X; \mathbb{Z})$ is homotopy finite.

(ii) If the dimensions are not equal then the definition of $\mathcal{A}_{\text{IPFun}}$ shows that f , and hence also $f \times f$, is a strong embedding, so they induce maps of intersection chains and the result follows. If the dimensions are equal then f is a PL homeomorphism so, by Propositions A.1(iii) and A.2, f and $f \times f$ preserve the filtrations and therefore induce maps of intersection chains. \square

Lemma 6.5 gives a functor

$$\text{sig} : \mathcal{A}_{\text{IPFun}} \rightarrow \mathcal{A}_{\geq 0, \text{rel}}^{\mathbb{Z}}.$$

Proposition 6.6. *If $F \in \text{ad}_{\text{IPFun}}^k(K)$ then $\text{sig} \circ F \in (\text{ad}_{\geq 0, \text{rel}}^{\mathbb{Z}})^k(K)$.*

Proof. We only need to check that $\text{sig} \circ F$ is well-behaved and nondegenerate. Write $F(\sigma, o) = (X_\sigma, \xi_{\sigma, o})$. First we show that the functor $IS_*^{\bar{n}}(X_\sigma; \mathbb{Z})$ is well-behaved. Let $\tau \subset \sigma$ be cells of K ; we want to show that the monomorphism

$$IS_i^{\bar{n}}(X_\tau; \mathbb{Z}) \rightarrow IS_i^{\bar{n}}(X_\sigma; \mathbb{Z})$$

is split for each i . For this it suffices to show that the quotient $IS_i^{\bar{n}}(X_\sigma; \mathbb{Z})/IS_i^{\bar{n}}(X_\tau; \mathbb{Z})$ is free, and this in turn follows from the fact that this quotient is a subgroup of the free abelian group $S_i(X_\sigma; \mathbb{Z})/S_i(X_\tau; \mathbb{Z})$. The proof of the other parts of the well-behavedness condition is similar.

For nondegeneracy, we need to show that the horizontal map in the following diagram is a quasi-isomorphism for each oriented simplex (σ, o) of K .

$$\begin{array}{ccc} \text{Hom}(IS_*^{\bar{n}}(X_\sigma; \mathbb{Z}), \mathbb{Z}) & \xrightarrow{\backslash \beta_*^{-1}([\varphi_{\sigma, o}])} & IS_{\dim \sigma - k - *}^{\bar{n}}(X_\sigma, \partial X_\sigma; \mathbb{Z}) \\ & \searrow \sim \Gamma_{X_\sigma} & \uparrow \\ & & IS_{\dim \sigma - k - *}^{\bar{m}}(X_\sigma, \partial X_\sigma; \mathbb{Z}) \end{array}$$

The construction of the cap product is given in Appendix B, and the fundamental class Γ_{X_σ} is given by [FM13a, Proposition 7.7]. Inspection of the definitions shows that the diagram commutes, and the slanted arrow is a quasi-isomorphism by Theorem B.5, so we only need to show that the vertical arrow is a quasi-isomorphism. For this it suffices to show that the maps

$$IH_*^{\bar{m}}(\partial X_\sigma; \mathbb{Z}) \rightarrow IH_*^{\bar{n}}(\partial X_\sigma; \mathbb{Z})$$

and

$$IH_*^{\bar{m}}(X_\sigma; \mathbb{Z}) \rightarrow IH_*^{\bar{n}}(X_\sigma; \mathbb{Z})$$

are isomorphisms. The first of these is an isomorphism by Proposition 2.10 and the argument in [GM83, Subsection 5.6.1]. To see that the second map is an isomorphism we observe that if Y denotes X_σ with a collar of the boundary removed then the maps $Y \rightarrow X_\sigma$ and $Y \rightarrow X_\sigma - \partial X_\sigma$ are stratified homotopy equivalences and therefore induce isomorphisms of intersection homology by [F03, Proposition 2.1], so it suffices to observe that the map

$$IH_*^{\bar{m}}(X_\sigma - \partial X_\sigma; \mathbb{Z}) \rightarrow IH_*^{\bar{n}}(X_\sigma - \partial X_\sigma; \mathbb{Z})$$

is an isomorphism by the argument in [GM83, Subsection 5.6.1]. \square

The proposition gives the maps

$$\text{sig} : \mathbf{Q}_{\text{IPFun}} \rightarrow \mathbf{Q}_{\geq 0, \text{rel}}^{\mathbb{Z}}$$

and

$$\text{sig} : \mathbf{M}_{\text{IPFun}} \rightarrow \mathbf{M}_{\geq 0, \text{rel}}^{\mathbb{Z}}$$

which are needed for Definition 6.1.

7. THE SYMMETRIC SIGNATURE OF AN IP-SPACE

For a compact oriented n -manifold M (and more generally for a Poincaré duality space) the symmetric signature $\sigma^*(M)$ is an element of the symmetric L -group $L^n(\mathbb{Z}[\pi_1 M])$. The symmetric signature was introduced by Miščenko as a tool for studying the Novikov conjecture, and since then it has become an important part of surgery theory (see [Ran92], for example). The symmetric signature has many useful properties, such as homotopy invariance, bordism invariance, and a product formula.

The paper [CSW91] has a brief description of a construction (using controlled topology) which assigns to a compact oriented Witt space X a symmetric signature in $L^n(\mathbb{Q}[\pi_1 X])$, with properties analogous to those of the classical symmetric signature (further information about this construction is given in [W94, pages 209-210]). A simpler construction of a symmetric signature $\sigma_{\text{Witt}}^*(X)$ with these properties was given in [FM13b, Section 5.4]. The two constructions are known to agree rationally by an argument due to Weinberger (cf. [ALMP12, Proposition 11.1]) and, independently, Banagl-Cappell-Shaneson [BCS03, Proposition 2].

In this section we show that when X is a compact oriented IP-space the construction in [FM13b] gives a symmetric signature

$$\sigma_{\text{IP}}^*(X) \in L^n(\mathbb{Z}[\pi_1 X])$$

with the usual properties.

Remark 7.1. Since we are not assuming that a basepoint for X is given, we must interpret the symbol $\pi_1 X$ as the fundamental groupoid of X , and the symbol $\mathbb{Z}[\pi_1 X]$ as a ringoid with involution, as in [WW95, Sections 2.1 and 2.3]. By a left (resp., right) module over $\mathbb{Z}[\pi_1 X]$ we mean a covariant (resp., contravariant) additive functor from $\mathbb{Z}[\pi_1 X]$ to the category of abelian groups. For a module \mathcal{M} and a point $x \in X$ we write \mathcal{M}_x for the restriction of \mathcal{M} to $\mathbb{Z}[\pi_1(X, x)]$. Then \mathcal{M} is determined up to canonical isomorphism by \mathcal{M}_x for any x , and because of this the results of [FM13b, LM13, LM] and Appendix B have routine extensions to this setting (cf. [WW95, Section 2.3]), so we will cite these results without further comment.

Now recall ([LM, Section 11]) that the relaxed symmetric Poincaré ad theory $\text{ad}_{\geq 0, \text{rel}}^{\mathbb{Z}}$ described in Subsection 5.2 has an analog $\text{ad}_{\geq 0, \text{rel}}^{\mathcal{R}}$ when \mathcal{R} is any ringoid with involution. By [LM, Proposition 12.3] there is a natural isomorphism of bordism groups

$$(7.1) \quad (\Omega_{\geq 0, \text{rel}}^{\mathbb{Z}[\pi_1 X]})_n \xrightarrow{\cong} (\Omega^{\mathbb{Z}[\pi_1 X]})_n = L^n(\mathbb{Z}[\pi_1 X]),$$

so we can construct $\sigma_{\text{IP}}^*(X)$ by giving a suitable element of $(\Omega_{\geq 0, \text{rel}}^{\mathbb{Z}[\pi_1 X]})_n$.

Notation 7.2. (i) Let π denote the fundamental groupoid of X .

(ii) Let \mathcal{Z} denote the $\mathbb{Z}[\pi]$ module which is the constant functor with value \mathbb{Z} .

Definition 7.3. For a right $\mathbb{Z}[\pi]$ module \mathcal{M} and a left $\mathbb{Z}[\pi]$ module \mathcal{N} , define

$$\mathcal{M} \otimes_{\mathbb{Z}[\pi]} \mathcal{N}$$

to be the abelian group

$$\left(\bigoplus_{x \in X} \mathcal{M}_x \otimes \mathcal{N}_x \right) / \sim,$$

where $(m \otimes fn) \sim (mf \otimes n)$ whenever f is a path from x to x' , m is an element of $\mathcal{M}_{x'}$, and n is an element of \mathcal{N}_x .

Remark 7.4. (i) This is canonically isomorphic to the tensor product of [Ran92, page 27] in this situation.

(ii) For each $x \in X$, $\mathcal{M} \otimes_{\mathbb{Z}[\pi]} \mathcal{N}$ is canonically isomorphic to $\mathcal{M}_x \otimes_{\mathbb{Z}[\pi_1(X,x)]} \mathcal{N}_x$.

Now we can construct the symmetric signature. For each $x \in X$ let \tilde{X}_x be the universal cover determined by x , and let \mathcal{M} be the left $\mathbb{Z}[\pi]$ module with

$$\mathcal{M}_x = IS_n^{\bar{0}}(\tilde{X}_x; \mathbb{Z}).$$

Choose an element $b \in \mathcal{Z} \otimes_{\mathbb{Z}[\pi]} \mathcal{M}$ which maps to a representative for the fundamental class $\Gamma_X \in IH_n^{\bar{0}}(X; \mathbb{Z})$; this is always possible by [FM13b, Proposition 6.1.3]. Let (C, D, β, φ) be defined as follows.

- C is the chain complex of $\mathbb{Z}[\pi]$ modules with $C_x = IS_*^{\bar{n}}(\tilde{X}_x; \mathbb{Z})$.
- Let \mathcal{N} be the left $\mathbb{Z}[\pi]$ module with

$$\mathcal{N}_x = IS_*^{Q_{\bar{n}, \bar{n}}}(\tilde{X}_x \times \tilde{X}_x; \mathbb{Z});$$

then D is the chain complex of abelian groups

$$\mathcal{Z} \otimes_{\mathbb{Z}[\pi]} \mathcal{N},$$

with the evident $\mathbb{Z}/2$ action.

- β is the map

$$C^t \otimes_{\mathbb{Z}[\pi]} C \rightarrow D$$

is induced by the composites

$$\begin{aligned} IS_*^{\bar{n}}(\tilde{X}_x; \mathbb{Z})^t \otimes_{\mathbb{Z}[\pi_1(X,x)]} IS_*^{\bar{n}}(\tilde{X}_x; \mathbb{Z}) &\cong \mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X,x)]} (IS_*^{\bar{n}}(\tilde{X}_x; \mathbb{Z}) \otimes IS_*^{\bar{n}}(\tilde{X}_x; \mathbb{Z})) \\ &\xrightarrow{1 \otimes \times} \mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X,x)]} IS_*^{Q_{\bar{n}, \bar{n}}}(\tilde{X}_x \times \tilde{X}_x; \mathbb{Z}). \end{aligned}$$

• $\varphi \in D^{\mathbb{Z}/2}$ is the image of b under the map $\mathcal{Z} \otimes_{\mathbb{Z}[\pi]} \mathcal{M} \rightarrow \mathcal{Z} \otimes_{\mathbb{Z}[\pi]} \mathcal{N}$ induced by the diagonal maps

$$IS_n^{\bar{0}}(\tilde{X}_x; \mathbb{Z}) \rightarrow IS_*^{Q_{\bar{n}, \bar{n}}}(\tilde{X}_x \times \tilde{X}_x; \mathbb{Z}).$$

Then β is a quasi-isomorphism by Proposition B.2, and C and D are homotopy finite over $\mathbb{Z}[\pi]$ by Proposition B.3, so (C, D, β, φ) is an object of $\mathcal{A}_{\geq 0, \text{rel}}^{\mathbb{Z}[\pi]}$.

Lemma 7.5. (C, D, β, φ) is an element of $\text{ad}_{\text{rel}}^{\mathbb{Z}[\pi_1 X]}(*)$.

Proof. This follows from the proof of [FM13b, Proposition 5.17], using Theorem B.4 in place of [FM13b, Theorem 4.1]. \square

The lemma says that (C, D, β, φ) represents an element of $(\Omega_{\geq 0, \text{rel}}^{\mathbb{Z}[\pi_1 X]})_n$ (see [LM13, Definition 4.2]).

Definition 7.6. Let X be a compact oriented IP-space. Then $\sigma_{\text{IP}}^*(X)$ is the image of the class of (C, D, β, φ) under the isomorphism $(\Omega_{\geq 0, \text{rel}}^{\mathbb{Z}[\pi_1 X]})_n \cong L^n(\mathbb{Z}[\pi_1 X])$.

Remark 7.7. The properties of the symmetric signature given in [FM13b, Section 5.5] remain valid for σ_{IP}^* , with the same proofs.

8. THE ASSEMBLY MAP AND THE L -THEORY FUNDAMENTAL CLASS

For an n -dimensional compact oriented topological manifold M , Ranicki constructs an L -theory fundamental class $[M]_{\mathbb{L}} \in L_n^\bullet(M)$ ([Ran92, Section 16]) which plays an important role in surgery theory. It is an oriented homeomorphism invariant whose image under the assembly map

$$L_n^\bullet(M) \rightarrow L^n(\mathbb{Z}[\pi_1 M])$$

is the symmetric signature $\sigma^*(M)$. The construction of $[M]_{\mathbb{L}}$ is not difficult to describe. There is an equivalence

$$M\text{STop} \rightarrow \mathbf{Q}_{\text{STop}}$$

in the stable category, where $M\text{STop}$ is the Thom spectrum and \mathbf{Q}_{STop} is the Quinn spectrum; see [LM13, Appendix B] for details. There is an ad theory $\text{ad}_{\text{STopFun}}$ which is related to ad_{STop} in the same way that ad_{IPFun} is related to ad_{IP} (see the end of [LM13, Section 8]), and the map given by forgetting the chain representative is an equivalence

$$\mathbf{Q}_{\text{STopFun}} \xrightarrow{\cong} \mathbf{Q}_{\text{STop}}.$$

The symmetric signature gives a map

$$\text{sig} : \mathbf{Q}_{\text{STopFun}} \rightarrow \mathbf{Q}_{\geq 0}^{\mathbb{Z}} \cong \mathbb{L}^\bullet(\mathbb{Z}),$$

so we have a map in the stable category

$$\text{Sig} : \mathbf{Q}_{\text{STop}} \rightarrow \mathbb{L}^\bullet(\mathbb{Z}).$$

Now if M is an n -dimensional compact oriented topological manifold then the identity map $M \rightarrow M$ represents an element

$$[M]_{\text{STop}} \in (\Omega_{\text{STop}})_n(M)$$

and the image of $[M]_{\text{STop}}$ under the composite
(8.1)

$$(\Omega_{\text{STop}})_n(M) \cong M\text{STop}_n(M) \rightarrow (\mathbf{Q}_{\text{STop}})_n(M) \xrightarrow{\text{Sig}} \mathbb{L}^\bullet(\mathbb{Z})_n(M) = L_n^\bullet(M)$$

is $[M]_{\mathbb{L}}$.

Remark 8.1. The composite

$$MSTop \rightarrow \mathbf{Q}_{STop} \xrightarrow{\text{Sig}} \mathbb{L}^\bullet(\mathbb{Z})$$

is the *Sullivan-Ranicki orientation*.

Our goal in this section is to prove

Theorem 8.2. *For an n -dimensional compact oriented IP-space X there is a fundamental class $[X]_{\mathbb{L}} \in \mathbb{L}_n^\bullet(X)$ with the following properties:*

- (i) $[X]_{\mathbb{L}}$ is an oriented PL homeomorphism invariant,
- (ii) The image of $[X]_{\mathbb{L}}$ under the assembly map is the symmetric signature $\sigma_{\text{IP}}^*(X)$ given by Definition 7.6.
- (iii) If X is a PL manifold then $[X]_{\mathbb{L}}$ is the same as the fundamental class constructed by Ranicki.

Remark 8.3. For Witt spaces, a different method for constructing a fundamental class is described in [CSW91].

The rest of the section is devoted to the proof of Theorem 8.2. We begin with the construction of $[X]_{\mathbb{L}}$. For a topological space Z , define a category $\mathcal{A}_{\text{IP},Z}$ as follows. An object of $\mathcal{A}_{\text{IP},Z}$ is an object X of \mathcal{A}_{IP} together with a map of topological spaces $X \rightarrow Z$. A morphism from $X \rightarrow Z$ to $X' \rightarrow Z$ is a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ & \searrow & \swarrow \\ & Z & \end{array}$$

for which the horizontal arrow is a morphism in \mathcal{A}_{IP} . There is a forgetful functor

$$\Upsilon : \mathcal{A}_{\text{IP},Z} \rightarrow \mathcal{A}_{\text{IP}},$$

and we define $\text{ad}_{\text{IP},Z}^k(K) \subset \text{pre}_{\text{IP},Z}^k(K)$ to be the set of functors F for which the composite $\Upsilon \circ F$ is in $\text{ad}_{\text{IP}}^k(K)$. The proof of Theorem 4.4 shows that this is an ad theory; in particular we obtain a functor Φ from spaces to spectra with

$$\Phi(Z) = \mathbf{Q}_{\text{IP},Z}.$$

By [LM13, Proposition 16.4(i), Remark 14.2(i), and Definitions 4.1 and 4.2], the homotopy groups of $\mathbf{Q}_{\text{IP},Z}$ are the same as the IP bordism groups $(\Omega_{\text{IP}})_*(Z)$ defined by Pardon ([Par90, Section 5]). Pardon proves that $(\Omega_{\text{IP}})_*$ is a homology theory, and in particular this shows that Φ is homotopy invariant in the sense of [WW95, Section 1]. We therefore have an assembly map

$$\alpha : Z_+ \wedge \Phi(*) \rightarrow \Phi(Z)$$

by [WW95, Theorem 1.1 and Observation 1.2].

Theorem 8.4. α is a weak equivalence.

The proof is deferred to Section 12. Now we have

Definition 8.5. Let X be an n -dimensional compact oriented IP-space.

(i) Let $[X]_{\text{IP}}$ be the class of the identity map $X \rightarrow X$ in $(\Omega_{\text{IP}})_n(X)$.

(ii) Let $[X]_{\mathbb{L}}$ be the image of $[X]_{\text{IP}}$ under the composite

$$(8.2) \quad (\Omega_{\text{IP}})_n(X) \xleftarrow[\cong]{\alpha} (\mathbf{Q}_{\text{IP}})_n(X) \xrightarrow{\text{Sig}} (\mathbf{Q}_{\geq 0, \text{rel}}^{\mathbb{Z}})_n(X) \xleftarrow[\cong]{} \mathbf{L}_n^{\bullet}(X),$$

where the last map is the isomorphism (7.1).

It remains to prove parts (i), (ii) and (iii) of Theorem 8.2. For part (i) it suffices to show that if $f : X \rightarrow X'$ is an oriented PL homeomorphism then $f_*([X]_{\text{IP}}) = [X']_{\text{IP}}$, and this in turn follows from the fact that the map

$$(I \times X) \cup_{1 \times X} X' \rightarrow X',$$

which is the identity on X' and takes (t, x) to $f(x)$, is a bordism between f and the identity map of X' .

For part (ii) let Ψ be the homotopy functor which takes Z to $\mathbf{Q}_{\geq 0, \text{rel}}^{\mathbb{Z}[\pi_1 Z]}$ (where $\mathbb{Z}[\pi_1 Z]$ denotes the fundamental ringoid), and observe that the symmetric signature gives a natural transformation

$$\nu : \Phi \rightarrow \Psi$$

in the stable category (cf. [LM, Section 13]). Now the naturality of the assembly map ([WW95, Theorem 1.1]) implies that the diagram

$$\begin{array}{ccc} & X_+ \wedge \mathbf{Q}_{\text{IP}} & \\ \alpha \swarrow & & \searrow 1 \wedge \text{Sig} \\ \mathbf{Q}_{\text{IP}, X} & & X_+ \wedge \mathbf{Q}_{\geq 0, \text{rel}}^{\mathbb{Z}} \\ & \searrow \nu & \swarrow \alpha \\ & \mathbf{Q}_{\geq 0, \text{rel}}^{\mathbb{Z}[\pi_1 X]} & \end{array}$$

commutes, and the result follows from the fact that the composite of ν with the isomorphism (7.1) takes $[X]_{\text{IP}}$ to $\sigma_{\text{IP}}^*(X)$.

For part (iii), we need to compare the composites (8.1) and (8.2) for X a PL manifold M . First we observe that there is a map

$$\text{Sig} : \mathbf{Q}_{\text{STop}} \rightarrow \mathbf{Q}_{\geq 0, \text{rel}}^{\mathbb{Z}}$$

(see [LM, Section 13]). By [LM, Proposition 13.3], (8.1) is equal to the composite

$$(8.3) \quad (\Omega_{\text{STop}})_n(M) \cong \text{MSTop}_n(M) \rightarrow (\mathbf{Q}_{\text{STop}})_n(M) \xrightarrow{\text{Sig}} (\mathbf{Q}_{\geq 0, \text{rel}}^{\mathbb{Z}})_n(M) \xleftarrow[\cong]{} \mathbf{L}_n^{\bullet}(M).$$

Next we observe that for each space Z there is an ad theory $\text{ads}_{\text{STop}, Z}$ defined analogously to $\text{ad}_{\text{IP}, Z}$. We get a functor Ξ from spaces to spectra by letting

$$\Xi(Z) = \mathbf{Q}_{\text{STop}, Z},$$

and we have

$$\pi_*\Xi(Z) = (\Omega_{\text{STop}})_*(Z).$$

Lemma 8.6. *The composite (8.3) is equal to the composite*

$$(\Omega_{\text{STop}})_n(M) \xleftarrow{\alpha} (\mathbf{Q}_{\text{STop}})_n(M) \xrightarrow{\text{Sig}} (\mathbf{Q}_{\geq 0, \text{rel}}^{\mathbb{Z}})_n(M) \xleftarrow{\cong} \mathbf{L}_n^\bullet(M)$$

where α is the assembly map for the functor Ξ .

We defer the proof for a moment. Now we observe that Ξ , \mathbf{Q}_{STop} , and Sig all have PL analogs; we write Ξ' for the PL analog of Ξ . To complete the proof of Theorem 8.2(iii) it suffices to show that the following diagram commutes.

$$\begin{array}{ccccc} \Xi(M) & \xleftarrow{\alpha} & \mathbf{Q}_{\text{STop}} \wedge M_+ & \xrightarrow{\text{Sig}} & \mathbf{Q}_{\geq 0, \text{rel}}^{\mathbb{Z}} \wedge M_+ \\ \uparrow & & \uparrow & & \uparrow = \\ \Xi'(M) & \xleftarrow{\alpha} & \mathbf{Q}_{\text{SPL}} \wedge M_+ & \xrightarrow{\text{Sig}} & \mathbf{Q}_{\geq 0, \text{rel}}^{\mathbb{Z}} \wedge M_+ \\ \downarrow & & \downarrow & & \downarrow = \\ \Phi(M) & \xleftarrow{\alpha} & \mathbf{Q}_{\text{IP}} \wedge M_+ & \xrightarrow{\text{Sig}} & \mathbf{Q}_{\geq 0, \text{rel}}^{\mathbb{Z}} \wedge M_+ \end{array}$$

The left-hand squares commute by the naturality of the assembly map, and the right-hand squares commute by the definition of the maps Sig . \square

Proof of Lemma 8.6. It suffices to show that the diagram

$$(8.4) \quad \begin{array}{ccc} (\Omega_{\text{STop}})_n(Z) & \xleftarrow[\cong]{i} & \text{MSTop}_n(Z) \\ & \searrow \alpha & \downarrow \cong j \\ & & (\mathbf{Q}_{\text{STop}})_n(Z) \end{array}$$

commutes, where Z is a space, i is the standard isomorphism and j is given by [LM13, Appendix B].

First we recall the definition of i (cf. [DK01, pages 224–5]). The k -th space of the spectrum $\text{MSTop} \wedge Z_+$ is $T\text{STop}_k \wedge Z_+$, where $T\text{STop}_k$ is the Thom space. The inclusion of the 0-section gives an embedding

$$B\text{STop}_k \rightarrow T\text{STop}_k.$$

Given a map $f : S^{n+k} \rightarrow T\text{STop}_k \wedge Z_+$, there is a homotopic map f' for which the composite

$$S^{n+k} \xrightarrow{f'} T\text{STop}_k \wedge Z_+ \xrightarrow{p_1} T\text{STop}_k$$

(where p_1 is the projection) is transverse to the 0-section. Then the oriented topological manifold $(p_1 \circ f')^{-1}(B\text{STop}_k)$ is equal to $(f')^{-1}(B\text{STop}_k \times Z)$, and i takes the homotopy class of f to the bordism class of the composite

$$(f')^{-1}(B\text{STop}_k \times Z) \xrightarrow{f'} B\text{STop}_k \times Z \xrightarrow{p_2} Z.$$

Now let $S_\bullet(TSTop_k \wedge Z_+)$ be the singular complex, and let $S_\bullet^\natural(TSTop_k \wedge Z_+)$ be the sub-semisimplicial set consisting of maps $g : \Delta^n \rightarrow TSTop_k \wedge Z_+$ for which the restriction of $p_1 \circ g$ to each face is transverse to the 0-section (cf. [LM13, Appendix B]). Let

$$(MSTop \wedge Z_+)^\natural$$

be the spectrum whose k -th space is the realization of $S_\bullet^\natural(TSTop_k \wedge Z_+)$. Transversality implies that the map

$$(MSTop \wedge Z_+)^\natural \rightarrow MSTop \wedge Z_+$$

is a weak equivalence. Hence the functor which takes Z to $(MSTop \wedge Z_+)^\natural$ is a homotopy functor, and there is an assembly map

$$MSTop^\natural \wedge Z_+ \xrightarrow{\alpha} (MSTop \wedge Z_+)^\natural.$$

Given a simplex g of $S_\bullet^\natural(TSTop_k \wedge Z_+)$, we obtain an element of $\text{ad}_{S\text{Top}, Z}^k(\Delta^n)$ by taking each oriented simplex (σ, o) to $(g|_\sigma)^{-1}(BSTop \times Z)$. This gives a natural transformation

$$J : (MSTop \wedge Z_+)^\natural \rightarrow \mathbf{Q}_{S\text{Top}, Z}.$$

Consider the diagram

$$\begin{array}{ccc} MSTop \wedge Z_+ & \xleftarrow{=} & MSTop \wedge Z_+ \\ \simeq \uparrow & & \simeq \uparrow \\ (MSTop \wedge Z_+)^\natural & \xleftarrow{\alpha} & MSTop^\natural \wedge Z_+ \\ J \downarrow & & J \downarrow \\ \mathbf{Q}_{S\text{Top}, Z} & \xleftarrow{\alpha} & \mathbf{Q}_{S\text{Top}} \wedge Z_+. \end{array}$$

This commutes by naturality of the assembly map (since the assembly map for the functor $MSTop \wedge Z_+$ is the identity map). On passage to homotopy groups, the left-hand vertical composite induces the map i of diagram (8.4), and the right-hand vertical composite induces the map j . Thus diagram (8.4) commutes as required. \square

9. THE STRATIFIED NOVIKOV CONJECTURE

Let G be a discrete group and BG its classifying space. Recall that the *strong Novikov conjecture* for G asserts that the assembly map

$$\alpha : L_n^\bullet(BG) \longrightarrow L^n(\mathbb{Z}[G])$$

is rationally injective. The symmetric L -spectrum splits rationally as a product of Eilenberg-MacLane spectra:

$$L^\bullet(\mathbb{Z}) \otimes \mathbb{Q} \simeq \prod_{j \geq 0} K(\mathbb{Q}, 4j),$$

and this splitting induces natural isomorphisms

$$S_X : \mathbf{L}_n^\bullet(X) \otimes \mathbb{Q} \xrightarrow{\cong} \bigoplus_{j \geq 0} H_{n-4j}(X; \mathbb{Q}).$$

An n -dimensional compact IP-space X possesses characteristic classes $L_j(X) \in H_j(X; \mathbb{Q})$, which are the Poincaré duals of the Hirzebruch L -classes when X is a smooth manifold. These classes have been introduced by Goresky and MacPherson in [GM80] (at least for spaces without odd codimensional strata, but the method works whenever one has a self-dual intersection chain sheaf, see [Ban07]). Goresky and MacPherson adapt a method of Thom, which exploits the bordism invariance of the signatures of transverse inverse images of maps to spheres. For singular X , these classes need not lift to the cohomology of X under capping with the (ordinary) fundamental class. We shall denote the total L -class by $L(X)$.

Lemma 9.1. *Let X be an n -dimensional compact IP-space. The isomorphism S_X maps the rational L -theory fundamental class $[X]_{\mathbb{L}}$ to the Goresky-MacPherson L -class $L(X)$.*

Proof. Let Y^m be a compact IP-space and $j : Y^m \hookrightarrow X^n$ a normally non-singular inclusion with trivial normal bundle ν . Let $D\nu = D^{n-m} \times Y$ be the total space of its disk bundle, $S\nu = S^{n-m-1} \times Y$ the total space of its sphere bundle. Note that $D\nu$ is a ∂ -IP space. Let $u \in H^{n-m}(D\nu, S\nu; \mathbb{Q})$ be the Thom class of the normal bundle. The composition

$$\begin{aligned} H_k(X; \mathbb{Q}) &\rightarrow H_k(X, X - Y; \mathbb{Q}) \xrightarrow{\cong} H_k(D\nu, S\nu; \mathbb{Q}) \\ &\xrightarrow{u \cap -} H_{k-n+m}(D\nu; \mathbb{Q}) \xrightarrow{\pi_*} H_{k-n+m}(Y; \mathbb{Q}) \xrightarrow{(-1)^s} H_{k-n+m}(Y; \mathbb{Q}), \end{aligned}$$

where $s = \frac{1}{2}(n - m + 1)(n - m)$ and $\pi : D\nu \rightarrow Y$ is the bundle projection, defines a map

$$j^! : H_k(X; \mathbb{Q}) \longrightarrow H_{k-n+m}(Y; \mathbb{Q}).$$

If Z is any topological space and R any coefficient ring, let $\epsilon_* : H_0(Z; R) \rightarrow R$ be the augmentation map. By the Thom-Goresky-MacPherson construction, the L -classes are uniquely characterized by the following two properties ([Ban07, Proposition 8.2.11]):

- If $j : Y^m \hookrightarrow X^n$ is a normally nonsingular inclusion with trivial normal bundle, then

$$L_{k-n+m}(Y) = j^! L_k(X).$$

- $\epsilon_* L_0(X) = \sigma(X)$, the signature of X .

Thus the lemma is proven if we show

- (1) If $j : Y^m \hookrightarrow X^n$ is a normally nonsingular inclusion with trivial normal bundle, then

$$(S_Y[Y]_{\mathbb{L}})_{k-n+m} = j^!(S_X[X]_{\mathbb{L}})_k;$$

and

$$(2) \quad \epsilon_*(S_X[X]_{\mathbb{L}})_0 = \sigma(X).$$

We turn to (1). Let ν be the normal bundle of j . We write S^0 as $S^0 = \{y_-, y_+\}$. Let $e \in H^1(D^1, S^0; \mathbb{Q})$ be the element obtained as the image of the unit $1 \in H^0(y_+; \mathbb{Q})$ under the composition

$$H^0(y_+; \mathbb{Q}) \xleftarrow{\cong} H^0(S^0, y_-; \mathbb{Q}) \xrightarrow{\delta} H^1(D^1, S^0; \mathbb{Q}),$$

where the left arrow is an excision isomorphism and δ is the connecting homomorphism of the triple (D^1, S^0, y_-) . The $(n-m)$ -fold cross-product $e \times \cdots \times e$ yields an element $e^{n-m} \in H^{n-m}(D^{n-m}, S^{n-m-1}; \mathbb{Q})$. The Thom class u arising in the definition of the map $j^!$ is then given by $u = e^{n-m} \times 1 \in H^{n-m}(D\nu, S\nu; \mathbb{Q})$, where $1 \in H^0(Y; \mathbb{Q})$. Analogous classes in \mathbb{L}^\bullet -homology and Ω_{IP} can be constructed in a similar fashion: If A is any abelian group, we shall briefly write $A_{\mathbb{Q}}$ for $A \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $e_{\mathbb{L}} \in (\mathbb{L}^\bullet)^1(D^1, S^0)_{\mathbb{Q}}$ be the element obtained as the image of the unit $1 \in (\mathbb{L}^\bullet)^0(y_+)_{\mathbb{Q}}$ under the composition

$$(\mathbb{L}^\bullet)^0(y_+)_{\mathbb{Q}} \xleftarrow{\cong} (\mathbb{L}^\bullet)^0(S^0, y_-)_{\mathbb{Q}} \xrightarrow{\delta} (\mathbb{L}^\bullet)^1(D^1, S^0)_{\mathbb{Q}}.$$

The $(n-m)$ -fold cross-product $e_{\mathbb{L}} \times \cdots \times e_{\mathbb{L}}$ yields an element $e_{\mathbb{L}}^{n-m} \in (\mathbb{L}^\bullet)^{n-m}(D^{n-m}, S^{n-m-1})_{\mathbb{Q}}$. Set $u_{\mathbb{L}} = e_{\mathbb{L}}^{n-m} \times 1 \in (\mathbb{L}^\bullet)^{n-m}(D\nu, S\nu)_{\mathbb{Q}}$, where $1 \in (\mathbb{L}^\bullet)^0(Y)_{\mathbb{Q}}$ is the unit. Let $e_{\text{IP}} \in (\Omega_{\text{IP}})^1(D^1, S^0)_{\mathbb{Q}}$ be the element obtained as the image of the unit $1 \in (\Omega_{\text{IP}})^0(y_+)_{\mathbb{Q}}$ under the composition

$$(\Omega_{\text{IP}})^0(y_+)_{\mathbb{Q}} \xleftarrow{\cong} (\Omega_{\text{IP}})^0(S^0, y_-)_{\mathbb{Q}} \xrightarrow{\delta} (\Omega_{\text{IP}})^1(D^1, S^0)_{\mathbb{Q}}.$$

The $(n-m)$ -fold cross-product $e_{\text{IP}} \times \cdots \times e_{\text{IP}}$ yields an element $e_{\text{IP}}^{n-m} \in (\Omega_{\text{IP}})^{n-m}(D^{n-m}, S^{n-m-1})_{\mathbb{Q}}$. Set $u_{\text{IP}} = e_{\text{IP}}^{n-m} \times 1 \in (\Omega_{\text{IP}})^{n-m}(D\nu, S\nu)_{\mathbb{Q}}$, where $1 \in (\Omega_{\text{IP}})^0(Y)_{\mathbb{Q}}$ is the unit. The cap-product of $\mu := [\text{id}_{(D^{n-m}, S^{n-m-1})}] \in (\Omega_{\text{IP}})_{n-m}(D^{n-m}, S^{n-m})_{\mathbb{Q}}$ with e_{IP}^{n-m} is given by

$$(9.1) \quad e_{\text{IP}}^{n-m} \cap \mu = (-1)^s [\text{pt} \hookrightarrow D^{n-m}] \in (\Omega_{\text{IP}})_0(D^{n-m})_{\mathbb{Q}},$$

as we shall now verify. Set $\mu_1 = [\text{id}_{(D^1, S^0)}] \in (\Omega_{\text{IP}})_1(D^1, S^0)_{\mathbb{Q}}$. Then μ is the $(n-m)$ -fold cross product

$$\mu = \mu_1 \times \cdots \times \mu_1$$

and thus

$$\begin{aligned} e_{\text{IP}}^{n-m} \cap \mu &= (e_{\text{IP}} \times \cdots \times e_{\text{IP}}) \cap (\mu_1 \times \cdots \times \mu_1) \\ &= (-1)^{s_1} (e_{\text{IP}} \cap \mu_1) \times \cdots \times (e_{\text{IP}} \cap \mu_1), \end{aligned}$$

where $s_1 = \frac{1}{2}(n-m)(n-m-1)$. Let $i : (S^0, \emptyset) \hookrightarrow (S^0, y_-)$ be the canonical inclusion. To compute $e_{\text{IP}} \cap \mu_1$, we consider the diagram

$$\begin{array}{ccccc}
\Omega_{\text{IP}}^1(D^1, S^0)_{\mathbb{Q}} & \otimes & (\Omega_{\text{IP}})_1(D^1, S^0)_{\mathbb{Q}} & \xrightarrow{\cap} & (\Omega_{\text{IP}})_0(D^1)_{\mathbb{Q}} \\
\delta' \uparrow & & \partial \downarrow & & \iota \uparrow \\
\Omega_{\text{IP}}^0(S^0)_{\mathbb{Q}} & \otimes & (\Omega_{\text{IP}})_0(S^0)_{\mathbb{Q}} & \xrightarrow{\cap} & (\Omega_{\text{IP}})_0(S^0)_{\mathbb{Q}} \\
i^* \uparrow & & i_* \downarrow & & \parallel \\
\Omega_{\text{IP}}^0(S^0, y_-)_{\mathbb{Q}} & \otimes & (\Omega_{\text{IP}})_0(S^0, y_-)_{\mathbb{Q}} & \xrightarrow{\cap} & (\Omega_{\text{IP}})_0(S^0)_{\mathbb{Q}} \\
\cong \downarrow \text{exc} & & \cong \uparrow \text{exc} & & \iota_+ \uparrow \\
\Omega_{\text{IP}}^0(y_+)_{\mathbb{Q}} & \otimes & (\Omega_{\text{IP}})_0(y_+)_{\mathbb{Q}} & \xrightarrow{\cap} & (\Omega_{\text{IP}})_0(y_+)_{\mathbb{Q}},
\end{array}$$

whose middle and bottom portion commute, while the top portion anti-commutes, since

$$(\delta' a) \cap \alpha = (-1)^{\deg(\delta' a)} a \cap \partial \alpha = -a \cap \partial \alpha$$

for an IP-cobordism class a of degree 0. The image of μ_1 under $i_* \partial$ is $[i : (S^0, \emptyset) \hookrightarrow (S^0, y_-)]$, while the image of $[\text{id}_{y_+}] \in (\Omega_{\text{IP}})_0(y_+)$ under the excision isomorphism is $[(y_+, \emptyset) \hookrightarrow (S^0, y_-)]$. Now

$$[(y_+, \emptyset) \hookrightarrow (S^0, y_-)] = [i] \in (\Omega_{\text{IP}})_0(S^0, y_-)_{\mathbb{Q}}$$

via the bordism $W = I \sqcup I$ (disjoint union of two intervals) and $F : W \rightarrow S^0$ defined by mapping the first copy of I by the constant map to y_+ and mapping the second copy of I to y_- . Then the disjoint union $\{y_+\} \sqcup S^0$ is contained in ∂W , F restricted to $\{y_+\} \sqcup S^0$ agrees with the disjoint union of the inclusion $y_+ \hookrightarrow S^0$ and the identity map $S^0 \rightarrow S^0$, while F maps $\partial W - (\{y_+\} \sqcup S^0)$ to y_- . Hence (W, F) is a valid bordism. Consequently,

$$\begin{aligned}
e_{\text{IP}} \cap \mu_1 &= \delta' i^* \text{exc}^{-1}(1) \cap \mu_1 \\
&= -\iota_+(1 \cap \text{exc}^{-1} i_* \partial(\mu_1)) \\
&= -\iota_+(\text{exc}^{-1} i_* \partial(\mu_1)) \\
&= -\iota_+[\text{id}_{y_+}] \\
&= -[\{y_+\} \hookrightarrow D^1]
\end{aligned}$$

and so, with $s_2 = n - m$,

$$\begin{aligned}
e_{\text{IP}}^{n-m} \cap \mu &= (-1)^{s_1} (-1)^{s_2} [\{y_+\} \hookrightarrow D^1] \times \cdots \times [\{y_+\} \hookrightarrow D^1] \\
&= (-1)^s [\text{pt} \hookrightarrow D^{n-m}].
\end{aligned}$$

For any pair (W, V) of IP-spaces, $n = \dim W$, let $\text{Si} : (\Omega_{\text{IP}})_n(W, V)_{\mathbb{Q}} \rightarrow \text{L}_n^{\bullet}(W, V)_{\mathbb{Q}}$ be the composition

$$(\Omega_{\text{IP}})_n(W, V) \xleftarrow{\cong} (\mathbf{Q}_{\text{IP}})_n(W, V) \xrightarrow{\text{Sig}} (\mathbf{Q}_{\geq 0, \text{rel}}^{\mathbb{Z}})_n(W, V) \xleftarrow{\cong} \text{L}_n^{\bullet}(W, V)$$

(which is just (8.2) in the absolute case), tensored with $\text{id}_{\mathbb{Q}}$. The following diagram commutes:

$$\begin{array}{ccccc}
(\Omega_{\text{IP}})_n(X)_{\mathbb{Q}} & \xrightarrow{\text{Si}} & L_n^{\bullet}(X)_{\mathbb{Q}} & \xrightarrow[\cong]{S_X} & \bigoplus_q H_{n-4q}(X; \mathbb{Q}) \\
\downarrow & & \downarrow & & \downarrow \\
(\Omega_{\text{IP}})_n(X, X-Y)_{\mathbb{Q}} & \xrightarrow{\text{Si}} & L_n^{\bullet}(X, X-Y)_{\mathbb{Q}} & \xrightarrow[\cong]{S_{(X, X-Y)}} & \bigoplus_q H_{n-4q}(X, X-Y; \mathbb{Q}) \\
\cong \uparrow \text{exc} & & \cong \uparrow \text{exc} & & \cong \uparrow \text{exc} \\
(\Omega_{\text{IP}})_n(D\nu, S\nu)_{\mathbb{Q}} & \xrightarrow{\text{Si}} & L_n^{\bullet}(D\nu, S\nu)_{\mathbb{Q}} & \xrightarrow[\cong]{S_{(D\nu, S\nu)}} & \bigoplus_q H_{n-4q}(D\nu, S\nu; \mathbb{Q}) \\
u_{\text{IP}} \cap - \downarrow & & u_{\mathbb{L}} \cap - \downarrow & & u \cap - \downarrow \\
(\Omega_{\text{IP}})_m(Y \times D^{n-m})_{\mathbb{Q}} & \xrightarrow{\text{Si}} & L_m^{\bullet}(Y \times D^{n-m})_{\mathbb{Q}} & \xrightarrow[\cong]{S_{Y \times D^{n-m}}} & \bigoplus_q H_{m-4q}(Y \times D^{n-m}; \mathbb{Q}) \\
\cong \downarrow \pi_* & & \cong \downarrow \pi_* & & \cong \downarrow \pi_* \\
(\Omega_{\text{IP}})_m(Y)_{\mathbb{Q}} & \xrightarrow{\text{Si}} & L_m^{\bullet}(Y)_{\mathbb{Q}} & \xrightarrow[\cong]{S_Y} & \bigoplus_q H_{m-4q}(Y; \mathbb{Q})
\end{array}$$

The left column, multiplied by $(-1)^s$, defines a map $j_{\text{IP}}^! : (\Omega_{\text{IP}})_n(X)_{\mathbb{Q}} \rightarrow (\Omega_{\text{IP}})_m(Y)_{\mathbb{Q}}$. The image of $[X]_{\text{IP}}$ in $(\Omega_{\text{IP}})_n(X, X-Y)_{\mathbb{Q}}$ equals the image of $[\text{id}_{(D\nu, S\nu)}] \in (\Omega_{\text{IP}})_n(D\nu, S\nu)_{\mathbb{Q}}$ under the excision isomorphism; the required bordism is given by the ∂ -IP-space W obtained from gluing the cylinder $X \times I$ to the cylinder $D\nu \times I$ along the canonical inclusion $D\nu \times \{1\} \hookrightarrow X \times \{0\}$. The map $F : W \rightarrow X$ is defined by $F(x, t) = x$ for $(x, t) \in X \times I$ and $(x, t) \in D\nu \times I$. Note that F maps $\partial W - (X \times \{1\} \sqcup D\nu \times \{0\})$ to $X - Y$, whence (W, F) is indeed a viable bordism. Using the cross product on IP-bordism

$$(\Omega_{\text{IP}})_{n-m}(D^{n-m}, S^{n-m-1})_{\mathbb{Q}} \otimes (\Omega_{\text{IP}})_m(Y)_{\mathbb{Q}} \xrightarrow{\times} (\Omega_{\text{IP}})_n(D\nu, S\nu)_{\mathbb{Q}},$$

we may express the element $[\text{id}_{(D\nu, S\nu)}]$ as

$$[\text{id}_{(D\nu, S\nu)}] = \mu \times [Y]_{\text{IP}}.$$

Thus, using (9.1), we find that

$$\begin{aligned}
\pi_*(u_{\text{IP}} \cap [\text{id}_{(D\nu, S\nu)}]) &= \pi_*((e_{\text{IP}}^{n-m} \times 1) \cap (\mu \times [Y]_{\text{IP}})) \\
&= \pi_*((-1)^{\deg(1) \deg(\mu)} (e_{\text{IP}}^{n-m} \cap \mu) \times (1 \cap [Y]_{\text{IP}})) \\
&= \pi_*((e_{\text{IP}}^{n-m} \cap \mu) \times (1 \cap [Y]_{\text{IP}})) \\
&= (-1)^s \pi_*([\text{pt} \hookrightarrow D^{n-m}] \times [Y]_{\text{IP}}) \\
&= (-1)^s [Y]_{\text{IP}}.
\end{aligned}$$

This proves that

$$j_{\text{IP}}^![X]_{\text{IP}} = [Y]_{\text{IP}}.$$

By the commutativity of the above diagram,

$$j^! S_X [X]_{\mathbb{L}} = j^! S_X \text{Si}[X]_{\text{IP}} = S_Y \text{Si} j_{\text{IP}}^![X]_{\text{IP}} = S_Y \text{Si}[Y]_{\text{IP}} = S_Y [Y]_{\mathbb{L}},$$

which proves property (1).

It remains to establish property (2). The map $f : X \rightarrow \text{pt}$ from X to a point induces a homomorphism

$$f_* : \bigoplus_q H_{n-4q}(X; \mathbb{Q}) \longrightarrow \bigoplus_q H_{n-4q}(\text{pt}; \mathbb{Q})$$

such that $(S_X[X]_{\mathbb{L}})_0 = f_* S_X[X]_{\mathbb{L}}$. If the dimension n is not divisible by 4, then $\bigoplus_q H_{n-4q}(\text{pt}; \mathbb{Q}) = 0$ and thus $\epsilon_* f_* S_X[X]_{\mathbb{L}} = 0 = \sigma(X)$, that is, (2) holds. Assume that n is divisible by 4, so that $\bigoplus_q H_{n-4q}(\text{pt}; \mathbb{Q}) = H_0(\text{pt}; \mathbb{Q})$. Using the commutative diagram

$$\begin{array}{ccc} L_n^\bullet(X)_{\mathbb{Q}} & \xrightarrow[\cong]{S_X} & \bigoplus_q H_{n-4q}(X; \mathbb{Q}) \\ f_* \downarrow & & \downarrow f_* \\ L_n^\bullet(\text{pt})_{\mathbb{Q}} & \xrightarrow[\cong]{S_{\text{pt}}} & \bigoplus_q H_{n-4q}(\text{pt}; \mathbb{Q}) \end{array}$$

we can write

$$(9.2) \quad f_* S_X[X]_{\mathbb{L}} = S_{\text{pt}} f_* [X]_{\mathbb{L}}.$$

Let $\{1\} = \pi_1(\text{pt})$ denote the trivial fundamental group of the point. The associated assembly map

$$L_n^\bullet(\text{pt}) = L_n^\bullet(B\{1\}) \xrightarrow{\alpha_{\{1\}}} L^n(\mathbb{Z}\{\{1\}\})$$

is an isomorphism. Recall that when n is divisible by 4, there is an isomorphism $\sigma : L^n(\mathbb{Z}\{\{1\}\}) \cong \mathbb{Z}$ given by the signature σ . The diagram

$$\begin{array}{ccc} L_n^\bullet(\text{pt})_{\mathbb{Q}} & \xrightarrow[\cong]{S_{\text{pt}}} & \bigoplus_q H_{n-4q}(\text{pt}; \mathbb{Q}) \\ \parallel & & \parallel \\ L_n^\bullet(B\{1\})_{\mathbb{Q}} & & H_0(\text{pt}; \mathbb{Q}) \\ \alpha_{\{1\}} \downarrow \cong & & \cong \downarrow \epsilon_* \\ L^n(\mathbb{Z}\{\{1\}\})_{\mathbb{Q}} & \xrightarrow[\cong]{\sigma} & \mathbb{Q} \end{array}$$

commutes, as the calculation

$$\begin{aligned} \epsilon_* S_{\text{pt}}[\text{pt}]_{\mathbb{L}} &= \epsilon_*(L^*(\text{pt}) \cap [\text{pt}]_{\mathbb{Q}}) = \epsilon_*(1 \cap [\text{pt}]_{\mathbb{Q}}) = \epsilon_*[\text{pt}]_{\mathbb{Q}} = 1 \\ &= \sigma(\sigma^*(\text{pt})) = \sigma(\alpha_{\{1\}}[\text{pt}]_{\mathbb{L}}), \end{aligned}$$

using e.g. [Ran92], shows. (The formula $S_M[M]_{\mathbb{L}} = L^*(M) \cap [M]_{\mathbb{Q}}$ holds for any closed smooth oriented n -manifold, where $L^*(M) \in H^*(M; \mathbb{Q})$ is the Hirzebruch L -class and $[M]_{\mathbb{Q}} \in H_n(M; \mathbb{Q})$ the rational fundamental class.) Using this diagram, we obtain

$$(9.3) \quad \epsilon_* S_{\text{pt}} f_* [X]_{\mathbb{L}} = \sigma \alpha_{\{1\}} f_* [X]_{\mathbb{L}}.$$

The diagram

$$\begin{array}{ccc}
L_n^\bullet(X)_\mathbb{Q} & \xrightarrow{f_*} & L_n^\bullet(\text{pt})_\mathbb{Q} \\
r_* \downarrow & & \parallel \\
L_n^\bullet(BG)_\mathbb{Q} & & L_n^\bullet(B\{1\})_\mathbb{Q} \\
\alpha_G \downarrow & & \downarrow \alpha_{\{1\}} \\
L^n(\mathbb{Z}[G])_\mathbb{Q} & \xrightarrow{f_*} & L^n(\mathbb{Z}[\{1\}])_\mathbb{Q}
\end{array}$$

commutes by the naturality of the assembly map. Consequently,

$$(9.4) \quad f_* \alpha_G r_* [X]_\mathbb{L} = \alpha_{\{1\}} f_* [X]_\mathbb{L}.$$

The ordinary signature information is contained in the symmetric signature by

$$(9.5) \quad \sigma f_* \sigma_{\text{IP}}^*(X) = \sigma(X).$$

Putting equations (9.2), (9.3), (9.4) and (9.5) together, we compute

$$\begin{aligned}
\epsilon_*(S_X[X]_\mathbb{L})_0 &= \epsilon_* f_* S_X[X]_\mathbb{L} = \epsilon_* S_{\text{pt}} f_* [X]_\mathbb{L} = \sigma \alpha_{\{1\}} f_* [X]_\mathbb{L} \\
&= \sigma f_* \alpha_G r_* [X]_\mathbb{L} = \sigma f_* \sigma_{\text{IP}}^*(X) = \sigma(X),
\end{aligned}$$

as was to be shown. \square

Let $G = \pi_1(X)$ be the fundamental group and $r : X \rightarrow BG$ a classifying map for the universal cover of X . The map r induces a homomorphism

$$H_*(X; \mathbb{Q}) \longrightarrow H_*(BG; \mathbb{Q})$$

on homology. The *higher signatures* of X are the rational numbers

$$\langle a, r_* L(X) \rangle, \quad a \in H^*(BG; \mathbb{Q}).$$

Theorem 9.2. *Let X be an n -dimensional compact IP-space whose fundamental group $G = \pi_1(X)$ satisfies the strong Novikov conjecture. Then the higher signatures of X are stratified homotopy invariants.*

Proof. Let X and X' be n -dimensional compact IP-spaces with fundamental group G and $f : X' \rightarrow X$ an orientation preserving stratified homotopy equivalence. If $r : X \rightarrow BG$ is a classifying map for the universal cover of X , then $r' = r \circ f : X' \rightarrow BG$ is a classifying map for the universal cover of X' . We must prove that

$$r'_* L(X') = r_* L(X) \in H_*(BG; \mathbb{Q}).$$

By Theorem 8.2(ii), the assembly map

$$L_n^\bullet(X) \longrightarrow L^n(\mathbb{Z}[G])$$

maps $[X]_{\mathbb{L}}$ to $\sigma_{\text{IP}}^*(X)$. Thus, using the factorization

$$\begin{array}{ccc} \mathbf{L}_n^\bullet(X) & \longrightarrow & L^n(\mathbb{Z}[G]) \\ r_* \downarrow & \nearrow \alpha & \\ \mathbf{L}_n^\bullet(BG) & & \end{array}$$

(similarly for X'), we may write

$$\alpha r_*[X]_{\mathbb{L}} = \sigma_{\text{IP}}^*(X), \quad \alpha r'_*[X']_{\mathbb{L}} = \sigma_{\text{IP}}^*(X').$$

The symmetric signature is known to be a stratified homotopy invariant, see [FM13b]. Therefore,

$$\sigma_{\text{IP}}^*(X) = \sigma_{\text{IP}}^*(r) = \sigma_{\text{IP}}^*(rf) = \sigma_{\text{IP}}^*(r') = \sigma_{\text{IP}}^*(X').$$

As α is by assumption rationally injective, it follows that

$$r_*[X]_{\mathbb{L}} = r'_*[X']_{\mathbb{L}} \in \mathbf{L}_n^\bullet(BG) \otimes \mathbb{Q}.$$

Using the commutative diagram

$$\begin{array}{ccc} \mathbf{L}_n^\bullet(X) \otimes \mathbb{Q} & \xrightarrow{r_*} & \mathbf{L}_n^\bullet(BG) \otimes \mathbb{Q} \\ S_X \downarrow \cong & & \cong \downarrow S_{BG} \\ \bigoplus_j H_{n-4j}(X; \mathbb{Q}) & \xrightarrow{r_*} & \bigoplus_j H_{n-4j}(BG; \mathbb{Q}) \end{array}$$

(and the analogous diagram for X'), together with Lemma 9.1, we deduce

$$\begin{aligned} r_*L(X) &= r_*S_X[X]_{\mathbb{L}} = S_{BG}r_*[X]_{\mathbb{L}} \\ &= S_{BG}r'_*[X']_{\mathbb{L}} = r'_*S_{X'}[X']_{\mathbb{L}} = r'_*L(X'). \end{aligned}$$

□

An analytic version of Theorem 9.2 has been proven by Albin-Leichtnam-Mazzeo-Piazza in [ALMP13]. The scope of their theorem is in fact larger, as it applies even to those non-Witt spaces, for which a so-called analytic self-dual mezzoperversity exists. It was shown in [ABLMP13] that such perversity data corresponds topologically to the Lagrangian structures of Banagl as introduced in [Ban02]. A comparison of the analytic argument to our argument shows that the role of our $\mathbf{L}_n^\bullet(X)$ is played in the analytic context by $K_*(X)$. The role of the isomorphisms S_X is played by the Chern character. The group $L^n(\mathbb{Z}[G])$ corresponds to $K_*(C_r^*G)$, while our assembly map α corresponds to the assembly map $K_*(BG) \rightarrow K_*(C_r^*G)$ used in the analytic argument.

10. MULTIPLICATIVITY AND COMMUTATIVITY

Recall from Definition 6.1 that the symmetric signature map

$$\text{Sig} : \mathbf{M}_{\text{IP}} \rightarrow \mathbf{M}_{\geq 0, \text{rel}}^{\mathbb{Z}}$$

is the following composite in the homotopy category of spectra:

$$\mathbf{M}_{\text{IP}} \xleftarrow{\simeq} \mathbf{M}_{\text{IPFun}} \xrightarrow{\text{sig}} \mathbf{M}_{\geq 0, \text{rel}}^{\mathbb{Z}}.$$

In this section we show that this composite is weakly equivalent to a composite of ring maps between commutative ring spectra. Specifically, we show

Theorem 10.1. *There are symmetric ring spectra \mathbf{A} , \mathbf{B} and \mathbf{C} , a commutative symmetric ring spectrum \mathbf{D} , and a strictly commutative diagram*

$$(10.1) \quad \begin{array}{ccccc} \mathbf{M}_{\text{IP}} & \xleftarrow{\simeq} & \mathbf{A} & \xrightarrow{\simeq} & \mathbf{M}_{\text{IP}}^{\text{comm}} \\ \uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq \\ \mathbf{M}_{\text{IPFun}} & \xleftarrow{\simeq} & \mathbf{B} & \xrightarrow{\simeq} & \mathbf{M}_{\text{IPFun}}^{\text{comm}} \\ \text{sig} \downarrow & & \downarrow & & \downarrow \\ \mathbf{M}_{\geq 0, \text{rel}}^{\mathbb{Z}} & \xleftarrow{\simeq} & \mathbf{C} & \xrightarrow{\simeq} & \mathbf{D} \end{array}$$

in which the horizontal arrows, the upper vertical arrows and the lower right vertical arrow are ring maps.

Remark 10.2. \mathbf{D} is weakly equivalent to $(\mathbf{M}_{\geq 0, \text{rel}}^{\mathbb{Z}})^{\text{comm}}$ by [LM, Remark 17.3].

The rest of this section is devoted to the proof of Theorem 10.1. The top half of the diagram has already been constructed in Remark 6.4. For the lower half we will use the method of the proof of [LM, Theorem 1.3] (it will be straightforward to check that the maps $\mathbf{M}_{\text{IPFun}} \leftarrow \mathbf{B} \rightarrow \mathbf{M}_{\text{IPFun}}^{\text{comm}}$ given by the proof of [LM, Theorem 1.1] are the same as those given by the proof of [LM, Theorem 1.3]).

Remark 10.3. In order to apply the proof of [LM, Theorem 1.3] without change we would need to know (by analogy with the paragraph before [LM, Definition 14.5]) that the cross product gave a natural quasi-isomorphism from the functor

$$(\mathcal{A}_{\text{IPFun}})^{\times l} \xrightarrow{\text{sig}^{\times l}} (\mathcal{A}_{\text{rel}}^{\mathbb{Z}})^{\times l} \xrightarrow{\otimes} \mathcal{A}_{\text{rel}}^{\mathbb{Z}}$$

to the functor

$$(\mathcal{A}_{\text{IPFun}})^{\times l} \rightarrow \mathcal{A}_{\text{IPFun}} \xrightarrow{\text{sig}} \mathcal{A}_{\text{rel}}^{\mathbb{Z}}$$

(where the unmarked arrow is the product in $\mathcal{A}_{\text{IPFun}}$). But this is not the case, for the simple reason that the cross product does not give a map

$$IS_*^{Q_{\bar{n}, \bar{n}}}(X \times X; \mathbb{Z})^{\otimes l} \rightarrow IS_*^{Q_{\bar{n}, \bar{n}}}(X \times X; \mathbb{Z})$$

(cf. [F, Lemma 6.43]). Our first task is to provide a suitable substitute, which will be given in Proposition 10.11.

Definition 10.4. Let Y_1, \dots, Y_k be stratified PL ∂ -pseudomanifolds and give $Y_1 \times \dots \times Y_k$ the product stratification. Define a perversity Q_k on $Y_1 \times \dots \times Y_k$ by

$$Q_k(S_1 \times \dots \times S_k) = 2s - 2 + \sum \bar{n}(S_i),$$

where the S_i are strata and s is the number of S_i that are singular.

In particular, $Q_1 = \bar{n}$ and $Q_2 = Q_{\bar{n}, \bar{n}}$.

Lemma 10.5. *The cross product induces a quasi-isomorphism*

$$IS_*^{Q_j}(Y_1 \times \dots \times Y_j; \mathbb{Z}) \otimes IS_*^{Q_k}(Y_{j+1} \times \dots \times Y_{j+k}; \mathbb{Z}) \rightarrow IS_*^{Q_{j+k}}(Y_1 \times \dots \times Y_{j+k}; \mathbb{Z}).$$

This is immediate from [F, Lemma 6.43 and Theorem 6.45]. We need a more general version of this.

Definition 10.6. (i) Let A be a finite totally ordered set. A *partition* ρ of A is a collection B_1, \dots, B_k of disjoint subsets of A such that $\cup B_i = A$ and $a < a'$ whenever $a \in B_i, a' \in B_{i'}$ with $i < i'$.

(ii) Let X_1, \dots, X_l be stratified PL ∂ -pseudomanifolds and let

$$\rho = \{B_1, \dots, B_k\}$$

be a partition of $\{1, \dots, l\}$. Let $Y_i = \prod_{j \in B_i} X_j$ and give Y_i the product stratification. Define

$$IS_*^\rho(X_1 \times \dots \times X_l; \mathbb{Z})$$

to be

$$IS_*^{Q_k}(Y_1 \times \dots \times Y_k; \mathbb{Z}).$$

Lemma 10.7. *The cross product induces a quasi-isomorphism*

$$IS_*^\rho(X_1 \times \dots \times X_l; \mathbb{Z}) \otimes IS_*^{\rho'}(X_{l+1} \times \dots \times X_{l+m}; \mathbb{Z}) \rightarrow IS_*^{\rho \cup \rho'}(X_1 \times \dots \times X_{l+m}; \mathbb{Z}).$$

This is immediate from Lemma 10.5.

Definition 10.8. Let $\rho = \{B_1, \dots, B_j\}$ and $\rho' = \{C_1, \dots, C_k\}$ be two partitions of a set A . Then ρ' is a *refinement* of ρ if each C_i is contained in some B_i .

Lemma 10.9. *Let ρ and ρ' be partitions of $\{1, \dots, l\}$. If ρ' is a refinement of ρ then*

$$IS_*^\rho(X_1 \times \dots \times X_l; \mathbb{Z}) \subset IS_*^{\rho'}(X_1 \times \dots \times X_l; \mathbb{Z})$$

and the inclusion is a quasi-isomorphism.

Proof. The inclusion follows from the fact that the perversity that gives IS_*^ρ is \leq the perversity that gives $IS_*^{\rho'}$. To show the quasi-isomorphism it suffices to show that

$$IS_*^{\bar{n}}(X_1 \times \dots \times X_l; \mathbb{Z}) \hookrightarrow IS_*^\rho(X_1 \times \dots \times X_l; \mathbb{Z})$$

is a quasi-isomorphism for every ρ . This in turn follows by induction from the following commutative diagram, where we let $\rho = \{B_1, \dots, B_k\}$ and $\rho_1 = \{B_2, \dots, B_k\}$.

$$\begin{array}{ccc} IS_*^{\bar{n}}(\prod_{i \in B_1} X_i; \mathbb{Z}) \otimes IS_*^{\bar{n}}(\prod_{i \notin B_1} X_i; \mathbb{Z}) & \xrightarrow{\times} & IS_*^{\bar{n}}(X_1 \times \dots \times X_l; \mathbb{Z}) \\ \downarrow & & \downarrow \\ IS_*^{\bar{n}}(\prod_{i \in B_1} X_i; \mathbb{Z}) \otimes IS_*^{\rho_1}(\prod_{i \notin B_1} X_i; \mathbb{Z}) & \xrightarrow{\times} & IS_*^{\rho}(X_1 \times \dots \times X_l; \mathbb{Z}) \end{array}$$

Here the horizontal arrows are quasi-isomorphisms by [F, Theorem 6.45] and the left vertical arrow is a quasi-isomorphism by the inductive hypothesis, so the right vertical arrow is a quasi-isomorphism as required. \square

Next is the analogue of Lemma 6.5 for this situation. Recall [LM, Definition 14.3(i)].

Lemma 10.10. (i) *Let $l \geq 1$ and let $\rho = \{B_1, \dots, B_k\}$ be a partition of $\{1, \dots, l\}$. Let $\bar{\rho}$ be the partition $\{B_1, \dots, B_k, B_1, \dots, B_k\}$ of $\{1, \dots, l\} \amalg \{1, \dots, l\}$. Let (X_i, ξ_i) for $1 \leq i \leq l$ be objects of $\mathcal{A}_{\text{IPFun}}$. Give each X_i the stratification of Proposition A.2 and give $X_1 \times \dots \times X_l$ and $(X_1 \times \dots \times X_l) \times (X_1 \times \dots \times X_l)$ the product stratifications. Then (C, D, β, φ) is an object of $\mathcal{A}_{\text{rel}}^{\mathbb{Z}}$, where*

$$\begin{aligned} C &= IS_*^{\bar{n}}(X_1 \times \dots \times X_l; \mathbb{Z}), \\ D &= IS_*^{\bar{\rho}}((X_1 \times \dots \times X_l) \times (X_1 \times \dots \times X_l); \mathbb{Z}), \\ \beta &\text{ is the cross product, and} \\ \varphi &\text{ is the image of } \xi_1 \times \dots \times \xi_l \text{ under the composite} \end{aligned}$$

$$\begin{aligned} IS_*^{\bar{0}}(X_1 \times \dots \times X_l; \mathbb{Z}) &\rightarrow IS_*^{\text{Q}\bar{n}, \bar{n}}((X_1 \times \dots \times X_l) \times (X_1 \times \dots \times X_l); \mathbb{Z}) \\ &\hookrightarrow IS_*^{\bar{\rho}}(X_1 \times \dots \times X_l \times X_1 \times \dots \times X_l; \mathbb{Z}), \end{aligned}$$

where the first map is induced by the diagonal and the second is given by Lemma 10.9.

(ii) *For $1 \leq i \leq l$, let $f_i : (X_i, \xi_i) \rightarrow (X'_i, \xi'_i)$ be a morphism in $\mathcal{A}_{\text{IPFun}}$. Let (C, D, β, φ) and $(C', D', \beta', \varphi')$ be the objects of $\mathcal{A}_{\text{rel}}^{\mathbb{Z}}$ corresponding to the l -tuples $\{(X_i, \xi_i)\}$ and $\{(X'_i, \xi'_i)\}$. Then the f_i induce a morphism $(C, D, \beta, \varphi) \rightarrow (C', D', \beta', \varphi')$.*

Proof. Part (i) follows from Lemma 10.9 and the fact (shown in the proof of Lemma 6.5(i)) that $IS_*^{\bar{n}}(X_1 \times \dots \times X_l; \mathbb{Z})$ is homotopy finite. Part (ii) follows from the proof of Lemma 6.5(ii). \square

Let

$$\text{sig}_{\rho} : (\mathcal{A}_{\text{IPFun}})^{\times l} \rightarrow \mathcal{A}_{\text{rel}}^{\mathbb{Z}}$$

be the functor given by Lemma 10.10. Now we can give the statement promised in Remark 10.3.

Proposition 10.11. *Let $\{B_1, \dots, B_k\}$ be a partition of $\{1, \dots, l\}$, let ρ_i be a partition of B_i for $1 \leq i \leq k$, and let ρ be a refinement of $\rho_1 \cup \dots \cup \rho_k$. The cross product gives a natural quasi-isomorphism from*

$$(\mathcal{A}_{\text{IPFun}})^{\times l} \xrightarrow{\prod \text{sig}_{\rho_i}} (\mathcal{A}_{\text{rel}}^{\mathbb{Z}})^{\times k} \xrightarrow{\otimes} \mathcal{A}_{\text{rel}}^{\mathbb{Z}}$$

to

$$(\mathcal{A}_{\text{IPFun}})^{\times l} \xrightarrow{\text{sig}_{\rho'}} \mathcal{A}_{\text{rel}}^{\mathbb{Z}}$$

This is immediate from Lemmas 10.7 and 10.9. Next we need the analogue of [LM, Definition 14.4]. Recall [LM, Definition 14.3(ii)].

Definition 10.12. Let $j \geq 0$ and let $r : \{1, \dots, j\} \rightarrow \{u, v\}$. Let \mathcal{A}_i denote $\mathcal{A}_{\text{IPFun}}$ if $r(i) = u$ and $\mathcal{A}_{\text{rel}}^{\mathbb{Z}}$ if $r(i) = v$.

(i) Let $1 \leq m \leq j$. A surjection

$$h : \{1, \dots, j\} \rightarrow \{1, \dots, m\}$$

is *adapted to r* if r is constant on each set $h^{-1}(i)$ and h is monic on $r^{-1}(v)$.

(ii) Given a surjection

$$h : \{1, \dots, j\} \rightarrow \{1, \dots, m\}$$

which is adapted to r , and a partition ρ_i of $h^{-1}(i)$ for $1 \leq i \leq m$, define

$$(h, \rho_1, \dots, \rho_m)_{\blacklozenge} : \mathcal{A}_1 \times \dots \times \mathcal{A}_j \rightarrow (\mathcal{A}_{\text{rel}}^{\mathbb{Z}})^{\times m}$$

by

$$(h, \rho_1, \dots, \rho_m)_{\blacklozenge}(x_1, \dots, x_j) = (i^\epsilon y_1, \dots, y_m),$$

where i^ϵ is the sign that arises from putting the objects x_1, \dots, x_j into the order $x_{\theta(h)^{-1}(1)}, \dots, x_{\theta(h)^{-1}(j)}$ and

$$y_i = \begin{cases} \text{sig}_{\rho_i}(\{x_p\}_{p \in h^{-1}(i)}) & \text{if } h^{-1}(i) \subset r^{-1}(u), \\ x_{h^{-1}(i)} & \text{if } h^{-1}(i) \in r^{-1}(v). \end{cases}$$

(iii) A *datum* of type r is a tuple

$$(h, \rho_1, \dots, \rho_m, \eta),$$

where h is a surjection which is adapted to r , ρ_i is a partition of $h^{-1}(i)$, and η is an element of Σ_j with the property that $h \circ \eta = h$.

(iv) Given a datum

$$\mathbf{d} = (h, \rho_1, \dots, \rho_m, \eta),$$

of type r , define

$$\mathbf{d}_{\blacksquare} : \mathcal{A}_1 \times \dots \times \mathcal{A}_j \rightarrow \mathcal{A}_{\text{rel}}^{\mathbb{Z}}$$

to be the composite

$$\begin{aligned} \mathcal{A}_1 \times \dots \times \mathcal{A}_j &\xrightarrow{\eta} \mathcal{A}_{\eta^{-1}(1)} \times \dots \times \mathcal{A}_{\eta^{-1}(j)} = \mathcal{A}_1 \times \dots \times \mathcal{A}_j \\ &\xrightarrow{(h, \rho_1, \dots, \rho_m)_{\blacklozenge}} (\mathcal{A}_{\text{rel}}^{\mathbb{Z}})^{\times m} \xrightarrow{\otimes} \mathcal{A}_{\text{rel}}^{\mathbb{Z}}, \end{aligned}$$

where η permutes the factors with the usual sign.

Finally, we have the analogue of [LM, Definition 14.5].

Definition 10.13. For data of type r , define

$$(h, \rho_1, \dots, \rho_m, \eta) \leq (h', \rho'_1, \dots, \rho'_{m'}, \eta')$$

if for each $i \in \{1, \dots, m\}$ there is a $p \in \{1, \dots, m'\}$ such that $\eta^{-1}(h^{-1}(i))$ is contained in $\eta'^{-1}(h'^{-1}(p))$ and $\eta'\eta^{-1}$ takes each piece of the partition ρ_i to a union of pieces of the partition ρ'_p .

With these changes, the proof of [LM, Theorem 1.3] goes through to construct the lower half of Diagram (10.1). This completes the proof of Theorem 10.1.

11. MULTIPLICATIVITY OF THE L -THEORY FUNDAMENTAL CLASS

In this section we prove

Theorem 11.1. *Let X and Y be compact oriented IP spaces. Then*

$$[X \times Y]_{\mathbb{L}} = [X]_{\mathbb{L}} \times [Y]_{\mathbb{L}}.$$

The first step in the proof is to observe that we can replace the spectra \mathbf{Q} in Definition 8.5 by the equivalent symmetric spectra \mathbf{M} . Each of the symmetric spectra $\mathbf{M}_{\text{IP}, Z}$, $Z_+ \wedge \mathbf{M}_{\text{IP}}$, $Z_+ \wedge \mathbf{M}_{\geq 0, \text{rel}}^{\mathbb{Z}}$ and $Z_+ \wedge \mathbf{M}_{\geq 0}^{\mathbb{Z}}$ is semistable ([HSS00, Definition 5.6.1]) by [LM13, Corollary 17.9(i)] and [Sch08, Examples 4.2 and 4.7], and hence their “true” (i.e., derived) homotopy groups agree with their homotopy groups by [Sch08, Example 5.5]. Thus for a compact oriented IP space Z of dimension l the class $[Z]_{\mathbb{L}}$ is the image of $[Z]_{\text{IP}}$ under the composite

$$(11.1) \quad (\Omega_{\text{IP}})_l(Z) \xrightarrow{\cong} \pi_l \mathbf{M}_{\text{IP}, Z} \xleftarrow[\cong]{\alpha} \pi_l(Z_+ \wedge \mathbf{M}_{\text{IP}}) \xrightarrow{\text{Sig}} \pi_l(Z_+ \wedge \mathbf{M}_{\geq 0, \text{rel}}^{\mathbb{Z}}) \xleftarrow[\cong]{} \pi_l(Z_+ \wedge \mathbf{M}_{\geq 0}^{\mathbb{Z}}).$$

Next we observe that the functors in (11.1) have product operations. For the first functor (and for any spaces X and Y), Cartesian product induces a map

$$(\Omega_{\text{IP}})_m(X) \otimes (\Omega_{\text{IP}})_n(Y) \rightarrow (\Omega_{\text{IP}})_{m+n}(X \times Y).$$

For the second functor, Cartesian product induces

$$\mathcal{A}_{\text{IP}, X} \times \mathcal{A}_{\text{IP}, Y} \rightarrow \mathcal{A}_{\text{IP}, X \times Y}$$

and this induces a map

$$\mathbf{M}_{\text{IP}, X} \wedge \mathbf{M}_{\text{IP}, Y} \rightarrow \mathbf{M}_{\text{IP}, X \times Y}$$

which gives the desired product. The third, fourth and fifth functors in (11.1) have products because \mathbf{M}_{IP} , $\mathbf{M}_{\geq 0, \text{rel}}^{\mathbb{Z}}$ and $\mathbf{M}_{\geq 0}^{\mathbb{Z}}$ are ring spectra.

It therefore suffices to show that the maps in the composite (11.1) preserve products. For the second map this follows from Proposition C.1, for the third map from Theorem 10.1, and for the fourth map from [LM, Remark 12.2]. We will denote the first map by χ , so it remains to show

Lemma 11.2. *The map*

$$\chi : (\Omega_{\text{IP}})_*(Z) \xrightarrow{\cong} \pi_* \mathbf{M}_{\text{IP},Z}$$

preserves products.

The rest of this section gives the proof of this lemma.

Recall that for a spectrum \mathbf{Q} or a symmetric spectrum \mathbf{M} we write Q_k and M_k for the k -th space. The map χ can be written as the composite

$$(11.2) \quad (\Omega_{\text{IP}})_l(Z) = \pi_l(Q_{\text{IP},Z})_0 \rightarrow \pi_{l+k}(Q_{\text{IP},Z})_k \rightarrow \pi_{l+k}(M_{\text{IP},Z})_k \rightarrow \pi_l \mathbf{M}_{\text{IP},Z}$$

for $k \geq 1$ (the first arrow is described in [LM13, Section 15], but one should use the signs in Appendix D below; the second arrow is described in [LM13, Section 17]).

If $f : W \rightarrow Z$ is a map from an l -dimensional compact oriented IP space to Z , and if $p \leq l+2$, let us write $f^{[l+1]}$ (resp., $f^{[p, l+2-p]}$) for the Δ^{l+1} -ad (resp., $\Delta^p \times \Delta^{l+2-p}$ -ad) which takes the top cell with its canonical orientation to f and all other cells to $\emptyset \rightarrow Z$. Then $f^{[l+1]}$ (resp., $f^{[p, l+2-p]}$) determines an element of $\pi_{l+1}(M_{\text{IP},Z})_1$ (resp., $\pi_{l+2}(M_{\text{IP},Z})_2$) which we will denote by $\underline{f}^{[l+1]}$ (resp., $\underline{f}^{[p, l+2-p]}$). From [LM13, Sections 15 and 17] (but using the signs in Appendix D) we see that

$$(11.3) \quad \chi([f]) \text{ is represented by } \underline{f}^{[l+1]} \in \pi_{l+1}(M_{\text{IP},Z})_1$$

and

$$(11.4) \quad \chi([f]) \text{ is represented by } -\underline{f}^{[0, l+2]} \in \pi_{l+2}(M_{\text{IP},Z})_2.$$

Now let $g : U \rightarrow X$ and $h : V \rightarrow Y$ be maps from compact oriented IP spaces of dimensions m and n respectively; we need to show that

$$(11.5) \quad \chi([g])\chi([h]) = \chi([g \times h]).$$

By (11.3) and the proof of [Sch, Theorem I.4.54], $\chi([g])\chi([h])$ is represented by the composite

$$\begin{aligned} S^1 \wedge S^1 \wedge S^m \wedge S^n &\rightarrow S^1 \wedge S^m \wedge S^1 \wedge S^n \xrightarrow{\underline{g}^{[m+1]} \wedge \underline{h}^{[n+1]}} (M_{\text{IP},X})_1 \wedge (M_{\text{IP},Y})_1 \\ &\rightarrow (M_{\text{IP},X \times Y})_2 \end{aligned}$$

(cf. [Sch, I.4.55]). By [LM13, Section 18] this composite is equal to $-(\underline{g \times h})^{[m+1, n+1]}$, so by (11.4) the proof of (11.5) reduces to showing that $(\underline{g \times h})^{[m+1, n+1]} = (\underline{g \times h})^{[0, m+n+2]}$, and for this in turn it suffices to show for $0 \leq l \leq m$ that

$$(11.6) \quad (\underline{g \times h})^{[m+1-l, n+1+l]} = (\underline{g \times h})^{[m-l, n+2+l]}.$$

To prove (11.6), let F be the $(\Delta^{m+1-l} \times \Delta^{n+2+l})$ -ad which takes

- the top cell (with its canonical orientation) to the composite

$$U \times V \times I \rightarrow U \times V \xrightarrow{g \times h} X \times Y$$

(where the first map is the projection),

- the cell $\partial_0 \Delta^{m+1-l} \times \Delta^{n+2+l}$ (with its canonical orientation) to $(-1)^{m+n}$ times

$$U \times V \times \{1\} \rightarrow U \times V \xrightarrow{g \times h} X \times Y,$$

- the cell $\Delta^{m+1-l} \times \partial_0 \Delta^{n+2+l}$ (with its canonical orientation) to $(-1)^{n-l}$ times

$$U \times V \times \{0\} \rightarrow U \times V \xrightarrow{g \times h} X \times Y,$$

- and all other cells to the map $\emptyset \rightarrow X \times Y$.

Then F determines a map

$$\Phi : \Delta^{m+1-l} \times \Delta^{n+1+l+1} \rightarrow (M_{\text{IP}, X \times Y})_2,$$

and the restriction of Φ to the boundary of $\Delta^{m+1-l} \times \Delta^{n+2+l}$ (which is nullhomotopic) is easily seen (using the signs in Appendix D below) to be $(-1)^{m+n} \overline{(g \times h)^{[m-l, n+2+l]}} + (-1)^{m+1+n} \overline{(g \times h)^{[m+1-l, n+1+l]}}$; this proves (11.6) and completes the proof of Lemma 11.2 \square

12. PROOF OF THEOREM 8.4

By [WW95, Observation 1.3] it suffices to show that Φ is strongly excisive. The fact that $(\Omega_{\text{IP}})_*$ is a homology theory implies that Φ preserves arbitrary coproducts, so it suffices to show that Φ preserves homotopy cocartesian squares. First we observe that Φ takes monomorphisms to cofibrations in the level model structure given by [MMSS01, Theorem 6.5]. For a based space W , let

$$\bar{\Phi}(W) = \Phi(W)/\Phi(*).$$

Then the natural map

$$\Phi(W) \rightarrow \bar{\Phi}(W_+)$$

(where $+$ denotes a disjoint basepoint) is a weak equivalence because $(\Omega_{\text{IP}})_*$ is a homology theory. It therefore suffices to show that $\bar{\Phi}$ takes homotopy cocartesian squares of based spaces to homotopy cocartesian squares of spectra.

As a first step we give a relationship between $\Sigma \bar{\Phi}(W)$ and $\bar{\Phi}(\Sigma W)$. Let CW be the cone $I \wedge W$, where 1 is the basepoint of I , and let $S(W)$ denote the pushout of the diagram

$$\bar{\Phi}(CW) \leftarrow \bar{\Phi}(W) \rightarrow \bar{\Phi}(CW).$$

Since $\bar{\Phi}(CW)$ is contractible, and since $\bar{\Phi}$ takes monomorphisms to cofibrations in the level model structure, $S(W)$ is weakly equivalent to $\Sigma \bar{\Phi}(W)$. Since ΣW is the pushout of

$$CW \leftarrow W \rightarrow CW,$$

there is an evident map

$$\mathfrak{S} : S(W) \rightarrow \bar{\Phi}(\Sigma W).$$

Lemma 12.1. *\mathfrak{S} is a weak equivalence for all W .*

We defer the proof for a moment. Let

$$B \xleftarrow{i} A \xrightarrow{j} C$$

be a diagram of based spaces. The homotopy pushout of this diagram is the pushout of the diagram

$$Mi \hookrightarrow A \hookrightarrow Mj,$$

where Mi and Mj are the mapping cylinders; we will denote this pushout by D . The homotopy pushout of

$$\bar{\Phi}(B) \xleftarrow{\bar{\Phi}(i)} \bar{\Phi}(A) \xrightarrow{\bar{\Phi}(j)} \bar{\Phi}(C)$$

is (up to weak equivalence) the pushout, which we will denote by E , of

$$\bar{\Phi}(Mi) \hookrightarrow \bar{\Phi}(A) \hookrightarrow \bar{\Phi}(Mj).$$

It therefore suffices to show that the map $E \rightarrow \bar{\Phi}(D)$ is a weak equivalence. Consider the diagram

$$\begin{array}{ccccccccc} \pi_i \bar{\Phi}(A) & \rightarrow & \pi_i(\bar{\Phi}(B) \vee \bar{\Phi}(C)) & \longrightarrow & \pi_i E & \longrightarrow & \pi_i S(A) & \rightarrow & \pi_i(S(B) \vee S(C)) \\ \downarrow = & & \downarrow & & \downarrow & & \downarrow \mathfrak{S} & & \downarrow \\ \pi_i \bar{\Phi}(A) & \longrightarrow & \pi_i \bar{\Phi}(B \vee C) & \longrightarrow & \pi_i \bar{\Phi}(D) & \rightarrow & \pi_i \bar{\Phi}(\Sigma A) & \rightarrow & \pi_i \bar{\Phi}(\Sigma B \vee \Sigma C), \end{array}$$

where the rightmost vertical arrow is induced by the maps

$$S(B) \xrightarrow{\mathfrak{S}} \bar{\Phi}(\Sigma B) \rightarrow \bar{\Phi}(\Sigma B \vee \Sigma C)$$

and

$$S(C) \xrightarrow{\mathfrak{S}} \bar{\Phi}(\Sigma C) \rightarrow \bar{\Phi}(\Sigma B \vee \Sigma C).$$

The top row of the diagram is exact because it is π_* of a cofiber sequence. The fact that $(\Omega_{\text{IP}})_*$ is a homology theory implies that the second row of the diagram is exact, and also that the second vertical arrow is an isomorphism. The fourth and fifth vertical arrows are isomorphisms by Lemma 12.1, and hence the middle vertical arrow is an isomorphism as required.

It remains to prove Lemma 12.1. We begin by describing a suspension map

$$s : \pi_i \bar{\Phi}(W) \rightarrow \pi_{i+1} S(W).$$

Let $a \in \pi_i \bar{\Phi}(W)$. Let $\kappa_i(W)$ denote the kernel of the map $\pi_i \bar{\Phi}(W) \rightarrow \pi_i \bar{\Phi}(*)$; since $(\Omega_{\text{IP}})_*$ is a homology theory, the map $\kappa_i(W) \rightarrow \pi_i \bar{\Phi}(W)$ is an isomorphism, so a comes from an element $\tilde{a} \in \kappa_i(W)$. Let $\Phi_0(W)$ denote the 0-th space of the spectrum $\Phi(W)$; see [LM13, Definitions 15.8 and 15.4]. Since $\Phi_0(W)$ is a Kan complex ([LM13, Lemma 15.12]), \tilde{a} is represented by an i -simplex σ of $\Phi_0(W)$ with all faces at the basepoint. Since $\kappa_i(CW) = 0$, there is an $(i+1)$ -simplex τ of $\Phi_0(CW)$ with $\partial_0(\tau) = \sigma$ and all other faces

at the basepoint. Let τ_1 and τ_2 be the images of τ under the two inclusions of $\bar{\Phi}_0(CW)$ into $S_0(W)$ and let

$$(12.1) \quad D = \Delta^i \cup_{d^0 \Delta^i} \Delta^i;$$

then τ_1 and τ_2 give a map

$$\mathbf{a} : D/\partial D \rightarrow S(W)$$

which represents $s(a)$. Combining the map s with the weak equivalence between $S(W)$ and $\Sigma\bar{\Phi}(W)$ gives the usual suspension map $\pi_i\bar{\Phi}(W) \rightarrow \pi_{i+1}\Sigma\bar{\Phi}(W)$, as the reader can verify, so s is an isomorphism.

It therefore suffices to show that the composite

$$(12.2) \quad (\Omega_{\text{IP}})_i(W, *) \cong \kappa_i(W) \cong \pi_i\bar{\Phi}(W) \xrightarrow{s} \pi_{i+1}S(W) \\ \xrightarrow{\cong} \pi_{i+1}\bar{\Phi}(\Sigma W) \cong (\Omega_{\text{IP}})_{i+1}(\Sigma W, *)$$

is an isomorphism. We will show that this composite is equal to the suspension isomorphism

$$(12.3) \quad s' : (\Omega_{\text{IP}})_i(W, *) \rightarrow (\Omega_{\text{IP}})_{i+1}(\Sigma W, *)$$

of the homology theory $(\Omega_{\text{IP}})_*$. First we give an explicit description of the composite (12.2). Let $b \in (\Omega_{\text{IP}})_i(W, *)$. Then b is represented by a map $f : M \rightarrow W$, where M is an i -dimensional manifold which is a boundary; say $M = \partial N$. Recall that k -simplices of $\Phi_0(W)$ are the same thing as elements of $\text{ad}_{\text{IP}, W}^k(\Delta^k)$. Let σ be the i -simplex of $\Phi_0(W)$ which takes Δ^i to f and all faces of Δ^i to $\emptyset \rightarrow W$. We can construct an $(i+1)$ -simplex τ of $\Phi_0(CW)$ with $\partial_0(\tau) = \sigma$ and all other faces at the basepoint as follows. Let P be

$$(I \times M) \cup_{1 \times M} N,$$

let $g : P \rightarrow CW$ take $(t, x) \in I \times M$ to $[t, f(x)] \in CW$ and N to $[1, *]$, and finally let τ take Δ^{i+1} to g , $d^0 \Delta^{i+1}$ to f , and the remaining faces to $\emptyset \rightarrow W$. Let τ_1 and τ_2 be the images of τ under the two maps $\Phi_0(CW) \rightarrow \bar{\Phi}_0(\Sigma W)$; then (with the notation of Equation (12.1)) τ_1 and τ_2 give a map

$$\mathbf{b} : D/\partial D \rightarrow \bar{\Phi}_0 \Sigma W$$

and the image of b in $\pi_{i+1}\bar{\Phi}(\Sigma W)$ is the element represented by \mathbf{b} .

Next we show that this description of the image of b in $\pi_{i+1}\bar{\Phi}(\Sigma W)$ can be simplified. Let the two copies of CW in ΣW be $[-1, 0] \wedge W$ and $[0, 1] \wedge W$, where the basepoints of $[-1, 0]$ and $[0, 1]$ are -1 and 1 . Let

$$Q = N \cup_{-1 \times M} ([-1, 1] \times M) \cup_{1 \times M} N,$$

and let $h : Q \rightarrow \Sigma W$ take (t, x) to $[t, f(x)]$ and both copies of N to the basepoint. Let τ_3 be the $(i+1)$ -simplex of $\Phi_0(\Sigma W)$ which takes Δ^{i+1} to h and all faces of Δ^{i+1} to $\emptyset \rightarrow \Sigma W$; then τ_3 gives a map

$$\mathbf{b}' : \Delta^{i+1}/\partial \Delta^{i+1} \rightarrow \bar{\Phi}(\Sigma W),$$

and we claim that \mathbf{b} and \mathbf{b}' represent the same element of $\pi_{i+1}\bar{\Phi}(\Sigma W)$. To see this, let v be the $(i+2)$ -simplex of $\Phi_0(\Sigma W)$ such that

- $v(\Delta^{i+2})$ is the composite

$$I \times Q \xrightarrow{p} Q \xrightarrow{h} \Sigma W$$

(where p is the projection),

- $v(d^0 \Delta^{i+2}) = h \circ p|_{0 \times Q}$,
- $v(d^1 \Delta^{i+2}) = h \circ p|_{1 \times (N \cup_{-1 \times M}([-1, 0] \times M))}$,
- $v(d^2 \Delta^{i+2}) = h \circ p|_{1 \times (([0, 1] \times M) \cup_{1 \times M} N)}$,
- v takes all other faces to $\emptyset \rightarrow \Sigma W$.

Then v gives a homotopy between \mathbf{b} and \mathbf{b}' which verifies the claim.

The element of $(\Omega_{\text{IP}})_{i+1}(\Sigma W, *)$ corresponding to \mathbf{b}' is represented by the map $h : Q \rightarrow \Sigma W$; this completes our calculation of the image of b under the composite (12.2).

Next we claim that the image of b under the map (12.3) is represented by $h|_{Q'}$, where

$$Q' = ([-1, 1] \times M) \cup_{1 \times M} N.$$

To see this, recall that the suspension map s' is defined to be the inverse of the composite

$$(\Omega_{\text{IP}})_{i+1}(\Sigma W, *) \xleftarrow{q} (\Omega_{\text{IP}})_{i+1}(C'W, W) \xrightarrow{\partial} (\Omega_{\text{IP}})_i(W, *),$$

where $C'W = [-1, 1] \wedge W$ (with the basepoint of $[-1, 1]$ at 1), q is the quotient map, and ∂ is the boundary map of the homology theory $(\Omega_{\text{IP}})_*$. Now consider the map

$$k : Q' \rightarrow C'W$$

which takes (t, x) to $[t, f(x)]$ and N to the basepoint. Recall from [Par90, Section 5] that the boundary map ∂ is defined as in [C79, Section 4]; thus $\partial[k] = [f] = b$. We also have $q \circ k = h|_{Q'}$, so $s'(b)$ is represented by $h|_{Q'}$ as claimed.

To complete the proof of Lemma 12.1 we observe that h and $h|_{Q'}$ represent the same element of $(\Omega_{\text{IP}})_{i+1}(\Sigma W, *)$ because the composite

$$I \times Q \xrightarrow{p} Q \xrightarrow{h} \Sigma W$$

(where p is the projection) is a bordism, in the sense of [C79, Section 4], between h and $h|_{Q'}$.

APPENDIX A. THE INTRINSIC FILTRATION OF A FINITE-DIMENSIONAL PL SPACE

Let X be a PL space. Say that two points $x_1, x_2 \in X$ are *equivalent* if there are neighborhoods U_1 of x_1 and U_2 of x_2 with a PL homeomorphism of pairs $(U_1, x_1) \approx (U_2, x_2)$. Let X be finite dimensional. Choose a triangulation of X , and let $X(i)$ be the i -skeleton of this triangulation. The *intrinsic filtration* of X is the filtration X^i for which $x \in X^i$ if and only if all points equivalent to x are in $X(i)$. This filtration is independent of the

chosen triangulation because it is the coarsest PL CS stratification² of X ([F, Remark 2.97]). We will use the notation X^i throughout this appendix to denote the intrinsic filtration of X .

Proposition A.1. *Let X be a finite-dimensional PL space.*

- (i) *If U is an open subset of X then $U^i = X^i \cap U$.*
- (ii) *If M is a PL manifold of dimension m then*

$$(X \times M)^i = \begin{cases} X^{i-m} \times M & m \leq i \leq \dim X + m, \\ \emptyset & \text{otherwise.} \end{cases}$$

- (iii) *If $f : X \rightarrow Y$ is a PL homeomorphism then $f(X^i) = Y^i$.*
- (iv) *If X is a PL pseudomanifold then the intrinsic filtration on X is a stratification in the sense of Definition 2.5.*

Proof. Part (i) is [F, Lemma 2.107], part (ii) is [F, Lemma 2.108], part (iii) is immediate from the definition of the intrinsic filtration, and part (iv) is [F, Proposition 2.104]. \square

The following fact is contained in [F, Corollary 2.111 and Proposition 2.110].

Proposition A.2. *Let X be a PL ∂ -pseudomanifold, and define subsets $X[i]$ by letting $X[i] \cap (X - \partial X) = (X - \partial X)^i$ and*

$$X[i] \cap \partial X = \begin{cases} (\partial X)^{i-1} & 1 \leq i \leq \dim X, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then the filtration $X[i]$ gives X the structure of a stratified PL ∂ -pseudomanifold. \square

APPENDIX B. GENERALIZATIONS AND VARIANTS OF SOME RESULTS FROM [FM13b]

We begin by showing that the results of [FM13b, Subsections 6.2 and 6.3] all remain valid with the field F replaced by a PID R . We will use the notation of those subsections, except that we denote singular chains by S_* instead of C_* ; in particular in this appendix we use $I^{\bar{p}}S_*$ (instead of $IS_*^{\bar{p}}$ as in previous sections) for intersection chains. First we have the analog of [FM13b, Proposition 6.4].

Proposition B.1. (i) *If X has a finite covering by evenly covered open sets (in particular, if X is compact) then $I^{\bar{p}}S_*(\tilde{X}, \tilde{A}; R)$ is chain homotopy equivalent over $R[\pi]$ to a nonnegatively-graded chain complex of free $R[\pi]$ -modules.*

(ii) *For all X , $I^{\bar{p}}S_*(\tilde{X}, \tilde{A}; R)$ is chain homotopy equivalent over $R[\pi]$ to a nonnegatively-graded chain complex of flat $R[\pi]$ -modules.*

²The definition of CS stratification is a weaker version of Definition 2.5.

Proof of Proposition B.1. The only place in the proof of [FM13b, Proposition 6.4] where the hypothesis that F is a field is used is in the proof of Lemma 6.6, where it is asserted that $B(V)$ is free. But the hypothesis is not needed there, because $\tilde{V} \approx V \times \pi$ and hence $B(V)$ is isomorphic to the tensor product of $R[\pi]$ with the cokernel of

$$\operatorname{colim} I^{\bar{p}}S_*(W; R) \rightarrow I^{\bar{p}}S_*(V; R),$$

which by [FM13b, Proposition 6.3] is the cokernel of

$$I_{\mathcal{D}}^{\bar{p}}S_*(V; R) \hookrightarrow I^{\bar{p}}S_*(V; R),$$

(where \mathcal{D} is given in the statement of [FM13b, Proposition 6.6], and $I_{\mathcal{D}}^{\bar{p}}S_*(V; R)$ denotes $\sum_{W \in \mathcal{D}} I^{\bar{p}}S_*(W; R)$), and by Lemma B.6 this cokernel embeds in the cokernel of

$$\sum_{W \in \mathcal{D}} S_*(W; R) \rightarrow S_*(V; R),$$

which is freely generated over R by the simplices that don't land in any W . \square

Now the proof of [FM13b, Proposition 6.5] goes through without change to show

Proposition B.2. *If X is a ∂ -IP space then the cross product*

$$I^{\bar{p}}S_*(\tilde{X}, \tilde{A}; R) \otimes_R I^{\bar{q}}S_*(\tilde{X}, \tilde{A}; R) \rightarrow I^{Q_{\bar{p}, \bar{q}}}S_*(\tilde{X} \times \tilde{X}, \tilde{A} \times \tilde{X} \cup \tilde{X} \times \tilde{A}; R)$$

induces a quasi-isomorphism

$$\begin{aligned} R \otimes_{R[\pi]} (I^{\bar{p}}S_*(\tilde{X}, \tilde{A}; R) \otimes_R I^{\bar{q}}S_*(\tilde{X}, \tilde{A}; R)) \\ \rightarrow R \otimes_{R[\pi]} I^{Q_{\bar{p}, \bar{q}}}S_*(\tilde{X} \times \tilde{X}, \tilde{A} \times \tilde{X} \cup \tilde{X} \times \tilde{A}; R). \end{aligned}$$

Next we have the analogue of [FM13b, Proposition 5.15], except that we require the perversity to be classical.

Proposition B.3. *Let \bar{p} be a classical perversity. Then $I^{\bar{p}}S_*(\tilde{X}; R)$ is quasi-isomorphic over $R[\pi]$ to a finite $R[\pi]$ chain complex.*

Proof. It suffices to show that [FM13b, Lemma 6.7] remains valid with F replaced by R . The only place in the proof of that result where the hypothesis that F is a field is used is in the second paragraph of the proof of Lemma 6.9, as part of the verification that $I^{\bar{p}}S_*(\operatorname{St}(s); F)$ is homotopy finite over F , so we need to prove that $I^{\bar{p}}S_*(\operatorname{St}(s); R)$ is homotopy finite over R . Because the perversity is classical, we may assume that $\operatorname{St}(s)$ has the intrinsic filtration (because the intersection chain complexes of $\operatorname{St}(s)$ with the original filtration and with the intrinsic filtration are quasi-isomorphic by the proof of [F, Theorem 5.50], and since they are free over R they are chain homotopy equivalent). Since X satisfies conditions (a) and (b) of Definition 2.2, it follows that $\bar{\operatorname{St}}(s)$ also satisfies them. Since $\bar{\operatorname{St}}(s) = s * \operatorname{Lk}(s)$, it follows that $\operatorname{Lk}(s)$ is a PL pseudomanifold, and hence that $(\partial s) * \operatorname{Lk}(s)$ is a

PL pseudomanifold. Let us write A for $(\partial s) * \text{Lk}(s)$ and A^* for the intrinsic filtration of A . It is shown in the proof of [FM13b, Lemma 6.9] that $\text{St}(s)$ is homeomorphic as a PL space to the open cone $c^\circ A$, so it suffices to show that the intersection chain complex of the latter, with its intrinsic filtration $(c^\circ A)^*$, is homotopy finite over R . Now by Proposition A.1 the restriction of $(c^\circ A)^*$ to $A \times (0, 1)$ is the same as the Cartesian product of A^* with the trivial filtration on $(0, 1)$, and hence $c^\circ A$ is stratified homotopy equivalent to

$$A' = ([0, \frac{1}{2}] \times A) / (0 \times x \sim 0 \times y)$$

with the filtration inherited from $c^\circ A$. A' is a stratified PL ∂ -pseudomanifold, so by [F, Corollary 5.49] the intersection chain complex of A' is quasi-isomorphic (and hence chain homotopy equivalent) to the simplicial intersection chain complex $I^{\bar{p}}C_*^T(A'; R)$ for any suitable triangulation T . Since A' is compact, $I^{\bar{p}}C_*^T(A'; R)$ is a finite chain complex, which completes the proof. \square

Now we can apply the method of [FM13b, Section 3] to construct a cap product

$$I_{\bar{q}}\bar{H}^i(\tilde{X}; R) \otimes I^{\bar{r}}H_j(X; R) \rightarrow I^{\bar{p}}H_{j-i}(\tilde{X}; R)$$

when X is a ∂ -IP-space and $D\bar{r} \geq D\bar{p} + D\bar{q}$; the only case needed in the present paper is $\bar{p} = \bar{m}$, $\bar{q} = \bar{n}$, $\bar{r} = \bar{0}$. As in [FM13b, Section 4], we obtain a map

$$\mathcal{D} : \text{colim}_K I_{\bar{p}}\bar{H}^i(\tilde{X}, \tilde{X} - \tilde{K}; R) \rightarrow I^{\bar{q}}H_{n-i}(\tilde{X}; R),$$

where K runs through the compact subsets of X , and we have

Theorem B.4 (Universal Poincaré duality). *Let X be an R -oriented stratified IP-space, let $p : \tilde{X} \rightarrow X$ be a regular covering of X , and let \bar{p}, \bar{q} be complementary perversities. Then \mathcal{D} is an isomorphism.*

The proof is the same as for [FM13b, Theorem 4.1]. Finally, we have the analog of Lefschetz duality.

Theorem B.5 (Universal Lefschetz Duality). *Let X be an n -dimensional compact ∂ -stratified IP-space with an R -orientation of $X - \partial X$. Let $p : \tilde{X} \rightarrow X$ be a regular covering of X , and let \bar{p}, \bar{q} be complementary perversities. Then the cap product with Γ_X gives isomorphisms*

$$I_{\bar{p}}\bar{H}^i(\tilde{X}, p^{-1}(\partial X); F) \rightarrow I^{\bar{q}}H_{n-i}(\tilde{X}; F)$$

and

$$I_{\bar{p}}\bar{H}^i(\tilde{X}; F) \rightarrow I^{\bar{q}}H_{n-i}(\tilde{X}, p^{-1}(\partial X); F).$$

The proof is the same as for [FM13b, Theorem 4.5]. To complete the proof of Proposition B.1 it remains to show:

Lemma B.6. *Let Z be a filtered space, let \mathcal{U} be a finite open cover of Z , let \bar{p} be a perversity which is $\leq \bar{t}$, and let $i \in \mathbb{Z}$. Then the intersection of $I^{\bar{p}}S_i(Z; R)$ with $\sum_{W \in \mathcal{U}} S_i(W; R)$ (considered as subgroups of $S_i(Z; R)$) is $\sum_{W \in \mathcal{U}} I^{\bar{p}}S_i(W; R)$.*

Remark B.7. See the beginning of [F, Section 6.2] for the fact that $I^{\bar{p}}S_i(Z; R)$ can be thought of as a subgroup of $S_i(Z; R)$ (which is obvious for classical perversities).

Proof of Lemma B.6. Let $\mathcal{U} = \{W_1, \dots, W_n\}$; the proof is by induction on n . Let

$$\xi \in I^{\bar{p}}S_*(Z; R) \cap \sum_{W \in \mathcal{U}} S_*(W; R).$$

Write

$$\xi = \sum k_m \sigma_m,$$

where $k_m \in R$ and σ_m is a singular simplex. Let

$$\eta = \sum_{\text{supp}(\sigma_m) \subset W_1} k_m \sigma_m.$$

The construction in the proof of [F07, Proposition 2.9], applied to the ordered covering (W_1, Z) (but without the preliminary subdivision in line –10 of page 1992), gives a singular chain

$$\theta \in \sum_{i=2}^n S_*(W_1 \cap W_i; R)$$

for which $\eta + \theta$ is an element of $I^{\bar{p}}S_*(W_1; R)$. Let $Z' = \cup_{i=2}^n W_i$ and let $\mathcal{U}' = \{W_2, \dots, W_n\}$. Then $\xi - \eta - \theta$ is an element of

$$I^{\bar{p}}S_*(Z'; R) \cap \sum_{W \in \mathcal{U}'} S_*(W; R),$$

so by the inductive hypothesis

$$\xi - \eta - \theta \in \sum_{W \in \mathcal{U}'} I^{\bar{p}}S_*(W; R),$$

and therefore

$$\xi \in \sum_{W \in \mathcal{U}} I^{\bar{p}}S_*(W; R)$$

as required. \square

APPENDIX C. MULTIPLICATIVITY OF THE ASSEMBLY MAP

This appendix gives the proof of

Proposition C.1. *Let \mathbf{F} be a homotopy invariant functor from spaces to spectra, and suppose that there is a natural transformation*

$$\mu : \mathbf{F}(X) \wedge \mathbf{F}(Y) \rightarrow \mathbf{F}(X \times Y).$$

Then the diagram

$$\begin{array}{ccc} (X_+ \wedge \mathbf{F}(*)) \wedge (Y_+ \wedge \mathbf{F}(*)) & \xrightarrow{\mu} & (X \times Y)_+ \wedge \mathbf{F}(*)) \\ \alpha \wedge \alpha \downarrow & & \alpha \downarrow \\ \mathbf{F}(X) \wedge \mathbf{F}(Y) & \xrightarrow{\mu} & \mathbf{F}(X \times Y) \end{array}$$

commutes up to homotopy, where α denotes the assembly map.

Let us recall the definition of the assembly map from [WW95, page 334]. For a space X , let \mathcal{C}_X (which is denoted $\text{simp}(X)$ in [WW95]) be the category whose objects are maps $\Delta^n \rightarrow X$, and whose morphisms are commutative triangles

$$\begin{array}{ccc} \Delta^m & \xrightarrow{f_*} & \Delta^n \\ & \searrow & \swarrow \\ & X & \end{array}$$

where f_* is the map induced by a monotone injection³ from $\{0, \dots, m\}$ to $\{0, \dots, n\}$. There is a natural equivalence in the homotopy category

$$\lambda : X \rightarrow \text{hocolim}_{\mathcal{C}_X} *$$

(see below). Let D be the functor from \mathcal{C}_X to spaces which takes $\Delta^n \rightarrow X$ to Δ^n . The assembly map is the following composite in the homotopy category of spectra (where \wedge is the derived smash product)

$$\begin{aligned} X_+ \wedge \mathbf{F}(*)) &\xrightarrow{\lambda \wedge 1} (\text{hocolim}_{\mathcal{C}_X} *)_+ \wedge \mathbf{F}(*)) \cong \text{hocolim}_{\mathcal{C}_X} \mathbf{F}(*)) \\ &\xleftarrow{\cong} \text{hocolim}_{\mathcal{C}_X} \mathbf{F} \circ D \rightarrow F(X). \end{aligned}$$

Our first task is to give an explicit description of λ (this was left as an exercise for the reader in [WW95]). We need a lemma.

Lemma C.2. *The map*

$$\text{hocolim}_{\mathcal{C}_X} D \rightarrow \text{colim}_{\mathcal{C}_X} D$$

is a weak equivalence.

Proof. The category \mathcal{C}_X is a Reedy category ([H03, Def 15.1.2]) which has fibrant constants ([H03, Definition 15.10.1]) by the proof of [H03, Proposition 15.10.4(1)]. The functor D is Reedy cofibrant ([H03, Definition 15.3.3(2)]) and the result follows by [H03, Theorem 19.9.1(1)]. \square

³It is not clear why [WW95] does not use all monotone maps in this definition. The reader can check that using all monotone maps would give the same assembly map, and would allow us to work with simplicial rather than semisimplicial sets in what follows.

Now let SX be the semisimplicial set whose n -simplices are the maps $\Delta^n \rightarrow X$. Then λ can be chosen⁴ to be the composite

$$X \xleftarrow{\cong} |SX| = \operatorname{colim}_{\mathcal{C}_X} D \xleftarrow{\cong} \operatorname{hocolim}_{\mathcal{C}_X} D \xrightarrow{\cong} \operatorname{hocolim}_{\mathcal{C}_X} *$$

Our next lemma gives a multiplicative property of λ . Let

$$d : \mathcal{C}_{X \times Y} \rightarrow \mathcal{C}_X \times \mathcal{C}_Y$$

be the functor which takes $f : \Delta^n \rightarrow X \times Y$ to the pair $(p_1 \circ f, p_2 \circ f)$, where p_1 and p_2 are the projections.

Lemma C.3. (i) *The diagram*

$$\begin{array}{ccc} X \times Y & \xrightarrow{\lambda \times \lambda} & (\operatorname{hocolim}_{\mathcal{C}_X} *) \times (\operatorname{hocolim}_{\mathcal{C}_Y} *) \\ \lambda \downarrow & & \cong \uparrow \\ \operatorname{hocolim}_{\mathcal{C}_{X \times Y}} * & \xrightarrow{\delta} & \operatorname{hocolim}_{\mathcal{C}_X \times \mathcal{C}_Y} * \end{array}$$

commutes, where δ is induced by d and the vertical arrow is induced by the projections $\mathcal{C}_{X \times Y} \rightarrow \mathcal{C}_X$ and $\mathcal{C}_{X \times Y} \rightarrow \mathcal{C}_Y$.

(ii) *The map δ in part (i) is a weak equivalence.*

The proof of part (i) is left to the reader. Part (ii) follows from (i) and the fact that λ is a weak equivalence.

Next we need some notation. For a space Z let Λ denote the composite

$$Z_+ \wedge \mathbf{F}(\ast) \xrightarrow{\lambda \wedge 1} (\operatorname{hocolim}_{\mathcal{C}_Z} *)_+ \wedge \mathbf{F}(\ast) \cong \operatorname{hocolim}_{\mathcal{C}_Z} \mathbf{F}(\ast).$$

Let

$$\delta' : \operatorname{hocolim}_{\mathcal{C}_{X \times Y}} \mathbf{F}(\ast) \rightarrow \operatorname{hocolim}_{\mathcal{C}_X \times \mathcal{C}_Y} \mathbf{F}(\ast)$$

be the map induced by d ; this is a weak equivalence by Lemma C.3(ii). Let E be the functor from $\mathcal{C}_X \times \mathcal{C}_Y$ to spaces that takes $(\Delta^m \rightarrow X, \Delta^n \rightarrow Y)$ to $\Delta^m \times \Delta^n$, and let

$$\delta'' : \operatorname{hocolim}_{\mathcal{C}_{X \times Y}} \mathbf{F} \circ D \rightarrow \operatorname{hocolim}_{\mathcal{C}_X \times \mathcal{C}_Y} \mathbf{F} \circ E$$

be the map induced by d and the diagonal maps $\Delta^n \rightarrow \Delta^n \times \Delta^n$.

⁴To know that this is a correct choice, one simply has to check that it gives an assembly map with the properties in [WW95, Theorem 1.1].

To complete the proof of Proposition C.1, we need only observe that the following diagram is commutative:

$$\begin{array}{ccccc}
X_+ \wedge \mathbf{F}(\ast) \wedge Y_+ \wedge \mathbf{F}(\ast) & \xrightarrow{\hspace{10em}} & (X \times Y)_+ \wedge \mathbf{F}(\ast) \\
\Lambda \wedge \Lambda \downarrow & & \Lambda \downarrow \\
\mathrm{hocolim}_{\mathcal{C}_X} \mathbf{F}(\ast) \wedge \mathrm{hocolim}_{\mathcal{C}_Y} \mathbf{F}(\ast) & \xrightarrow{\hspace{2em}} & \mathrm{hocolim}_{\mathcal{C}_X \times \mathcal{C}_Y} \mathbf{F}(\ast) & \xleftarrow{\delta'} & \mathrm{hocolim}_{\mathcal{C}_X \times \mathcal{C}_Y} \mathbf{F}(\ast) \\
\uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq \\
\mathrm{hocolim}_{\mathcal{C}_X} \mathbf{F} \circ D \wedge \mathrm{hocolim}_{\mathcal{C}_Y} \mathbf{F} \circ D & \xrightarrow{\hspace{2em}} & \mathrm{hocolim}_{\mathcal{C}_X \times \mathcal{C}_Y} \mathbf{F} \circ E & \xleftarrow{\delta''} & \mathrm{hocolim}_{\mathcal{C}_X \times \mathcal{C}_Y} \mathbf{F} \circ D \\
\downarrow & & \searrow & & \downarrow \\
\mathbf{F}(X) \wedge \mathbf{F}(Y) & \xrightarrow{\hspace{10em}} & \mathbf{F}(X \times Y)
\end{array}$$

APPENDIX D. CORRECTIONS TO SOME SIGNS IN [LM13]

All references in this appendix are to [LM13].

In Definition 15.4(i), the i -th face map $\mathrm{ad}^k(\Delta^n) \rightarrow \mathrm{ad}^k(\Delta^{n-1})$ should be $(-1)^i$ times the composite with the map induced by the i -th coface map $\Delta^{n-1} \rightarrow \Delta^n$. There should also be an analogous sign in Definition 17.2.

In order for Proposition 17.8 to be true, the paragraph that comes after Lemma 15.7 needs to be replaced by the following.

“Next observe that for each n there is an isomorphism of \mathbb{Z} -graded categories

$$\nu : \mathcal{C}ell(\Delta^{n+1}, \partial_0 \Delta^{n+1} \cup \{0\}) \rightarrow \mathcal{C}ell(\partial_0 \Delta^{n+1})$$

which lowers degrees by 1, defined as follows: a simplex σ of Δ^{n+1} which is not in $\partial_0 \Delta^{n+1} \cup \{0\}$ contains the vertex 0. Let ν take σ (with its canonical orientation) to the simplex of $\partial_0 \Delta^n$ spanned by the vertices of σ other than 0 (with its canonical orientation). Let

$$\theta : \mathcal{C}ell(\Delta^{n+1}, \partial_0 \Delta^{n+1} \cup \{0\}) \rightarrow \mathcal{C}ell(\Delta^n)$$

be the composite of η with the isomorphism induced by the face map $\Delta^n \rightarrow \partial_0 \Delta^{n+1}$. θ is incidence-compatible, so by part (e) of Definition 3.10 it induces a bijection

$$\theta^* : \mathrm{ad}^k(\Delta^n) \rightarrow \mathrm{ad}^{k+1}(\Delta^{n+1}, \partial_0 \Delta^{n+1} \cup \{0\}).”$$

This change leads to corresponding changes in Section 16 and Lemma 17.11, which we leave to the interested reader.

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