

# SUBALGEBRAS OF GROUP COHOMOLOGY DEFINED BY INFINITE LOOP SPACES

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ABSTRACT. We study natural subalgebras  $Ch_E(BG; R)$  of group cohomology  $H^*(BG; R)$  defined in terms of the infinite loop spaces in spectra  $E$  and give representation theoretic descriptions of those based on  $QS^0$  and the Johnson-Wilson theories  $E(n)$ . We describe the subalgebras arising from the Brown-Peterson spectra  $BP$  and as a result give a simple reproof of Yagita's theorem that the image of  $BP^*(BG)$  in  $H^*(BG; \mathbb{F}_p)$  is  $F$ -isomorphic to the whole cohomology ring; the same result is shown to hold with  $BP$  replaced by any complex oriented theory  $E$  with a non-trivial map of ring spectra  $E \rightarrow H\mathbb{F}_p$ . We also extend our constructions to define subalgebras of  $H^*(X; R)$  for any space  $X$ ; when  $X$  is a finite CW complex we show that the subalgebras  $Ch_{E(n)}(X; R)$  give a natural unstable chromatic filtration of  $H^*(X; R)$ .

## 1. INTRODUCTION

The chromatic point of view in stable homotopy theory, with its filtration of homotopy and associated families of cohomology theories used to detect such filtered information, is well established. The main object of the current article is to examine aspects of how such families of theories systematically give rise also to naturally defined subalgebras of ordinary cohomology. For reasons discussed further below, our main interest is in subalgebras of group cohomology.

Perhaps the first example of such a subalgebra of group cohomology defined by a generalised cohomology theory was Thomas' *Chern subring*,  $Ch(G)$ , the subring of  $H^*(BG; \mathbb{Z})$  generated by the Chern classes of the irreducible complex representations of the finite group  $G$  [27]. It was observed in [9] that one can view  $Ch(G)$  as the image of the  $K$ -theory of  $BG$  in cohomology, in a sense made precise below and in §2, but we note for now that this description relies on the Atiyah completion theorem [2].

More generally (and more explicitly), given a cohomology theory  $E^*(-)$  and space  $X$ , there is a subalgebra of  $H^*(X; R)$ , which we denote  $Ch_E(X; R)$ , generated by the images of certain unstable cohomology operations  $E^*(-) \rightarrow H^*(-; R)$  (see §2 for its formal definition) and which is natural in maps of  $X$ . In the case  $X = BG$ ,  $E = K$  and  $R = \mathbb{Z}$  this coincides with Thomas' Chern subring; for general  $E$  and  $R = \mathbb{F}_p$ , this subalgebra is an unstable algebra over the Steenrod algebra. When  $R = \mathbb{F}_p$  and  $E = \widehat{E(n)}$  it was shown in [9] that the resulting subalgebras defined, up to inseparable isogeny, a 'chromatic' filtration of group cohomology. The varieties of these subalgebras were characterised in terms of colimits over certain categories  $\mathcal{A}^{(n)}$  as introduced by Green and Leary [10].

In this paper we examine a number of questions relating to the general nature of subrings generated in this fashion, with particular interest in their dependency

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on the spectrum  $E$  considered. Our principal example of the space  $X$  is that of a classifying space  $BG$  of a group, usually finite. Apart from historical reasons (this paper may be read as a sequel to [10] and [9]), this owes much to the apparent scope for good descriptions of  $Ch_E(X; R)$  when  $X = BG$ : given the nature of the construction of our subalgebras, it is reasonable to expect a good description of  $Ch_E(BG; R)$  whenever there is a good theory for  $E^*(BG)$  (cf. the case of  $K$ -theory above). More remarkably, we obtain below good descriptions of  $Ch_E(BG; R)$  in cases of  $E$  where there is as yet no good understanding of  $E^*(BG)$ .

The topics considered here are related to diverse results already in the literature. For one example Yagita [32] and others studied the image of Brown-Peterson theory  $BP$  in mod- $p$  cohomology using the Thom map from  $BP$  to  $H\mathbb{F}_p$ . His was a stable question, whereas our construction is firmly rooted in the unstable world, but we shall see that in many cases (such as  $BP$ ) the questions coincide. Another example is that considered in [26, 11], where permutation representations were used to pull back classes from the cohomology of symmetric groups; we shall see in the final section that this too fits as a special example of our construction, this time using the infinite loop space  $QS^0$ .

We offer three sets of results. After a discussion in §2 of our basic construction we consider first, in §3, the case of  $BP$ -theory. In fact our main result applies pretty much to any connective, complex oriented ring theory  $T$  with appropriate coefficients (see §3 for details). We prove

**Theorem 3.1** *Let  $T$  be an even ring spectrum with a ‘Thom’ map of ring spectra  $\Theta: T \rightarrow H\mathbb{F}_p$  and suppose  $G$  a finite group. Then the subalgebras  $Ch_T(BG; \mathbb{F}_p)$  and  $Im \Theta_*: T^*(BG) \rightarrow H^*(BG; \mathbb{F}_p)$  are each  $F$ -isomorphic to  $H^*(BG; \mathbb{F}_p)$ .*

This reproduces Yagita’s results [32] in the cases  $T = BP$  and  $k(n)$ . Moreover, for  $T = BP$  or  $MU$ , there is an equality  $Ch_T(BG; \mathbb{F}_p) = Im \Theta_*$ .

Our second main set of results concern periodic complex oriented theories, and in particular the Johnson-Wilson theories  $E(n)$ . The main theorem of [9] concerned subrings  $Ch_{\widehat{E(n)}}(BG; \mathbb{F}_p)$  using the  $I_n$ -adically completed theories  $\widehat{E(n)}$ . There we identified the varieties of the subrings and demonstrated a series of ‘ $F$ -inclusions’

$$\cdots \overset{F}{\subset} Ch_{\widehat{E(n)}}(BG; \mathbb{F}_p) \overset{F}{\subset} Ch_{\widehat{E(n+1)}}(BG; \mathbb{F}_p) \overset{F}{\subset} \cdots \subset H^*(BG; \mathbb{F}_p). \quad (1.1)$$

The need for the completed theories was forced by our reliance on the description of  $\widehat{E(n)}^*(BG)$  in the work of [14]. These results begged the questions as to whether this completion was really necessary for the results of [9] (despite there being no available theory for  $E(n)^*(BG)$ ), and whether the  $F$ -inclusions were actually strict inclusions. Here we present two results. The first concerns the relationship between  $Ch_E(X; R)$  and  $Ch_{\widehat{E}}(X; R)$ . For  $E = E(n)$  we obtain

**Theorem 4.1 & Corollary 4.2** *For  $X$  with the homotopy type of a finite type CW complex, there is an equivalence*

$$Ch_{E(n)}(X; R) = Ch_{\widehat{E(n)}}(X; R).$$

*Hence for a finite group  $G$  there is a homeomorphism of varieties*

$$\text{var}(Ch_{E(n)}(BG; \mathbb{F}_p)) \cong \text{colim}_{V \in \mathcal{A}^{(n)}(G)} \text{var}(H^*(BV; \mathbb{F}_p)).$$

We note also (see Theorem 4.6) a similar relationship between  $Ch_E(X; R)$  and  $Ch_{\widehat{E}}(X; R)$  for other spectra  $E$ . Results such as these suggest a number of further questions. For one, we would be very interested in obtaining a direct proof of 4.2 without recourse to [14]. For another, together with other observations in this paper and (unpublished) calculations, we conjecture that the subalgebra  $Ch_E(X; R)$  depends for given  $X$  and  $R$  just on the underlying Bousfield class of the spectrum  $E$ .

Furthermore, Section 5 approaches the issue of whether the F-inclusions in 1.1 might be true inclusions. For finite CW complexes we obtain

**Theorem 5.1** *Let  $X$  be a finite CW complex and  $R$  a  $p$ -local ring. Then for every  $n = 1, 2, \dots$  there is an inclusion  $Ch_{E(n)}(X; R) \subset Ch_{E(n+1)}(X; R)$  as subalgebras of  $H^*(X; R)$ .*

After considering both connective and periodic complex oriented theories  $E$ , our final section offers a non-complex oriented example, that of the subalgebra based on the stable cohomotopy group  $\pi_S^0(BG)$ . We obtain

**Theorem 6.1** *Let  $G$  be a finite group. Then  $Ch_{QS^0}(BG; \mathbb{F}_p) \cong S_h(G)$  as algebras.*

Here  $S_h(G)$  is the subalgebra of  $H^*(BG; \mathbb{F}_p)$  generated by permutation representations as considered in [11] and closely related to the work of [26].

**Notational conventions** Throughout  $p$  will denote a fixed prime.  $H^*(X; R)$  denotes the singular cohomology of  $X$  with constant coefficients in the ring  $R$ . By the *variety* of an  $\mathbb{F}_p$ -algebra  $S$ , denoted  $\text{var}(S)$ , we mean the set of algebra homomorphisms from  $S$  to an algebraically closed field  $k$  of characteristic  $p$ , topologised with the Zariski topology. In general  $S$  will be graded and, as discussed in [10] and [9],  $\text{var}(S)$  will be homeomorphic to  $\text{var}(S^{\text{even}})$  (even when  $p = 2$ ). A map  $\psi: S \rightarrow S'$  between  $\mathbb{F}_p$ -algebras will be called an F-isomorphism (or inseparable isogeny) if it induces a homeomorphism of the associated varieties. This is equivalent to saying that for every  $x \in \text{Ker}(\psi)$  there is a  $k$  such that  $x^{p^k} = 0$ , and for every  $y \in S'$  there is an  $l$  such that  $y^{p^l} \in \text{Im}(\psi)$ .

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## 2. THE BASIC CONSTRUCTION

We present our basic construction as a generalisation of the classical *Chern subring* of a group  $G$ , originally defined [27] as the subring of  $H^{\text{even}}(G; \mathbb{Z})$  generated by the Chern classes of all irreducible complex representations of  $G$ , whence the name.

**Definition 2.1.** *Let  $\{Y_i\}$  be a family of spaces and  $\mathcal{F}$  a set of maps of the form  $f: X \rightarrow Y_i$ . Consider cohomology with coefficients in some ring  $R$ . We define  $Ch_{\mathcal{F}}(X; R)$  to be the subring of  $H^*(X; R)$  generated by all elements of the form  $f^*(y)$  as  $f$  runs over the elements of  $\mathcal{F}$  and  $y$  over the homogeneous elements of the  $H^*(Y_i; R)$ .*

**Example 2.2.** Let  $G$  be a compact Lie group. Consider the single space  $Y_1 = BU$  and take  $\mathcal{F}$  to be the set of all maps  $BG \rightarrow BU$ . Then  $Ch_{\mathcal{F}}(BG; \mathbb{Z})$  is equivalent to the classical Chern subring [27] of  $H^*(BG; \mathbb{Z})$  generated by Chern classes of irreducible representations. This is proved for  $\mathbb{F}_p$  coefficients in [9, Prop. 1.6] and the integral version noted in [*loc. cit.* 1.7].

**Remark 2.3.** An element  $y \in H^k(Y_i; R)$  may of course be represented by a map  $Y_i \rightarrow \underline{H}_k$ , where  $\underline{H}_k$  denotes an Eilenberg-Mac Lane space of type  $K(R, k)$ . It is thus equivalent to declare  $Ch_{\mathcal{F}}(X; R)$  to be the subalgebra of  $H^*(X; R)$  generated by maps  $X \rightarrow \underline{H}_k$  which factor through some  $f \in \mathcal{F}$ .

Our main interest lies in examples of the construction in which the spaces  $Y_i$  are infinite loop spaces, thus representing generalised cohomology theories, and the maps  $f: X \rightarrow Y_i$  run over all elements of the corresponding cohomology group. For notational purposes, let  $E$  denote a spectrum [1] representing a generalised cohomology theory  $E^*(-)$ . For convenience we shall often suppose the coefficients  $E^*$  are concentrated in even degrees; such a spectrum we call *even*. Note that an even ring spectrum is automatically complex oriented [1]. We shall moreover follow [17, 25] and write  $\underline{E}_r$  for the  $r^{\text{th}}$  space in the  $\Omega$ -spectrum for  $E$ , so  $\underline{E}_r = \Omega^{\infty-r} E$  and  $\underline{E}_r$  represents  $E$ -cohomology in degree  $r$ , that is  $E^r(X) = [X, \underline{E}_r]$  for any space  $X$ . For reasons discussed further in [9] we restrict attention to the even graded spaces  $\underline{E}_{2r}$ . Note that each  $\underline{E}_r$  is an H-space with product  $\underline{E}_r \times \underline{E}_r \rightarrow \underline{E}_r$  representing addition in  $E$ -cohomology and arising from the loop space structures  $\Omega \underline{E}_{r+1} = \underline{E}_r$ ; moreover, each  $\underline{E}_r$  is an infinite loop space.

We shall make a standing assumption on the ring  $R$  that the spaces  $\underline{E}_r$  considered satisfy a Künneth isomorphism in cohomology with  $R$  coefficients,

$$H^*(\underline{E}_r \times \underline{E}_s; R) \cong H^*(\underline{E}_r; R) \otimes H^*(\underline{E}_s; R)$$

(understanding the completion of the tensor product where appropriate). This is satisfied, for example, whenever  $R$  is a field, such as  $\mathbb{F}_p$ , or for any coefficients  $R$  if  $E$  is Landweber exact [4].

**Definition 2.4.** For a spectrum  $E$  and a space  $X$ , define  $Ch_E(X; R)$  to be the subalgebra of  $H^*(X; R)$  obtained by taking  $\mathcal{F}$  to be the set of all homotopy classes of maps  $X \rightarrow \underline{E}_{2r}$ , allowing all  $r \in \mathbb{Z}$ . Equivalently, this is the subring of  $H^*(X; R)$  generated by the images of  $E^{2*}(X)$  under all (unstable) homogeneous cohomology operations  $E^{2*}(-) \rightarrow H^*(-; R)$ .

Thus the example 2.2 identifies the subring  $Ch_K(BG; \mathbb{Z})$ , for  $K$  the spectrum representing complex  $K$ -theory, with the classical Chern subring of  $H^*(BG; \mathbb{Z})$ . In the final part of this article we consider the analogous construction made using the single infinite loop space  $QS^0$ .

We note some basic properties of the subrings  $Ch_E(X; R)$ .

It is clear that this construction is natural in maps of the space  $X$ : a map  $\phi: X \rightarrow Y$  gives a homomorphism  $Ch_E(Y; R) \rightarrow Ch_E(X; R)$ . It is also clear that the construction behaves well with respect to disjoint unions of spaces, *i.e.*,  $Ch_E(\coprod_j X_j; R) = \prod_j Ch_E(X_j; R)$ . With this observation we shall largely restrict ourselves to connected spaces.

In the case  $R = \mathbb{F}_p$ , it is readily checked that  $Ch_E(X; \mathbb{F}_p)$  is closed under the action of the Steenrod algebra  $\mathcal{A}_p$ ; in fact  $Ch_E(-; \mathbb{F}_p)$  may be considered as a

functor from spaces to unstable algebras over  $\mathcal{A}_p$ , a point considered further by Powell [21].

However, at first sight, there appear to be a lot of maps  $f: X \rightarrow \underline{E}_{2r}$  that have to be considered in the construction of  $Ch_E(X; R)$ . The following lemma substantially reduces the number of maps needed. For a CW complex  $X$  we write  $X^{(m)}$  for its  $m$ -skeleton and recall Milnor's short exact sequence

$$0 \longrightarrow \lim^1 E^{2r-1}(X^{(m)}) \longrightarrow E^{2r}(X) \longrightarrow \lim E^{2r}(X^{(m)}) \longrightarrow 0.$$

Recall also that the skeleta  $\{X^{(m)}\}$  define a topology (which we shall refer to as the *skeletal topology*) on the group  $\lim E^{2r}(X^{(m)})$  where the open neighbourhoods of zero are given by

$$F_m E^{2r}(X) = \ker \left( E^{2r}(X) \longrightarrow E^{2r}(X^{(m)}) \right).$$

**Lemma 2.5.** *Suppose  $X$  is a finite type CW complex. Then  $Ch_E(X; R)$  is generated by elements  $f^*(x)$  where  $f: X \rightarrow \underline{E}_{2r}$  run over a set of topological  $E^{2*}$ -module generators of  $E^{\text{even}}(X)$  modulo phantom maps.*

*Proof.* We first show that if  $f_1, f_2 \in E^{2r}(X)$  then any element of the form  $(f_1 + f_2)^*(x)$  lies in the subring generated using elements of the form  $f_1^*(y)$  and  $f_2^*(z)$ . Now, addition in  $E^{2r}(X)$  is represented by a map

$$\sigma: \underline{E}_{2r} \times \underline{E}_{2r} \longrightarrow \underline{E}_{2r}.$$

For  $x \in H^t(\underline{E}_{2r}; R)$ , the element of  $(f_1 + f_2)^*(x) \in Ch_E(X; R)$  can be written as the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f_1 \times f_2} \underline{E}_{2r} \times \underline{E}_{2r} \xrightarrow{\sigma} \underline{E}_{2r} \xrightarrow{x} \underline{H}_t.$$

If we write  $\sigma^*(x)$  as  $\sum x' \otimes x'' \in H^*(\underline{E}_{2r}) \otimes H^*(\underline{E}_{2r})$  using the Künneth isomorphism, we obtain  $(f_1 + f_2)^*(x) = \sum f_1^*(x') f_2^*(x'')$ , an element in the ring generated using just  $f_1$  and  $f_2$  (it is a finite sum of non-trivial terms by virtue of the finite type assumption on  $X$ ).

Similarly, we may restrict attention to  $E^*$ -module generators as follows. The  $E^*$ -action on  $E^*(X)$  may be represented by maps

$$\mu: E^{2s} \times \underline{E}_{2r} \rightarrow \underline{E}_{2(r+s)}$$

where the group  $E^{2s}$  is regarded as a space with the discrete topology. Consider an element of  $Ch_E(X; R)$  given by  $(\alpha f)^*(x)$  where  $f \in E^{2r}(X)$  and  $\alpha \in E^{2s}$ . The map  $\alpha f$  is represented by the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{\alpha \times f} E^{2s} \times \underline{E}_{2r} \xrightarrow{\mu} \underline{E}_{2(r+s)}$$

but this is equivalent to a composite

$$X \xrightarrow{f} \underline{E}_{2r} \cong \{\alpha\} \times \underline{E}_{2r} \xrightarrow{\mu_\alpha} \underline{E}_{2(r+s)}$$

where we write  $\mu_\alpha$  for the restriction of  $\mu$  to this specific component. Thus we have the equality

$$(\alpha f)^*(x) = f^*(x \mu_\alpha).$$

Next note that we can disregard any phantom map  $f: X \rightarrow \underline{E}_{2r}$ . Suppose such a map gave rise to an element  $f^*(x) \in H^t(X)$  via some element  $x \in H^t(\underline{E}_{2r})$ . As

$H^t(X) \cong H^t(X^{(m)})$  for any  $m > t + 1$  with the isomorphism induced by restriction  $\rho: X^{(m)} \rightarrow X$ , the element  $f^*(x)$  is non-zero if and only if the composite

$$X^{(m)} \xrightarrow{f\rho} \underline{E}_{2r} \xrightarrow{x} \underline{H}_t$$

is non-zero. However, by definition of  $f$  being a phantom map,  $f\rho$  is trivial and the supposed element  $f^*(x) = 0$ . A similar argument to the first above now shows that if  $f_1, f_2 \in E^{2r}(X)$  and  $f_2$  is phantom, then  $(f_1 + f_2)^*(x) = f_1^*(x)$  for any  $x \in H^t(\underline{E}_{2r}; R)$ . Thus we need only use  $E^*$ -module generators of  $E^{\text{even}}(X)$  modulo phantom maps in order to generate the whole of  $Ch_E(X; R)$ .

Finally, we show that in fact we need only consider topological generators. Once again consider a generator  $f^*(x) \in Ch_E(X; R)$  and let  $x \in H^t(\underline{E}_{2r}; R)$ . Suppose  $f \in E^{2r}(X)$  is represented by a sequence of elements  $f_m \in E^{2r}(X)$ ,  $m = 1, 2, \dots$ , where  $f - f_m \in F_m E^{2r}(X)$  and  $f_m$  is in the (algebraic)  $E^*$ -span of a given set of topological generators. As  $H^t(X; R) \cong H^t(X^{(m)}; R)$  for all  $m > t + 1$ , the elements  $f^*(x)$  and  $(f_m)^*(x)$  are identical for all  $m > t + 1$ . This concludes the proof.  $\square$

**Remark 2.6.** If  $E$  is a ring spectrum and so  $E^*(X)$  is an  $E^*$ -algebra, a similar argument to the one above for sums shows that we can reduce the maps  $f \in E^*(X)$  further to include only topological *ring* generators of  $E^*(X)$  mod phantoms.

The result (2.5) however does not imply that  $Ch_E(X; R)$  may be computed for a CW complex  $X$  in terms of  $Ch_E(X^{(m)}; R)$ . While an element of  $Ch_E(X; R)$  certainly gives rise, by restriction, to an element of  $\lim Ch_E(X^{(m)}; R)$ , it is possible to have a tower of elements in  $\lim Ch_E(X^{(m)}; R)$  which do not lift to  $Ch_E(X; R)$ .

Below we consider various examples of Landweber exact spectra [19]. This class includes  $MU$ , the Brown-Peterson theories  $BP$ , the Johnson-Wilson spectra  $E(n)$  and their various completions such as  $\widehat{E}(n)$ , it also contains complex  $K$ -theory and many examples of elliptic spectra. The topology of the spaces in the  $\Omega$ -spectrum for a Landweber exact theory is well understood [4, 13, 16, 17, 25]; in particular,  $H_*(\underline{E}_{2r}; \mathbb{Z})$  is torsion free and concentrated in even dimensions.

**Proposition 2.7.** *Let  $E$  be a Landweber exact theory. Then there is an inclusion  $Ch_K(X; R) \subset Ch_E(X; R)$  for any space  $X$ . If  $G$  is a finite group, then  $Ch_E(BG; \mathbb{F}_p)$  is a finitely generated  $\mathbb{F}_p$ -algebra and  $H^*(BG; \mathbb{F}_p)$  is a finitely generated  $Ch_E(BG; \mathbb{F}_p)$  module.*

*Proof.* The first part follows from [9, Prop. 2.8]; the result there is for  $\mathbb{F}_p$  coefficients, but the surjection  $H^*(\underline{E}_2; \mathbb{F}_p) \rightarrow H^*(BU; \mathbb{F}_p)$  lifts to any coefficients and the general case follows.

As we can identify (2.2)  $Ch_K(BG; \mathbb{F}_p)$  with the mod- $p$  Chern subring, over which we know  $H^*(BG; \mathbb{F}_p)$  is finitely generated as a module [27], the inclusion  $Ch_K(BG; \mathbb{F}_p) \subset Ch_E(BG; \mathbb{F}_p)$  immediately tells us that  $H^*(BG; \mathbb{F}_p)$  is a finitely generated module over  $Ch_E(BG; \mathbb{F}_p)$ . The finite generation of  $Ch_E(BG; \mathbb{F}_p)$  as an  $\mathbb{F}_p$ -algebra now follows from standard commutative algebra [3, Prop. 7.8] and the finite generation of  $H^*(BG; \mathbb{F}_p)$  as an  $\mathbb{F}_p$ -algebra [8, 23, 28, 29].  $\square$

We conclude this section with some remarks about the relation of  $Ch_E(X; R)$  to Bousfield localisation. Recall [5] that for any homology theory  $E_*(-)$  there is a localisation functor  $L_E$  with a natural transformation  $\eta: \text{id} \rightarrow L_E$ .

**Proposition 2.8.** *For any space  $X$ , the subalgebra  $Ch_E(X; R) \subset H^*(X; R)$  is contained in the image of  $\eta_X^*: H^*(L_E X; R) \rightarrow H^*(X; R)$ .*

*Proof.* The spaces  $\underline{E}_{2r}$  are themselves  $E$ -local. Hence any map  $f: X \rightarrow \underline{E}_{2r}$  factors through the localisation  $\eta_X: X \mapsto L_EX$ . Thus any generator of  $Ch_E(X; R)$  lies in  $\eta_X^*$ . As  $\eta_X^*$  is an algebra homomorphism it is closed under sums and products and the result follows.  $\square$

The inclusion  $Ch_E(X; R) \subset \text{Im}(\eta_X^*)$  of (2.8) will usually be strict. For example, if  $E$  is Landweber exact,  $H^*(\underline{E}_{2r; R})$  lies entirely in even dimensions [4] and hence  $Ch_E(X; R) \subset H^{\text{even}}(X; R)$ ; there are many examples where  $\text{Im}(\eta_X^*)$  contains odd dimensional elements, *e.g.* for  $X = B\mathbb{Z}/p$ ,  $R = \mathbb{F}_p$  and  $E = BP$ .

However, even the inclusion  $Ch_E(X; R) \subset \text{Im}(\eta_X^{\text{even}})$  for such spectra will also likely be strict. For an example here, take  $R = \mathbb{F}_p$  and let  $X$  be the classifying space of an elementary abelian group of rank 3 then (as  $X$  is then a nilpotent space) [5] tells us that  $L_{BP}X$  is just the  $p$ -localisation of  $X$  and so  $\text{Im}(\eta_X^{\text{even}})$  is the whole of  $H^{\text{even}}(X; \mathbb{F}_p)$ . We shall see by Corollary 3.8 that  $Ch_{BP}(X; \mathbb{F}_p)$  can be identified with the image of  $BP^*(X)$  under the Thom map, and for this particular  $X$  this image is not the whole of  $H^{\text{even}}(X; \mathbb{F}_p)$ .

### 3. STABILISATION AND $BP$ -THEORY

The purpose of this section is an analysis of  $Ch_E(X; R)$  in the case  $E = BP$ , the Brown-Peterson spectrum. In the process we prove its equivalence to various ‘stable’ versions of the construction. In fact, our main result applies to any even ring spectrum  $T$  equipped with a *Thom map*, a map of ring spectra  $\Theta: T \rightarrow H$  which is onto in homotopy.

**Theorem 3.1.** *Let  $T$  be an even ring spectrum with a Thom map  $\Theta$  and  $G$  a finite group. Then the subalgebras  $Ch_T(BG; \mathbb{F}_p)$  and  $\text{Im}\Theta_*: T^*(BG) \rightarrow H^*(BG; \mathbb{F}_p)$  are each  $F$ -isomorphic to  $H^*(BG; \mathbb{F}_p)$ .*

**Remark 3.2.** If we take  $T = BP$  or  $k(n)$  in Theorem 3.1 we recover Yagita’s results [32, (4.2), (4.3)]. As noted at the end of the previous section, the inclusion  $Ch_{BP}(BG; \mathbb{F}_p) \subset H^{\text{even}}(BG; \mathbb{F}_p)$  can however be strict, for example when  $G$  is elementary abelian of rank 3. In the final section we shall show that the result (3.1) can fail when  $T$  is not complex oriented.

**Corollary 3.3.** *Suppose  $T$  is an even ring spectrum with a Thom map and  $G$  is a finite group. Let  $\mathcal{A}(G)$  be the category of elementary abelian  $p$ -subgroups of  $G$  with morphisms those inclusions  $V \rightarrow W$  given by conjugation by an element of  $G$ . Then there is a homeomorphism of varieties*

$$\text{var}(Ch_T(BG; \mathbb{F}_p)) \cong \text{colim}_{V \in \mathcal{A}(G)} \text{var}(H^*(BV; \mathbb{F}_p)).$$

*Proof.* By Theorem 3.1 we can identify  $\text{var}(Ch_T(BG; \mathbb{F}_p))$  with  $\text{var}(H^*(BG; \mathbb{F}_p))$ , while Quillen’s theorem [23] describes  $\text{var}(H^*(BG; \mathbb{F}_p))$  as the colimit of the varieties of the  $H^*(BV; \mathbb{F}_p)$  over the category indicated.  $\square$

We start the main work of the section with the introduction of a stable version of the  $Ch_E$  construction. The subalgebra  $Ch_E(X; R)$  is an unstable construction in the sense that the maps  $x: \underline{E}_{2r} \rightarrow \underline{H}_k$  were not assumed to commute with the loop space structure on these spaces; equivalently,  $Ch_E(X; R)$  is the subalgebra generated by the image of  $E^*(X)$  under all unstable operations  $E^*(-) \rightarrow H^*(-; R)$ .

**Definition 3.4.** The stable  $E$ -Chern subalgebra  $Ch_E^s(X; R)$  is the subalgebra generated by elements  $f^*(x)$  where  $f \in E^{2r}(X)$  and  $x: \underline{E}_{2r} \rightarrow \underline{H}_k$  is a map of infinite loop spaces. Equivalently, it is that generated by the image of  $E^*(X)$  under all stable operations  $E^*(-) \rightarrow H^*(-; R)$ .

**Example 3.5.** Since there are no non-trivial stable maps  $K \rightarrow H\mathbb{F}_p$ , one has  $Ch_K^s(X; \mathbb{F}_p) = 0$  for all spaces  $X$ . The  $K$ -theory subalgebra  $Ch_K(X; \mathbb{F}_p)$  however is in general non-trivial, for example  $Ch_K(B\mathbb{Z}/p; \mathbb{F}_p) = H^{\text{even}}(B\mathbb{Z}/p; \mathbb{F}_p)$ .

Suppose now we have a ring spectrum  $T$  equipped with a Thom map as introduced above. Our principal examples are  $MU$  with  $R = \mathbb{Z}$  or  $\mathbb{F}_p$  and  $BP$  with  $R = \mathbb{Z}_{(p)}$  or  $\mathbb{F}_p$ , but other  $p$ -local spectra such as  $k(n)$ ,  $P(n)$  or  $BP\langle n \rangle$  also qualify, whereas the periodic spectra  $K(n)$  and  $E(n)$  obviously do not. When we have such a map, we shall write  $\Theta_k$  for the corresponding infinite loop maps  $\underline{T}_k \rightarrow \underline{H}_k$ .

**Definition 3.6.** For a ring spectrum  $T$  with Thom map  $\Theta$ , let  $Ch_T^\Theta(X; R)$  denote the subalgebra of  $H^*(X; R)$  given by elements of the form  $f^*(\Theta)$  where  $f \in T^{2r}(X)$  (note that this is closed under products as  $\Theta$  is assumed to be a ring map). Equivalently,  $Ch_E^\Theta(X)$  is the image of  $E^*(X)$  under the ring homomorphism

$$\Theta_*: T^*(X) \rightarrow H^*(X; R).$$

We have natural inclusions

$$\text{Im } \Theta_* = Ch_T^\Theta(X) \subset Ch_T^s(X) \subset Ch_T(X) \subset H^*(X; R).$$

**Proposition 3.7.** Let  $T$  be an even ring spectrum with a Thom map  $\Theta$ . Then for any space  $X$  and any Landweber exact spectrum  $E$  there is an inclusion of subalgebras

$$Ch_E(X; R) \subset Ch_T^\Theta(X; R).$$

*Proof.* The Thom map  $\Theta: T \rightarrow H$  induces a map of Atiyah-Hirzebruch spectral sequences for each space  $\underline{E}_{2r}$  which on  $E_2$ -pages is a surjection

$$H^*(\underline{E}_{2r}; T^*) \longrightarrow H^*(\underline{E}_{2r}; R).$$

By the properties of  $H_*(\underline{E}_{2r}; \mathbb{Z})$  for a Landweber exact spectrum  $E$  mentioned above, both these spectral sequences collapse and we conclude that any map  $x: \underline{E}_{2r} \rightarrow \underline{H}_k$  can be lifted through the Thom map  $\underline{E}_{2r} \xrightarrow{l} \underline{T}_k \xrightarrow{\Theta_k} \underline{H}_k$ . (Note that only even values of  $k$  arise as  $H^*(\underline{E}_{2r})$  is concentrated in even dimensions.) Thus any element  $f^*(x) \in Ch_E(X; R)$  with  $f: X \rightarrow \underline{E}_{2r}$  and  $x \in H^k(\underline{E}_{2r}; R)$  may be written as  $(lf)^*(\Theta) \in Ch_T^\Theta(X; R)$ .  $\square$

Thus if the spectrum is both Landweber exact and has a Thom map, nothing is lost by stabilising. Assuming appropriate coefficients  $R$ , this applies in particular to both  $MU$  and  $BP$ .

**Corollary 3.8.** For any space  $X$  we have equalities

$$Ch_{BP}^\Theta(X; R) = Ch_{BP}^s(X; R) = Ch_{BP}(X; R)$$

$$Ch_{MU}^\Theta(X; R) = Ch_{MU}^s(X; R) = Ch_{MU}(X; R). \quad \square$$

**Remark 3.9.** This result should be contrasted with the example 3.5 which, with Proposition 2.7, observes that for  $E$  any of the periodic Landweber exact spectra listed above, the subalgebras  $Ch_E(X)$  and  $Ch_E^s(X)$  are quite different – the former, for  $X = BG$ , containing the whole Chern subring, the latter being trivial.



Similarly, assuming the prime  $p$  to be that of the version of  $BP$ -theory considered, choosing one of  $E$  or  $T$  to be  $MU$  and the other to be  $BP$ , Proposition 3.7 gives results such as

**Corollary 3.10.** *For  $X$  any space,  $Ch_{BP}(X; \mathbb{F}_p) = Ch_{MU}(X; \mathbb{F}_p)$ .  $\square$*

Specialising to classifying spaces of finite groups and  $\mathbb{F}_p$  coefficients, we now prove the main theorem, 3.1, of the section. In fact, we offer two proofs, both resting on the stabilisation results just proved. The first uses also the main theorem of [9], while the second uses instead work of Carlson [6] or of Green and Leary [10] on corestriction of Chern classes.

*First Proof.* By Proposition 3.7, for every  $n = 1, 2, \dots$ , there are inclusions

$$Ch_{\widehat{E(n)}}(BG; \mathbb{F}_p) \subset Ch_T^\ominus(BG; \mathbb{F}_p) \subset Ch_T(BG; \mathbb{F}_p) \subset H^*(BG; \mathbb{F}_p).$$

However, by [9, Theorem 0.2] the inclusion  $Ch_{\widehat{E(n)}}(BG; \mathbb{F}_p) \subset H^*(BG; \mathbb{F}_p)$  is an  $F$ -isomorphism if  $n$  is not less than the  $p$ -rank of  $G$ .  $\square$

*Second Proof.* By Proposition 3.7 there are inclusions

$$Ch_K(BG; \mathbb{F}_p) \subset Ch_T^\ominus(BG; \mathbb{F}_p) \subset Ch_T(BG; \mathbb{F}_p) \subset H^*(BG; \mathbb{F}_p).$$

We follow [10] and write  $\overline{R(G)}$  for the closure of a functorial subring  $R(G) \subset H^*(BG; \mathbb{F}_p)$  under corestriction of elements from  $R(H)$  for all subgroups  $H$  of  $G$ . By [10, §8] or [6] (see [10] for detailed discussion),  $\overline{Ch_K(BG; \mathbb{F}_p)}$  is  $F$ -isomorphic to  $H^*(BG; \mathbb{F}_p)$ , implying  $\overline{Ch_T^\ominus(BG; \mathbb{F}_p)}$  is also  $F$ -isomorphic to  $H^*(BG; \mathbb{F}_p)$ . As corestriction is a stable construction,  $\overline{Ch_T^\ominus(BG; \mathbb{F}_p)} = Ch_T^\ominus(BG; \mathbb{F}_p)$  and so  $Ch_T^\ominus(BG; \mathbb{F}_p)$  and hence  $Ch_T(BG; \mathbb{F}_p)$  is  $F$ -isomorphic to  $H^*(BG; \mathbb{F}_p)$ .  $\square$

#### 4. EXACT SPECTRA AND COMPLETIONS

The principal result of [9] was to identify the varieties  $\text{var}(Ch_{\widehat{E(n)}}(BG; \mathbb{F}_p))$  associated to suitably complete versions of the height  $n$  Johnson-Wilson spectra  $\widehat{E(n)}$  for  $G$  a finite group. As in Corollary 3.3, these varieties were described as colimits over certain categories  $\mathcal{A}^{(n)}(G)$  of elementary abelian subgroups of  $G$ , giving a partial confirmation of the conjecture in [10]. The requirement for the spectra considered to be complete reflects the use in the main proof of results from the Hopkins-Kuhn-Ravenel theory [14] of  $E^*(BG)$ , but begs the question as to the real role (if any) of completion in the structure of the subalgebras  $Ch_E(BG; \mathbb{F}_p)$ . While the completeness properties of  $E$  are relevant to the discussion of homotopy classes of maps  $BG \rightarrow \underline{E}_{2r}$ , our main result below shows the cohomological image of such maps tells a different story.

Explicitly, our main result identifies subalgebras defined using the classical (incomplete) Johnson-Wilson theories  $E(n)$  with those associated to their Baker-Würgler completions  $\widehat{E(n)}$ . As we note in Theorem 4.6, analogues hold for similar completion pairs  $E, \widehat{E}$ .

**Theorem 4.1.** *For  $X$  with the homotopy type of a finite type CW complex, there is an equivalence*

$$Ch_{E(n)}(X; R) = Ch_{\widehat{E(n)}}(X; R).$$

The main result of [9] now gives the following, which corresponds to the original conjecture of [10].

**Corollary 4.2.** *For a finite group  $G$  there is a homeomorphism of varieties*

$$\mathrm{var}(Ch_{E(n)}(BG; \mathbb{F}_p)) \cong \operatorname{colim}_{V \in \mathcal{A}^{(n)}(G)} \mathrm{var}(H^*(BV; \mathbb{F}_p)). \quad \square$$

The categories  $\mathcal{A}^{(n)}(G)$  of elementary abelian subgroups of  $G$  are as defined in [10]: the objects are the elementary abelian subgroups of  $G$  and the morphisms are those injective homomorphisms whose restrictions to rank at most  $n$  subgroups are given by conjugation by some element of  $G$ .

We approach results of this form by considering the more general question of the relation between subalgebras  $Ch_F(X; R)$  and  $Ch_E(X; R)$  when  $E$  is a ring spectrum and  $F$  an  $E$ -module spectrum which is *exact* over  $E$ , by which we mean that for any space  $X$  there is a natural equivalence

$$F_*(X) \cong F_* \otimes_{E_*} E_*(X).$$

Under suitable finiteness conditions we show that there is an inclusion

$$Ch_F(X; R) \subset Ch_E(X; R).$$

In general however these subalgebras differ: for example, complex  $K$ -theory is exact over  $BP$ , but the inclusion  $Ch_K(X; R) \subset Ch_{BP}(X; R)$  is certainly strict in many cases.

We continue with the assumption that the ring  $R$  is such that Künneth isomorphisms hold in cohomology with  $R$  coefficients for products of spaces from the  $E$  or  $F$   $\Omega$ -spectra. This puts no constraints on  $R$  if  $E$  and  $F$  are both Landweber exact, such as the pair  $E(n)$  and  $\widehat{E(n)}$ .

**The proof of Theorem 4.1** follows from the next two lemmas with  $E = E(n)$  and  $F = \widehat{E(n)}$ : recall that  $\widehat{E(n)}$  is exact over  $E(n)$  and, following from [16], that the map  $H^*(\widehat{E(n)}_*; R) \rightarrow H^*(E(n)_*; R)$  induced by completion is onto.

**Lemma 4.3.** *Suppose  $F$  is an exact  $E$ -module spectrum and the space  $X$  is of finite type. Then  $Ch_F(X; R) \subset Ch_E(X; R)$ .*

*Proof.* Assume  $X$  is connected, for else we can argue by connected components. If  $X$  had the homotopy type of a finite CW complex then  $F^*(X) = F^* \otimes_{E^*} E^*(X)$ ; if  $X$  is only of finite type then this statement is only true after completion of the tensor product. The finite type hypothesis however tells us that the completion is with respect to the skeletal topology (as in Section 2):

$$F^*(X) = F^* \widehat{\otimes}_{E^*} E^*(X) = \lim_{m \rightarrow \infty} \left( F^* \otimes_{E^*} E^*(X^{(m)}) \right).$$

Thus topological generators of  $F^*(X)$  may be taken from elements of the (un-completed) tensor product  $F^* \otimes_{E^*} E^*(X)$ . Such an element, considered as a map  $X \rightarrow \underline{E}_{2r}$ , may thus be lifted as a finite sum of maps through spaces  $F^* \times \underline{E}_{2s_i}$ , where  $F^*$  is regarded as a discrete space, and thus we can represent  $f$  as a composite

$$X \longrightarrow \prod_i (F^* \times \underline{E}_{2s_i}) \longrightarrow \underline{E}_{2r}.$$

By the connectedness assumption on  $X$ , such a map actually factors through a single connected component of  $\prod_i (F^* \times \underline{E}_{2s_i})$ , *i.e.*, a finite product of spaces  $\underline{E}_{2s_i}$ .

Now consider a generator  $f^*(x) \in Ch_F(X; R)$  as usual. Factoring  $f$  as above,  $f^*(x)$  is represented by a composite

$$X \longrightarrow \prod_i \underline{E}_{2s_i} \longrightarrow \underline{E}_{2r} \longrightarrow \underline{H}_t$$

The composite of the final two arrows denotes an element of  $H^t(\prod_i \underline{E}_{2s_i}; R)$  and as such we can write it, using the assumed Künneth isomorphism, in the form  $\sum y^{(1)} \otimes \cdots \otimes y^{(q)}$  (where the sum is potentially infinite). If we write  $f_i$  for the projection of the initial map  $X \longrightarrow \underline{E}_{2s_i}$  to the  $i^{\text{th}}$  factor, we obtain

$$f^*(x) = \sum f_1^*(y^{(1)}) \cdots f_q^*(y^{(q)}),$$

an element of  $Ch_E(X; R)$ : note that this sum will contain only a finite number of non-zero terms by virtue of the finite type assumption on  $X$ , even if the first sum  $\sum y^{(1)} \otimes \cdots \otimes y^{(q)} \in \bigotimes_i H^*(\underline{E}_{2s_i}; R)$  is infinite.  $\square$

The second lemma is a direct consequence of the definitions:

**Lemma 4.4.** *Suppose  $\phi: E \rightarrow F$  is a map of spectra inducing a surjection in the cohomology of the corresponding  $\Omega$ -spectra,  $\phi^*: H^*(\underline{F}_*; R) \rightarrow H^*(\underline{E}_*; R)$ . Then  $Ch_E(X; R) \subset Ch_F(X; R)$  for any space  $X$ .  $\square$*

**Remark 4.5.** Similar arguments to those in the proof of Lemma 4.3, applied to the universal examples  $\underline{E}_{2r}$ , yield an inclusion  $Ch_F(X; R) \subset Ch_E(X; R)$  for any space  $X$  under the alternative assumption that the spaces  $\underline{E}_{2r}$  are finite type. Applying this observation to  $E = MU$  and  $F = K$  we recover the inclusion  $Ch_K(X; R) \subset Ch_{MU}(X; R)$  noted earlier. New examples however are

$$\begin{aligned} Ch_{KO}(X; R) &\subset Ch_{MSP}(X; R) & Ch_{KO}(X; R) &\subset Ch_{MSpin}(X; R) \\ Ch_{KO}(X; R) &\subset Ch_{El}(X; R) & Ch_K(X; R) &\subset Ch_{MSpin^c}(X; R) \\ Ch_{El}(X; R) &\subset Ch_{MSpin}(X; R) \end{aligned}$$

where  $MSpin^c$  is the self-conjugate spin cobordism of [12] and  $El$  is the (non-complex oriented) *integral* elliptic theory of Kreck and Stolz [18]; see [15]. With care, the arguments above may be extended to prove also the inclusion

$$Ch_{K(n)}(X; R) \subset Ch_{P(n)}(X; R)$$

using Yagita's  $I_n$  version of the Landweber exact functor theorem [31].

The proof of Theorem 4.1 establishes a much more general identity between sub-algebras defined by spectra and their completions. We refer to [16] for a discussion of specific examples and details of the hypotheses required.

**Theorem 4.6.** *For  $X$  with the homotopy type of a finite type CW complex, and for a spectrum  $E$  with  $I$ -adic completion  $\hat{E}$  satisfying the hypotheses 1.2 of [16], there is an equivalence*

$$Ch_E(X; R) = Ch_{\hat{E}}(X; R). \quad \square$$

## 5. UNSTABLE CHROMATIC FILTRATIONS

In [9] it was shown for a finite group  $G$  the subalgebras  $Ch_{\widehat{E(n)}}(BG; \mathbb{F}_p)$  (which by Corollary 4.1 we can now identify with  $Ch_{E(n)}(BG; \mathbb{F}_p)$ ) formed an “ $F$ -filtration” of  $H^*(BG; \mathbb{F}_p)$ , in the sense that their associated varieties assemble to a sequence of quotient spaces

$$\text{var}(H^*(BG; \mathbb{F}_p)) \twoheadrightarrow \cdots \twoheadrightarrow \text{var}(Ch_{\widehat{E(n+1)}}(BG; \mathbb{F}_p)) \twoheadrightarrow \text{var}(Ch_{\widehat{E(n)}}(BG; \mathbb{F}_p)) \twoheadrightarrow \cdots \twoheadrightarrow k.$$

The question remains however as to whether there are underlying actual inclusions  $Ch_{E(n)}(BG; \mathbb{F}_p) \subset Ch_{E(n+1)}(BG; \mathbb{F}_p)$ .

In this section we examine the subalgebras  $Ch_{E(n)}(X; R)$  for a space  $X$  with the homotopy type of a finite CW complex. Under this hypothesis we obtain the stronger result which shows the subalgebras  $Ch_{E(n)}(X; R)$  form an actual filtration of the algebra  $H^{\text{even}}(X; R)$ ; by the work of Section 2 for  $R = \mathbb{F}_p$  this is as subalgebras over the Steenrod algebra and compatible with the (unstable) chromatic filtration of the space  $X$  (cf. [24, Def. 7.5.3] and see Remark 5.2 below). We shall assume throughout this section that the coefficient ring  $R$  is  $p$ -local. Our main result is as follows.

**Theorem 5.1.** *Let  $X$  be a finite CW complex. Then for every  $n \geq 1$  there is an inclusion  $Ch_{E(n)}(X; R) \subset Ch_{E(n+1)}(X; R)$  as subalgebras of  $H^{\text{even}}(X; R)$ .*

*Proof.* We use the associated spectra  $BP\langle n \rangle$  and Wilson’s Splitting Theorem of [30]. Specifically, the spectrum  $BP\langle n \rangle$  has homotopy  $\mathbb{Z}_{(p)}[v_1, \dots, v_n]$  with  $v_i$  of dimension  $|v_i| = 2(p^i - 1)$ , and for  $r < (p^n + \dots + p + 1)$

$$\underline{BP}\langle n+1 \rangle_{2r} \simeq \underline{BP}\langle n \rangle_{2r} \times \underline{BP}\langle n+1 \rangle_{2r+|v_{n+1}|}$$

as H-spaces [30, Corollary 5.5]. Recall also that we can define the spectrum  $E(n)$  as  $v_n^{-1}BP\langle n \rangle$  and the space  $\underline{E}(n)_{2r}$  can be constructed as the homotopy colimit of

$$\underline{BP}\langle n \rangle_{2r} \xrightarrow{v_n} \underline{BP}\langle n \rangle_{2r-|v_n|} \xrightarrow{v_n} \underline{BP}\langle n \rangle_{2r-2|v_n|} \longrightarrow \cdots$$

where the maps, as indicated, represent in homotopy multiplication by  $v_n$ .

Now consider a generator of  $Ch_{E(n)}(X; R)$  which we take to be represented by a map  $f: X \rightarrow \underline{E}(n)_{2r}$  and an element  $x \in H^*(\underline{E}(n)_{2r}; R)$ . It suffices to show that  $f^*(x) \in Ch_{E(n+1)}(X; R)$ . Viewing  $\underline{E}(n)_{2r}$  as the homotopy colimit as mentioned, compactness of  $X$  means that the map  $f$  factors through some intermediate space  $\underline{BP}\langle n \rangle_{2r-k|v_n|}$  for  $k$  sufficiently large. Perhaps taking  $k$  to be even larger, view this space as a factor in

$$\underline{BP}\langle n+1 \rangle_{2r-k|v_n|} = \underline{BP}\langle n \rangle_{2r-k|v_n|} \times \underline{BP}\langle n+1 \rangle_{2r-k|v_n|+|v_{n+1}|}.$$

Then we have a factorisation of  $f$  as

$$X \xrightarrow{f_1} \underline{BP}\langle n+1 \rangle_{2r-k|v_n|} \xrightarrow{f_2} \underline{E}(n)_{2r}$$

where  $f_2$  is projection on the first factor followed by localisation with respect to powers of  $v_n$ . Note that  $f_2^*(x)$  will be an element of the form  $y \otimes 1$  in

$$H^*(\underline{BP}\langle n+1 \rangle_{2r-k|v_n|}; R) = H^*(\underline{BP}\langle n \rangle_{2r-k|v_n|}; R) \otimes H^*(\underline{BP}\langle n+1 \rangle_{2r-k|v_n|+|v_{n+1}|}; R)$$

(the Künneth isomorphism holds for  $k$  sufficiently large, again by [30]).

Now consider the localisation of  $\underline{BP}\langle n+1 \rangle_{2r-k|v_n|}$  with respect to powers of  $v_{n+1}$ . For  $s$  sufficiently small, the map

$$\underline{BP}\langle n+1 \rangle_s = \underline{BP}\langle n \rangle_s \times \underline{BP}\langle n+1 \rangle_{s+|v_{n+1}|}$$

$$\downarrow v_{n+1}$$

$$\underline{BP}\langle n+1 \rangle_{s-|v_{n+1}|} = \underline{BP}\langle n \rangle_{s-|v_{n+1}|} \times \underline{BP}\langle n \rangle_s \times \underline{BP}\langle n+1 \rangle_{s+|v_{n+1}|}$$

is the inclusion as the last two factors. In particular (again assuming  $s$  sufficiently small), the  $v_{n+1}$ -localisation map  $\underline{BP}\langle n+1 \rangle_s \rightarrow \underline{E}\langle n+1 \rangle_s$  gives an epimorphism in cohomology. We thus have a diagram

$$X \xrightarrow{f_1} \underline{BP}\langle n+1 \rangle_{2r-k|v_n|} \xrightarrow{f_2} \underline{E}\langle n \rangle_{2r},$$

$$\downarrow g$$

$$\underline{E}\langle n+1 \rangle_{2r-k|v_n|}$$

where  $g$  is the  $v_{n+1}$ -localisation map, and a class  $z \in H^*(\underline{E}\langle n+1 \rangle_{2r-k|v_n|}; R)$  with  $g^*(z) = f_2^*(x)$ . Then  $f^*(x) = (gf_1)^*(z) \in Ch_{E(n+1)}(X; R)$  as claimed.  $\square$

**Remark 5.2.** Recall the ‘‘algebraic’’ chromatic filtration of a finite spectrum  $X$  (for example, [24, Def. 7.5.3]). This is given by a tower of  $p$ -local spectra

$$L_0X \longleftarrow L_1X \longleftarrow L_2X \longleftarrow \cdots \longleftarrow X$$

and is defined as the corresponding filtration of  $\pi_*(X)$  by the kernels of the homomorphisms  $(\eta_X)_*: \pi_*(X) \rightarrow \pi_*(L_nX)$ . Here  $L_n$  is localisation with respect to the spectra  $E(n)$ . The corresponding tower in the category of spaces gives the filtration of  $H^*(X; R)$  by images of the  $\eta_X^*$ . Proposition 2.8 shows the set of subalgebras  $\{Ch_{E(n)}(X; R)\}$  to be compatible with this filtration, but what is not immediate for general spaces  $X$  is that the subalgebras  $Ch_{E(n)}(X)$  are nested. Theorem 5.1 shows that, for finite complexes  $X$ , there is a filtration

$$Ch_{E(1)}(X; R) \subset \cdots \subset Ch_{E(n)}(X; R) \subset Ch_{E(n+1)}(X; R) \subset \cdots \subset H^*(X; R)$$

with a levelwise inclusion in the unstable algebraic chromatic filtration.

## 6. SUBALGEBRAS DEFINED BY $QS^0$

Suppose throughout this section that  $G$  is a finite group.

Recall that  $QS^0 = \lim_{n \rightarrow \infty} \Omega^n \Sigma^n S^0$  is the infinite loop space that represents stable cohomotopy in degree zero. We restrict attention here to the zero space  $QS^0$  as, by the Segal conjecture [7], there are no non-trivial maps  $BG \rightarrow QS^n$  for  $n > 0$ , and we have nothing to say about the cases  $n < 0$ . Owing to the torsion in  $H^*(QS^0; \mathbb{Z})$ , we shall restrict our coefficients  $R$  in this section to the case  $R = \mathbb{F}_p$ .

We begin by recalling the algebra  $S_h = S_h(G) \subset H^*(BG; \mathbb{F}_p)$  of [11, Def. 1.2]. For a finite  $G$ -set  $X$  of cardinality  $n$ , a choice of bijection between  $X$  and the set  $\{1, \dots, n\}$  induces a homomorphism  $\rho_X$  from  $G$  to the symmetric group  $\Sigma_n$  and thus an algebra homomorphism  $\rho_X^*: H^*(B\Sigma_n; \mathbb{F}_p) \rightarrow H^*(BG; \mathbb{F}_p)$ . For a fixed space  $X$  two choices of  $\rho_X$  differ only by an inner automorphism of  $\Sigma_n$  and so the

homomorphism  $\rho_X^* : H^*(B\Sigma_n; \mathbb{F}_p) \rightarrow H^*(BG; \mathbb{F}_p)$  depends only on  $X$  and not on the choice of bijection. Then  $S_h = S_h(G)$  is defined as the subalgebra of  $H^*(BG; \mathbb{F}_p)$  generated by elements of the form  $\rho_X^*(x)$  as  $X$  runs over all finite  $G$  sets and where the  $x$  are homogeneous elements of  $H^*(B\Sigma_n; \mathbb{F}_p)$ .

**Theorem 6.1.** *Let  $G$  be a finite group. Then  $Ch_{QS^0}(BG; \mathbb{F}_p) \cong S_h(G)$  as algebras.*

*Proof.* By definition,  $S_h(G)$  is generated by elements of the form  $f^*(x)$  where  $f$  is drawn from some particular class of maps  $BG \rightarrow B\Sigma_n$  and  $x \in H^*(B\Sigma_n; \mathbb{F}_p)$ . Given such an element, consider the composite

$$BG_+ \xrightarrow{f} B\Sigma_{n+} \xrightarrow{i_n} B\Sigma_{\infty+} \xrightarrow{D} QS^0 \quad (*)$$

where  $i_n$  is induced by the inclusion of  $\Sigma_n$  in the infinite symmetric group, and  $D$  is the Dyer-Lashof map. By [22]  $D$  induces an isomorphism in cohomology, while by Nakaoka [20, §7] the induced map  $i_n^* : H^*(B\Sigma_{\infty}; \mathbb{F}_p) \rightarrow H^*(B\Sigma_n; \mathbb{F}_p)$  is a surjection. Thus there is an element  $y \in H^*(QS^0; \mathbb{F}_p)$  with  $x = (Di_n)^*(y)$ . Hence  $f^*(x) = (Di_n f)^*(y) \in Ch_{QS^0}(BG; \mathbb{F}_p)$ .

To show the converse we use the Segal conjecture. Recall that this identifies the homotopy classes of maps  $BG_+ \rightarrow QS^0$  with the elements of the Burnside ring, completed at the augmentation ideal. Specifically, the map that assigns to a finite  $G$ -set  $X$  the homotopy class of  $D \circ i_n \circ B\rho_{X+}$  extends to the Burnside ring  $A(G)$ . The resulting map is continuous with respect to the topology given by powers of the augmentation ideal and the skeletal topology, respectively; since  $\pi_s^0(BG_+)$  is already complete, this map factors through the completion of  $A(G)$ , yielding an isomorphism  $\widehat{A(G)} \cong \pi_s^0(BG_+)$ . By virtue of Lemma 2.5, it is enough to show that any *topological* generator of  $Ch_{QS^0}(BG; \mathbb{F}_p)$  lies in  $S_h(G)$ . Such element may be represented by a  $G$ -set  $X$  and an element of  $H^*(QS^0; \mathbb{F}_p)$  as above; the claim follows.  $\square$

**Remark 6.2.** A representation theoretic description of the variety associated to  $S_h(G)$  is given in [11]. Combined with Theorem 6.1 one obtains a natural homeomorphism

$$\text{var}(Ch_{QS^0}(BG; \mathbb{F}_p)) \cong \text{colim}_{V \in \mathcal{A}_h(G)} \text{var}(H^*(BV; \mathbb{F}_p))$$

where  $\mathcal{A}_h(G)$  denotes the category whose objects are elementary abelian subgroups of  $G$  and whose morphisms are those injective group homomorphisms  $f : V \rightarrow W$  for which  $f(U)$  is conjugate in  $G$  to  $U$  for every subgroup  $U$  of  $V$ .

**Remark 6.3.** In general neither  $S_h(G)$  nor  $Ch(BG; \mathbb{F}_p)$ , the classical Chern subring, are contained in the other. See [11] for worked examples.

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