H a b i l i t a t i o n s s c h r i f t

eingereicht bei der
Fakultät für Mathematik
der
Ruprecht - Karls - Universität
Heidelberg

Vorgelegt von
Dr. rer. nat. (USA) Gerd Laures
aus Koblenz

Juni 2001

An E_∞-splitting of spin bordism with applications to real K-theory and topological modular forms

Contents

Introduction

This work originated from the question:

what is the E_∞-structure of the K-local spin bordism theory?

The work determines the structure by giving a multiplicative splitting of the spin bordism into E_∞-cells. The answer to the question leads to a construction of the Witten orientation for topological modular forms in the K-local E_∞-setting and creates new isomorphisms of Conner-Floyd type. The work consists of two major parts, one in which the splitting formula is developed and one which deals with its applications to real K-theory and to local topological modular forms.

The splitting of the spin bordism. One of the most important results about spin bordism goes back to Anderson, Brown and Peterson. They showed that two spin manifolds are spin bordant if and only if all Stiefel-Whitney and KO-characteristic numbers coincide. Moreover, all spin bordism groups can be computed from the additive 2-local splitting

$$MSpin \cong \bigvee_{\substack{n(J)even \\ 1\notin J}} ko\,\langle 4n(J)\rangle \vee \bigvee_{\substack{n(J)odd \\ 1\notin J}} ko\,\langle 4n(J)+2\rangle \vee \bigvee_{i\in I} \Sigma^{d_i} H\mathbb{Z}/2.$$

Here, J is a finite sequence of non negative integers, $n(J)$ is the sum over all entries and I is some index set. However, the formula does not capture the ring structure of the spin bordism. This problem has been around for the last 30 years because the Eilenberg Mac Lane part is very hard to track in practice. Fortunately, when we restrict our attention to the part which can be detected by K-theory this term disappears. More precisely, the localization of spin bordism with respect to mod 2 K-theory $K(1)$ is a sum of KO-theories.

The generator ζ of the -1st homotopy group of the $K(1)$-local sphere vanishes in KO and so it does in $MSpin$. These considerations lead to an E_∞-map from the E_∞-cone T_ζ over ζ to the $K(1)$-local $MSpin$. One may hope that this map already is an isomorphism since the homotopy groups appear to be the same. However, when taking into account the Dyer-Lashof operations we see that in KO-homology we get a free θ-algebra in one generator for T_ζ and one in infinitely many generators for $MSpin$. When analyzing the action of the Adams operations and collecting all results we end up in the multiplicative splitting:

$$MSpin \cong T_\zeta \wedge \bigwedge_{i=1}^{\infty} TS^0.$$

Here, TS^0 is the free E_∞-spectrum generated by the sphere spectrum. The proof of this splitting formula takes the major part of the work. The difficulty is the determination of the θ-algebra structure and a good control of the behaviour of the map from T_ζ. For that, we need to understand the ABP-splitting map in KO-homology and to conduct some 2-adic analysis.

Consequences of the splitting formula. As a first immediate corollary we obtain the θ-algebra structure of the spin bordism itself:

$$\pi_* MSpin \cong \pi_* KO \otimes T\{f_1, f_2, \dots\}$$

That is, the homotopy of $MSpin$ is the free θ-algebra over $\pi_* KO$ in infinitely many generators. Moreover, we give an algorithm for constructing the generating classes.

Dispite this beautiful formula for the θ-algebra structure it turns out that $MSpin$ can not be made into a KO-algebra even in the $K(1)$-local world. (The question if it can be made into an KO-module spectrum in the category of spectra was asked by Mahowald and answered by Stolz in [**Sto94**].)

The splitting formula allows a cellular decomposition of KO-theory when viewed as a relative E_∞-CW complex via the $\hat{A}$-map. Moreover, we give a new proof of the formula

$$MSpin_* X \otimes_{MSpin_*} KO_* \cong KO_* X$$

which was obtained by Hovey and Hopkins in [**HH92**]. It is not hard to see that the difficulties in proving this Conner-Floyd isomorphism appear in the $K(1)$-local setting. Thus the splitting formula can be applied and promptly furnishes the result.

Finally, the splitting is applied to topological modular forms. To motivate the results we first recall some facts about elliptic cohomology theories. For a more detailed treatment the reader is referred to the overview articles of Segal [**Seg88**] and Hopkins [**Hop95**].

A brief review of elliptic cohomology theories. In his analysis of the Dirac operator on loop spaces [**Wit88**] Witten introduced a bordism invariant for spin manifolds whose Pontryagin class $p_1/2$ vanishes. This so called Witten genus associates to each $O\langle 8\rangle$-manifold an integral modular form. One may hope that the theory allows a generalization to families of $O\langle 8\rangle$-manifolds. From the topological point of view, this means that there is a multiplicative transformation of generalized cohomology theories from the bordism theory $MO\langle 8\rangle$ to some new cohomology theory E. For spin bordism KO-theory plays the role of a family index theory via the $\hat{A}$-genus. Hence the theory E can be thought of as a higher version of KO-theory.

There are several candidates for such a theory E. The first was introduced by Landweber, Ravenel and Stong in [**LRS95**]. It has coefficients in the ring of modular forms of level 2 over $\mathbb{Z}[1/2]$. For this and other elliptic theories in which 2 is invertible the Witten orientation was first constructed by the author in [**Lau99**]. The difficulty appears at the prime 2 since $MO\langle 8\rangle$ is not well understood yet.

There are elliptic cohomology theories for which 2 is not a unit (see [**Fra92**] et al.) These theories E share the property that they are complex orientable and the associated formal group comes from the formal completion of an elliptic curve. The homotopy inverse limit of all these so called elliptic spectra is denoted by tmf since its coefficients allow an interpretation in terms of topological modular forms. This new theory tmf is not complex oriented but it maps to any elliptic spectrum in a canonical way. Hence, an $O\langle 8\rangle$ orientation of tmf furnishes the desired orientation of any other elliptic spectrum.

The applications to topological modular forms. In [**HAS99**] Hopkins, Ando and Strickland show that the construction of a Witten orientation can be reduced to finding one for the $K(1)$ and $K(2)$ localizations of tmf. Here, the theory $K(n)$ is the nth Morava K-theory at the prime 2. In [**Hop98b**] Hopkins shows that tmf can be obtained from T_ζ by attaching one more E_∞-cell. In particular, there is an E_∞-map from T_ζ to tmf. It thus suffices to relate the cone T_ζ to the bordism theory $MO\langle 8\rangle$. Since in the $K(1)$-local category $MO\langle 8\rangle$ coincides with $MSpin$ this is done by the splitting formula.

During the time when this work had been finished Hopkins found another approach to the Witten orientation which even works in the unlocalized setting. However, a splitting formula for the unlocalized $O\langle 8\rangle$-bordism seems to be out of reach from the today's point of view.

We use our splitting formula to explicitly list the E_∞-cells which are needed to obtain tmf from $MO\langle 8\rangle$: loosely speaking, the cells kill the free part of $MO\langle 8\rangle$ and turn the Adams operation ψ into the Atkin-Lehner operator for trivialized elliptic curves. Furthermore, we are able to establish the Conner-Floyd isomorphism

$$MO\langle 8\rangle_* X \otimes_{MO\langle 8\rangle_*} tmf_* \cong tmf_* X$$

in the $K(1)$-local category. In the last chapter deeper results on the structure of tmf are necessary which we reproduce from the work of Hopkins [**Hop98b**].

How this work is organized. In the first chapter we supply the technical framework for the splitting formula. We recall the basic properties of the $K(1)$-local category and then turn to E_∞-spectra. We take the classical point of view and work with operads. However, all constructions can be made in the brave new world of [**EKMM97**] or in the category of symmetric spectra of Smith [**HSS00**] as well. As a consequence calculations become more manageable. However, the invariance of the E_∞-structure under localization is not available in the literature and we felt to provide a proof here.

In the second chapter we review Dyer-Lashof operations in K-theory. The θ-algebra structure of TS^0 was first computed by McClure by using Dyer-Lashof operations in singular homology. We give a new proof of his result by looking at the representations of the symmetric groups and using results of Hodgkin and Atiyah.

In the third chapter we construct the map from the cone T_ζ to $MSpin$ and look at his behaviour in KO-homology. For that, we give explicit generators for the homology rings. Then we use Bott's canabalistic classes to compute the image of the generators under the ABP-splitting map.

The fourth chapter is devoted to the computation of the θ-algebra structure of various bordism theories. Partial results were obtained earlier by Snaith in [**HS75**] by looking at representations of the symmetric groups. Unfortuately, all calculations of Snaith were made in characteristic 2 and hence are not of any use for us. Moreover, it seemed to be difficult to generalize his long calculations to higher orders of 2. Hence, we take a different approach and look at tom Dieck operations instead. A fixed point formula of Quillen and a 'change of suspension formula' help to compute the action of θ for the spin bordism.

In the fifth chapter we compute the action of the Adams operations to decribe spherical classes. The splitting theorem is then a consequence of earlier results and some 2-adic analysis.

In the last two chapter the promised applications are given.

Acknowledgements. The author would like to thank Mike Hopkins for posing the question at the Oberwolfach Homotopietagung 1998 and for sharing the notes [**Hop98b**] that freely. He is indebted to Matthias Kreck and Haynes Miller for their encouragement and for interesting discussions. He is also grateful to Matthew Ando, Alexander Schmidt, Pete Bousfield and Peter May for their help in getting various points straight.

CHAPTER 1

E_∞-spectra and localizations

This chapter reviews the basics of the category of E_∞-spectra and their properties under localization. We will work in the category of May, Puppe et al. which is described in [**LMS86**] or [**Lau93**]. We use operads to describe E_∞-structures. This classical approach to E_∞-spectra has the advantage that every object is fibrant which keeps calculations easy to survey.

1. The category of spectra

In this work a topological space is a compactly generated weak Hausdorff space. We write $\mathcal{T}$ for the category of topological spaces and $\mathcal{T}_*$ for its based pendant. To talk about the category of spectra we first fix a real inner product space $\mathcal{U}$ of countable dimension, a so called universe. A prespectrum X is a collection of pointed spaces X_V indexed by the finite dimensional subspaces $V \subset \mathcal{U}$ equipped with maps

$$\sigma_{W,V} : S^{W-V} \wedge X_V \longrightarrow X_W$$

for all $V \subset W$. Here, the space $W - V$ is the orthogonal complement of V in W and S^{W-V} denotes its one-point compactification. One requires that $\sigma_{V,V}$ is the identity and that the associativity condition $(1 \wedge \sigma_{V,U})\sigma_{W,V} = \sigma_{W,U}$ holds whenever $U \subset V \subset W$. A spectrum is a prespectrum with the additional property that the adjoint maps

$$\tau_{W,V} : X_V \longrightarrow \Omega^{W-V} X_W$$

are homeomorphisms. A map of (pre-) spectra $f : X \longrightarrow Y$ is a collection of maps $f_U : X_U \longrightarrow Y_U$ commuting with the structure maps.

A construction of Lewis shows that the forgetful functor from the category of spectra $S_{\mathcal{U}}$ to prespectra admits a left adjoint L, called the specification. The specification is easy to visualize in case that each $\sigma_{W,V}$ is a closed inclusion:

$$LX = (V \mapsto \operatorname*{colim}_{W \supset V} \Omega^{W-V} X_W).$$

The morphism set $\mathcal{S}_\mathcal{U}(X,Y)$ has a natural topology as subspace of the product of mapping spaces $\mathcal{T}(X_V,Y_V)$. Moreover, for each pointed space Q we may form the spectrum

$$X^Q : V \mapsto \mathcal{T}_*(Q,X_V)$$

and its adjoint

$$X \wedge Q = L(V \mapsto X_V \wedge Q).$$

These constructions are natural with respect to all variables and we have the relationship

$$T_*(Q,\mathcal{S}_\mathcal{U}(X,Y)) \cong \mathcal{S}_\mathcal{U}(X\wedge Q,Y) \cong \mathcal{S}_\mathcal{U}(X,Y^Q).$$

The category $\mathcal{S}_\mathcal{U}$ becomes a closed model category if we define the fibrations and weak equivalences spacewise. That is, a map f from X to Y is a fibration (or w.e.) if for for all V the maps $f_V : X_V \longrightarrow Y_V$ are so. The resulting homotopy category is equivalent to the stable category of Adams for any infinte dimensional $\mathcal{U}$.

This coordinate free approach to spectra enables us to change universes in a continuous way. Suppose $\mathcal{U}$ and $\mathcal{V}$ are universes and let $\mathcal{L}(\mathcal{U},\mathcal{V})$ denote the (contractible) space of linear isometries from $\mathcal{U}$ to $\mathcal{V}$. Any $f \in \mathcal{L}(\mathcal{U},\mathcal{V})$ defines an adjoint pair of functors:

$$f_*X = L(W \mapsto S^{W-f(U)} \wedge X_U) \qquad f^*X : V \mapsto X_{fV}$$

with $U = f^{-1}W$. Since the functors continuously depend on the isometry f this construction can be generalized as follows. A map from a space A to $\mathcal{L}(\mathcal{U},\mathcal{V})$ gives rise to a functor $A \rtimes \quad : \mathcal{S}_\mathcal{U} \longrightarrow \mathcal{S}_\mathcal{V}$ and its adjoint $F[A, \quad) : \mathcal{S}_\mathcal{V} \longrightarrow \mathcal{S}_\mathcal{U}$. The construction is natural in A and reduces to the above if A is a point. Moreover, the half smash product has the following properties:

(i) the identity map $id_\mathcal{U} \in \mathcal{L}(\mathcal{U},\mathcal{U})$ serves as a unit

$$id_*X = \{id_\mathcal{U}\} \rtimes X \cong X$$

(ii) for any $X \in \mathcal{S}_\mathcal{U}$ and for any $A \longrightarrow \mathcal{L}(\mathcal{U},\mathcal{V})$, $B \longrightarrow \mathcal{L}(\mathcal{V},\mathcal{W})$ the map

$$B \times A \longrightarrow \mathcal{L}(\mathcal{V},\mathcal{W}) \times \mathcal{L}(\mathcal{U},\mathcal{V}) \longrightarrow \mathcal{L}(\mathcal{U},\mathcal{W})$$

satisfies

$$(B\times A) \rtimes X \cong B \rtimes (A \rtimes X).$$

(iii) for any $X \in \mathcal{S}_\mathcal{U}$, $Y \in S_\mathcal{V}$ and for any $A \longrightarrow \mathcal{L}(\mathcal{U},\mathcal{U}')$, $B \longrightarrow \mathcal{L}(\mathcal{V},\mathcal{V}')$ the map

$$A \times B \longrightarrow \mathcal{L}(\mathcal{U},\mathcal{U}') \times \mathcal{L}(\mathcal{V},\mathcal{V}') \longrightarrow \mathcal{L}(\mathcal{U} \oplus \mathcal{V},\mathcal{U}' \oplus \mathcal{V}')$$

satisfies

$$(A \times B) \rtimes (X \wedge Y) \cong (A \rtimes X) \wedge (B \rtimes Y).$$

Elmendorf took all spectra over all universes together into a big category $\mathcal{S}$. A morphism from a spectrum X over $\mathcal{U}$ into a Y over $\mathcal{V}$ is given by a pair, an isometry $f : \mathcal{U} \longrightarrow \mathcal{V}$ and a map of spectra from f_*X to Y. It is convenient to topologize $\mathcal{S}(X, Y)$ in a way that we have

$$\mathcal{T}/\mathcal{L}(\mathcal{U},\mathcal{V})(A, \mathcal{S}(X, Y)) \cong \mathcal{S}_\mathcal{V}(A \rtimes Y).$$

Here, $\mathcal{T}/\mathcal{L}(\mathcal{U},\mathcal{V})$ denotes the category of spaces over $\mathcal{L}(\mathcal{U},\mathcal{V})$.

These constructions are useful once it comes to smash products. If X is indexed over $\mathcal{U}$ and Y over $\mathcal{V}$ then the specification of the (partial) prespectrum

$$U \times V \mapsto X_U \wedge Y_V$$

is indexed over $\mathcal{U} \times \mathcal{V}$. This product is associative and symmetric up to coherent equivalences and turns $\mathcal{S}$ into a symmetric monoidal category with unit $I = (S^0, 0)$. To get an internal product in $\mathcal{S}_\mathcal{U}$ we can choose an isometry $f : \mathcal{U} \times \mathcal{U} \longrightarrow \mathcal{U}$ and take the pushforward $f_*(X \wedge Y)$. A more canonical object is the spectrum

$$\mathcal{L}(\mathcal{U} \times \mathcal{U},\mathcal{U}) \rtimes (X \wedge Y).$$

However, it still is only associative up to coherent equivalences in the homotopy category. To talk about 'ring-like' objects in $\mathcal{S}_\mathcal{U}$ we need to introduce the concept of an operad.

2. Operads and E_∞-spectra

An operad in a symmetric monoidal category $\mathcal{M} = (\mathcal{M}, I, \otimes)$ is a family of objects $T_0, T_1, \ldots \in \mathcal{M}$ together with right Σ_n-actions on each T_n and $\Sigma_n \times \Sigma_{i_1} \times \ldots \Sigma_{i_n}$-equivariant structure maps

$$T_n \otimes T_{i_1} \otimes \ldots \otimes T_{i_n} \longrightarrow T_{i_1+i_2+\ldots+i_n}.$$

These maps should satisfy certain associativity axioms. A pointed operad is an operad together with a map $I \longrightarrow T_1$ which behaves like a unit when composed with structure maps.

Instead of dealing with the details here we give the most important class of examples of pointed operads: the endomorphism operad $\mathrm{End}(X)$ for each object X in $\mathcal{M}$. Its n'th object is the morphism set $\mathcal{M}(X^{\otimes n}, X)$ and the structure maps are given by compositions

$$f_n \otimes f_{i_1} \otimes \cdots \otimes f_{i_n} \mapsto f_n(f_{i_1} \otimes \cdots \otimes f_{i_n}).$$

If $\mathcal{M}$ is enriched over $\mathcal{M}'$ we obtain an operad in $\mathcal{M}'$. For instance, the linear isometry operad $\mathcal{L} = \mathrm{End}(\mathcal{U})$ is an operad of spaces and so is $\mathrm{End}(X)$ for any spectrum X indexed over U. Moreover, the canonical projection

$$p : \mathrm{End}(X) \longrightarrow \mathcal{L}$$

is a map of pointed operads, meaning a collection of maps which is compatible with all structure maps. It associates to a map of spectra $f : X^{(n)} \longrightarrow X$ the underlying isometry $p(f) : \mathcal{U}^{(n)} \longrightarrow \mathcal{U}$.

An operad $\mathcal{T}$ over $\mathcal{L}$ is called an E_∞-operad if each of the spaces is contractible and the symmetric groups act in a free fashion. This way, the operad $\mathcal{L}$ becomes an E_∞ operad itself. An E_∞-ring spectrum is a spectrum X together with a map $T \longrightarrow \mathrm{End}(X)$ of pointed operads over $\mathcal{L}$.

There is another way to decribe the action of T on X which will proof useful later. Let G be a group which acts on $\mathcal{U}$ and X be a G-spectrum. Then for any G-equivariant map $A \longrightarrow \mathcal{L}(\mathcal{U}, \mathcal{V})$ let $A \rtimes_G X$ be the coequalizer of the two maps

$$(A \times G) \rtimes X \xrightarrow{\mu \times 1} A \rtimes X \quad \text{and} \quad (A \times G) \rtimes X \cong A \rtimes (G \rtimes X) \xrightarrow{1 \rtimes \mu} A \rtimes X.$$

The spectrum

$$TX = \bigvee_{n \geq 0} T_n \rtimes_{\Sigma_n} X^{(n)}$$

is the free E_∞-algebra generated by X over the operad T. This way, T becomes an endofunctor of $\mathcal{S}_{\mathcal{U}}$. More precisely, T is a triple (or a 'monad') since it comes with a unit $\eta : id \longrightarrow T$ and a natural transformation $\mu : T^2 \longrightarrow T$. The latter is built from the structure maps data of the operad T. An algebra over T is a spectrum X together with a morphism $\xi : TX \longrightarrow X$ making the diagrams

$$\begin{tikzcd} T^2X \arrow[r, "T\xi"] \arrow[d, "\mu_X"'] & TX \arrow[d, "\xi"] \\ TX \arrow[r, "\xi"] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "="'] & TX \arrow[d, "\xi"] \\ & X \end{tikzcd}$$

commute. An E_∞-map between E_∞-spectra is a map of spectra which commutes with the action of T. Later we will also use the notion of an H_∞-spectrum which is defined in the same way but the diagrams need only to commute up to homotopy.

Examples 2.1.

(i) The sphere spectrum is an E_∞-spectrum since it is the free object $T(*)$ generated by a point.

(ii) Let G be one of the classical groups $O, SO, Pin, Spin$ or one of their complex analogues $U, SU, Spin^c$ or Sp ect. Then the geometric bar construction [**May77**] gives simplicial toplogical spaces $B_*(GV, SV)$ and $B_*(GV)$ for each $V \subset \mathcal{U}$. Their geometric realizations BGV ($B(GV, SV)$ resp.) allow multiplication maps

$$\mu : BGV \times BGW \longrightarrow B(GV \times GW) \longrightarrow BG(V \oplus W)$$

which are commutative and associative on the nose. The spectification of

$$Th : V \mapsto B(GV, SV)/BGV$$

is called the Thom spectrum MG. For $f \in \mathcal{L}_n$ and for subspaces $V_1, V_2, \ldots, V_n$ define the structure maps

$$Th(V_1) \wedge \ldots \wedge Th(V_n) \xrightarrow{Th\mu} Th(V_1 \oplus \ldots \oplus V_n) \xrightarrow{Th(f)} Th(f(V_1 \oplus \ldots \oplus V_n))$$

and spectify to obtain an E_∞-structure on MG.

(iii) Other examples are less elementary. An investigation of the infinite loop spaces shows that connective real and complex K-theory ko and k are represented by E_∞-ring spaces and hence are E_∞-ring spectra. Moreover, the stable Adams operations ψ^r act as E_∞-maps after completion. Recently it was proved that the periodic theories KO and K are commutative S-algebras in [**EKMM97**] and symmetric spectra in [**Joa01**]. Since we work in a different category we will go through an argument below.

We write $\mathcal{S}_{\mathcal{U}}^T$ for the category of E_∞-spectra. The free functor T is left adjoint to the forgetful functor $U : \mathcal{S}_{\mathcal{U}}^T \longrightarrow \mathcal{S}_{\mathcal{U}}$. Hence, inverse limits in $\mathcal{S}_{\mathcal{U}}^T$ are inherited from $\mathcal{S}_{\mathcal{U}}$. Colimits are more difficult. It can be shown that for any diagram (X_α) of T-algebras the ordinary coequalizer of

$$T(\operatorname{colim} UTX_\alpha) \underset{d_1}{\overset{d_0}{\rightrightarrows}} T(\operatorname{colim} UX_\alpha)$$

admits a T-algebra structure which satisfies the universal property of a colimit in $\mathcal{S}_{\mathcal{U}}^T$. Here, d_0 is induced by the T-algebra structure on each X_α whereas d_1 comes from the natural transformation μ. The argument uses the fact that UT preserves reflexive coequalizers which can be checked spacewise. Alternatively, the existence of colimits follows from a general fact about categories of algebras over a triple (compare [**BW85**]).

The E_∞-category $\mathcal{S}_{\mathcal{U}}^T$ admits a closed model category structure with the following data: a map is fibration (resp. weak equivalence) if and only if Uf is one.

Finally, note that for E_∞-spectra E and F the product $E \wedge F \in \mathcal{S}_{U\times U}$ is an E_∞-spectrum in the following way: $E \wedge F$ is an algebra over the product E_∞-operad $T \times T$ via

$$\begin{aligned}(\mathcal{L}_n \times \mathcal{L}_n) \ltimes_{\Sigma_n} (E \wedge F)^n &\cong (\mathcal{L}_n \ltimes E^n \wedge \mathcal{L}_n \ltimes F^n)/\Sigma_n \\ &\longrightarrow \mathcal{L}_n \ltimes_{\Sigma_n} E^n \wedge \mathcal{L}_n \ltimes_{\Sigma_n} F^n \longrightarrow E \wedge F.\end{aligned}$$

3. The Bousfield localization with respect to $K(1)$

The concept of localization comes up in various branches of mathematics as an instrument of simplification. It can be applied to problems which itself are local in nature. For example, an abelian group G may be localized at a prime p if one is only interested in the p-torsion of G. Also, sometimes it happens that a non local problem can be localized in different ways and the local solutions lead to a global solution of the original problem. In this section we consider the localization with respect to some homology theory E. It is devoted to attack problems in stable homotopy which can be solved by means of the theory E.

A map $f : X \longrightarrow Y$ is called an E-equivalence if the induced map f_* in E-homology is an isomorphism. Since E does not distinguish between honest isomorphisms and E-equivalences it is useful to have a category in which E-equivalences are invertible and no other information is lost. The existence of this E-local category was proved by Bousfield in [**Bou79**]. He even showed that it can be realized as a full subcategory of the stable category $Ho\mathcal{S}_{\mathcal{U}}$.

In detail, a spectrum Z is E-local if each E-equivalence f induces a bijection in cohomology $f^* : Z^*Y \longrightarrow Z^*X$. The inclusion functor of the full subcategory $\mathcal{C}$ of E-local objects in $Ho\mathcal{S}_{\mathcal{U}}$ admits a left adjoint $L_E : Ho\mathcal{S}_{\mathcal{U}} \longrightarrow \mathcal{C}$, called the Bousefield localization functor. Hence, the unit of the adjunction $\eta : X \longrightarrow L_E X$ is an E-equivalence and associates an E-local spectrum to any spectrum X.

The Bousfield localization has the following elementary properties:

(i) L_E takes E-equivalences to isomorphisms.
(ii) L_E is idempotent: $L_E\, L_E \cong L_E$.
(iii) L_E preserves homotopy inverse limits.
(iv) L_E preserves cofibre sequences.
(v) If E is a ring spectrum and X is a module spectrum over E then X is E-local: $X \cong L_E X$.

We want to take E to be the first Morava K-theory. At the prime 2 the theory $K(1)$ coincides with mod 2 K-theory $K\mathbb{Z}/2 = K \wedge S\mathbb{Z}/2$ for the following reason: the 2-typicalization of the multiplicative formal group law $\hat{G}_m$ is the Honda formal group as the 2-series are the same. Hence, the localization with respect to $K(1)$ can be obtained as a process in two stages:

$$L_{K(1)} \cong L_{S\mathbb{Z}/2} L_{K_{(2)}}.$$

Note that for the localization it makes no difference to work with complex or real K-theory ([**Mei79**]). The latter was investigated by Adams, Baird and Ravenel. They showed that the KO-local sphere is closely related to the image of J spectrum: Let $J_{(2)}$ be the fibre of

$$\psi^3 - 1 : KO\mathbb{Z}_{(2)} \longrightarrow KO\mathbb{Z}_{(2)}.$$

Here, ψ^3 is the third stable Adams operation. Then there is a fibration of the form

$$L_{KO\mathbb{Z}_{(2)}} S \longrightarrow J_{(2)} \longrightarrow \Sigma^{-1} S\mathbb{Q}.$$

The rational part vanishes once we localize with respect to the Moore spectrum $S\mathbb{Z}/2$: for any X the localization $L_{S\mathbb{Z}/2}X$ is the function spectrum

$F(\Sigma^{-1}S\mathbb{Z}/2^\infty, X)$. One should think of the localization with respect to $S\mathbb{Z}/2$ as a completion since there is an exact sequence [**Bou79**]

$$0 \longrightarrow \mathrm{Ext}(\mathbb{Z}/2^\infty, \pi_* X) \longrightarrow \pi_* L_{S\mathbb{Z}/2} X \longrightarrow \mathrm{Hom}(\mathbb{Z}/2^\infty, \pi_{*-1} X) \longrightarrow 0.$$

Summarizing, we see that the $K(1)$-local sphere coincides with the completed image of J-spectrum $L_{S\mathbb{Z}/2} J_{(2)}$. Moreover, we get the fibration

$$L_{K(1)} S \longrightarrow L_{K(1)} KO \xrightarrow{\psi^3 - 1} L_{K(1)} KO.$$

which enables us to calculate the homotpoy groups of the K(1)-local sphere. In addition, it turns out that $KO_{(2)}$-theory is smashing. This means that for any X the spectrum $L_{KO_{(2)}} S \wedge X$ is a $KO_{(2)}$-localization of X. Hence, smashing the sequence above with X gives the fibre sequence

$$X \longrightarrow KO \wedge X \xrightarrow{\psi^3 - 1} KO \wedge X$$

in the $K(1)$-local category.

4. E_∞-structures on localizations

We now show that the localization functor with respect to an arbitrary theory E can be chosen to preserve E_∞-structures. For the category of S-modules this fact can be found in [**EKMM97**]. Since for the category $\mathcal{S}_\mathcal{U}$ it is not available in the literature we sketch a proof here.

To make the statement plausible let E be a spectrum. Then one usually constructs L_E as follows: Let $\{i_\alpha : X_\alpha \longrightarrow Y_\alpha\}$ be the set of stable CW-inclusions such that each i_α is an E_*-equivalence and the number of cells of Y_α is smaller or equal to the number c of elements in the coefficients of E. Then the first approximation of L_E is the pushout

$$\begin{tikzcd}
\bigvee_\alpha \bigvee_{f: X_\alpha \to X} X_\alpha \arrow[r, "(f)"] \arrow[d, "(i_\alpha)"] & X \arrow[d] \\
\bigvee_\alpha \bigvee_{f: X_\alpha \to X} Y_\alpha \arrow[r] & L_E^1 X
\end{tikzcd}$$

If L_E^1 is applied often enough we set $L_E X = (L_E^1)^t X$. More precisely, we have to take colimits when passing through limit ordinals and let t be the first cardinality greater than c.

Now if X is an E_∞-ring spectrum we simply replace the maps i_α by the free E_∞-maps Ti_α and take all colimits in the E_∞-category. In order to see that this strategy works we need the

LEMMA 4.1. (i) *T preserves E_*-equivalences.*

(ii) *T preserves h-cofibrantly generated maps between h-cofibrantly generated spectra.*

The second statement means the following: given a map $g : A \longrightarrow B$ of prespectra with the property that each of the maps g_U and each of the structure maps of the prespectra A and B are h-cofibrations. Then the application of T to the spectification of g is generated by a map of prespectra with the same properties. An h-cofibration is a map which has the homotopy extension property up to homotopy for all spaces.

SKETCH OF A PROOF. We may assume that T is the linear isometry operad. For (i) let $f : X \longrightarrow Y$ be an E_*-equivalence. Then clearly $\mathcal{L}_n \rtimes f^{\wedge n}$ is so. Moreover, the projection onto the second factor $E\Sigma_n \times \mathcal{L}_n \longrightarrow \mathcal{L}_n$ is a Σ_n-equivariant homotopy equivalence. Hence, we have natural equivalences

$$\mathcal{L}_n \rtimes_{\Sigma_n} X^{\wedge n} \cong (E\Sigma_n \times \mathcal{L}_n) \rtimes_{\Sigma_n} X^{\wedge n} \cong E\Sigma_{n+} \wedge_{\Sigma_n} (\mathcal{L}_n \rtimes X^{\wedge n})$$

The Lerray-Serre spectral sequence applied to the right (relative) bundle over $B\Sigma_n$ shows that $T_n f = \mathcal{L}_n \rtimes_{\Sigma_n} f^{\wedge n}$ is an E_*-equivalence.

For (ii) observe that a map $f : X \longrightarrow Y$ is a h-cofibration iff the projection p from the mapping cylinder M_f to Y is a homotopy equivalence under X [**tDKP70**]. Hence we see that $T_n p : T_n M_g \longrightarrow T_n B$ is a homotopy equivalence under $T_n A$. We are left to show that

$$T_n M_g = \operatorname{colim}(T_n(A \wedge I_+) \longleftarrow T_n A \longrightarrow T_n B)$$

is homotopy equivalent under $T_n A$ to

$$M_{T_n g} = \operatorname{colim}(T_n A \wedge I_+ \longleftarrow T_n A \longrightarrow T_n B)$$

This is provided by the maps

$$\begin{array}{rcl} T_n A \wedge I_+ & \cong & T_n(A \wedge I_+) \\ (f, x_1, \dots, x_n, t) & \to & (f, x_1, t, \dots, x_n, t) \\ (f, x_1, \dots, t_1 \cdots t_n) & \leftarrow & (f, x_1, t_1, \dots, x_n, t_n) \end{array}$$

which are homotopy inverse to each other under $T_n A \wedge \{0\}_+$. $\square$

Let D be the above diagram with Ti_α instead of i_α. Then each of the maps Ti_α is h-cofibrantly generated and an E_*-equivalence. Hence the usual pushout in $\mathcal{S}_\mathcal{U}$ is a homotopy pushout. Thus the new localization map

$$\eta : X \longrightarrow L_E^1 X \cong \operatorname*{colim}_{\mathcal{S}_\mathcal{U}^T} D$$

is an E_*-equivalence as it is a direct summand of

$$TX \longrightarrow T\operatorname*{colim}_{\mathcal{S}_\mathcal{U}} D = \operatorname*{colim}_{\mathcal{S}_\mathcal{U}{}^T} TD.$$

The latter is an E_*-equivalence by the lemma and excision. Similarly, passing through limit ordinals will preserve E_*-equivalences since it does so after appying T.

We are left to show that $L_E X$ is E_*-local. Let us be given an extension problem

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "f"] & Z \arrow[dl, dashed] \\ L_E X & \end{tikzcd}$$

in which g bs an E_*-equivalence. Without loss of generality we may assume g is a CW-inclusion of a subcomplex Y in a CW-spectrum Z. Consider first the case that the number of cells of Z is not greater than c. Then one verifies that f already maps to $L_E^s X$ for some $s < t$. Moreover, Tg is part of the glueing maps for $L_E^{s+1} X$. Hence there is an obvous extension. For the general case one decomposes the map g into smaller pieces and uses Zorns lemma as in [**Bou79**] to get the desired result.

The construction actually shows a bit more: if a map $g : Y \longrightarrow Z$ is an E_∞-map between E_∞-ring spectra then clearly $L_E g$ is so.

EXAMPLE 4.2. In order to show that KO is an E_∞-spectrum in the $K(1)$-local category we merely observe that the map

$$ko \longrightarrow ko[\beta^{-1}] = KO$$

is a $K(1)$-equivalence (compare 3.1.3) and the completed KO is $K(1)$-local. Here $\beta \in \pi_8 ko$ is the Bott class. Furthermore, the stable Adams operations act on KO by E_∞-maps. A similar result holds for the complex K-theory.

CHAPTER 2

θ-Algebras

This chapter is devoted to the algebraic objects which come up as the homotopy of $K(1)$-local E_∞-ring spectra. A θ-algebra is an algebra together with a single operation θ. The classical theory of such θ-algebras goes back to Grothendieck and Atiyah who investigated the exterior power operations in the representation theory and in K-cohomology rings. These power operations were later generalized by McClure and Hopkins to Dyer-Lashof operations for arbitrary $K(1)$-local E_∞-spectra. Their general properties were recently studied by Bousefield in [**Bou99**][**Bou96b**][**Bou96a**] from an axiomatic point of view.

We first review Atiyah's and Hodgkin's work on power operations. Then we give a new proof of McClure's result on the structure of TS^0. Finally, we turn to the spectrum T_ζ and work out the computations of Hopkins. There is hardly any new result in this chapter. However, the new treatments and proofs provide a convenient framework for things to come.

We work in the category $\mathcal{C}$ of $K(1)$-local spectra and omit the localization functor from the notation.

1. The K-homology ring of TS^0

Since Atiyah's work on power operations we know that there is a close relationship between operations in K-theory and the K-homology ring of

$$TS^0 \simeq \bigvee_{n=0}^{\infty} B\Sigma_{n+}.$$

The latter can be computed.

Theorem 1.1. *There is an isomorphism of rings*

$$\pi_*K \wedge TS^0 \cong \pi_*K[\theta_1, \theta_2, \theta_4, \dots]$$

The ring of the right hand side is to be understood as object in the category of 2-complete π_*K-algebras. For instance, the power series $\sum_n \theta_{2^n} 2^n$ is a valid class. The proof of the theorem uses the following result of Hodgkin which can be found in [**Hod72**]:

PROPOSITION 1.2. *Let $R\Sigma_n$ be the represenation ring and $I\Sigma_n$ the augmentation ideal. Equip $R\Sigma_n \subset R\Sigma_n^\wedge$ with the $I\Sigma_n$-adic topology. Then we have*

$$\begin{aligned} \pi_* K\mathbb{Z}/2^r \wedge B\Sigma_{n+} &\cong Hom_{cts}(R\Sigma_n, \pi_* K\mathbb{Z}/2^r) \\ \pi_* K\mathbb{Z}/2^r \wedge TS^0 &\cong \pi_* K\mathbb{Z}/2^r[\theta_1, \theta_2, \theta_4, \dots]. \end{aligned}$$

Here, Hom_{cts} *denotes the group of continuous homomorphisms.*

Hence, the theorem follows from the

LEMMA 1.3.

$$\begin{aligned} \pi_* K \wedge B\Sigma_{n+} &\cong \lim_r \pi_* K\mathbb{Z}/2^r \wedge B\Sigma_{n+} \cong \mathrm{Hom}_{cts}(R\Sigma_n, \pi_* K) \\ \pi_* K \wedge TS^0 &\cong \lim_r \pi_* K\mathbb{Z}/2^r \wedge TS^0 \end{aligned}$$

PROOF. For any spectrum X the $K(1)$-local $K \wedge X$ can be written as the homotopy inverse limit of the sequence

$$K\mathbb{Z}/2 \wedge X \longleftarrow K\mathbb{Z}/4 \wedge X \longleftarrow K\mathbb{Z}/8 \wedge X \longleftarrow \dots$$

Hence, there is a short exact sequence

$$0 \to \lim_r{}^1 \pi_{*+1} K\mathbb{Z}/2^r \wedge X \to \pi_* K \wedge X \to \lim_r \pi_* K\mathbb{Z}/2^r \wedge X \to 0$$

and it suffices to show that the $\lim^1$-term vanishes. This is obvious for the case $X = TS^0$ from Hodgkin's result and follows for the classifying spaces since they are direct summands of TS^0. $\square$

We are going to describe the elements θ_k in more detail. The class θ_1 comes from the unit of the operad T

$$S^0 \simeq T_1 S^0 \longrightarrow TS^0 \simeq S^0 \wedge TS^0 \longrightarrow K \wedge TS^0.$$

For the others we consider the dual representation ring $\mathrm{Hom}(R\Sigma_n, \mathbb{Z}_2)$ of all homomorphisms. It admits an interpretation as the group of elements of degree n in the ring of symmetric polynomials in indeterminants t_i of degree 1: let

$$\Delta_{k,n} \in \mathbb{Z}_2[t_1, t_2, \dots, t_k]_n^{\Sigma_k} \otimes R\Sigma_n$$

be the function

$$\Delta_{k,n}(t_1, t_2, \dots, t_k, g) = \mathrm{Trace}(gT^{\otimes n}).$$

In this notation we regard $R\Sigma_n$ as the character ring and g lies in Σ_n. The letter T denotes here the diagonal matrix $(t_1, t_2, \dots, t_k)$ acting on $\mathbb{C}^k$. Atiyah showed that the map

$$\Delta : \sum_{n\geq 0} \mathrm{Hom}(R\Sigma_n, \mathbb{Z}_2) \longrightarrow \sum_{n\geq 0} \lim_k [t_1, t_2, \dots, t_k]_n^{\Sigma_k}$$

given by $\sum f_n \mapsto \sum (1 \otimes f_n)\Delta_{k,n}$ is a ring isomorphism.

Example 1.4. Σ_2 admits two irreducible representations: the trivial 1 and the sign representation σ. One readily verifies

$$\Delta_{k,2} = e_2 \otimes \sigma + (e_1^2 - e_2) \otimes 1$$

for all $k \geq 2$. Here, the e_i's denote the elementary symmetric functions. The augmentation ideal of $R\Sigma_2 = \mathbb{Z}[\sigma]/\sigma^2 - 1$ is generated by $1 - \sigma$. We claim that all homomorphisms are continuous. Obviously e_1^2 is so. Hence it suffices to check e_2:

$$e_2(1-\sigma)^n = -\sum_{k\,odd} \binom{n}{k} = -2^{n-1}$$

In these terms, we can inductively define the θ_i's by declaring the powers sums $\sigma_{2^k} = \sum t_i^{2^k}$ to be the k-th Witt polynomial in the θ_i's

$$\sigma_{2^k} = \theta_1^{2^k} + \ldots + 2^k\theta_{2^k}$$

or, equivalently,

$$\prod_{i=0}^{\infty}(1 - t_i x) = \prod_{i=0}^{\infty}(1 - \theta_n x^n).$$

Atiyah explained how such an element θ_k leads to an operation

$$\theta^k : KX \xrightarrow{()^{\otimes n}} K_{\Sigma_n}X \cong KX \otimes \mathrm{Rep}\Sigma_n \xrightarrow{1\otimes\theta_k} KX$$

For instance, the power sums σ_k give rise to the Adams operation ψ^k. Hence, the formula for θ_2 may be interpreted as

$$\psi^2(x) = x^2 + 2\theta^2(x)$$

for all $x \in KX$. This was the first topological example of what is called a θ-algebra. Later McClure [**BMMS86**] has shown how the θ_i come up by an operation of Dyer-Lashof type.

2. The category of θ-algebras

We now turn to the algebraic picture and work in the category of 2-complete groups.

DEFINITION 2.1. *A θ-algebra is a commutative algebra A over a ring R with unit together with a function $\theta : A \longrightarrow A$ such that*

$$\begin{aligned} \theta(1) &= 0 \\ \theta(a+b) &= \theta(a)+\theta(b)-a\,b \\ \theta(a\,b) &= \theta(a)b^2+a^2\,\theta(b)+2\theta(a)\theta(b). \end{aligned}$$

For a θ-algebra A we define the operation $\psi : A \longrightarrow A$ by the equation $\psi(x) = x^2 + 2\theta(x)$. One easily checks the

PROPOSITION 2.2. *ψ is a ring homomorphism and commutes with θ.*

EXAMPLES 2.3. (i) $\mathbb{Z}_2$ is a θ-algebra via

$$\theta(x) = \frac{x - x^2}{2} \qquad \text{and} \qquad \psi(x) = x.$$

Similarly, the ring $C = \mathcal{T}(\mathbb{Z}_2, \mathbb{Z}_2)$ of continuous functions on the 2-adics is a θ-algebra with $\psi(f) = f$.

(ii) There is not any θ-algebra of characteristic 2: when setting $(a, b) = (1, 1)$ (and $(1, 0)$ resp.) in the addition formula we see that 1 equals 0 for any of such.

(iii) For any space X the ring KX is a θ-algebra via the operation θ^2 as explained before. The properties of θ immediately follow from the naturality and the fact that the Adams operation $\psi^2 = \psi$ is a ring homomorphism. We will see more examples from topology below.

The following observation carries the name Wilkerson criterion [**Bou96a**]:

PROPOSITION 2.4. *Let A be torsion free and ψ an algebra endomorphism of A with the property that $\psi(a) = a^2$ mod 2. Then A has a unique θ-algebra structure with $\psi x = x^2 + 2\theta x$ for all $x \in A$.*

The forgetful functor from θ-algebras $\mathcal{F}$ to complete p-modules admits a left adjoint T: if M is free on one generator x we define

$$TM = R[x, x_1, x_2, \ldots]$$

and set $\theta(x_i) = x_{i+1}$, $\theta(x) = x_1$. This algebra will be denoted by $R \otimes T\{x\}$ in the sequel. If M is free on generators $\{x_i\}_{i\in I}$ we set

$$TM = R \otimes \bigotimes_{i\in I} T\{x_i\} \cong R \otimes T\{x_i\}_{i\in I}.$$

For the general case we first observe that $\mathcal{F}$ has coequalizers and tensor products and thus colimites: if $f, g : A \longrightarrow B$ are θ-maps, then the ideal generated by the set

$$\{f(a) - g(a)\,; a \in A\}$$

is closed under the operation of θ:

$$\theta(f(a) - g(a)) = (f(\theta a) - g(\theta a)) + (f(a) - g(a))g(a).$$

Hence, the quotient ring is a coequalizer in $\mathcal{F}$. Tensor products of θ-algebras are obtained by choosing free representations and taking cokernels successively. Similarly, the free functor T of a general module M is obtained by presenting M as the cokernel of a map of free modules.

The free algebra $T\{x\}$ has another basis which is constructed as follows. In each θ-algebra A there is family of natural operations θ_n which satisfies

$$\psi^n a = (\theta_0 a)^{2^n} + 2(\theta_1 a)^{2^{n-1}} + \cdots + 2^n \theta_n a$$

Here, ψ^n is the iteration of ψ. These operations can be inductively defined by the equations (compare [**Bou96a**])

$$\begin{aligned} \theta_n a &= \theta^{\sim n}(\theta_0 a) + \theta^{\sim n-1}(\theta_1 a) + \cdots + \theta^{\sim 1}(\theta_{n-1} a). \\ \theta^{\sim n} a &= \sum_{i=1}^{2^{n-1}} (-1)^{i+1} 2^{i-n} \binom{2^{n-1}}{i} (\psi a)^{2^{n-1}-i} (\theta a)^i. \end{aligned}$$

For instance, for our θ-ring KX the classes θ_n coincide with Hodgkin's operations θ_{2^n} considered earlier.

LEMMA 2.5. *$\theta(\theta_k) = \theta_{k+1} + \epsilon$ with a polynomial ϵ depending only on $\theta_0, \theta_1, \dots, \theta_k$. In particular, we have*

$$T\{\theta\} \cong \mathbb{Z}_2[\theta_0, \theta_1, \theta_2, \dots]$$

PROOF. Compute

$$\sum_{i=0}^{k+1} 2^i \theta_i^{2^{k+1-i}} = \psi\Big(\sum_{i=0}^{k} 2^i \theta_i^{2^{k-i}}\Big) = \sum_{i=0}^{k} 2^i (\psi\theta_i)^{2^{k-i}} = \sum_{i=0}^{k} 2^i (\theta_i^2 + 2\theta(\theta_i))^{2^{k-i}}$$

□

3. Dyer-Lashof Operations for $K(1)$-local E_∞-spectra

We have seen that the ring $\pi_0 K \wedge TS^0$ carries a free θ-algebra structure in a generator θ_1 by interpreting the elements as operations in K-theory. There is a more intrinsic description of the θ-algebra structure which works for arbitrary $K(1)$-local E_∞-spectra E.

An E_∞-structure ξ on E determines a power operation

$$P : E^0 X \longrightarrow E^0 T_n X$$

by setting

$$P(x) : T_n X \longrightarrow TX \xrightarrow{Tx} TE \xrightarrow{T\xi} E$$

for each $x \in E^0 X$. For $X = S^0$ and $n = 2$ this gives a map $P(x) : B\Sigma_{2+} \longrightarrow E$ for each $x \in \pi_0 E$. The classifying space $B\Sigma_{2+}$ reduces to two copies of S^0 in the $K(1)$-local world. To see this, consider the map

$$(\epsilon, Tr) : B\Sigma_{2+} \longrightarrow pt_+ \vee (E\Sigma_2)_+ \cong S^0 \vee S^0$$

which consists of the constant map $\epsilon = const_+$ and the transfer Tr. It is a weak equivalence in $\mathcal{C}$: the transfer of a one dimensional trivial bundle is the bundle corresponding to the representation in which Σ_2 acts on $\mathbb{C}^2$ by permuting coordinates. Since this bundle comes from $1 + \sigma$ we obtain the isomorphism

$$(1 \wedge \epsilon, 1 \wedge Tr)_* : \pi_0 K \wedge B\Sigma_{2+} \cong \mathbb{Z}_2\sigma_2 \oplus \mathbb{Z}_2 e_2 \xrightarrow{1} \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

as one easily checks.

We follow the lines of [**Hop98b**] and define maps

$$\theta, \psi : S^0 \longrightarrow B\Sigma_{2+}$$

by requiring

$$\begin{pmatrix} Tr \\ \epsilon \end{pmatrix} \begin{pmatrix} \theta & \psi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $e : S^0 \cong Be_+ \longrightarrow B\Sigma_{2+}$ we have $\epsilon e = 1$, $Tr\, e = 2$ and thus

$$e = \psi - 2\theta.$$

With $\theta(x) = P(x)\,\theta$ and $\psi(x) = P(x)\,\psi$ the last equation gives

$$\psi(x) - 2\theta(x) = P(x)\, e = x^2$$

which is Atiyah's equation. This also justifies the sign in the definition of θ. It is needless to mention that θ is natural with respect to E_∞-maps.

PROPOSITION 3.1. *The operation θ turns $\pi_0 E$ into a θ-algebra.*

PROOF. First assume that $\pi_0 E$ is torsion free. Then it suffices to check that ψ is a ring homomorphism. Recall from [**BMMS86**] the formulae

$$\begin{aligned} P(x+y) &= Px + Py + Tr^*(x\,y) \\ P(x\,y) &= Px\,Py. \end{aligned}$$

Thus we must show that ψ induces a ring map in E-cohomology. That is, the stable map ψ should commute with the diagonal map

$$\Delta_+\psi = (\psi \wedge \psi)\Delta_+ \in \pi_0 B\Sigma_{2+} \wedge B\Sigma_{2+}.$$

Since ϵ commutes with Δ_+ we have

$$(\epsilon \wedge \epsilon)\Delta_+\psi = \Delta_+\epsilon\psi = \Delta_+ = (\epsilon \wedge \epsilon)(\psi \wedge \psi)\Delta_+.$$

Moreover, the map $f = (Tr \wedge Tr)\Delta_+\psi$ is null: in K-theory we have

$$f^*1 = \psi^*(Tr(1)^2) = \psi^*((1+\sigma)^2) = \psi^*(2(1+\sigma)) = 2\psi^* Tr(1) = 0.$$

Also the composite $\Delta_+ Tr\,\psi$ vanishes for trivial reasons. Since (ϵ, Tr) is a $K(1)$-equivalence and E is local we have established the commutativity of ψ and Δ_+. This finishes the proof for the torsion free case.

For general E let x, y be classes in $\pi_0 E$. Consider the E_∞-map

$$T(x,y) : T(S^0 \vee S^0) \longrightarrow E.$$

In order to establish the addition and multiplication laws it suffices to show that $\pi_0 T(S^0 \vee S^0)$ is torsion free. There are many ways to see that the Hurewicz map

$$\pi_0 T(S^0 \vee S^0) \longrightarrow \pi_0 K \wedge T(S^0 \vee S^0) \cong \pi_0 K \otimes T\{x, y\}$$

is injective. For instance, the computation in the Adams-Novikov spectral sequence based on K at the end of this section gives an argument. $\square$

With 2.3 we have

COROLLARY 3.2. *Any $K(1)$-local E_∞- spectrum with coefficients in a $\mathbb{F}_2$-algebra is trivial.*

Example 3.3. For a pointed space X consider the function spectrum K^X. Since it is an E_∞-ring spectrum we have a θ-algebra structure on $\pi_0 X = KX$. From the definition of the power operations we see that the operations ψ and θ coincide with the second Adams operation ψ^2 and Atiyah's operation θ^2.

The ring of all operations contains the subring $\pi_0 K \wedge TS^0$ as explained earlier. Its θ-algebra structure is determined by the

Theorem 3.4. *In $\pi_0 K \wedge TS^0$ we have the equality $\psi(\sigma_{2^k}) = \sigma_{2^{k+1}}$. In particular, $\pi_* K \wedge TS^0$ is the free θ-algebra in θ_1.*

Proof. The equation looks like the relation among Adams operations $\psi^2\psi^{2^k} = \psi^{2^{k+1}}$. Indeed the formula will follow once we have established

$$(\psi f)(x) = \psi^2(f(x))$$

for all $x \in KX$ and $f \in \pi_0 K \wedge B\Sigma_{n+}$. In this context f is regarded as an operation as explained earlier. The equality is part of the following commutative diagram

$$\begin{tikzcd}
X \arrow[d, "\psi\wedge 1"] \arrow[dddd, dashed, bend right] & & \\
B\Sigma_{2+} \wedge X \arrow[r, "\Delta"] \arrow[d, "1\wedge f\wedge 1"] & E\Sigma_{2+} \underset{\Sigma_2}{\wedge} X^2 \arrow[r, "1\wedge(fx)^2"] \arrow[d, "1\wedge(f\wedge 1)^2"] & E\Sigma_{2+} \underset{\Sigma_2}{\wedge} K^2 \arrow[dddd, dashed, bend left] \\
E\Sigma_{2+} \underset{\Sigma_2}{\wedge} B\Sigma_{n+}^2 \wedge X \arrow[r, "1\wedge\Delta"] \arrow[dd, "\xi"] & E\Sigma_{2+} \underset{\Sigma_2}{\wedge} (B\Sigma_{n+} \wedge X)^2 \arrow[d, "1\wedge\Delta"] & \\
& E\Sigma_{2+} \underset{\Sigma_2}{\wedge} (E\Sigma_{n+} \underset{\Sigma_n}{\wedge} X^n)^2 \arrow[r, "1\wedge x^{2n}"] \arrow[d, "\xi"] & E\Sigma_{2+} \underset{\Sigma_2}{\wedge} (E\Sigma_{n+} \underset{\Sigma_n}{\wedge} K^n)^2 \arrow[uu, "\xi"] \arrow[d, "\xi"] \\
B\Sigma_{2n+} \wedge X \arrow[r, "1\wedge\Delta"] & E\Sigma_{2n+} \underset{\Sigma_{2n}}{\wedge} X^{2n} \arrow[r, "1\wedge x^{2n}"] & E\Sigma_{2n+} \underset{\Sigma_{2n}}{\wedge} K^{2n} \arrow[d, "\xi"] \\
& & K
\end{tikzcd}$$

Here, we assumed that f lies in the K-Hurewicz image which is allowed by induction. Then the composite over the left curved side of the outer square is $(\psi f)(x)$ whereas the right side gives $\psi^2(f(x))$ by what we said earlier. The last statement is immediate from 1.1 and 2.5. □

An alternative proof of the proposition is given in [**BMMS86**]. McClure used the ordinary Dyer-Lashof operations in singular homology instead of represenation theory to obtain the result.

COROLLARY 3.5. *For every K(1)-local E_∞-spectrum $\pi_*E \wedge TS^0$ is the free θ-algebra on the generator θ_1.*

Before proving the corollary we need a lemma which is easily checked.

LEMMA 3.6. *Suppose E, F are E_∞-ring spectra and $x \in \pi_0 E \wedge F$ lies in the Hurewicz image of $\pi_0 E$. Then we have*

$$\theta_E x = \theta_{E\wedge F} x \in \pi_0 E \wedge F.$$

PROOF OF 3.5: The case $E = K$ of the corollary was shown in the theorem. For $E = S^0$ we consider the Adams-Novikov spectral sequences

$$\begin{aligned} \mathrm{Ext}_{K_*K}(\pi_*K, \pi_*K) &\implies \pi_*S^0 \\ \mathrm{Ext}_{K_*K}(\pi_*K, \pi_*K \wedge TS^0) &\implies \pi_*TS^0 \end{aligned}$$

They converge by the theorem 6.10 of [**Bou79**]. To compute the E_2-term of the second observe that for the isomorphism

$$\pi_*K \wedge TS^0 \cong \pi_*K \otimes T\{\theta_1\}$$

all classes θ_n are spherical by the lemma. Hence, the spectral sequence for π_*TS^0 takes the form

$$\mathrm{Ext}_{K_*K}(\pi_*K, \pi_*K) \otimes T\{\theta_1\} \Longrightarrow \pi_*S^0 \otimes T\{\theta_1\}$$

and we are done.

Finally, the general statement follows from the Kuenneth isomorphism:

$$\pi_*E \wedge TS^0 \cong \pi_*E \otimes_{\pi_*S^0} (\pi_*S^0 \otimes T\{\theta_1\}) \cong \pi_*E \otimes T\{\theta_1\}$$

for arbitrary $K(1)$-local E_∞-spectra E. $\square$

The corollary can be found without proof in [**Hop98b**].

4. The spectrum T_ζ

In the last paragraph we computed the θ-algebra structure associated to the homotopy of the free spectrum generated by the sphere. Now we proceed with our basic calculations and investigate the sphere spectrum with one more E_∞-cell attached to. The cone is taken over a class in the homotopy of S^0 which plays an important in many contexts in topology.

Recall from section 1.3 that we have a cofibre sequence

$$S^0 \longrightarrow KO \xrightarrow{\psi^3-1} KO.$$

Since ψ^3 acts trivially in $\pi_0 KO$ the element $1 \in \pi_0 KO \cong \mathbb{Z}_2$ gives rise to a non trivial class $\zeta \in \pi_{-1}S^0$. Obviously, ζ topologically generates $\pi_{-1}S^0 \cong \mathbb{Z}_2$.

Remark 4.1. The class $\zeta = \zeta_1$ belongs to a family of homotopy classes

$$\zeta_n : S^{-1} \longrightarrow L_{K(n)}S^0$$

which play an important role in the reassembling of spectra from their monochromatic parts, that is, in Hopkins' chromatic splitting conjecture. The higher ζ_n correspond to the determinant map on the Morava stabilizer group S_n under the homotopy fixed point spectral sequence

$$E_2 = H^{*,*}(S_n; E_{n*})^{\mathbb{Z}/n} \Longrightarrow \pi_* L_{K(n)}S^0.$$

The interested reader is referred to [**Hov95**].

Definition 4.2. *We define T_ζ to be the homotopy pushout of the diagram*

$$\begin{tikzcd} TS^{-1} \arrow[r, "T*"] \arrow[d, "\zeta"] & T* = S^0 \arrow[d] \\ S^0 \arrow[r] & T_\zeta \end{tikzcd}$$

The theory T_ζ corepresents the functor which associates to any E in $\mathcal{C}^T$ the set of all null homotopies of ζ in E. This set is non empty for KO-theory. A choice of a null homotopy defines a map of cofibre sequences

$$\begin{tikzcd} S^0 \arrow[r, "\gamma"] \arrow[d, "1"] & C_\zeta \arrow[r, "\delta"] \arrow[d, "\iota"] & S^0 \arrow[d, "1"] \\ S^0 \arrow[r, "1"] & KO \arrow[r, "\psi^3-1"] & KO \end{tikzcd}$$

This diagram immediately gives

LEMMA 4.3. $Ho\mathcal{C}^T(T_\zeta, KO) \cong \mathbb{Z}_2$

PROOF. Any two choices of a map ι can only differ by a multiple of $1\,\delta$. $\square$

We assume from now on that we made a choice of a map ι. Then this map defines a splitting (ι_*, ι'_*) of the exact sequence

$$0 \longrightarrow \pi_0 KO \xrightarrow{\gamma_*} \pi_0 KO \wedge C_\zeta \xrightarrow{\delta_*} \pi_0 KO \longrightarrow 0.$$

Let b be the image of $1 \in \pi_0 KO$ under the composite

$$\pi_0 KO \xrightarrow{\iota'_*} \pi_0 KO \wedge C_\zeta \xrightarrow{\eta_{C_\zeta *}} \pi_0 KO \wedge TC_\zeta \xrightarrow{\xi_{S^0 *}} \pi_0 KO \wedge T_\zeta$$

COROLLARY 4.4. *(compare* [**Hop98b**]*) The KO-linear extension of b*

$$b_* : \pi_* KO \wedge TS^0 \longrightarrow \pi_* KO \wedge T_\zeta$$

is an isomorphism of θ-algebras. Thus $\pi_* KO \wedge T_\zeta$ is the free θ-algebra in the generator b.

PROOF. Our choice of a null homotopy defines an E_∞-map from T_ζ to $KO \wedge TS^0$. Its KO-linear extension is the inverse of b_* as one easily checks. $\square$

The class $f = \psi(b) - b$ is fixed under ψ^3 since

$$\begin{aligned} \psi^3(\psi(b) - b)) &= \psi^3\psi(b) - \psi^3(b) \\ &= \psi\psi^3(b) - \psi^3(b) \\ &= \psi(b+1) - (b+1) = \psi(b) - b. \end{aligned}$$

In fact, it turns out that f is represented by a unique spherical class and we have

THEOREM 4.5. *(compare* [**Hop98b**]*) There is an isomorphism of θ-algebras*

$$\pi_* T_\zeta \cong \pi_* KO \otimes T\{f\}.$$

This result will not be used for the splitting theorem. For the proof the reader is referred to the work of Hopkins. He also shows in [**Hop98b**] how the Bott class behaves under the KO-Hurewicz map

$$i : \pi_* KO \otimes T\{f\} \cong \pi_* T_\zeta \longrightarrow \pi_* KO \wedge T_\zeta \cong \pi_* KO \otimes T\{b\}.$$

We have

$$i(v^4) = v^4 9^{-2b} = v^4 \sum_{n=0}^{\infty} \binom{-2b}{n} 2^{3n} \in T\{b\}.$$

In particular,

$$i(v^4) \equiv v^4 \mod 2$$

Note that the image of η is clear since it is spherical.

CHAPTER 3

The ABP-splitting and 2-adic functions

In this chapter we construct an E_∞-map from the cone T_ζ to the $K(1)$-local spin cobordism theory $MSpin$ and investigate its behaviour in KO-homology. Using the Anderson-Brown Peterson splitting we determine the 2-adic functions associated to the class b and to generators of the KO-homology ring of $MSpin$. The established formulae will play an important role in the proof of the multiplicative structure of $MSpin$.

1. The 2-adic functions associated to the class b

As a start we remind ourselves of the definition of the KO-characterstic classes. The first KO-characteristic class of a complex line bundle L over a space X is defined by

$$\pi^1(L) = L - 2 \in \widetilde{KO}(X).$$

This class is only part of a series of characteristic classes

$$\pi_s(\xi) = \sum_{s=0}^{\infty} \pi^j(\xi) s^j \in KO(X)[\![s]\!]$$

which are naturally defined for arbitrary oriented stable bundles and which are multiplicative:

$$\pi_s(\xi + \eta) = \pi_s(\xi)\pi_s(\eta)$$

For the complex bundle L we have $\pi_s(L) = 1 + s\pi^1(L)$. In fact, these properties determine π_s because the group $KO\,BSO(m)$ injects into $K(B\mathbb{T}^{[\frac{m}{2}]})$ under the complexification of the map which is induced by the restriction to the maximal torus $\mathbb{T}^{[\frac{m}{2}]}$ (compare [**ABP66**]).

It is possible to express $\pi^j(\xi)$ in the exterior powers of ξ. Explicitly, the equation

$$\pi_s(\xi) = \sum_{i=0}^{\infty} \Lambda^i(\xi - \mathrm{dim}\xi)t^i = (1+t)^{-dim\,\xi} \sum_{i=0}^{\infty} (\Lambda^i \xi) t^i$$

is easily verified for the new generator t of $KO(X)[\![s]\!]$ which is given by the equation $s = t/(1+t)^2$.

We write $\pi^j \in KO\, BSpin$ for the jth KO-characteristic class of the universal stable Spin bundle. Without changing the notation we also consider the same class as an object of $\widetilde{KO}MSpin$ via the Thom isomorphism. For any non ordered sequence of positive numbers (partition) $J = (j_1, \ldots, j_n)$ we set

$$\pi^J = \pi^{j_1} \cdots \pi^{j_n} : MSpin \longrightarrow KO.$$

In these terms the Anderson-Brown Peterson splitting says

THEOREM 1.1. *(compare* [**ABP66**]*) There is a countable set of cohomology classes $z_i \in H^*(MSpin, \mathbb{Z}/2)$ such that the map*

$$(\pi^J, z_i) : MSpin \longrightarrow \bigvee_{\substack{n(J)even \\ 1 \notin J}} ko\,\langle 4n(J)\rangle \vee \bigvee_{\substack{n(J)odd \\ 1 \notin J}} ko\,\langle 4n(J)+2\rangle \vee \bigvee_{i \in I} \Sigma^{|z_i|} H\mathbb{Z}/2$$

is a 2-local homotopy equivalence.

COROLLARY 1.2. *The map $(\pi^J) : MSpin \xrightarrow{\cong} \bigvee_{1 \notin J} KO$ is a $K(1)$-equivalence.*

PROOF. This is an immediate consequence of the the unlocalized ABP-splitting, 1.3 below and the vanishing of the group $K(1)_* H\mathbb{Z}/2$ [**Ada74**]. □

PROPOSITION 1.3. *The $(n-1)$-connected cover $ko\,\langle n\rangle \longrightarrow KO$ is a $K(1)$-equivalence for all $n \in \mathbb{N}$.*

PROOF. We first look at complex K-theory and set $n = 0$. Then 2-locally we may equip k with the BP-orientation which induces the 2-typicalization of the multiplicative formal group law. The BP-formula (compare [**Rav86**])

$$\eta_R(v_1) = \eta_L(v_1) + 2t_1$$

may be transported via the orientation map from $\pi_2 BP \wedge BP$ to $\pi_2 K(1) \wedge k$. Hence, v_1 coincides with the Bott class v in $\pi_2 K(1) \wedge k$. Since v_1 is invertible we conclude

$$K(1) \wedge k \cong K(1) \wedge k[v^{-1}] \cong K(1) \wedge K.$$

Next we turn to the real case. It is not hard to check that the well known cofibre sequence [**Mei79**]

$$\Sigma KO \xrightarrow{\eta} KO \longrightarrow K$$

also exists in the connective world. Furthermore, observe that the Hurewicz map from $\pi_1 S^0$ to $\pi_1 K(1) \wedge ko$ factorizes over the complexification map $\pi_1 KO \to \pi_1 K$ and hence annihalates η. Thus we obtain a commutative diagram

$$\begin{array}{ccccc} \pi_* K(1) \wedge ko & \rightarrowtail & \pi_* K(1) \wedge k & \twoheadrightarrow & \pi_{*-2} K(1) \wedge ko \\ \downarrow & & \downarrow \cong & & \downarrow \\ \pi_* K(1) \wedge KO & \rightarrowtail & \pi_* K(1) \wedge K & \twoheadrightarrow & \pi_{*-2} K(1) \wedge KO \end{array}$$

in which the first vertical arrow is injective and the last one surjective.

For a general n just observe that the fibre of $ko\langle n\rangle \to ko\langle n-1\rangle$ is a suitably suspended copy of the integer or mod 2 Eilenberg-MacLane spectrum and hence vanishes $K(1)$-locally. $\square$

Corollary 1.4. *ζ is null in $\pi_{-1} MSpin$.*

Proof. 2-locally the ABP-splitting gives a map from ko to $MSpin$ which induces an isomorphism in π_0. Hence it suffices to show that ζ vanishes in ko. The latter coincides with the periodic KO in the $K(1)$-local category by what we said before. In KO the class ζ vanishes by its definition. $\square$

In the last section we have chosen a null homotopy ι of ζ in KO. Hence the last corollary supplies us with an E_∞-map

$$\varphi : T_\zeta \longrightarrow MSpin$$

which will be the object of study for the rest of this section. We are interested in the image of the class b under the induced map

$$\varphi_* : \pi_* KO \wedge T_\zeta \longrightarrow \pi_0 KO \wedge MSpin \cong \bigoplus_{1 \notin J} \pi_* KO \wedge KO.$$

To describe each component of its image we recall the

Proposition 1.5. [**Rav84**][**Hop98b**] *Let Φ be the map*

$$\pi_* KO \wedge KO \longrightarrow \mathcal{T}(\mathbb{Z}_2^\times / \pm 1, \pi_* KO)$$

which associates to a class $f \in \pi_k KO \wedge KO$ the continuous 2-adic function

$$f(\lambda) : S^k \xrightarrow{f} KO \wedge KO \xrightarrow{1\wedge\psi^\lambda} KO \wedge KO \xrightarrow{\mu} KO.$$

Then Φ is an isomorphism.

Remark 1.6. The map Φ even becomes an isomorphism of θ-algebras if we set $\psi = id$ for the ring of continuous functions as we did before.

We are now able to state the result.

Proposition 1.7. *The 2-adic function $\pi^J\varphi_* b$ is the unique continuous homomorphism h which sends 3 to 1 for $J = \emptyset$ and vanishes for all other J.*

Proof. The commutative diagram

$$\begin{tikzcd}
& C_\zeta \arrow[r, "\iota"] \arrow[dl] \arrow[d] & KO \arrow[d] \arrow[dr, "1"] & \\
TC_\zeta \arrow[r] & T_\zeta \arrow[r, "\varphi"] & MSpin \arrow[r, "\pi^\emptyset"] & KO
\end{tikzcd}$$

tells us that all $\pi^J\varphi_* b$ must vanish except for $J \neq \emptyset$. For $J = \emptyset$ we compute with $a = \iota'_*(1)$

$$\begin{aligned}
\Phi(\pi^\emptyset_*\varphi_* b)(3^n) &= \Phi((1 \wedge \iota)a)(3^n) = \mu(1 \wedge \psi^{3^n})(1 \wedge \iota)a \\
&= \left\langle \psi^{3^n}\iota, a \right\rangle = \langle \iota + n(1\,\delta), a\rangle = n
\end{aligned}$$

□

Since φ induces a map of θ-algebras the last result completely determines its behaviour in KO-homology. Unfortunately, the ABP-splitting does not tell us anything about the θ-algebra structure of the spin bordism. Hence other methods are required in things to come.

2. The KO-homology ring of $BSpin$

We now provide polynomial generators of the real and complex K-homology of $MSpin$. We first need the Thom isomorphism for homology.

Lemma 2.1. *Let E be one of the theories $ko, k, KO, K, K\mathbb{Z}/2^r$. Then the Thom isomorphism in cohomology induces the ring isomorphism*

$$\tau_* : \pi_* E \wedge MSpin \xrightarrow{\cong} \pi_* E \wedge BSpin_+.$$

Proof. Since the homology commutes with direct limits it is enough to show that

$$\tau_* : \pi_{*+8n} E \wedge MSpin(8n) \longrightarrow \pi_* E \wedge BSpin(8n)_+$$

is an isomorphism. This is well known for the connective theories and follows for the non connective by inverting the Bott class. □

Hence, it suffices to look at the classifying space $BSpin$. A further simplification is given by the following result of Snaith.

LEMMA 2.2. *(compare* [**HS75**]*) The canonical map from BSpin to BSO is a* $K(1)$*-equivalence.*

Let L be the canonical line bundle over $\mathbb{C}P^\infty$. Equip K-theory with the orientation in the way that the Euler class of a line bundle L is given by $x = v^{-1}(1 - L^*)$ [1]. As usual, define additive generators $\beta_i \in \pi_{2i} K \wedge \mathbb{C}P^\infty_+$ as the dual of the classes x^i.

LEMMA 2.3. *Let* $f : S^1 \longrightarrow S^1$ *be the map which sends a complex number* z *to its square. Then* $Bf : BS^1 \longrightarrow BS^1$ *has the following impact on the generators:*

$$Bf_*(\beta_k) = \sum_j (-1)^{k-j} \binom{j}{2j-k} 2^{2j-k} \beta_j.$$

Here, we omitted the Bott class from the notation.

PROOF.

$$\begin{aligned} \left\langle Bf_*\beta_k, x^j \right\rangle &= \left\langle \beta_k, Bf^* x^j \right\rangle = \left\langle \beta_k, [2](x)^j \right\rangle = \left\langle \beta_{k-j}, (2-x)^j \right\rangle \\ &= \left\langle \beta_{k-j}, \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} 2^i x^{j-i} \right\rangle = (-1)^{k-j} \binom{j}{2j-k} 2^{2j-k} \end{aligned}$$

□

The main result of this paragraph is the

PROPOSITION 2.4. *Let* S^1 *be the maximal torus of* $Spin(2)$ *and let* u_j *be the image of* β_j *in* $\pi_0 K \wedge BSpin$*. Then*

$$\pi_* K \wedge BSpin_+ \cong \pi_* K[u_4, u_8, u_{12}, \ldots].$$

[1] There is quite an ambiguity about the standard orientation in K-theory in the literature. Many authors prefer taking $1 - L$ rather than its dual. We use Bott's definition [**Bot69**] here and let $v = L^* - 1 \in \pi_2 K$ be the periodicity class. Since we are only interested in real bundles the difference is not essential in the sequel. Our approach merely has the advantage that the Todd series then describes the change of orientations: when surpressing the Bott class from the notation we have

$$td(x_H) = \frac{x_H}{1 - e^{-x_H}} = \frac{x_H}{ch\, x_K} = \frac{x_H}{\exp_{\hat{G}_m} x_H}.$$

Moreover, let $b_k \in \pi_0 K \wedge BSO_+ \cong \pi_0 K \wedge BSpin_+$ *be the image of the class* $\beta_k \in \pi_0 K \wedge BSO(2)_+$. *Then we have*

$$u_k = \sum_j (-1)^{k-j} \binom{j}{2j-k} 2^{2j-k} b_j$$

Hence, $\pi_* K \wedge BSpin_+$ *is a polynomial algebra in* $b_2, b_4, b_6 \ldots$. *Finally, since*

$$\pi_0 KO \wedge BSpin_+ \cong \pi_0 K \wedge BSpin_+$$

the same classes also freely generate $\pi_* KO \wedge BSpin_+$.

Before proving 2.4 we need some preperation. A 2-adic number λ can be written in its 2-adic expansion

$$\lambda = \sum_k \alpha_k(\lambda)\, 2^k.$$

This way, each α_k becomes a continuous function with values in $\{0,1\} \subset \mathbb{Z}_2$. It turns out

LEMMA 2.5. *(compare* [**Hop98b**]*) The map*

$$\mathbb{Z}_2[\alpha_0, \alpha_1, \ldots]/(\alpha_k^2 - \alpha_k) \longrightarrow \mathcal{T}(\mathbb{Z}_2, \mathbb{Z}_2)$$

is an isomorphism of rings.

Hence one readily verifies with 1.2 and 1.5 the

LEMMA 2.6. (i) $\pi_* K \wedge MSpin \cong \lim_r \pi_* K\mathbb{Z}/2^r \wedge MSpin$.

(ii) *The Bockstein sequences*

$$\pi_* K\mathbb{Z}/2^i \wedge MSpin \rightarrowtail \pi_* K\mathbb{Z}/2^{i+1} \wedge MSpin \twoheadrightarrow \pi_* K\mathbb{Z}/2^i \wedge MSpin\ .$$

are short exact for all $i \geq 1$.

PROOF OF 2.4. By the first part of the lemma it suffices to check the corresponding result mod 2^i for all $i > 0$. Using the second part of the lemma and the 5-lemma we only need to show the statement for mod 2 K-theory. This is a well known result of Snaith [**HS75**]8.5 .

The second statement follows from 2.3 and the commutative diagram

$$\begin{tikzcd} Spin(2) \supset S^1 \arrow[r, "z^2"] \arrow[d] & S^1 \cong SO(2) \arrow[d] \\ Spin \arrow[r] & SO \end{tikzcd}$$

□

We mention another set of generators which by birth comes with a lift to the ring $\pi_* K \wedge BSU_+$. The hasty reader my skip to the next section since these generators will not be needed for the proof of the main theorem. Let

$$f : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow BSU$$

be the map which classifies the product $(1 - L_1)(1 - L_2)$. For each natural number k and $1 \leq i \leq k-1$ choose integers n_k^i such that

$$\sum_{i=1}^{k-1} n_k^i \binom{k}{i} \quad = \quad \text{g.c.d.}\{\binom{k}{1}, \dots, \binom{k}{k-1}\}.$$

Then we show in the appendix

THEOREM 2.7. *For any complex oriented E define elements*

$$d_k \quad = \quad \sum_{i=1}^{k-1} n_k^i f_*(\beta_i \otimes \beta_{k-i}) \in \pi_{2k} E \wedge BSU_+.$$

Then we have

$$\pi_* E \wedge BSU_+ \quad \cong \quad \pi_* E[d_2, d_3, d_4, \dots].$$

It is interesting to note

THEOREM 2.8. *(compare* [**Lau00**]*) Let ω be the canonical quaternian line bundle over $\mathbb{H}P^\infty$ and let $z_k \in \pi_* K \wedge \mathbb{H}P^\infty_+$ be dual to $c_2(\omega)^j$. Set $d'_{2k} = d_{2k} + z_k \in \pi_* K \wedge BSpin_+$ for all k. Then we have*

$$\pi_* K \wedge BSpin_+ \quad \cong \quad \pi_* K[d_{2k} | k \neq 2^s] \otimes_{\pi_* K} \pi_* K[d'_4, d'_8, d'_{16}, \dots].$$

Moreover, each z_k is decomposable in $\pi_0 K \wedge BSpin_+$.

3. Stable canabalistic classes in KO

In order to express the class $\varphi_* b$ in any set of generators of $\pi_0 KO \wedge MSpin$ we are going to compute the 2-adic functions associated to each generator explicitly. Recall from [**Bot69**] the definition of the canabalistic classes $\theta^k(\xi) \in KX$. They are defined for complex vector bundles ξ over compact spaces X. These classes are characterized by the following two properties

(i) $\theta^k(L) = 1 + L^* + \cdots + (L^*)^{k-1} = \frac{1-(L^*)^k}{1-L^*}$ for all line bundles L

(ii) $\theta^k(\xi \times \xi') = \theta^k(\xi)\, \theta^k(\xi')$ for all complex bundles ξ, ξ'.

In particular, we have the equality

$$\theta^k(\xi + n) = k^n \theta^k(\xi).$$

Assume in the following that k is an odd number. Then we can turn each θ^k into a stable operation by setting

$$\hat{\theta}^k(\xi) \overset{def}{=} \frac{\theta^k(\xi)}{k^{dim_{\mathbb{C}}\xi}} \in KX$$

The formulae for line bundles and sums of vector bundles stay the same. The complex classes θ^k do have a real counterpart for spin bundles ξ which we also denote by $\theta^k(\xi) \in KO\,X$. If the underlying spin bundle admits a reduction to the special unitary group then the complexification of these real classes coincide with the complex classes [**Bot69**]p.87f.

In the following we write $\hat{\theta}^k \in \widetilde{KO}\,BSpin$ for the universal canabalistic classes. We will see in a moment that they come up in the ABP-splitting map.

Lemma 3.1. *The diagram*

$$\begin{tikzcd}
\pi_0 KO \wedge MSpin \arrow[r, "\Xi"] \arrow[d, "(1\wedge\pi^J)_*"'] & \mathrm{Hom}_{cts}(\widetilde{KO}\,MSpin, \mathbb{Z}_2) \arrow[d, "\pi^{J*}"'] \\
\pi_0 KO \wedge KO \arrow[r, "\Phi"] & \mathcal{T}(\mathbb{Z}_2^\times/\pm 1, \mathbb{Z}_2)
\end{tikzcd}$$

commutes. Here, the upper horizontal arrow is the duality map. The right horizontal arrow takes a homomorphism $\alpha : \widetilde{KO}\,MSpin \longrightarrow \mathbb{Z}_2$ to the map

$$\lambda \mapsto \alpha(MSpin \xrightarrow{\pi^J} KO \xrightarrow{\psi^\lambda} KO).$$

The lemma is easily checked. The above diagram may be composed with the diagram of Thom isomorphisms

$$\begin{tikzcd}
\pi_0 KO \wedge BSpin_+ \arrow[r, "\Xi"] & \mathrm{Hom}_{cts}(KO\,BSpin, \mathbb{Z}_2) \\
\pi_0 KO \wedge MSpin \arrow[r, "\Xi"] \arrow[u, "\cong"] & \mathrm{Hom}_{cts}(\widetilde{KO}\,MSpin, \mathbb{Z}_2) \arrow[u, "\cong"']
\end{tikzcd}$$

Consider the J-component of the ABP-splitting map

$$\Theta_J \overset{def}{=} \pi^{J*}(\tau^*)^{-1}\Xi = \Phi(1\wedge\pi^J)_*\tau_*^{-1} : \pi_0 KO \wedge BSpin_+ \longrightarrow \mathcal{T}(\mathbb{Z}_2^\times/\pm 1, \mathbb{Z}_2).$$

Proposition 3.2. *For all $a \in \pi_0 KO \wedge BSpin_+$ we have*

$$\Theta_J(a)(k) = \left\langle a, \hat{\theta}^k \psi^k(\pi^J) \right\rangle.$$

PROOF. Let ξ_{8n} be the universal spin bundle over $BSpin(8n)$. Then we have the well known relation

$$\psi^k(z_n) = \theta^k(\xi_{8n})z_n$$

between the Adams operations and the canabalistic classes (compare [**Bot69**] p.89.) Here, $z_n \in \widetilde{KO}\, MSpin(8n)$ is the Thom class. Let $\beta \in \pi_8 KO$ be the Bott class and for any $a \in \pi_0 BSpin(8n)$ set $g = \tau^{*-1}\Xi(a)$. Then compute

$$\begin{aligned}\Theta_J(a)(k) &= \pi^{J^*}(g)(k) = g(\psi^k(\beta^{-n}z_n\pi^J(\xi_{8n}))) \\ &= g(k^{-4n}\beta^{-n}\theta^k(\xi_{8n})z_n\psi^k(\pi^J(\xi_{8n}))) \\ &= \tau^* g(\hat{\theta}^k(\xi_{8n})\psi^k(\pi^J(\xi_{8n})) \\ &= \left\langle a, \hat{\theta}^k(\xi_{8n})\psi^k(\pi^J(\xi_{8n}))\right\rangle.\end{aligned}$$

Thus the claim follows after stabilization. □

We are now well prepared to compute the 2-adic functions which correspond to the generators u_n defined in the last section.

THEOREM 3.3. *The 2-adic function $\Theta_J(u_n)$ vanishes for all non empty J which do not contain 1. For $\Theta = \Theta_\emptyset$ we have the formula*

$$k\sum_n \Theta(u_n)(k)x^n = \frac{(1-x)^k - (1-x)^{-k}}{(1-x)-(1-x)^{-1}}$$

or, equivalently,

$$\Theta(u_n)(k) = \frac{(-1)^n}{k}\sum_{i=1}^{k}\binom{2i-k-1}{n}.$$

We first need the

LEMMA 3.4. *Let L be the canonical line bundle over $\mathbb{C}P^\infty$. Then the complexified real canabalistic classes satisfy*

$$k\,\hat{\theta}^k(L^2-1)\otimes\mathbb{C} = \frac{(1-x)^k - (1-x)^{-k}}{(1-x)-(1-x)^{-1}}.$$

PROOF. We decompose the spin bundle $1-L^2$ into a sum of bundles which admit a reduction to the special unitary group by writing

$$(1-L^2) = (L-\bar{L}) + 1 - L^2 = (1-L)(1-\bar{L}) - (1-L)^2$$

To determine the real canabalistis classes of the latter we compute the complex canabalistic classes in $K\,\mathbb{C}P^\infty \times \mathbb{C}P^\infty \cong \mathbb{Z}_2[\![x,y]\!]$

$$\begin{aligned}\hat\theta^k((1-L_1)(1-L_2)) &= \hat\theta^k(1-L_1-L_2+L_1L_2)\\ &= \frac{\hat\theta^k(L_1L_2)}{\hat\theta^k(L_1)\hat\theta^k(L_2)} = \frac{(1-L_1^*)(1-L_2^*)(1-(L_1^*L_2^*)^k)}{(1-L_1^{*k})(1-L_2^{*k})(1-L_1^*L_2^*)} = \frac{q_k(x+_{\hat G_m}y)}{q_k(x)q_k(y)}\end{aligned}$$

Here, $q_k(x)$ is the polynomial

$$q_k(x) = \frac{1-(1-x)^k}{x} = (1-x)+(1-x)^2+\cdots+(1-x)^{k-1}$$

Thus we obtain

$$\hat\theta^k(1-L^2)\otimes\mathbb{C} = \frac{k}{q_k(x)q_k(-_{\hat G_m}x)}\Big(\frac{q_k([2](x))}{q_k(x)}\Big)^{-1} = k\frac{q_x(x)}{q_k(-_{\hat G_m}x)q_k([2](x))}.$$

An elementary calculation finishes the proof. □

Proof of 3.3: The proposition gives

$$\begin{aligned}\Theta_J(u_n)(k) &= \left\langle u_n, \hat\theta^k\psi^k(\pi^J)\otimes\mathbb{C}\right\rangle\\ &= \left\langle \beta_n, \hat\theta^k(L^2-1)\psi^k(\pi^J(L^2-1))\otimes\mathbb{C}\right\rangle.\end{aligned}$$

Hence, Θ_J vanishes for all non empty J which do not contain 1. Moreover, for $J=\emptyset$ we get with the lemma

$$\sum_n \Theta(u_n)(k)x^n = \sum_n \left\langle \beta_n, \hat\theta^k(L^2-1)\otimes\mathbb{C}\right\rangle x^n = k^{-1}\frac{(1-x)^k-(1-x)^{-k}}{(1-x)-(1-x)^{-1}}.$$

□

Using our calculations we can determine the 2-adic functions which correspond to the other generators very easily. For instance, consider the map

$$f : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow BSU \longrightarrow BSpin$$

which classifies the product $(1-L_1)(1-L_2)$. Then we have for the generators

$$a_{ij} = f_*(\beta_i\otimes\beta_j) \in \pi_{2(i+j)}K\wedge BSpin_+$$

Corollary 3.5.

$$\sum_{i,j}\Theta(a_{ij})(k)x^iy^j = \frac{q_k(x+_{\hat G_m}y)}{q_k(x)q_k(y)}.$$

PROOF. Compute

$$\begin{aligned}\Theta(a_{ij})(k) &= \left\langle a_{ij}, \hat{\theta}_k \otimes \mathbb{C} \right\rangle = \left\langle \beta_i \otimes \beta_j, f^* \hat{\theta}_k \otimes \mathbb{C} \right\rangle \\ &= \left\langle \beta_i \otimes \beta_j, \hat{\theta}_k((1-L_1)(1-L_2)) \otimes \mathbb{C} \right\rangle = \left\langle \beta_i \otimes \beta_j, \frac{q_k(x +_{\hat{G}_m} y)}{q_k(x)q_k(y)} \right\rangle .\end{aligned}$$

□

The computations of the KO-homology of the ABP-map enable us to compare the generators to the class $\varphi_* b$. At this point, the reader may easily verify that $\varphi_* b$ corresponds to u_4 and to $a_{1,2}$ modulo 2. We will not go through the calculation here since we will work out a closer relationship later.

CHAPTER 4

The θ-algebra structures of bordism theories

In this chapter we determine the θ-algebra structure of the unitary, special unitary and spin bordism theories. This problem was partially answered by Snaith in [**HS75**] who used group theoretical methods to compute the action of θ modulo 2 for the classifying spaces. However, in order to give a complete description of the θ-algebra structure we need integral information: each time when applying θ we loose a power of 2. That is, the image of θ on a mod 2^n class is only well defined modulo 2^{n-1}. Hence, in order to show that a class generates a free summand we must know θ integrally. Unfortunately, Snaith's (and Priddy's [**Pri75**]) method does not generalize that easily to the integral situation. So we use a completely different approach here.

1. The θ-algebra structure of $\pi_0 K \wedge MU$

We start by investigating the H_∞-structure of the spectrum $(K \wedge MU)^{BS^1_+}$ by computing the value of $\tilde{x}_{MU} = v\, x_{MU}$ under the operation

$$P : \pi_0 (K \wedge MU)^{BS^1_+} = (K \wedge MU)^0 BS^1 \longrightarrow (K \wedge MU)^0 B\Sigma_2 \times BS^1.$$

Here, x_{MU} denotes the MU-Euler class and $v \in \pi_2 K$ is the Bott class. In this section it is important not to surpress the Bott class from the notation.

Our strategy is to calculate the operation on each factor of the product $\tilde{x}_{MU}$ separately. This means the following: the complex bordism theory and the K-theory admit H^2_∞-structures or, equivalently, H_∞-structures on the wedge $\bigvee_i \Sigma^{2i} MU$ and $\bigvee_i \Sigma^{2i} K$ respectively. Hence so do the corresponding function spectra. The associated operation P is the well known tom Dieck-Steenrod operation

$$P : MU^n BS^1 \longrightarrow MU^{2n} B\Sigma_2 \times BS^1.$$

for complex bordism and the Atiyah power operation for K-theory [**BMMS86**]. Hence, we know how to compute the operation on each factor and only have to relate their product to the class $P\tilde{x}_{MU}$. To state the result, we use the standard notation and write DX for the gadget $E\Sigma_2 \wedge_{\Sigma_2} X^2$. We also write

$$\delta : D(X \wedge Y) \longrightarrow DX \wedge DY$$

for the diagonal map. Then we have the following "change of suspension" formula:

LEMMA 1.1. *Suppose E is a H_∞^d-ring spectrum and F is a H_∞-ring spectrum. Then for any based space X, $\alpha : X \longrightarrow \Sigma^d E$ and $\beta : \Sigma^d X \longrightarrow F$ we have*

$$y_E\, P(\alpha\,(\Sigma^{-d}\beta)) = P(\alpha)\,\Sigma^{-d}P(\beta) : X \wedge B\Sigma_{2+} \longrightarrow \Sigma^d E \wedge F$$

with $y_E = P(\Sigma^d 1) \in E^d B\Sigma_{2+}$.

PROOF. It is easy to check that the diagram

$$\begin{tikzcd}
D\Sigma^d X \arrow[d, "D\Sigma^d\Delta_X"'] & B\Sigma_{2+} \wedge \Sigma^d X \arrow[l, "\Delta_X"'] \arrow[r, "\Sigma^d\Delta"] & B\Sigma_{2+} \wedge X \wedge B\Sigma_{2+} \wedge \Sigma^d X \arrow[d, "\Delta\wedge\Delta"] \\
D(X \wedge \Sigma^d X) \arrow[rr, "\delta"] & & DX \wedge D\Sigma^d X
\end{tikzcd}$$

commutes. Moreover, when we write $\mathcal{P}$ for the external Steenrod operation and give $E \wedge F$ the H_∞^d-ring structure which is induced from the isomorphism

$$\textstyle\bigvee_i \Sigma^{di}(E \wedge F) \cong (\bigvee_i \Sigma^{di}E) \wedge F.$$

then we get

$$\begin{aligned}
y_E\, P(\alpha\,\Sigma^{-d}\beta) &= P(\Sigma^d(\alpha\,\Sigma^{-d}\beta)) = P(\Sigma^d\Delta_X^*(\alpha \wedge \Sigma^{-d}\beta)) \\
&= \Delta^*\mathcal{P}((\Sigma^d\Delta_X)^*(\alpha \wedge \beta)) = \Delta^*(D\Sigma^d\Delta_X)^*\mathcal{P}(\alpha \wedge \beta) \\
&= \Delta^*(D\Sigma^d\Delta_X)^*\delta^*(\mathcal{P}(\alpha) \wedge \mathcal{P}(\beta)) \\
&= (\Sigma^d\Delta_{B\Sigma_{2+}\wedge X})^*(\Delta \wedge \Delta)^*(\mathcal{P}(\alpha) \wedge \mathcal{P}(\beta)) \\
&= P(\alpha)\,\Sigma^{-d}P(\beta)
\end{aligned}$$

In the first and third line we used [**BMMS86**] p.250f. □

LEMMA 1.2. *Let E be a H_∞^2-ring spectrum which is complex oriented by a H_∞^2-map $f : MU \longrightarrow E$. Then $P(\Sigma^2 1)$ is the Euler class y_E of the sign representation.*

PROOF. This immediately follows from [**BMMS86**]p.257. □

We next show how the operation works in complex bordism.

LEMMA 1.3. *For all $n > r > 0$ we have the formula*

$$y_{MU}^{n-1} P(x_{MU}) = y_{MU}^{n-1} x_{MU}(x_{MU} +_{\hat{G}_u} y_{MU}) \quad \in MU^{2n+2} B\Sigma_2 \times \mathbb{C}P^r.$$

Here, $\hat{G}_u$ is the universal formal group law.

PROOF. For the sake of simplicity we omit the index MU from the notation. Choose s arbitrary and recall from [**Qui71**]3.17) the formula

$$y^{n-1} P(x) = \sum_{l(\alpha) \leq n} y^{n-l(\alpha)} a(y)^\alpha s_\alpha(x) \quad \in MU^{2n+2} \mathbb{R}P^s \times \mathbb{C}P^r$$

which relates the tom Dieck-Steenrod operation to the Landweber-Novikov operations s_α. Here, α is a sequence of non negative integers and $l(\alpha) = \sum \alpha_i$. The power series $a^\alpha = a_1^{\alpha_1} a_2^{\alpha_2} \cdots$ are defined by the equation

$$x +_{\hat{G}_u} y = x + \sum_{j \geq 1} a_j(x)\, y^j.$$

Let $i : \mathbb{C}P^{r-1} \hookrightarrow \mathbb{C}P^r$ be the inclusion and c_t be the total Conner-Floyd Chern class. Then we have for the Euler class $z = e(L^*)$

$$s_t(z) = s_t(i_!(1)) = i_! c_t(L^*) = i_!(\sum_{j \geq 1} t_j z^j) = \sum_{j \geq 1} t_j z^{j+1}.$$

Since the Landweber-Novikov operations are natural the same formula holds for x instead of z. Hence we get

$$\begin{aligned} y^{n-1}\, P(x) &= y^n\, x + y^{n-1} \sum_{j \geq 1} a_j(y) x^{j+1} \\ &= y^n\, x + y^{n-1}\, x\, ((x +_{\hat{G}_u} y) - y) = y^{n-1} x\, (x +_{\hat{G}_u} y). \end{aligned}$$

The claim follows by passing to the limit

$$MU^* \mathbb{C}P^r \times B\Sigma_2 = \lim_s MU^* \mathbb{C}P^r \times \mathbb{R}P^s.$$

□

In the following let $g(x)$ be the invertible power series

$$g(x) = \sum_{i=0}^{\infty} b_i x^i; \quad b_0 = 1$$

with coefficients in $\pi_0 K \wedge MU \cong \mathbb{Z}_2[b_1, b_2, \ldots]$. It is well known (compare [**Ada74**]) that the two Euler classes $\tilde{x}_{MU}$ and $x = \tilde{x}_K$ are related in $(K \wedge MU)^0 BS^1$ by the formula

$$\tilde{x}_{MU} = x\, g(x).$$

A similar relation holds for $\tilde{y}_{MU}$ and $y = \tilde{y}_K$ in $(K \wedge MU)^0 B\Sigma_2$. Observe that in the notation of 2.1.4, the class y corresponds to $1-\sigma$ under the isomorphism

$$K^0 B\Sigma_2 \cong R\Sigma_2^\wedge \cong (\mathbb{Z}_2[\sigma]/\sigma^2 - 1)^\wedge \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2(1-\sigma).$$

LEMMA 1.4. *The power sum $\sigma_2 = \sum t_i^2 \in \pi_0 K \wedge B\Sigma_{2+}$ satisfies*

$$\langle \sigma_2, y^n \rangle = 2^n.$$

PROOF. Since $y^{n+1} = 2^n y$ the formula follows from

$$\langle \sigma_2, y \rangle = \left\langle e_1^2 - e_2, 1-\sigma \right\rangle - \langle e_2, 1-\sigma \rangle = 2$$

Here we used the explicit description of $\Delta_{k,2}$ in 2.1.4. □

We are now able to state the main result.

THEOREM 1.5. *The θ-algebra structure of $\pi_0 K \wedge MU$ is determined by the equations*

$$\sum_{i \geq 0} \psi(b_i) x^i (2-x)^i = \psi(g(x)) = \frac{g(x) g(2-x)}{g(2)}.$$

PROOF. The first equation is clear since for K-theory the operation ψ coincides with the second Adams operation and

$$\psi^2 x = [2]_{\hat{G}_m}(x) = 2x - x^2.$$

To do the second consider the curve $b(x) = x\, g(v\, x)$ and regard $K \wedge MU$ as a H_∞^2-ring spectrum via the equivalence $K \wedge \bigvee_i \Sigma^{2i} MU \cong \bigvee_i \Sigma^{2i} K \wedge MU$. Then we have the formula

SUBLEMMA 1.6. $\psi(x_{MU}) = b(x_K)\, b(2v^{-1} - x_K)$.

PROOF. First note that $b(x_K)$ is the MU Euler class. Moreover, the K-Hurewicz map $\pi_* MU \longrightarrow \pi_* K \wedge MU$ classifies the formal group law ${}^b\hat{G}_m$ since

$$\begin{aligned} e_{MU}(L_1 \otimes L_2) &= b(e_K(L_1 \otimes L_2)) = b(e_K(L_1) +_{\hat{G}_m} e_K(L_2)) \\ &= \; b(b^{-1}(e_{MU}(L_1)) +_{\hat{G}_m} b^{-1}(e_{MU}(L_2)) = e_{MU}(L_1) +_{{}^b\hat{G}_m} e_{MU}(L_2) \end{aligned}$$

Hence when pairing the equality of 1.3 with σ_2 we get with 1.4

$$\begin{aligned} b(2v^{-1})^{n-1}\,\psi(x_{MU}) &= \left\langle y_{MU}^{n-1}\,P(x_{MU}),\sigma_2\right\rangle \\ &= \left\langle y_{MU}^{n-1}x_{MU}(x_{MU}+_{b\hat{G}_m}y_{MU}),\sigma_2\right\rangle \\ &= b(2v^{-1})^{n-1}x_{MU}(x_{MU}+_{b\hat{G}_m}b(2v^{-1})) \\ &= b(2v^{-1})^{n-1}b(x_K)\,b(2v^{-1}-x_K). \end{aligned}$$

Since the coefficient ring $\pi_* K \wedge MU$ is a division algebra and the module $(K \wedge MU)^* B\Sigma_2 \wedge \mathbb{C}P^r$ is free we may cancel the term $b(2v^{-1})^{n-1}$ on both sides. The claim follows by passing to the limit. $\square$

Our change of suspension formula 1.1 reads for $X = BS^1_+$, $\alpha = x_{MU}$ and $\beta = \Sigma^2 v$

$$y_{MU}P(v\,x_{MU}) = P(x_{MU})\,P(\Sigma^2 v) = P(x_{MU})\,y_K P(v) = P(x_{MU})\,y_K\,v^2$$

In the last two equations we used [**BMMS86**] p.274f. Hence we obtain with the sublemma

$$\begin{aligned} \psi(x)\psi(g(x)) = \psi(\tilde{x}_{MU}) &= \psi(x_{MU})\left\langle \frac{y_K}{y_{MU}}v^2,\sigma_2\right\rangle \\ = \quad b(x_K)b(2v^{-1}-x_K)\frac{2v^{-1}}{b(2v^{-1})}v^2 &= \psi(x)\frac{g(x)g(2-x)}{g(2)}. \end{aligned}$$

The result follows by canceling $\psi(x)$ on both sides. $\square$

COROLLARY 1.7. *In $\pi_0 K \wedge MU$ we have the formula mod 2*

$$\theta(b_r) = (1+b_1)b_r^2 + \sum_{i=0}^{r} b_i(b_{2r-i}+b_{2r-i+1})$$

In particular, for $r > 0$ this gives modulo 2 and decomposables

$$\theta(b_r) = b_{2r} + b_{2r+1}$$

PROOF. Since $\pi_0 K \wedge MU$ is torsion free and $2\theta(x) = \psi(x) - x^2$ it is enough to compute the action of ψ on the generators. Let D_k be the operator $(d^k/k!\,dx^k)_{|x=0}$. Then we get mod 4

$$\begin{aligned} (-1)^r\psi(b_r) &= D_{2r}\psi(g(x)) = D_{2r}\frac{g(x)g(2-x)}{g(2)} \\ &= (1+2b_1)(\sum_{i+j=2r}(-1)^j b_i b_j + 2\sum_{i+j-1=2r} j\,b_i b_j) \end{aligned}$$

The result follows after some elementary transformations which are left to the reader. $\square$

The last formula was already obtained by Snaith for the θ-algebra $\pi_0 K \wedge BU_+$ modulo 2 and decomposables in [**HS75**] 6.3.6. To get his result from the above one we need the

PROPOSITION 1.8. *The Thom isomorphism*

$$\tau_* : \pi_0 K \wedge MU \xrightarrow{\cong} \pi_0 K \wedge BU_+$$

is a map of θ-algebras.

PROOF. Consider BU as the direct limit of all $BU(V)$ for finite dimensional subspaces V of the universe. Then the diagonal map

$$Th(V) \xrightarrow{\Delta} Th(V) \wedge BU(V)_+ \longrightarrow Th(V) \wedge BU_+$$

induces a map of spectra $\Delta : MU \longrightarrow MU \wedge BU_+$ which is an E_∞-ring map by its construction. Moreover, the Thom class

$$\tau : MU \longrightarrow MSpin^c \longrightarrow K$$

is well known to be an H_∞-ring map by [**BMMS86**]p.280. Thus the composite

$$K \wedge MU \xrightarrow{1\wedge\Delta} K \wedge MU \wedge BU_+ \xrightarrow{1\wedge\tau\wedge 1} K \wedge K \wedge BU_+ \xrightarrow{\mu\wedge 1} K \wedge BU_+$$

is an H_∞-ring map. This map induces the Thom isomorphism in homotopy. $\square$

2. The θ-algebra structure of $\pi_0 K \wedge MSU$

We next turn to the special unitary bordism theory. The result will not be needed for the proof of the splitting theorem. Once more let

$$f : \mathbb{C}P^\infty_+ \wedge \mathbb{C}P^\infty_+ \longrightarrow BSU_+ \longrightarrow K \wedge BSU_+$$

be the map which classifies $(1 - L_1)(1 - L_2)$ and

$$f(x, y) = \sum_{i,j} a_{ij} x^i y^j$$

be the associated power series.

THEOREM 2.1. *The θ-algebra structure of $\pi_0(K \wedge MSU_+)$ is determined by the equations*

$$\sum_{i,j} \psi(a_{ij})(x(2-x))^i (y(2-y))^j = \psi f(x, y) = \frac{f(x, y)\, f(2-x, y)}{f(2, y)}.$$

PROOF. The first equation is clear. To see the second, observe that the decomposition $(1-L_1)(1-L_2) = (L_1L_2 - 1) + (1-L_1) + (1-L_2)$ implies

$$\iota_* \, f(x,y) = \frac{g(x)\, g(y)}{g(x +_{\hat{G}_m} y)}.$$

Here the map $g : \mathbb{C}P^\infty_+ \longrightarrow K \wedge BU_+$ corresponds to $1-L$ and $\iota : BSU \longrightarrow BU$ is the inclusion. Since ι_* is an injection (compare appendix A) it may as well be omitted from the notation. By naturality and 1.5 we compute

$$\psi(g(x +_{\hat{G}_m} y)) = \mu^* \psi(g(x)) = \frac{g(x +_{\hat{G}_m} y)\, g(2 - (x +_{\hat{G}_m} y))}{g(2)}.$$

and hence

$$\begin{aligned} \psi f(x,y) &= \frac{\psi g(x)\, \psi g(y)}{\psi g(x +_{\hat{G}_m} y)} = \frac{g(x)g(2-x)g(y)g(2-y)}{g(2)g(x +_{\hat{G}_m} y)g(2-(x +_{\hat{G}_m} y))} \\ &= f(x,y)\, f(2-x,y)\, \frac{g(2-y)}{g(y)g(2)} = \frac{f(x,y)\, f(2-x,y)}{f(2,y)}. \end{aligned}$$

Here we used the identities

$$2 - (x +_{\hat{G}_m} y)) = (2-x) +_{\hat{G}_m} y; \qquad 2 +_{\hat{G}_m} y = 2 - y.$$

which are easily checked. $\square$

COROLLARY 2.2. *In $\pi_0 K \wedge MSU$ we have modulo 2 and decomposables of lower index*

$$\theta(a_{ij}) = a_{2i,2j} + a_{2i+1,2j}.$$

PROOF. In view of the theorem it is clear how to proceed. We compute mod 4 and decomposables

$$\begin{aligned} \psi f(x,y) &= \frac{f(x,y)f(2-x,y)}{f(2-y)} \\ &= (1 + \sum_{i,j\geq 1} a_{ij}x^iy^j)(1 + \sum_{i,j\geq 1} (-1)^i a_{ij}x^iy^j + 2a_{ij}x^{i-1}y^j)(1 + 2\sum_{j\geq 1} a_{1j}y^j)) \\ &= 1 + \sum_{i,j\geq 1} ((1 + (-1)^i)a_{ij} + 2a_{i+1,j})x^iy^j \end{aligned}$$

The result follows by looking at the coefficient in front of $x^{2i}y^{2j}$. $\square$

REMARK 2.3. It is not hard to give an explicit formula for the action of θ on the nose but we will not go through the tedious calculations here.

3. The θ-algebra structure of $\pi_0 KO \wedge MSpin$

The θ-algebra structure of $\pi_0 KO \wedge MSpin$ is determined by 2.1 and the surjective map of θ-algebras

$$\pi_0 K \wedge MSU \longrightarrow \pi_0 K \wedge MSpin \cong \pi_0 KO \wedge MSpin.$$

Alternatively, we may use the Thom isomorphism 1.8 and look at the surjective θ-algebra map

$$\pi_0 K \wedge BU_+ \longrightarrow \pi_0 K \wedge BSO_+ \cong \pi_0 K \wedge BSpin_+ \cong \pi_0 K \wedge MSpin.$$

In each case the analysis of the spin θ-algebra structure is ultimately based on the θ-algebra $\pi_0 K \wedge MU$. In 1.5 we found an implicit formula for the action of θ on the free generators b_i. To identify free θ-algebra summands we also need to know the action of the powers θ^j which we do next.

DEFINITION 3.1. *For a monomial $m = b_{i_1} \cdots b_{i_k}$ in $\pi_0 K \wedge MU$ we define its length $l(m)$ to be the maximum of the set of indices $\{i_1, \dots, i_k\}$. For a general element x of the form $\sum_{s \in S} m_s 2^{i_s}$ with $m_s \neq m_{s'}$ whenever $i_s = i_{s'}$ and $s \neq s'$ we define lengths*

$$\begin{aligned} l_1(x) &= sup\{l(m_s) - i_s |\, s \in S\} \\ l_2(x) &= sup\{l(m_s) 2^{-i_s} |\, s \in S\}. \end{aligned}$$

LEMMA 3.2. *The two lengths l_k , $k = 1, 2$ have the following properties*

(i) $max\{l_s(a\,b), l_s(a + b)\} \leq \ max\{l_s(a), l_s(b)\}$.

(ii) $l_2(a) \leq l_1(a)$ *if* $l_1(a) > 0$

PROOF. The easy proof is left to the reader. $\square$

Next we consider the action of θ and ψ on the generators. In 1.7 we have seen that the l_1-length of $\theta(b_i)$ (and $\psi(b_i)$) is at least $2i + 1$ (and $2i$ respectively.) In fact, the following result shows the equality.

LEMMA 3.3. *For each i we have*

(i) $l_1(\psi(b_i)) = 2i$

(ii) $l_1(\theta(b_i)) = 2i + 1$

(iii) $l_2(\theta(x)) < 2i$ *if* $l_2(x) < i$.

PROOF. Since $l(\psi(b_0)) = l(1) = 0$ we may assume that the equality is true for all numbers lower than i. Then we have with 1.5

$$\begin{aligned} l_1(2^i\psi(b_i)) &= l_1(g(2)\sum_{k+l=i}\binom{l}{k}2^{l-k}\psi(b_l)) \\ &= l_1(\sum_{k+l=i,j}\binom{j}{k}2^{j-k}b_l b_j) = i \end{aligned}$$

The second estimation follows from the first

$$l_1(2\theta(b_i)) = l_1(b_i^2 + 2\theta(b_i)) = l_1(\psi(b_i)) = 2i.$$

To show the last statement let $x = \sum_s m_s 2^{i_s}$ be an element with $l_2(x) < i$. Then there is a $N > 0$ with the property that the length of each m_s is strict smaller than $2^{i_s}(i - 2^{-N})$. Hence we have with the multiplication formula 2.2.1 for θ

$$l_2(\theta(m_s)) \leq l_1(\theta(m_s)) \leq 2^{i_s+1}(i - 2^{-N}).$$

Using $\theta(2a) = 2\theta(a) - a^2$ we conclude

$$\begin{aligned} l_2(\theta(x)) &\leq \sup_s l_2(\theta(m_s 2^{i_s})) \leq \sup_s \max\{l_2(2^{i_s}\theta(m_s)), 2i - 2^{1-N}\} \\ &= \max\{\sup_s 2^{-i_s} l_2(\theta(m_s)), 2i - 2^{1-N}\} = 2i - 2^{1-N} < 2i. \end{aligned}$$

□

It is convenient to work with Landau symbols. We let $o_i(k)$ represent classes whose l_i-length is strict smaller than k.

LEMMA 3.4. *For all $i > 0$ we have the formula*

$$\theta(b_i + o_2(i)) = b_{2i+1} + (1 + b_1)b_{2i} + o_2(2i).$$

PROOF. By 1.7 the class

$$a = \theta(b_i) - (b_{2i+1} + (1 + b_1)b_{2i})$$

is a sum of monomials of length at most $2i - 1$ modulo 2. Moreover, by the previous lemma we have

$$l_2(a) = l_2(o_2(2i) + 2o_1(2i - 1)) < \max\{2i, \frac{2i-1}{2}\} = 2i.$$

Thus the claim follows from the third part of the lemma. □

The last formula is particularly nice when it comes to the spin groups. Before stating the result we work out the relations that appear by passing from the unitary to the special orthogonal groups.

LEMMA 3.5. *Let $i : \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$ be the map which classifies the conjugate tautological bundle $\bar{L}$. Then we have the formula*

$$i_*\beta_s = \sum_{t=1}^{s} (-1)^t \binom{s-1}{t-1} \beta_t.$$

PROOF. Compute

$$\begin{aligned} \langle i_*\beta_s, x^t \rangle &= \langle \beta_s, (i^*x)^t \rangle = \langle \beta_s, x^t(x-1)^{-t} \rangle \\ &= \left\langle \beta_{s-t}, (-1)^t \sum_k \binom{t+k-1}{k} x^k \right\rangle = (-1)^t \binom{s-1}{s-t} \end{aligned}$$

□

LEMMA 3.6. *In $\pi_0 K \wedge BSO_+$ we have for all k*

$$b_{2k+1} = k\, b_{2k} + \textit{terms with lower index} \in \pi_0 K \wedge BSO_+.$$

PROOF. Since L and $\bar{L}$ are isomorphic as stable oriented real bundles the previous lemma gives

$$\begin{aligned} 0 &= i_* b_{2k+2} - b_{2k+2} = \sum_{j=1}^{2k+1} (-1)^j \binom{2k+1}{j-1} b_j \\ &= -(2k+1) b_{2k+1} + (2k+1) k b_{2k} + \text{terms with lower index.} \end{aligned}$$

□

We have seen earlier that the map

$$\Psi : \mathbb{Z}_2[b_2, b_4, \ldots] \longrightarrow \pi_0 K \wedge BU_+ \longrightarrow \pi_0 K \wedge BSO_+$$

is an isomorphism. Hence we can define for all $x \in \pi_0 K \wedge BSO_+$

$$l_i(x) = l_i(\Psi^{-1} x).$$

PROPOSITION 3.7. *In $\pi_0 K \wedge BSO_+$ we have the formula for all $i > 0$*

$$\theta^j(b_{2i} + o_2(2i)) = b_{2^{j+1}i} + o_2(2^{j+1}i).$$

PROOF. Since b_1 vanishes in $\pi_0 K \wedge BSO_+$ the formula inductively follows from 3.6 and 3.4. □

Now we can state the

THEOREM 3.8. *$\pi_0 KO \wedge MSpin$ is the free θ-algebra generated by the set of all u_{8k+4} for $k \geq 0$. Moreover, each generator u_{8k+4} can be altered by any element of strict smaller l_2-length.*

PROOF. It is enough to show that the ring homomorphism

$$\mathbb{Z}_2[\theta^j u_{8k+4} |\, k, j \geq 0] \longrightarrow \mathbb{Z}_2[b_2, b_4, b_6, \dots]$$

is an isomorphism modulo 2. We know from 3.2.4 that u_{4k} coincides with b_{2k} up to a class of l_2-length strict smaller than $2k$. Hence, the proposition 3.7 gives

$$\theta^j u_{8k+4} = b_{2^{j+1}(2k+1)} + o_2(2^{j+1}(2k+1)).$$

Since each even number can uniquely be written in the form $2 \cdot 2^j(2k+1)$ for some j, k there is an obvious correspondence of the highest terms of the generators. This finishes the proof of the theorem. $\square$

REMARK 3.9. We could have proved the theorem without using the Thom isomorphism. For that we merely observe that the class $b_{2k} \in \pi_0 K \wedge MSpin$ can be lifted to a class of the form $b_{2k} + x \in \pi_0 K \wedge MSU$ with $l_2(x) < 2k$ and proceed as above.

CHAPTER 5

The splitting theorem

In the previous chapters we computed the θ-algebra $\pi_0 KO \wedge MSpin$ and determined the values of the free generators under the ABP-map. Now we show that all except of one generator can be chosen to be spherical. The only missing generator is hit by the class b under the map coming from the cone. These results lead us to the proof of the splitting theorem.

1. Spherical classes

The spherical classes can be identified with the elements of $\pi_0 KO \wedge MSpin$ which are invariant under the action of the Adams operation with the help of the exact sequence

$$0 \longrightarrow \pi_0 MSpin \longrightarrow \pi_0 KO \wedge MSpin \xrightarrow{\psi^3 - 1} \pi_0 KO \wedge MSpin \longrightarrow 0.$$

Note that it is enough to look for classes which are invariant under ψ^g for any topological generator g of $\mathbb{Z}_2^\times / \pm 1 \cong \mathbb{Z}_2$.

Unlike the θ-operation the action of ψ^g is not compatible with the Thom isomorphism. We denote the operation on the base $\pi_0 KO \wedge BSpin_+$ by ψ^g_B and the one on the Thom spectrum $\pi_0 KO \wedge MSpin$ by ψ^g_M in the sequel. Before describing these we need the

Lemma 1.1. *For all $k \in \mathbb{Z}_2^\times / \pm 1$ the two selfmaps $\psi^k \wedge 1$ and $k^{4n}\,(1 \wedge \psi^{k^{-1}})$ of $\pi_{8n} KO \wedge KO$ coincide.*

Proof. It is enough to check the corresponding statement for complex K-theory. Since $\pi_{2n} K \wedge K$ is torsion free we even may rationalize. A general element of $\pi_{2n} K \wedge K \otimes \mathbb{Q}$ takes the form $a = \sum_s a_s u^s v^{n-s}$ if u, v denote the left and right Bott classes. Hence we compute

$$(\psi^k \wedge 1)(a) = \textstyle\sum_s a_s (k\,u)^s v^{n-s} = k^n \sum_s a_s u^s (k^{-1} v)^{n-s} = k^n\,(1 \wedge \psi^{k^{-1}})(a).$$

$\square$

LEMMA 1.2. *The operation* $\psi_B^{3^{-1}}$ *is given by the formula*

$$\psi_B^{3^{-1}} u_i = \sum_{j=0}^{i} \sum_{s+t=i-j} \binom{j}{s}\binom{s}{t} 3^{j-t}(-1)^{i-j} u_j.$$

PROOF. It suffices to show the equation in $\pi_0 K \wedge BS^1_+$ after replacing the classes u_k with β_k. The previous lemma tells us that for all i, j the equality

$$\left\langle \psi^{3^{-1}} \beta_i, x^j \right\rangle = \left\langle \beta_i, \psi^3 x^j \right\rangle$$

holds. Hence we obtain

$$\begin{aligned} \left\langle \psi^{3^{-1}} \beta_i, x^j \right\rangle &= \left\langle \beta_i, (1-(1-x)^3)^j \right\rangle = \left\langle \beta_{i-j}, (x^2-3x+3)^j \right\rangle \\ = & \left\langle \beta_{i-j}, \sum_{s=0}^{j} \binom{j}{s} 3^{j-s} x^s (x-3)^s \right\rangle = \sum_{s+t=i-j} \binom{j}{s}\binom{s}{t} (-1)^{i-j} 3^{j-t} \end{aligned}$$

□

PROPOSITION 1.3. *We have the formula*

$$\psi_M^{3^{-1}} u_i = \sum_{j=0}^{i} a_j \psi_B^{3^{-1}} u_{i-j}.$$

Here the numbers a_j *are determined by the equation*

$$\sum_{j=0}^{\infty} a_j x^j = \frac{3-6x+7x^2-4x^3+x^4}{3-6x+3x^2}.$$

PROOF. Since the map

$$\pi_0 KO \wedge MSpin \xrightarrow{\Xi} \mathrm{Hom}_{cts}(\widetilde{KO}\, MSpin, \mathbb{Z}_2) \xrightarrow{(\pi^J)^*} \prod_{1 \notin J} \mathcal{T}(\mathbb{Z}_2^\times/\pm, \mathbb{Z}_2)$$

is injective by 1.2, 1.5 and 3.1 of chapter 3 so is the duality map Ξ. Hence it suffices to show the equality after pairing each side with an arbitrary class $a = \tau^* b \in \widetilde{KO}\, MSpin$. Let $f : BS^1 \longrightarrow BSpin$ be the inclusion of the maximal torus of $Spin(2)$. Then we compute with 3.3.4:

$$\begin{aligned} \left\langle \psi_M^{3^{-1}} u_i, a \right\rangle &= \left\langle u_i, \psi_M^3(\tau^* b) \right\rangle = \left\langle \tau_*^{-1} u_i, \hat{\theta}^3\, \tau\, \psi_B^3(b) \right\rangle \\ &= \left\langle u_i, \hat{\theta}^3\, \psi_B^3(b) \right\rangle = \left\langle \beta_i, (\hat{\theta}^3(L^2-1)\otimes \mathbb{C}) f^*(\psi_B^3(b)) \right\rangle \\ &= \sum_{j=0}^{\infty} a_j \left\langle \beta_{i-j}, f^*(\psi_B^3(b)) \right\rangle = \left\langle \sum_{j=0}^{\infty} a_j \psi_B^{3^{-1}} u_{i-j}, b \right\rangle \end{aligned}$$

□

It will prove useful to introduce another measure for the monomials of the ring

$$\mathbb{Z}/2[u_{4k};\ k \geq 1] \cong \mathbb{Z}/2[b_{2k};\ k \geq 1].$$

DEFINITION 1.4. *Let the degree d of a monomial $u_{4i_1} \cdot u_{4i_2} \cdots u_{4i_k}$ be $\sum_k i_k$ and let the degree of a sum of such be the maximum degree of the monomials. We will write $o(n)$ for terms of degree strict smaller than n.*

PROPOSITION 1.5. *We have modulo 2 for all i*

(i) $\psi_B^{3^{-1}} u_{4i} = u_{4i} + o(i-1)$

(ii) $\psi_M^{3^{-1}} u_{4i} = u_{4i} + u_{4i-4} + o(i-1)$

PROOF. Consider first the case that $i = 2k$ is even. Then we obtain modulo 2 and $o(i-1)$ with 3.2.4 and 4.3.6

$$\begin{aligned} \psi_B^{3^{-1}} u_{8k} &= u_{8k} + \sum_{s+t=2} \binom{8k-2}{s}\binom{s}{t} u_{8k-2} + \sum_{s+t=4} \binom{8k-4}{s}\binom{s}{t} u_{8k-4} \\ &= u_{8k} + u_{8k-2} + u_{8k-4} = u_{8k}. \end{aligned}$$

Similarly, for $i = 2k+1$ we get mod 2 and $o(i-1)$

$$\begin{aligned} \psi_B^{3^{-1}} u_{8k+4} &= u_{8k+4} + \sum_{s+t=2} \binom{8k+2}{s}\binom{s}{t} u_{8k+2} + \sum_{s+t=4} \binom{8k}{s}\binom{s}{t} u_{8k} \\ &= u_{8k+4} + u_{8k+2} = u_{8k+4}. \end{aligned}$$

To see the second statement observe that $\sum_{j=0}^{\infty} a_j x^j = 1 + x^4 + \ldots$ and hence with the previous lemma

$$\psi_M^{3^{-1}} u_{4i} = \psi_B^{3^{-1}} u_{4i} + \psi_B^{3^{-1}} u_{4i-4} + o(i-1) = u_{4i} + u_{4i-4} + o(i-1).$$

□

Now we are well prepared to show the

THEOREM 1.6. *For each odd $k > 1$ there exists a $z_k \in \pi_0 KO \wedge MSpin$ which is invariant under the action of the Adams operations and which coincides with u_{4k} modulo elements of strict smaller l_2-length.*

PROOF. We first construct the class z_k modulo 2. When we write Δ for the homomorphism $\psi_M^{3^{-1}} - 1$ then the previous lemma reads

$$\Delta u_{4i} = u_{4i-4} + o(i-1).$$

Moreover, we have for all s, t

$$\begin{aligned} \Delta(u_{4s}\, u_{4t}) &= \Delta(u_{4s})\Delta(u_{4t}) + \Delta(u_{4s})u_{4t} + u_{4s}\,\Delta(u_{4t}) \\ &= u_{4s-4}u_{4t} + u_{4s}\,u_{4t-4} + o(s+t-1). \end{aligned}$$

In particular, we obtain modulo terms which are the Δ-image of classes with degree at most $n+m+1$ and with length at most $2\max(n, m+j)$

$$u_{4n}u_{4m} = u_{4n-4}u_{4m+4} + o(n+m) = \cdots = u_{4(n-j)}u_{4(m+j)} + o(n+m).$$

Setting $n = k-1, m = 0$ and $j = (k-1)/2$ we thus find a class x of length strict smaller than $2k$ with the property that

$$\Delta(u_{4k} + x) = u_{4j}^2 + o(k-1).$$

We can get rid of the highest term by adding u_{4j+4}^2:

$$\Delta(u_{4k} + u_{4j+4}^2 + x) = u_{4j}^2 + (\Delta u_{4j+4})^2 + o(k-1) = o(k-1).$$

Now we have won since the remaining terms are of the form $u_{4n}u_{4m}$ with degree strict smaller than $k-1$. They can be removed in the same fashion as above: Setting $j = n+1$ we see inductively that $u_{4n}u_{4m}$ lies in the Δ-image of classes with length strict smaller than $2k$.

Actually we have shown a bit more. Let S_r be the set of pairs (i, j) with $i + j < k + r + 2$. Then modulo 2 we can choose z_k to be of the form

$$z_k = \sum_{(i,j)\in I_0} u_{4i}u_{4j} = u_{4k} + o_2(2k)$$

for some $I_0 \subset S_0$. In the general situation it suffices to inductively construct sets $I_s \subset S_s$ such that

$$z_k^{(s)} = \sum_{r=0}^{s} 2^r \sum_{(i,j)\in I_r} u_{4i}u_{4j}$$

is invariant modulo 2^{s+1}. Suppose that we have already found $I_0, \ldots, I_{s-1}$. Then $\Delta z_k^{(s-1)}$ is a sum of terms of the form $2^t u_n u_m$ with $n + m < 4k + 4t + 3$. The lemma 1.8 below tells us that we may assume that n and m are multiples of 4. Since the monomials in the generators u_{4i} are linearly independent and $\Delta z_k^{(s-1)}$ vanishes modulo 2^s we are left with terms of the form $2^s u_{4i}u_{4j}$ with $i + j < k + s + 1$. These can be removed with the method above. $\square$

LEMMA 1.7. *Let R be a complete, local ring with maximal ideal m and $a \in m$ be given. Let $v_k = \sum_{s\geq 0} n_s^{(k)} w_{k+s}$ be a convergent series in $R[w_1, w_2, \dots]$ with $n_0^{(k)} \in R^\times$ and $a^s \mid n_s^{(k)}$ for all s, k. Then there are $m_t^{(k)}$ such that $w_k = \sum_{s\geq 0} m_s^{(k)} v_{k+s}$ and a^s divides $m_s^{(k)}$ for all s, t.*

PROOF. Suppose the elements $m_s^{(k)}$ are already constructed modulo m^j and let $n_s^{(k)} = a\, l_s^{(k)}$ with $a^{s-1} \mid l_s^{(k}$ for all $s > 0, k \geq 0$. Then we have modulo m^{j+1}

$$n_0^{(k)} w_k = v_k - a \sum_{s>0} l_s^{(k)} w_{k+s} = v_k - \sum_{r>0} (a \sum_{s+t=r, s>0} l_s^{(k)} m_t^{(k+s)}) v_{k+r}.$$

Since the coefficients $m_s^{(k)}$ are unique the claim follows. □

LEMMA 1.8. *Each u_n can be written as a convergent series of the form $\sum_s a_s u_{4s}$ with $2^{4s-n} \mid a_s$ for $4s \geq n$.*

PROOF. We know from 3.2.4 and 4.3.6 that we can write u_n in the form $\sum_j k_j b_{2j}$ with $2^{4j-n} \mid k_j$ for $4j \geq n$. Hence, the previous lemma gives with $a = 16$, $v_k = u_{4k}$ and $w_k = b_{2k}$

$$u_n = \sum_s (\sum_{j+t=s} k_j m_t^{(j)}) u_{4s} \qquad \text{and } 2^{4t} \mid m_t^{(j)}.$$

□

Note that the proof of the theorem created an algorithm which produces the spherical classes z_k. The first two ones can be chosen as follows modulo 2:

$$\begin{aligned} z_3 &= u_{12} + u_8^2 + u_8 u_4 + u_8 + u_4 \\ z_5 &= u_{20} + u_{12}^2 + u_{12} u_8 + u_{16} u_4 + u_{12} u_4 + u_8 u_4 + u_4^2 \end{aligned}$$

It is not possible to alter the class u_4 by terms of strict smaller l_2-length in a way that it becomes a spherical class. However, the sum $u_4 + u_4^2$ happens to be invariant modulo 2. Hence u_4 behaves in the same way as the class b which was defined earlier. A closer relationship between the two classes is established in the next section.

2. Some 2-adic analysis and the proof of the splitting theorem

In chapter 3 we constructed an E_∞-map $\varphi : T_\zeta \longrightarrow MSpin$ and investigated its behaviour in KO-theory. In 3.1.7 we calculated the image of the θ-algebra generator $b \in \pi_0 KO \wedge T_\zeta$ under the image of $\Phi\pi_*^J\varphi_*$ for all $1 \notin J$. The resulting continuous functions determine $\varphi_* b$ since the map

$$\pi_0 KO \wedge MSpin \stackrel{\Phi(1\wedge\pi^J)_*}{\longrightarrow} \bigoplus_{1\notin J} \mathcal{T}(\mathbb{Z}_2^\times / \pm 1, \mathbb{Z}_2)$$

is an isomorphism. In the same chapter we also calculated the 2-adic functions which correspond to the algebra generators u_{4k}. In this section we compare the 2-adic functions and prove the

THEOREM 2.1. $\varphi_* b = u_4 + o_2(2)$.

A weaker statement is shown in the following

LEMMA 2.2. *Mod 4 the class $\varphi_* b$ coincides with $-u_4$.*

PROOF. With $y = -2x + x^2$ the formula 3.3.3 reads

$$\begin{aligned} k\sum_n \Theta(u_n)(k)x^n &= (1+y)^{(1-k)/2}\frac{(1+y)^k - 1}{y} \\ &= \sum_{s,t} \binom{(1-k)/2}{s}\binom{k}{t+1} y^{s+t}. \end{aligned}$$

Moreover, observe that for all integers $n = \sum_s \alpha_s 2^s$ we have mod 16

$$3^n = 1 + 2\alpha_0 + 8\alpha_1$$

as one easily verifies. Thus we obtain for $k = 3^n$ mod 4

$$\begin{aligned} \Theta(u_4)(k) &= k^{-1}\sum_{s+t=2} \binom{(1-k)/2}{s}\binom{k}{t+1} \\ &= \binom{(1-k)/2}{2} + \frac{1-k}{2k}\binom{k}{2} + \frac{1}{k}\binom{k}{3} \\ &= \alpha_0 + 2\alpha_1 - \alpha_0 - \alpha_0 = -n. \end{aligned}$$

The result now follows from 3.1.7. □

It is clear that $\varphi_* b$ is some convergent series in the u_{4k}-monomials. One might hope to get along with the indecomposable classes u_{4k} itself. For this purpose, we mention that the group of continuous functions has a simple basis which is given by the binomial functions:

PROPOSITION 2.3. *(compare Mahler* [**Sch84**]*) Let $f : \mathbb{Z}_2 \longrightarrow \mathbb{Q}_2$ be continuous. Then there is a convergent series of the form*

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}.$$

Moreover, the null series $a_n \in \mathbb{Q}_2$ is unique.

Another basis is given by the family $x \mapsto \binom{2x}{2n}$ since it coincides with the binomial basis modulo 2.

PROPOSITION 2.4. *Let $f : \mathbb{Z}_2^\times/\pm 1 \longrightarrow \mathbb{Z}_2$ be an even continuous function. Then f admits an expansion of the form*

$$f = \sum_{n=0}^{\infty} a_n \Theta(u_{4n})$$

for some null series $a_n \in \mathbb{Z}_2$. Moreover, the expansion is unique.

PROOF. The continuous function $g(x) = f(2x-1)(2x-1) : \mathbb{Z}_2 \longrightarrow \mathbb{Q}_2$ admits a Mahler expansion

$$g(x) = \sum_m a_m \binom{2x}{2m} = \sum_m a_m (\binom{2x-1}{2m} + \binom{2x-1}{2m-1}).$$

Hence, there is a null series a'_m such that for all $k \in \mathbb{Z}_2^\times$

$$f(k) = \frac{f(k)+f(-k)}{2} = \sum_m a'_m k^{-1}(\binom{k}{m} - \binom{-k}{m}).$$

We claim that for each m the function

$$\varphi_m(k) = k^{-1}(\binom{k}{m} - \binom{-k}{m})$$

can be expressed in terms of the $\Theta(u_{4n})$. With $t(x) = (2x - x^2)(1-x)^{-1}$ the formula 1.7 reads

$$t(x)\sum_j \Theta(u_j)(k)x^j = k^{-1}((1-x)^k - (1-x)^{-k}) = \sum_m \varphi_m(k)x^m.$$

Using 1.8 we hence constructed an expansion of f with a null series $a_n \in \mathbb{Q}_2$. Its intergrality and uniqueness now easily follows from the isomorphism of groups

$$\mathbb{Z}_2[u_4, u_8, u_{12}, \ldots] \cong \bigoplus_{1 \notin J} \mathcal{T}(\mathbb{Z}_2^\times/\pm 1, \mathbb{Z}_2).$$

$\square$

The proof of the proposition provides an algorithm for the coefficients in the expansion of the function associated to b. Note that we may write the latter in the form

$$b(x) = \frac{\log(x)}{\log(3)} : \mathbb{Z}_2^\times / \pm 1 \longrightarrow \mathbb{Z}_2.$$

Here, the 2-adic logarithm is given by the formula

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \text{ for } |x| < 1.$$

It is elementary to check that the logarithm always is divisible by 4 and that $\log(3) = 4$ modulo 8. Hence, the quotient $\log(x)/\log(3)$ is well defined. By 3.1.7 it coincides with $b(x)$ since the 2-adic logarithm satisfies the usual properties. Before carrying out the program of expanding b we observe

LEMMA 2.5. *$l_1(\varphi_s) \leq [\frac{s}{2}] - 1$ for all $s \geq 1$.*

PROOF. It is easy to see with 1.7 that

$$\varphi_s = 2\Theta(u_{s-1}) + \sum_{j=0}^{s-2} \Theta(u_j)$$

Hence the assertion follows from 2.4 and 1.8. □

PROOF OF 2.1. Let α be the linear operator which takes a continuous function f on $\mathbb{Z}_2^\times$ to the even function

$$\alpha(f) : \mathbb{Z}_2^\times / \pm 1 \longrightarrow \mathbb{Z}_2;\ k \mapsto k^{-1}(f(k) - f(-k)).$$

Then we have for all $k = 2x - 1 \in \mathbb{Z}_2^\times$

$$\begin{aligned} \log(k) &= 2^{-1}\alpha(k \log(k)) \\ &= \sum_{n \geq 1} (-1)^{n+1} \frac{2^{n-1}}{n} \sum_{s=0}^{n} (-1)^{n-s} \binom{n}{s} (2\alpha(x^{s+1}) - \alpha(x^s)). \end{aligned}$$

It is well known that

$$x^n = \sum_{m \leq n} a_{mn} \binom{x}{m} \text{ with } a_{mn} = S(m,n)m!.$$

Here, $S(m,n)$ is the Stirling number of the second kind. Furthermore, the expansion

$$\binom{2x}{2m} = \sum_{l=0}^{\infty} 2^{2l} \binom{m+l}{2l} \binom{x}{m+l}$$

shows with 1.7, Pascal's equality and the lemma that

$$l_1(\alpha(\binom{x}{m})) \leq l_1(\varphi_{2m} + \varphi_{2m-1}) \leq m-1.$$

The number $m!$ is divisible by $2^{m-\sigma(m)}$. Here, $\sigma(m)$ is the sum of the number of digits in the 2-adic decomposition of m. This gives

$$l_2(a_{mn}\alpha(\binom{x}{m})) \leq 2^{-m+\sigma(m)}(m-1) \leq 1$$

for all m. Since for $n \geq 5$ the number $2^{n-1}/n$ is divisible by 4 all summands in the expansion have l_2-length at most $1/4$ or l_1-length at most 2. Hence b is an expression in terms with l_2-length strict smaller than 2 and terms with l_1-length at most 4. Thus the assertion follows from 2.1. $\square$

Now we can show in its full glory the main

THEOREM 2.6. *The E_∞-map*

$$(\varphi, z_3, z_5, z_7, \dots) : T_\zeta \wedge \bigwedge_{k=1}^{\infty} TS^0 \longrightarrow MSpin$$

is an isomorphism.

PROOF. The map is a KO-equivalence by 1.6, 2.1, 2.3.5, 2.4.4 and 4.3.8. $\square$

CHAPTER 6

The relations of spin bordism to real K-theory

1. The θ-algebra structure of $\pi_* MSpin$

A first consequence of the splitting formula is the

COROLLARY 1.1. *Let $f \in \pi_0 MSpin$ be the image of $f \in \pi_0 T_\zeta$. Then we have an isomorphism of θ-algebras*

$$\pi_* MSpin \cong \pi_* KO \otimes T\{f, z_3, z_5, z_7, \dots\}.$$

PROOF. This immediately follows from the theorem and 2.4.5. $\square$

REMARK 1.2. The formula for the homotopy ring of spin bordism evokes the hope that $MSpin$ can be made into a KO-algebra spectrum. We have seen earlier that $MSpin$ splits into a sum of KOs and hence is a KO-module spectrum (in contrary to the unlocalized $MSpin$ [**Sto94**].) However, there does not exist any map of ring spectra from KO to $MSpin$ even in the $K(1)$-local world: any such would give a self map of KO when composed with the $\hat{A}$-map $\pi^\emptyset$. The induced map in KO-homology factorizes over the free ring $\pi_0 KO \wedge MSpin$ and thus coincides with the augmentation

$$\epsilon_* : \pi_0 KO \wedge KO \longrightarrow \pi_0 KO \subset \pi_0 KO \wedge KO$$

by 3.1.5 and 3.2.5. Even rationally, there is no self ring map of KO which induces the augmentation map in KO-homology.

2. The E_∞-cellular structure of the $\hat{A}$-map

In the following we always assume that we have chosen z_k in a way that the constant $\hat{A} z_k$ is null. It is well known that the $\hat{A}$-map is an E_∞-ring map. It hence coincides with the E_∞-map

$$MSpin \cong T_\zeta \wedge \bigwedge TS^0 \xrightarrow{(\iota, *)} KO$$

since it does so when restricted to $C_\zeta \vee \bigvee TS^0$.

REMARK 2.1. The E_∞-property of $\hat{A}$ actually is a consequence of the splitting theorem and the H_∞-property of $\hat{A}$: it is enough to show that the map $(\iota, *)$ coincides with $\hat{A}$. In KO-homology they agree since they have the same values on the θ-algebra generators $b, z_3, z_5, \ldots$. Hence the claim follows from the following proposition which is interesting in its own right.

PROPOSITION 2.2. (i) *There are no phantom cohomology operations in KO-theory.*[1]

(ii) $KO^0KO \cong \mathcal{T}(\mathbb{Z}_2^\times/\pm, \mathbb{Z}_2)^\wedge \cong \mathrm{Hom}_{cts}(\pi_0 KO \wedge KO, \pi_0 KO)$.
In particular, any self map of KO is determined by its behaviour in KO-homology. That is, the map

$$KO^0KO \longrightarrow \mathrm{Hom}_{cts}(\pi_0 KO \wedge KO, \pi_0 KO \wedge KO)$$

is injective.

PROOF. Let $G^\#$ be the Pontrjagin dual $\mathrm{Hom}_{cts}(G, \mathbb{Z}_{2^\infty})$ of a 2-profinite abelian group G. Then the pairing

$$\pi_0(K \wedge KO, \mathbb{Z}_{2^\infty}) \otimes K^0KO \longrightarrow \mathbb{Z}_{2^\infty}$$

induces an isomorphism (compare 2.3 of [**Bou99**])

$$K^0KO \cong \pi_0(K \wedge KO, \mathbb{Z}_{2^\infty})^\# \cong \mathcal{T}(\mathbb{Z}_2^\times/\pm 1, \mathbb{Z}_2)^\wedge.$$

Hence any map from KO to K is determined by its behaviour in K-homology and there are no phantom maps. Thus it suffices to show that the complexification

$$KO^0KO \longrightarrow K^0KO$$

is an isomorphism. It is well known that the real and the complex (completed) representation ring of each group $Spin(8k)$ are the same (compare [**And64**]). Using 3.1.2 we just saw that the complex inverse system is $\lim^1$-free. Hence so is the real and we have

$$KO^0MSpin \cong K^0MSpin.$$

This gives the assertion. □

The corollary 1.1 suggests how to obtain KO-theory by attaching E_∞-cells to $MSpin$.

[1] For the unlocalized version of this statement see [**And83**].

COROLLARY 2.3. *The diagram*

$$\begin{tikzcd}
\bigwedge_{i=1}^{\infty} TS^0 \arrow[r, "*"] \arrow[d, "{(f,z_3,z_5,\dots)}"'] & T* \arrow[d] \\
MSpin \arrow[r, "\hat{A}"] & KO
\end{tikzcd}$$

is a homotopy pushout of E_∞-spectra.

Before deducing the corollary from the splitting theorem we need a tool which is basic in the calculation of homotopy pushouts.

THEOREM 2.4. *(Eilenberg, Moore)*
Let P be the homotopy pushout of the diagram

$$Y \longleftarrow X \longrightarrow Z$$

in $\mathcal{S}_U^T$. Then there is a natural spectral sequence of the form

$$E^2_{p,q} = \mathrm{Tor}^{\pi_*X}_{p,q}(\pi_*Y, \pi_*Z) \Longrightarrow \pi_{p+q}P.$$

PROOF. The spectral sequence is well known. The construction in the category of S-algebras [**EKMM97**] verbatim carries over to the category $\mathcal{S}_U^T$. Alternatively, one may use theorem IV.6.2 [**EKMM97**] and observe that the category $\mathcal{S}_U^T$ is Quillen equivalent to the category of S-algebras. □

LEMMA 2.5. *The $T\{f\}$-module $T\{b\}$ is free*[2] *with basis all monomials of the form*

$$(\theta_{i_1}b)(\theta_{i_2}b)\cdots(\theta_{i_k}b) \qquad \text{with } i_s \neq i_t \text{ for } s \neq t.$$

PROOF. Compute modulo $2, \theta b, \theta^2 b, \dots, \theta^{k-1}b$ with 2.2.5

$$\begin{aligned}
\theta^k f &= \theta^k(\psi(b) - b) = \psi\theta^k(b) - \theta^k b = (\theta^k b)^2 + \theta^k b \\
\theta_k f &= (\theta_k b)^2 + \theta_k b.
\end{aligned}$$

Hence, it is elementary to check that the map

$$\bigoplus_{I\subset\{1,\dots,n\}} \mathbb{F}_2[f, \theta_1 f, \dots, \theta_n f]\,\langle\theta_I b\rangle \longrightarrow \mathbb{F}_2[b, \theta_1 b, \dots, \theta_n b]$$

is an isomorphism. □

[2]In [**Hop98b**] Hopkins says that the map $T\{f\} \longrightarrow T\{b\}$ is étale but the author is unsure of the meaning of this property for the map between infinitely generated rings.

PROOF OF 2.3: Let P be the homotopy pushout of the diagram. We prove that the induced map from P to KO is a K-equivalence. The Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{p,q}^{\pi_* K\wedge\bigwedge TS^0}(\pi_* K \wedge MSpin, \pi_* K) \Longrightarrow \pi_{p+q} K \wedge P$$

collapses since by the splitting theorem and the lemma $\pi_* K \wedge MSpin$ is flat. This gives

$$\pi_* K \wedge P \cong \pi_* K \wedge MSpin \otimes_{\pi_* K\wedge\bigwedge TS^0} \pi_* K \cong \pi_* K \otimes T\{b\} \otimes_{T\{f\}} \mathbb{Z}_2.$$

Hence it suffices to show that the latter is the algebra

$$\pi_* K \wedge KO \cong \mathcal{T}(\mathbb{Z}_2^\times/\pm, \pi_* K) \cong \mathcal{T}(\mathbb{Z}_2, \pi_* K).$$

Observe that in the ring of continuous functions we have for all n

$$\sum_{k=0}^{\infty} 2^k \alpha_k = \varphi b = \psi^n \varphi b = \sum_{k=0}^{n} 2^k (\theta_k \varphi b)^{2^{n-k}}$$

and hence by induction on n modulo 2

$$\theta_n \varphi b = \alpha_n.$$

Hence the claim follows from 3.2.5 and the calculation made in the proof of the lemma.

There is a more direct proof of the corollary which uses a result of Hopkins: he shows that the right square of the diagram

$$\begin{tikzcd}
TS^0 \wedge \bigwedge TS^0 \arrow[r, "1\wedge *"] \arrow[d, "{(f,(z_3,z_5,\dots))}"'] & TS^0 \wedge T* \arrow[r, "*"] \arrow[d, "f"'] & T* \arrow[d] \\
MSpin \arrow[r] & T_\zeta \arrow[r] & KO
\end{tikzcd}$$

is a homotopy pushout. In fact, this is clear by the same argumentation as above. The splitting theorem gives the homotopy pushout property of the left square and hence furnishes the result. $\square$

3. Another additive splitting and the Conner-Floyd isomorphism

We have seen earlier that $MSpin$ additively splits into a sum of KO-theories. Using the multiplicative splitting theorem we are now able to write down an additive splitting which recovers more structure.

COROLLARY 3.1. *For all spectra X there is a natural isomorphism of $\pi_0 MSpin$-modules*

$$\pi_* MSpin \wedge X \cong \pi_* KO \wedge X \otimes T\{f, z_3, z_5, \dots\}.$$

Here, the module structure of the right hand side is given by the isomorphism of 1.1.

PROOF. Choose a projection pr of the free $T\{f\}$-module $T\{b\}$ onto the summand $T\{f\}$. Then the composite

$$\begin{aligned} \pi_* T_\zeta \wedge X &\longrightarrow \pi_* KO \wedge T_\zeta \wedge X \cong \pi_* KO \wedge T_\zeta \otimes_{\pi_* KO} \pi_* KO \wedge X \\ &\cong \pi_* KO \wedge X \otimes T\{b\} \xrightarrow{1 \otimes pr} \pi_* KO \wedge X \otimes T\{f\} \end{aligned}$$

is a natural transformation between cohomology theories. Hence the result follows from the splitting theorem. $\square$

This result immediately implies the well known

COROLLARY 3.2. *(Hopkins, Hovey* [**HH92**]*)*
For all spectra X the natural map

$$\pi_* MSpin \wedge X \otimes_{\pi_* MSpin} \pi_* KO \longrightarrow \pi_* KO \wedge X$$

induced by the $\hat{A}$-orientation is an isomorphism.

REMARK 3.3. Hopkins and Hovey prove the general Conner-Floyd isomorphism for $MSpin$ and KO by localizing at each prime. The essential work is done at the prime 2 since for odd primes the original method of Conner and Floyd applies. Let $\beta \in \pi_8 MSpin$ correspond to the Bott class of the first ko-summand in the ABP-splitting. Then they show that $\beta^{-1} MSpin$ is K-local. Hence there is a natural isomorphism

$$\begin{aligned} &(MSpin \wedge R)_* X \otimes_{(MSpin \wedge R)_*} (KO \wedge R)_* \\ \cong\ &(L_K MSpin \wedge R)_* X \otimes_{(L_K MSpin \wedge R)_*} (KO \wedge R)_* \end{aligned}$$

for all ring spectra R. When setting $R = S\mathbb{Z}/2^k$ we get with 3.1

$$\begin{aligned} &(L_K MSpin \wedge S\mathbb{Z}/2^k)_* X \otimes_{(L_K MSpin \wedge S\mathbb{Z}/2^k)_*} (KO \wedge S\mathbb{Z}/2^k)_* \\ \cong\ &\pi_* L_{K(1)} MSpin \wedge S\mathbb{Z}/2^k \wedge X \otimes_{\pi_* L_{K(1)} MSpin \wedge S\mathbb{Z}/2^k} \pi_* L_{K(1)} KO \wedge S\mathbb{Z}/2^k \\ \cong\ &\pi_* KO \wedge S\mathbb{Z}/2^k \wedge X \end{aligned}$$

The general statement now can be finished as in section 6 of [**HH92**].

CHAPTER 7

The relations of spin bordism to tmf

This section deals with the applications of the splitting formula to the $K(1)$-local topological modular forms. It is quite surprising that each result of the previous chapter completely carries over from real K-theory to tmf: we describe the Witten orientation by attaching E_∞-cells to $MO\langle 8\rangle$ and prove the Conner-Floyd isomorphism as we did for the $\hat{A}$-orientation.

1. The ring of divided congruences

An elliptic curve over a field F is a non singular curve defined by a Weierstraß equation

$$C_W : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with $a_1, a_2, a_3, a_4, a_6 \in F$. We will consider curves which are defined over rings rather than fields and which have mild singularities. Unfortunately, the theory of such generalized elliptic curves demands a great effort and becomes rather abstract. At this moment we put up with the following two examples.

Example 1.1. (i) The curve

$$C_j : y^2 + xy = x^3 - \frac{36}{j-1728}x - \frac{1}{j-1728}$$

is defined over $\mathbb{Z}_2[j^{-1}]$ since the series

$$\frac{1}{j-1728} = \sum_{s=0}^{\infty} 2^{6s}3^{3s}j^{-s-1}$$

converges 2-adically. It is the universal curve with prescribed j-invariant: for any elliptic curve over a field F in Weierstraß form there is a certain rational function j in the a_i which only depends on the isomorphism class of that curve. Moreover, for any given $1728 \neq j \in F$ this j-invariant of C_j is j (compare [**Sil86**].)

(ii) Another important curve is the Tate curve which is defined over $\mathbb{Z}[\![q]\!]$ by the equation

$$C_{Tate} : y^2 + xy = x^3 + B(q)x + C(q).$$

Here,

$$\begin{aligned} B(q) &= -1/48(E_4(q) - 1) \\ C(q) &= -1/496(E_4(q) - 1) + 1/864(E_6(q) - 1) \end{aligned}$$

with the Eisenstein series

$$E_{2k} = 1 - 4k/B_{2k} \sum_{n=1}^{\infty} (\sum_{d|n} d^{2k-1}) q^n$$

and the Bernoulli numbers B_{2k}.

Consider the ring homomorphism

$$\lambda : \mathbb{Z}_2[j^{-1}] \longrightarrow \mathbb{Z}_2[\![q]\!]; \qquad j^{-1} \mapsto \frac{E_4^3 - E_6^2}{1728 E_4^3}.$$

It has the following meaning: the rational functions in j are precisely the modular functions of weight zero and the map λ gives their Fourier expansions. We state for later purpose the

LEMMA 1.2. *The q-expansion map λ is an inclusion of a pure subgroup. That is, its cokernel is torsion free.*

PROOF. It suffices to show that λ is injective modulo 2. Let us be given a polynomial

$$p(j^{-1}) = \sum_{i \geq 0} \alpha_i j^{-i}$$

whose q-expansion λp is divisible by 2. Then it inductively follows from

$$\lambda j^{-1} = q + O(q^2).$$

that all coefficients α_i are even. □

There are some more quantities for elliptic curves in Weierstraß form which we are going to use (compare [**Sil86**]):

$$\begin{aligned} b_2 &= a_1^2 + 4a_2 \\ b_4 &= 2a_4 + a_1 a_3 \\ b_6 &= a_3^2 + 4a_6 \end{aligned}$$

$$\begin{aligned}
b_8 &= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2 \\
c_4 &= b_2^3 - 24b_4 \\
c_6 &= -b_2^3 + 36b_2b_4 - 216b_6 \\
\Delta &= -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6 \\
\omega &= dx/(2y + a_1x + a_3) = dy/(3x^2 + 2a_2x + a_4 - a_1y)
\end{aligned}$$

Two curves in Weierstraß form over a ring R define the same object iff there is a change of variables of the form

$$\begin{aligned}
x &= u^2x' + r \\
y &= u^3y' + u^2sx' + t
\end{aligned}$$

with $r, s, t \in R$ and $u \in R^\times$. Under these transformations we have

$$u^4c_4' = c_4;\ u^6c_6' = c_6;\ u^{-1}\omega' = \omega.$$

Thus the quantities c_4 and c_6 are invariances of elliptic curves together with their nowhere vanishing differentials ω. Such objects are called modular forms over R. For instance, we have

$$c_4(C_{Tate}) = E_4;\ c_6(C_{Tate}) = -E_6$$

and

$$c_4(\lambda C_j) = E_4^3/E_6^2;\ c_6(\lambda C_j) = -E_4^3/E_6^2.$$

LEMMA 1.3. *There is an isomorphism κ between the curves C_{Tate} and λC_j over $\mathbb{Z}_2[\![q]\!]$.*

PROOF. The q-expansion of E_4 and E_6 show that the quotient

$$E_4/E_6 \in 1 + 8q\mathbb{Z}_2[\![q]\!]$$

admits a root

$$u = \sqrt{E_4/E_6} \in 1 + 4q\mathbb{Z}_2[\![q]\!].$$

Hence the quantities

$$r = \frac{u^2 - 1}{12},\ s = \frac{u-1}{2},\ t = \frac{1-u^2}{24}$$

lie in $\mathbb{Z}_2[\![q]\!]$. Since

$$u = \sqrt[4]{c_4(\lambda C_j)/c_4(C_{Tate})} = \sqrt[6]{c_6(\lambda C_j)/c_6(C_{Tate})}$$

one easily verifies that this gives the desired isomorphism. $\square$

It is well known that the Eisenstein series E_4 and E_6 generate the graded ring of modular forms over the complex numbers. Over the 2-adic integers, the ring of modular forms is given by

$$mf_* \overset{def}{=} \mathbb{Z}_2[c_4, c_6, \Delta]/(1728\Delta - c_4^3 - c_6^2).$$

It is a subring of the ring D of divided congruences: the elements of D are those 2-adically convergent series $\sum f_i$ of (inhomogeneous) modular forms over $\mathbb{Q}_2$ such that the q-expansion $\sum f_i(q) = \lambda(\sum f_i)$ has coefficients in $\mathbb{Z}_2$[1]. This is the ring where all congruences between modular forms take place: for example, the congruence

$$E_4 \equiv 1 \mod 240$$

corresponds to the class

$$(E_4 - 1)/240 \in D.$$

The presence of congruences means that mf_* is not a pure subgroup of D. However, we can make it into one by introducing a new parameter v which keeps track of the grading:

$$\lambda_* : mf_* \longrightarrow D[v^{\pm}]; \quad f \mapsto f(v^{-1}C_{Tate}, v\,dx/2y + x).$$

That is,

$$\lambda_*(c_4) = E_4/v^4; \quad \lambda_*(c_6) = -E_6/v^6.$$

Moreover, λ_* admits a factorization

$$\lambda_* : mf_* \xrightarrow{\iota} \mathbb{Z}_2[j^{-1}, w^{\pm}] \xrightarrow{\lambda_v} D[v^{\pm}].$$

Here, we set

$$\begin{array}{rclrcl} \iota(c_4) & = & w^{-4} & \iota(c_6) & = & -\sqrt{1-1728/j}\,w^{-6} \\ \lambda_v(j^{-1}) & = & \lambda(j^{-1}) & \lambda_v(w) & = & v/\sqrt[4]{E_4}. \end{array}$$

LEMMA 1.4. *λ_v is pure.*

[1]Katz shows in [**Kat75**] that D is the coordinate ring of the moduli problem, which is given by elliptic curves together with isomorphisms of their formal groups with the multiplicative formal group. Its relation to the $K(1)$-local topological modular forms will become apparent later.

PROOF. The composite of λ_v with the isomorphism

$$D[v^{\pm}] \longrightarrow D[v^{\pm}],\ v \mapsto \sqrt[4]{E_4}\, v$$

is the map $\lambda[v^{\pm}]$ and hence is pure by 1.2. □

It is well known that an elliptic curve defines a formal group. If the curve is given in Weierstraß form the addition close to the origin gives a formal group law with coefficients in $\mathbb{Z}[a_1, a_2, \dots, a_6]$. The formula

$$\begin{aligned} \hat{F}_W(x,y) &= x + y - a_1xy - a_2(x^2y + xy^2) \\ &\quad -(2a_3x^3y - (a_1a_2 - 3a_3)x^2y^2 + 2a_3xy^3) + \cdots \end{aligned}$$

is taken from [**Sil86**]. In particular, there are group laws $\hat{G}_j$ and $\hat{G}_{Tate}$ attached to the curves C_j and C_{Tate}, respectively.

PROPOSITION 1.5. *The formal groups associated to C_j and C_{Tate} admit multiplicative reductions.*

PROOF. For the Tate curve this result is well known (compare [**Sil86**]): an isomorphism over the power series ring $\mathbb{Z}[\![q, u]\!]$ is given by

$$(u \in \mathbb{C}^*/q^{\mathbb{Z}}) \quad \mapsto \quad \begin{pmatrix} x &=& \sum_{n\in\mathbb{Z}} q^n u/(1-q^n u)^2 - 2\sum_{n\geq 1} nq^n/(1-q^n) \\ y &=& \sum_{n\in\mathbb{Z}} q^{2n} u^2/(1-q^n u)^3 - \sum_{n\geq 1} nq^n/(1-q^n) \end{pmatrix}.$$

Let $f : \hat{G}_m \longrightarrow \hat{G}_{Tate}$ be the resulting strict isomorphism and set

$$\chi(x) = x/\lambda_v(w) = \sqrt[4]{E_4}x/v, \qquad \rho(x) = \sqrt[4]{c_4(C_j)}x = x/\sqrt[4]{1 - 1728/j}.$$

Then by 1.3 we have a strict isomorphism over $D[v^{\pm}]$

$${}^{\chi}\hat{G}_m \xrightarrow{\chi} \hat{G}_m \xrightarrow{f} \hat{G}_{Tate} \xrightarrow{\kappa} \lambda\hat{G}_j \xrightarrow{\lambda\rho^{-1}} \lambda\, {}^{\rho}\hat{G}_j.$$

Hence, the strict isomorphism over $\mathbb{Q}_2[j^{-1}, w^{\pm}]$

$$\exp_{{}^{\rho}\hat{G}_j} \log_{{}^{\chi}\hat{G}_m} : {}^{\chi}\hat{G}_m \xrightarrow{\cong} {}^{\rho}\hat{G}_j$$

q-expands integrally. Thus the result follows from the lemma 1.4. □

2. Elliptic cohomology theories and Artin-Schreier classes

In this section we are interested in complex oriented theories whose formal group laws come from elliptic curves. We first recall some basic notions from [**Qui71**] or [**Ada74**].

Let E be a multiplicative cohomology theory. Then E is called complex orientable if there is a class $x \in \widetilde{E}^2\mathbb{C}P^\infty$ which restricts to $\Sigma^2 1 \in \widetilde{E}^2 S^2$ under the inclusion map of the bottom cell. Any choice of x is a complex orientation of E. An orientation of E supplies a system of Thom and Euler classes for complex vector bundles. The Euler class of a tensor product of line bundles defines a formal group law

$$\hat{G}_E(x,y) \overset{def}{=} e(L_1 \otimes L_2) \in E^*\mathbb{C}P^\infty \times \mathbb{C}P^\infty \cong \pi_* E[\![x,y]\!]$$

with $x = e(L_1)$,$y = e(L_2)$.

Definition 2.1. *(Hopkins)*
An elliptic spectrum is a triple (E, C, κ) consisting of

(i) *a ring spectrum E with $\pi_{odd}E = 0$ and for which there is a unit in $\pi_2 E$. (These assumptions guarantee that E is complex orientable as one can verify with the Atiyah-Hirzebruch spectral sequence.)*
(ii) *a generalized elliptic curve C over $\pi_0 E$*
(iii) *an isomorphism κ of the formal completion of this curve with the formal group G_E associated to E.*

Example 2.2. There are elliptic spectra $K[j^{-1}]$ and K_{Tate} for which the elliptic curves C_j and C_{Tate} and the isomorphisms from 1.5 are part of the data:

$$\begin{aligned} K[j^{-1}]_*X &\overset{def}{=} K_*(X, \mathbb{Z}_2[j^{-1}]) \cong \mathbb{Z}_2[j^{-1}, w^{\pm}] \otimes_{MU_*} MU_*X \\ K_{Tate*}X &\overset{def}{=} K_*(X, D) \cong D[v^{\pm}] \otimes_{MU_*} MU_*X. \end{aligned}$$

Note that the spectra come to us together with their real companians $KO[j^{-1}]$ and KO_{Tate} which are similarly defined with real K-theory.

There is another theory we like to mention. The 2-series of the formal group law associated to the universal Weierstraß curve C_W takes the form

$$[2](x) = 2x - a_1 x^2 + O(x^3)$$

and hence satisfies the Landweber exactness conditions (compare f.i.[**Fra92**]) if a_1 is inverted. Hence we obtain the theory

$$W_*X \overset{def}{=} \mathbb{Z}[a_1, \dots, a_6][a_1^{-1}] \otimes_{MU_*} MU_*X$$

Note that here the coefficients a_i of the curve C_W have the degree $2i$ and the transformation

$$x = a_1^2 x', \quad y = a_1^3 y'$$

gives a curve over $\pi_0 W$.

Let E be any elliptic theory and assume that its elliptic curve is given in Weierstraß form. Then similarly, we can choose a unit $u \in \pi_2 E$ and transform the curve C over $\pi_0 E$ into a curve over $\pi_* E$ with $a_i \in \pi_{2i} E$. The [2]-series of the formal group law then shows the relation

$$a_1 \equiv v \bmod 2 \quad \in \pi_2 K \wedge E.$$

Hence the localization map

$$E \longrightarrow E[a_1^{-1}]$$

is a $K(1)$-equivalence. Thus in the $K(1)$-local category a_1 becomes a unit and gives rise to the class

$$j^{-1} = \frac{\Delta}{c_4^3} \in \pi_0 E$$

which will prove useful in things to come.

DEFINITION 2.3. *Let E be a $K(1)$-local theory. A class $b \in \pi_0 KO \wedge E$ is an Artin-Schreier element if $\psi^3 b = b + 1$.*

In [**Hop98b**] Hopkins constructs an Artin-Schreier class for any elliptic E as follows: he first looks at complex K-theory and sets

$$b \overset{def}{=} -\frac{\log(c_4/v^4)}{\log(81)} \in \pi_0 K \wedge E.$$

The 2-adic logarithm is well defined by the relation

$$v \equiv a_1^4 \equiv c_4 \bmod 8 \qquad \in \pi_8 K \wedge E$$

which is easily verified. In fact, for the theory W the class b gives rise to a unique real class and hence it does so for any E. Moreover, it is an Artin-Schreier element:

$$\psi^3 b = -\frac{\log(c_4/(81v^4))}{\log(81)} = -\frac{\log(c_4/v^4)}{\log(81)} + \frac{\log(81)}{\log(81)} = b+1.$$

3. E_∞-elliptic spectra and the Atkin-Lehner operator

An E_∞-structure on a complex oriented theory E provides unstable operations (compare [**Hop98b**]:) let

$$f : \pi_0 E \longrightarrow R$$

be a ring map and $H \subset f^* G_E$ be a closed finite subgroup. Then there is a new map

$$\psi_H : \pi_0 E \longrightarrow R$$

and an isogeny $f^* G_E \longrightarrow \psi^* G_E$ with kernel H. If the formal group is isomorphic to the multiplicative and H is the canonical subgroup of order 2 then we may take f to be identity map. In this case ψ_H coincides with the operator ψ defined earlier.

Definition 3.1. *(Hopkins)*
An E_∞-elliptic spectrum is an E_∞-spectrum E with the following data and properties:

(i) *E is an elliptic theory*
(ii) *each isogeny described above extends to an isogeny of the elliptic curve associated to E.*

We would like to make the elliptic theories $K[j^{-1}]$ and K_{Tate} into E_∞-elliptic theories. Consider the quotient C_j/H of C_j by the subgroup scheme H of 2-torsion points. It is well known that C_j/H is an elliptic curve and there is an isogeny

$$\pi : C_j \longrightarrow C_j/H$$

with kernel H. This gives a ring map

$$\psi : \mathbb{Z}_2[j^{-1}] \longrightarrow \mathbb{Z}_2[j^{-1}]$$

which sends j^{-1} to $j^{-1}(C_j/H)$. This map carries the names Atkin-Lehner operator [**AL70**] or Frobenius operator [**Kat77**][**Gou88**]. For the Tate curve we can describe the operation ψ more explicitly: the isogeny π is given by

$$Tate(q) \cong G_m/q^{\mathbb{Z}} \longrightarrow G_m/q^{2\mathbb{Z}} \cong Tate(q^2); \quad x \mapsto x^2$$

and hence $\psi(q) = q^2$. It is obvious that there is a power series l with

$$\psi(j^{-1})(q) = j^{-1}(q^2) = l(j^{-1})(q).$$

Its geometric interpretation shows the 2-adic convergence of l [2].

LEMMA 3.2. *The formulae*

$$\psi(j^{-1}) = l(j^{-1}) \text{ and } \psi(q) = q^2$$

impart the structure of θ-algebras to $\mathbb{Z}_2[j^{-1}]$, D and $\mathbb{Z}_2[\![q]\!]$.

PROOF. For any power series a with coefficients in $\mathbb{Z}_2$ we have mod 2

$$\psi a = \sum a_i q^{2i} \equiv \sum a_i^2 q^{2i} \equiv a^2.$$

Hence the assertion follows from 1.2 and the Wilkerson criterion. □

In [**Hop98b**] Hopkins constructs a universal relation which must hold in the homotopy of any $K(1)$-local E_∞-elliptic theory E. This relation can be incorporated into an E_∞-cellular complex M which maps into E. It turns out that the K-theory of this complex imparts an E_∞-structure to K_{Tate} in a way that the associated θ-algebra is the one given above.

To construct M and the universal relation consider the element

$$f = \psi b - b \ \in \pi_0 K \wedge E.$$

Since it is invariant under the Adams operations it is a modular function. Hence we may consider f as a 2-adic analytic function in j^{-1} which maps to the original f under the canonical map

$$\mathbb{Z}_2[j^{-1}] \longrightarrow \pi_0 E.$$

By the naturality of the Frobenius operator the value of ψ on f is determined by the θ-algebra structure of $\mathbb{Z}_2[j^{-1}]$..

[2] A alternative proof for the convergence will be given in the next section with the help of formula 4.1.5.

LEMMA 3.3. *(Hopkins)*
The map

$$\mathbb{Z}_2[f] \longrightarrow \mathbb{Z}_2[j^{-1}]$$

is an isomorphism.

PROOF. The beautiful proof given in [**Hop98b**] is based on well known congruences for the Ramanujan τ-function. □

Let h be the 2-adically convergent power series with

$$h(f) = \theta(f) \in \mathbb{Z}_2[j^{-1}].$$

Then this gives a universal relation for the θ-algebra structure of any $K(1)$-local E_∞ ring spectrum.

PROPOSITION 3.4. *(Hopkins)*
Set

$$y = \theta(f) - h(f) \in \pi_0 T_\zeta$$

and let M be the E_∞-homotopy pushout of the diagram

$$T_\zeta \xleftarrow{y} TS^0 \xrightarrow{*} T* = S^0.$$

Then we have isomorphisms of rings

$$\pi_* KO \wedge M \cong \pi_* KO \wedge T_\zeta \otimes_{\pi_* KO \wedge TS^0;y} \pi_* KO$$

and

$$\pi_* M \cong \pi_* KO[j^{-1}].$$

PROOF. Hopkins shows that the map

$$y_* : \pi_0 KO \wedge TS^0 \longrightarrow \pi_0 KO \wedge T_\zeta$$

is flat. Hence the first claim follows from the Eilenberg-Moore spectral sequence. For the second the spectral sequence and the lemma give

$$\begin{aligned} \mathrm{Tor}^{\pi_* S^0 \otimes T\{y\}}(\pi_* KO \otimes T\{f\}, \pi_* S^0) &\cong \mathrm{Tor}^{T\{y\}}(\pi_* KO \otimes T\{f\}, \mathbb{Z}_2) \\ \cong \quad \pi_* KO \otimes T\{f\} \otimes_{T\{y\}} \mathbb{Z}_2 &\cong \pi_* KO[f] \cong \pi_* KO[j^{-1}] \end{aligned}$$

□

COROLLARY 3.5. *There is an additive $K(1)$-local homotopy equivalence between M and $KO[j^{-1}]$.*

PROOF. Consider the natural transformation of homology theories

$$\begin{aligned}\pi_*M\wedge X &\longrightarrow \pi_*M\wedge KO\wedge X\cong \pi_*KO\wedge M\otimes_{\pi_*KO}\pi_*KO\wedge X\\ &\cong \pi_*KO\wedge T_\zeta\otimes_{\pi_*KO\wedge TS^0;y}\pi_*KO\otimes_{\pi_*KO}\pi_*KO\wedge X\\ &\cong \pi_*KO\wedge X\otimes T\{b\}\otimes_{T\{f\}}T\{f\}\otimes_{T\{y\}}\mathbb{Z}_2\\ &\xrightarrow{1\otimes pr_+} \pi_*KO\wedge X[j^{-1}].\end{aligned}$$

It is an isomorphism on the coefficients. □

COROLLARY 3.6. *The canonical map*

$$\pi_0K\wedge M\longrightarrow D,\ b\mapsto b(K_{Tate})/v=1$$

is an isomorphism of θ-algebras. Hence, K_{Tate} is an E_∞-elliptic theory and the map

$$KO_{Tate}=KO\otimes D\longrightarrow KO\wedge M$$

is a homotopy equivalence.

PROOF. Rationally, the inverse of

$$\pi_0K\wedge M\otimes\mathbb{Q}\cong\mathbb{Q}_2[j^{-1},w^\pm]\xrightarrow{\lambda_v}D[v^\pm]\otimes\mathbb{Q}\xrightarrow{v=1}D\otimes\mathbb{Q}$$

is given by

$$D\otimes\mathbb{Q}\cong mf_*\otimes\mathbb{Q}\xrightarrow{\iota\otimes\mathbb{Q}}\pi_0K\wedge M\otimes\mathbb{Q}$$

as one easily checks with the formulae of the first section. Next consider the diagram of left $\mathbb{Z}_2[j^{-1}]$-modules

$$\begin{array}{ccc}\mathcal{T}(\mathbb{Z}_2,\mathbb{Z}_2)\otimes\mathbb{Z}_2[j^{-1}] & \xrightarrow{\mu(s\otimes i)} & \pi_0K\wedge M\\ \downarrow{\scriptstyle 1\otimes\lambda} & & \downarrow{\scriptstyle b\mapsto b(K_{Tate})}\\ \mathcal{T}(\mathbb{Z}_2,\mathbb{Z}_2)[\![q]\!] & \xrightarrow{\binom{n}{k}\mapsto c_k(K_{Tate})} & \mathcal{T}(\mathbb{Z}_2,\mathbb{Z}_2)[\![q]\!]\end{array}.$$

Here,

$$s:\mathcal{T}(\mathbb{Z}_2,\mathbb{Z}_2)\longrightarrow\pi_0K\wedge T_\zeta\longrightarrow\pi_0K\wedge M$$

is the ψ^3-equivaraint section of the projection

$$\pi_0K\wedge M\longrightarrow\pi_0K\wedge K\cong\mathcal{T}(\mathbb{Z}_2,\mathbb{Z}_2),\ f\mapsto 0$$

constructed in [**Hop98b**]. It sends the Mahler basis to certain classes

$$c_k\in\pi_0KO\wedge M\cong T\{b\}\otimes_{T\{y\}}\mathbb{Z}_2.$$

The right vertical arrow is the map of θ-algebras which sends b to the Artin-Schreier element of K_{Tate}. We claim that it is pure. This follows from the fact that the left vertical map is so and the top and bottom arrows are isomorphisms. Hence, $\pi_0 K \wedge M$ consists of inhomogeneous modular forms $\sum f_i$ with the property that for each $\lambda \in \mathbb{Z}_2^\times$ the sum $\sum \lambda^i f_i$ q-expands integrally. However, it is well known that the ring of divided congruences is invariant under these Adams operations (compare [**Kat75**]). This finishes the proof. $\square$

REMARK 3.7. Let tmf be the homotopy inverse limit of all E_∞-elliptic spectra over all maps which preserve the E_∞-elliptic structure. Then the canonical E_∞-map

$$M \longrightarrow tmf$$

is an E_∞-homotopy equivalence. This follows from a calculation of the homotopy of tmf [**Hop98a**]. Hence we will write tmf instead of M in the sequel. It might be possible to provide an E_∞-elliptic structure to the theory $K[j^{-1}]$ and to show the universal property of M by taking homomotpy fix points under complex conjugation. Note however, that the equivalence

$$tmf \cong KO[j^{-1}]$$

is only additive. This follows for instance from the fact that the idempotents of the ring D are trivial whereas the ring $\pi_0 KO \wedge KO[j^{-1}]$ has many idempotents. Hence, the situation is similar to the one of the elliptic cohomology which was defined by Kreck and Stolz in [**KS93**].

4. Witten's map of θ-algebras

Next, we are looking for orientations which induce maps of θ-algebras. For the theory $K[\![q]\!]$ we can do the following: let

$$\theta(q,u) = (1-u)\prod_{n\geq 1}\frac{(1-q^n u)(1-q^n u^{-1})}{(1-q^n)^2}$$

be the normalized θ-function. Then for $x_K = 1-u$ the power series

$$\theta(q,u) \in \pi_* K[\![q, x_K]\!] \cong K^*\mathbb{C}P^\infty[\![q]\!]$$

defines a new orientation t of $K[\![q]\!]$.

LEMMA 4.1. *Let Θ_t be the composite*

$$\pi_0 K \wedge MU \xrightarrow{t_*} \pi_0 K \wedge K[\![q]\!] \longrightarrow \mathcal{T}(\mathbb{Z}_2^\times, \mathbb{Z}_2)[\![q]\!].$$

Then for all $k \in \mathbb{Z}_2^\times$ we have

$$\sum_{i \geq 0} \Theta_t(b_i)(k) x^i = \theta(q, (1-x)^k)/x$$

PROOF. The proof is very close to the one of 3.3.3. So we only give a sketch here. Let z_K be the ordinary K-theoretical Thom class. Then for line bundles z_K is related to the new Thom class z_t by the formula

$$z_t = z_K \theta(q, 1-x_K)/x_K$$

and hence

$$\psi^k(z_t) = \frac{\hat{\theta}^k z_K \theta(q, (1-x_K)^k)}{1-(1-x_K)^k}.$$

This gives

$$\begin{aligned} \sum_{i \geq 0} \Theta_t(b_i)(k) x^i &= \sum_{i \geq 0} \left\langle \beta_i, \frac{\hat{\theta}^k \theta(q, (1-x_K)^k)}{1-(1-x_K)^k} \right\rangle x^i \\ &= \theta(q, (1-x)^k)/x. \end{aligned}$$

□

PROPOSITION 4.2. *The t-orientation induces a map of θ-algebras*

$$\pi_* K \wedge MU \longrightarrow \pi_* K \wedge K[\![q]\!].$$

PROOF. We need to show that the diagram

$$\begin{array}{ccc} \pi_* K \wedge MU & \xrightarrow{\psi} & \pi_* K \wedge MU \\ {\scriptstyle t_*}\downarrow & & \downarrow{\scriptstyle t_*} \\ \pi_* K \wedge K[\![q]\!] & \xrightarrow{q \mapsto q^2} & \pi_* K \wedge K[\![q]\!] \end{array}$$

commutes. Let

$$g(x) = \sum_{i=0}^{\infty} b_i x^i \in \pi_0 K \wedge MU[\![x]\!]$$

be the power series considered earlier. Then with 4.1.5 it suffices to check the equality

$$\sum_{i \geq 0} \psi(t_* b_i) x^i = t_*(\frac{g(1+\sqrt{1-x})g(1-\sqrt{1-x})}{g(2)}).$$

With the last lemma this equation reads

$$\frac{\theta(q^2,(1-x)^k)}{x} = \frac{\theta(q,-\sqrt{1-x}^k)\theta(q,\sqrt{1-x}^k)2}{(1+\sqrt{1-x})(1-\sqrt{1-x})\theta(q,-1)}$$

which is easily verified by expanding the θ-series. □

Unfortunately, the t-orientation does not lift to K_{Tate} without further restrictions on the bundles. To see this, recall from [**HBJ92**] or [**Sil98**] the relation of θ to the Weierstraß σ-function

$$\sigma(u,q) = e^{-G_2 x^2} u^{-1/2}\theta(u,q)$$

where $u = e^{2\pi i x}$ and

$$G_2 = -B_2/4 + \sum_{n=1}^{\infty}(\sum_{d|n} d)q^n.$$

Moreover, the σ-function admits an expansion of the form

$$\sigma(u,q) = x\ \exp(-\sum_{k=2}^{\infty}\frac{2}{(2k)!}G_{2k}x^{2k})$$

with the divided Eisenstein series

$$G_{2k} = (-B_{2k}/4k)E_{2k}.$$

and hence expands in $mf_*\otimes\mathbb{Q}[\![x]\!]$. We conclude that G_2 is the only term which keeps us from lifting t. Since this term disappears for bundles with vanishing first Pontryagin class p_1 we are lead to the following construction. Let ξ be a real bundle and

$$S_t\xi = \sum_{k=0}^{\infty}(S^k\xi)t^k$$

be the formal sum of its symmetric powers. If ξ is spin we have a Thom class

$$z(\xi) \in \widetilde{KO}\,Th(\xi) \subset \widetilde{KO}\,Th(\xi)[\![q]\!]$$

This Thom class can be altered by any unit. For instance,

$$z_W(\xi) \stackrel{def}{=} z(\xi)\otimes\bigotimes_{k=1}^{\infty}S_{q^k}(\dim\xi-\xi)$$

is another Thom class. Since z_W is natural and multiplicative it defines a ring map

$$W: MSpin \longrightarrow KO[\![q]\!].$$

The map W leads to the so called Witten genus in homotopy

$$W_*: \Omega_{4k}^{Spin} = \pi_{4k} MSpin \longrightarrow \pi_{4k} KO[\![q]\!]$$

which is by the topological Riemann-Roch formula

$$W_*(M) \otimes \mathbb{C} \cong v^{4k} \int_M \hat{A}(TM) ch(\bigotimes_{k=1}^{\infty} S_{q^k}(TM - \dim TM) \otimes \mathbb{C}).$$

LEMMA 4.3. *The diagram*

$$\begin{tikzcd} MSU \arrow[r] \arrow[d] & MSpin \arrow[r, "W"] & KO[\![q]\!] \arrow[d] \\ MU \arrow[rr, "t"] & & K[\![q]\!] \end{tikzcd}$$

commutes.

PROOF. This is readily verified with the Chern character (compare [**HBJ92**].) $\square$

In particular, our calculation in Chern classes shows that W lifts to a map

$$MO\langle 8\rangle \cong MSpin \longrightarrow KO_{Tate}$$

which we denote by the same letter.

LEMMA 4.4. *The map*

$$W_*: \pi_0 KO \wedge MO\langle 8\rangle \longrightarrow \pi_0 KO \wedge KO_{Tate} \longrightarrow \pi_0 KO_{Tate} \cong \pi_0 KO \wedge tmf$$

is a map of θ-algebras. Moreover, the $K(1)$-local Witten genus

$$W: \pi_0 MO\langle 8\rangle \longrightarrow D$$

lifts to $\pi_0 tmf$.

PROOF. The induced map from MSU to $MSpin$ is surjective in K-homology. Hence the map

$$W_*: \pi_0 KO \wedge MO\langle 8\rangle \longrightarrow \pi_0 KO \wedge KO_{Tate}$$

is a map of θ-algebras by 4.2 and 4.3. This shows the first claim. For the second, observe that the map

$$W: \pi_0 KO \wedge MO\langle 8\rangle \longrightarrow \pi_0 KO \wedge tmf \cong D$$

is compatible with the Adams operations. $\square$

REMARK 4.5. We will soon see that there is a ($K(1)$-local) $O\langle 8\rangle$-manifold with Witten genus j^{-1}. Hence, the lemma shows the existence of the Atkin-Lehner operator, or, equivalently, the 2-adic convergence of the series l.

5. The E_∞-cellular decomposition of the Witten map

In this section the Witten orientation becomes a map of E_∞-ring spectra. For that, an explicit lift of the Artin-Schreier class of tmf to $MO\langle 8\rangle$ is needed. This requires some geometry.

The ABP-splitting map induces an isomorphism

$$(\hat{A}, \pi^{(2)}) : \pi_8 MSpin \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Hence there are 8-dimensional Spin manifolds with $\hat{A}$-genus 1. We want to make a convenient choice of such a Bott manifold. The quaternian projective space $\mathbb{H}P^2$ admits a metric of positive scalar curvature and so its $\hat{A}$-genus vanishes. Hence any Bott manifold can be altered by a multiple of $\mathbb{H}P^2$. Since the signature $\operatorname{sig}(\mathbb{H}P^2)$ is 1 we can find a Bott manifold $\mathfrak{B}$ with vanishing signature.

To make $\mathfrak{B}$ explicit, consider the Kummer surface

$$\mathfrak{K} = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}P^3 |\, p(z) = 0\};\ p(z) = \sum_{i=0}^{3} z_i^4.$$

The homogeneous polynomial p defines a section in $(L^*)^4$ which transversally intersects the 0-section in $\mathfrak{K}$. Hence this line bundle gives the normal bundle of $\mathfrak{K}$ in $\mathbb{C}P^3$. The surface $\mathfrak{K}$ intersects $[\mathbb{C}P^1]$ in 4 points. Thus the image of its fundamental class $[\mathfrak{K}]$ is $4\beta_2 \in H_4\mathbb{C}P^3$ under the map induced by the inclusion. With these data it is elementary to check that

$$\hat{A}(\mathfrak{K}) = 2,\ \operatorname{sig}(\mathfrak{K}) = -16.$$

This gives the relation

$$\mathfrak{B} = \mathfrak{K}^2/4 - 64\,\mathbb{H}P^2.$$

The following observation the author learned from Matthias Kreck.

PROPOSITION 5.1. *The sequence*

$$\pi_8 MO\langle 8\rangle \longrightarrow \pi_8 MSpin \xrightarrow{p_1^2} \mathbb{Z}$$

is exact. Moreover, any $O\langle 8\rangle$-manifold is spin bordant to a multiple of

$$\mathfrak{M} \overset{def}{=} \mathfrak{B} - 224\,\mathbb{H}P^2.$$

PROOF. It is elementary to check that

$$p_1^2(\mathfrak{K}^2) = 4608;\; p_1^2(\mathbb{H}P^2) = 4$$

and hence

$$p_1^2(\mathfrak{M}) = 0;\; \mathrm{sig}(\mathfrak{M}) = -224.$$

Kervaire and Milnor [**KM63**] showed the existence of an almost parallelizable manifold $\mathfrak{M}'$ with signature 224. Hence, $\mathfrak{M}$ is spin bordant to an $O\langle 8\rangle$-manifold. Since its $\hat{A}$-genus is 1 any other class in the kernel must be a multiple of $\mathfrak{M}$ and the claim follows. □

PROPOSITION 5.2. *The class*

$$b^W = -\frac{\log(\mathfrak{M}/v^4)}{\log(81)} \in \pi_0 KO \wedge MSpin$$

is an Artin-Schreier element.

We first need to show that the quotient is well defined.

LEMMA 5.3. *The class $v^4 - \mathfrak{M}$ is divisible by 16 in $\pi_8 KO \wedge MSpin$.*

PROOF. Compute for all $\lambda \in \mathbb{Z}_2^\times$

$$\psi^\lambda \mu\, \pi_*^\emptyset (v^4 - \mathfrak{M}) = (\lambda^4 - 1)\, v^4 \equiv 0 \bmod 16.$$

and

$$\psi^\lambda \mu\, \pi_*^{(2)} (v^4 - \mathfrak{M}) = \pi^{(2)}(\mathfrak{M}).$$

To evaluate the latter we compute with the topological Riemann-Roch formula

$$\begin{aligned} \pi^{(2)}(\mathfrak{K}^2) &= \left\langle \pi^{(2)}(\nu_{\mathfrak{K}^2}), [\mathfrak{K}^2]_{KO} \right\rangle \\ &= \left\langle \hat{A}(T\mathfrak{K}^2) ch(\pi^{(2)}(\nu_{\mathfrak{K}^2}) \otimes \mathbb{C}), [\mathfrak{K}^2] \right\rangle = 2304. \end{aligned}$$

This gives

$$\pi^{(2)}(\mathfrak{M}) \equiv 576 \equiv 0 \bmod 16.$$

Hence, the assertion follows from the ABP-splitting and Adams' description of $\pi_* KO \wedge KO$ [**Ada74**]. □

PROOF OF 5.2: Since the quotient is well defined by 5.3 we can compute as before

$$\psi^3 b^W = -\frac{\log(\mathfrak{M}/(81v^4))}{\log(81)} = -\frac{\log(\mathfrak{M}/v^4)}{\log(81)} + \frac{\log(81)}{\log(81)} = b + 1.$$

□

REMARK 5.4. Since the Kummer surface admits an SU-structure on its stable tangent bundle one might hope to obtain an Artin-Schreier class in $\pi_0 KO \wedge MSU$ this way. For that, consider $\mathfrak{K}$ as an object of

$$\pi_* K \wedge MSU \subset \pi_* K \wedge MU \cong \pi_* K[b_1, b_2, \dots].$$

To determine the polynomial we can first work out its Chern numbers to obtain

$$\mathfrak{K} = 18(\mathbb{C}P^1)^2 - 16\mathbb{C}P^2.$$

Then Miscenko's formula gives

$$\sum_{n=0}^{\infty} \frac{\mathbb{C}P^n}{n+1} b(x)^{n+1} = \log_{\hat{G}_u}(b(x)) = \log_{\hat{G}_m}(x) = -\frac{\log(1-vy)}{v}.$$

A comparison of coefficients furnishes the equality

$$\mathfrak{K} = 48b_2 - 24b_1^2 - 24vb_1 + 2v^2 = -24a_{1,1} + 2v^2 \in \pi_4 K \wedge MSU.$$

Hence $v^4 - \mathfrak{K}^2/4$ is divisible by 8 and so

$$\mathfrak{M} \equiv v^4 \bmod 8 \ \in \pi_8 K \wedge MSU.$$

Thus we can set up the logarithm and lift the Artin-Schreier class to MSU. As a consequence of the splitting formula of $MSpin$ and the lemma below we have

COROLLARY 5.5. *T_ζ is a direct E_∞-summand in the $K(1)$-local MSU.*

At the prime 2 the theory MSU additively splits into a sum of suspensions of Pengelley's indecomposable theories BoPs [**Pen82**] and BPs. One summand BoP contains the unit and comes with a map into ko which is surjective in homotopy. It would be interesting to know the precise relationship between the $K(1)$-local BoP and the theory T_ζ.

LEMMA 5.6. *Let b and b' be two Artin Schreier elements of $\pi_0 KO \wedge MSpin$. Then there is an E_∞-self homotopy equivalence κ of $MSpin$ which carries b to b'.*

PROOF. We may assume that b is the Artin-Schreier element considered earlier. The short exact sequence

$$0 \longrightarrow \pi_0 MSpin \longrightarrow \pi_0 KO \wedge MSpin \xrightarrow{\psi^3-1} \pi_0 KO \wedge MSpin \longrightarrow 0$$

tells us that b and b' can only differ by a class $a \in \pi_0 MSpin$. Let κ be the E_∞-self map of

$$MSpin \cong T_\zeta \wedge \bigwedge TS^0$$

which is the identity on each TS^0 and restricts to

$$\iota + a\,\delta : C_\zeta \longrightarrow MSpin$$

on T_ζ. Then its inverse is defined in the same way with a replaced by $-a$. □

COROLLARY 5.7. *The Witten orientation can be realized as an E_∞-ring map*

$$W : MO\langle 8\rangle \longrightarrow tmf.$$

Moreover, there is a choice of classes $z_3, z_5, \ldots \in \pi_0 MO\langle 8\rangle$ such that the diagram

$$\begin{tikzcd} \bigwedge_{i=1}^\infty TS^0 \arrow[r, "*"] \arrow[d, "{(y,z_3,z_5,\ldots)}"'] & T* \arrow[d] \\ MO\langle 8\rangle \arrow[r, "W"] & tmf \end{tikzcd}$$

is a homotopy pushout of $K(1)$-local E_∞-ring spectra.

PROOF. The lemma gives a splitting

$$(\tilde{\varphi}, (\tilde{z}_k)) : T_\zeta \wedge \bigwedge TS^0 \xrightarrow{\cong} MO\langle 8\rangle$$

for which the Artin-Schreier class $b \in \pi_0 KO \wedge T_\zeta$ maps to b^W. Let $d_k \in \pi_0 tmf$ be the images of the free generators under the Witten genus. Consider the E_∞-map

$$MO\langle 8\rangle \cong T_\zeta \wedge \bigwedge TS^0 \longrightarrow tmf$$

which is the canonical map on T_ζ and is given by d_k on the free components. Its induced map of θ-algebras in KO-homology coincides with the Witten map on the generators. We have seen in 4.4 that W gives a map of θ-algebras and so the two coincide. Since any such map is determined by its behaviour in KO-homology (compare 2.2) we proved the first claim.

For the second one, observe that W induces a surjection by 3.3

$$\tilde{\varphi}_* : \pi_0 T_\zeta \longrightarrow \pi_0 MO\langle 8\rangle \xrightarrow{W_*} \pi_0 tmf.$$

Hence, there are classes $t_k \in \pi_0 T_\zeta$ with the property

$$\tilde{\varphi}_*(t_k) = W_*(\tilde{z}_k)$$

and the classes

$$z_k \stackrel{def}{=} \tilde{z}_k - t_k \in \pi_0 MO\langle 8\rangle$$

are annihilated by the Witten genus. Thus the claim follows from 3.4 once we have shown that the self map

$$(id, (z_k)) : T_\zeta \wedge \bigwedge TS^0 \longrightarrow T_\zeta \wedge \bigwedge TS^0$$

is an isomorphism. This is clear since its inverse is given by $(id, (\tilde{z}_k + t_k))$. $\square$

COROLLARY 5.8. *For all spectra X the natural map*

$$\pi_* MO\langle 8\rangle \wedge X \otimes_{\pi_* MO\langle 8\rangle} \pi_* tmf \longrightarrow \pi_* tmf \wedge X$$

induced by the Witten orientation W is an isomorphism.

PROOF. We have by 6.3.1, 3.3 and 5.7

$$\begin{aligned} &\pi_* MO\langle 8\rangle \wedge X \otimes_{\pi_* MO\langle 8\rangle} \pi_* tmf \\ &\quad\cong\quad \pi_* KO \wedge X \otimes T\{f, z_3, z_5, \dots\} \otimes_{T\{f, z_3, z_5, \dots\}} T\{f\}/\{\theta(f) - h(f)\} \\ &\quad\cong\quad \pi_* KO \wedge X[f] \end{aligned}$$

Hence it is homology theory. $\square$

REMARK 5.9. One may ask if the isomorphism of Conner-Floyd type still holds if $MO\langle 8\rangle$ is not localized. For that, one needs to know if $\mathfrak{M}^{-1} MO\langle 8\rangle$ is K-local. This question will be investigated somewhere else.

APPENDIX A

The homology ring of BSU and 2-structures on formal groups

Let L be the canonical line bundle over $\mathbb{C}P^\infty$ and let $\beta_i \in E_{2i}\mathbb{C}P^\infty$ be dual to $c_1(L)^i$. Let

$$f : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow BSU$$

be the map which classifies the product $(1-L_1)(1-L_2)$. For each natural number k and $1 \le i \le k-1$ choose integers n_k^i such that

$$\sum_{i=1}^{k-1} n_k^i \binom{k}{i} \quad = \quad \text{g.c.d.}\{\binom{k}{1}, \ldots, \binom{k}{k-1}\}. \tag{1}$$

Then we will show

THEOREM 1.10. *Define elements*

$$d_k \quad = \quad \sum_{i=1}^{k-1} n_k^i f_*(\beta_i \otimes \beta_{k-i}) \in E_{2k}BSU. \tag{2}$$

Then for any complex oriented E we have

$$E_*BSU \quad \cong \quad \pi_*E[d_2, d_3, d_4, \ldots]. \tag{3}$$

Recall from [**Ada74**] that for any complex oriented ring theory E we are given a class $x \in \tilde{E}^2\mathbb{C}P^\infty$ such that

$$E^*\mathbb{C}P^\infty \cong \pi^*E[\![x]\!].$$

The H-space structure of $BS^1 \cong \mathbb{C}P^\infty$ induces a comultiplication

$$\mu^* : E^*\mathbb{C}P^\infty \longrightarrow E^*\mathbb{C}P^\infty \hat{\otimes} E^*\mathbb{C}P^\infty;\ x \mapsto x +_F y$$

and a ring structure map

$$E_*\mathbb{C}P^\infty \otimes E_*\mathbb{C}P^\infty \longrightarrow E_*\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\mu_*} E_*\mathbb{C}P^\infty.$$

Let $\beta_0 = 1, \beta_1, \beta_2, \ldots$ be the additive basis of $E_*\mathbb{C}P^\infty$ dual to $x^0, x^1, x^2, \ldots$.

DEFINITION 1.11. *The binomial coefficients of the formal group law F*

$$\binom{k}{i,j}_F \in \pi_{2(i+j-k)}E$$

are defined by the equation

$$(x +_F y)^k = \sum_{i,j} \binom{k}{i,j}_F x^i y^j.$$

With this notation we easily see

LEMMA 1.12.

$$\beta_i \beta_j = \sum_{k=0}^{i+j} \binom{k}{i,j}_F \beta_k.$$

EXAMPLE 1.13. Let E be integral singular homology. Then F is the additive formal group law $\hat{G}_a$ and

$$\binom{k}{i,j}_{\hat{G}_a} = \begin{cases} \binom{k}{i} & \text{if } i+j=k \\ 0 & \text{else} \end{cases} .$$

Hence $H\mathbb{Z}_*\mathbb{C}P^\infty$ is the divided power algebra $\Gamma[\beta_1]$.

Next let E be K-theory with its standard orientation $F = \hat{G}_m$. Then

$$(x +_{\hat{G}_m} y)^k = (x + y - v^{-1}xy)^k = \sum_{s=0}^{k} \sum_{t=0}^{s} \binom{k}{s} \binom{s}{t} (-v)^{s-k} x^{k-s+t} y^{k-t}$$

and hence

$$\binom{k}{i,j}_{\hat{G}_m} = \binom{k}{2k-i-j} \binom{2k-i-j}{k-j} (-v)^{k-i-j}$$

Finally, let E be complex bordism MU. The coefficients of the universal formal group law FGL are the Milnor hypersurfaces $H_{i,j}$ of type $(1,1)$ in $\mathbb{C}P^i \times \mathbb{C}P^j$ and hence

$$\binom{k}{i,j}_{FGL} = \sum_{\substack{i_1 \cdots + i_k = i \\ j_1 + \cdots j_k = j}} \prod_{l=1}^{k} H_{i_l, j_l}.$$

In the following let $f : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow BSU$ be the map which classifies $(1 - L_1)(1 - L_2)$. Even though f is not a map of H-spaces we may use it to produce interesting classes in E_*BSU. Let $a_{ij} \in E_{2(i+j)}BSU$ be the image of $\beta_i \otimes \beta_j$ under the induced map f_*.

LEMMA 1.14. *The following relations hold for all* i, j, k

$$a_{0,0} = 1 \quad ; \quad a_{0i} = a_{i0} = 0 \text{ for all } i \neq 0 \tag{4}$$

$$a_{ij} = a_{ji} \tag{5}$$

$$\sum_{l,s,t} \binom{l}{s,t}_F a_{j-s,k-t} a_{il} = \sum_{l,s,t} \binom{l}{s,t}_F a_{lk} a_{i-s,j-t} \tag{6}$$

PROOF. The first relation is obvious. Let τ be the self map of $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ which switches the two factors. Then the second relation immedately follows from the fact that $f\tau$ is homotopic to f. To do the last consider the two maps g, h from $(\mathbb{C}P^\infty)^{\times 3}$ to $(\mathbb{C}P^\infty)^{\times 4}$ given by

$$\begin{aligned} g(x, y, z) &= (y, z, x, y\, z) \\ h(x, y, z) &= (x\, y, z, x, y) \end{aligned}$$

Their effect on our generators is

$$g_*(\beta_i \otimes \beta_j \otimes \beta_k) = \sum_{l,s,t} \binom{l}{s,t}_F \beta_{j-s} \otimes \beta_{k-t} \otimes \beta_i \otimes \beta_l \tag{7}$$

$$h_*(\beta_i \otimes \beta_j \otimes \beta_k) = \sum_{l,s,t} \binom{l}{s,t}_F \beta_l \otimes \beta_k \otimes \beta_{i-s} \otimes \beta_{j-t} \tag{8}$$

This can be verified by pairing the left hand side with the cohomological monomials in the the x_i's.The maps g and h become homotopic in BSU when composed with $\mu(f \times f)$ since

$$\begin{aligned} (\mu(f \times f)g)^* \xi_{univ} &= (1 - L_2)(1 - L_3) + (1 - L_1)(1 - L_2 L_3) \\ &= (1 - L_1 L_2)(1 - L_3) + (1 - L_1)(1 - L_2) = (\mu(f \times f)h)^* \xi_{univ} \end{aligned}$$

The desired relation now follows from the above by applying $\mu(f \times f)_*$ to the right hand side of (7) and (8). $\square$

There is another way to look at the classes a_{ij} and the relations of 1.14. First note that $E \wedge BSU_+$ is itself a complex oriented ring theory with

$$x_{E \wedge BSU_+} = (1 \wedge \eta)_* x_E$$

In abuse of the notation we will simply denote this orientation by x in the following. Hence, we may view

$$(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+ \xrightarrow{f_+} BSU_+ \xrightarrow{\eta \wedge 1} E \wedge BSU_+$$

as a power series

$$f(x,y) = 1 + \sum_{i,j\geq 1} b_{ij} x^i y^j \in (E \wedge BSU_+)^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

for some $b_{ij} \in E_{2(i+j)}BSU$. Of course, we have

$$\begin{aligned} b_{ij} &= \sum_{k,l} b_{ij}(1\wedge\eta)_* \left\langle \beta_i \otimes \beta_j, x^k y^l \right\rangle \\ &= \left\langle (1\wedge\eta)_*\beta_i \otimes (1\wedge\eta)_*\beta_j, 1 + \sum_{k,l} b_{kl} x^k y^l \right\rangle \\ &= \langle (1\wedge\eta)_*(\beta_i\otimes\beta_j), f^*(\eta\wedge 1)\rangle = (\mu f(1\wedge\eta))_*(\beta_i\otimes\beta_j) = a_{ij}. \end{aligned}$$

The power series f is a 2-structure on the fomal group law F in the sense of [**HAS98**]3.1.: This means that the relations

$$f(x,0) = f(0,y) = 1 \tag{9}$$

$$f(x,y) = f(y,x) \tag{10}$$

$$f(y,z)f(x, y +_F z) = f(x +_F y, z)f(x,y). \tag{11}$$

hold. In fact, one easily checks that these are equivalent to 1.14.

Definition 1.15. *For any complex oriented E let $C_2(E)$ be the graded ring freely generated by the a_{ij}'s subject to the relations of 1.14. We wirte α for the canonical map from $C_2(E)$ to E_*BSU.*

We will see below that α is an isomorphism. Equivalently, given any 2-structure f' on the formal group F over an π_*E-algebra S then there is a unique algebra homomorphism

$$\varphi : E_*BSU \longrightarrow S$$

with $\varphi f = f'$. Hence, E_*BSU carries the universal 2-structure on F which is the result of [**HAS98**].

Consider the canonical map of H-spaces $\iota : BSU \longrightarrow BU$. Writing g for the map from $\mathbb{C}P^\infty$ to BU which classifies $1 - L$ we see from the homotopy equivalence

$$\iota + g : BSU \times \mathbb{C}P^\infty \longrightarrow BU$$

that ι is an inclusion in homology. It is well known [**Ada74**] that E_*BU is a polynomial algebra with generators $b_i = g_*\beta_i$. Alternatively, let g' be the map which classifies $L - 1$. Then the classes $g'\beta_i = b'_i$ again give polynomial generators of E_*BU. Since the map $\mu_{BU}(g \times g')\Delta$ is null they are determined by the equation

$$\sum_{i=0}^{\infty} b_i' x^i = (\sum_{i=0}^{\infty} b_i x^i)^{-1}.$$

PROPOSITION 1.16. *We have the formula*

$$\iota_* a_{ij} = \sum_{\substack{s=0,\dots,i;\ t=0,\dots,j \\ k=0,\dots,s+t}} \binom{k}{s,t}_F b_k' b_{i-s} b_{j-t}$$

In particular, modulo decomposables in $\tilde{E}_*BU$

$$\iota_* a_{ij} = \sum_{k=0}^{i+j} \binom{k}{i,j}_F b_k'.$$

PROOF. Decompose f by writing

$$\begin{aligned} f^*\xi_{univ} &= 1 - L_1 - L_2 + L_1L_2 = (L_1L_2 - 1) + (1 - L_1) + (1 - L_2) \\ &= (g'\mu_{\mathbb{C}P^\infty} + g\,p_1 + g\,p_2)^*\xi_{univ} \end{aligned}$$

and calculate

$$\begin{aligned} \iota_* a_{ij} &= \iota_* f_* (\beta_i \otimes \beta_j) \\ &= \mu_*^{BU}(g'\mu_*^{\mathbb{C}P^\infty} \otimes g\,p_{1*} \otimes g\,p_{2*}) \sum_{\substack{i_1+i_2+i_3=i \\ j_1+j_2+j_3=j}} \beta_{i_1} \otimes \beta_{j_1} \otimes \beta_{i_2} \otimes \beta_{j_2} \otimes \beta_{i_3} \otimes \beta_{j_3} \\ &= \sum_{\substack{i_1+i_2=i \\ j_1+j_3=j}} \sum_{k=0}^{i_1+j_1} \binom{k}{i_1, j_2}_F b_k' b_{i_2} b_{j_3} \end{aligned}$$

□

Now choose n_k^i and d_k as in 1.10 and set

$$\epsilon(k) \stackrel{def}{=} \sum_{i=1}^{k-1} n_k^i \binom{k}{i} = \text{g.c.d.} \left\{ \binom{k}{1}, \dots, \binom{k}{k-1} \right\} = \begin{cases} p & \text{for } k = p^s \\ 1 & \text{else} \end{cases}.$$

For a graded ring R we write R_+ for the elements in positive degrees and

$$Q(R) = R/R_+^2.$$

COROLLARY 1.17. *In $Q(E_*BU)$ we have $\iota_* d_k = \epsilon(k) b_k'$.*

PROOF. Using the identity

$$\binom{s+t}{s,t}_F = \binom{s+t}{s}$$

we compute with the proposition

$$\iota_* d_k = \sum_{i+j=k} n_k^i \iota_* a_{ij} = \sum_{i=1}^{k-1} n_k^i \binom{k}{i} b'_k = \epsilon(k) b'_k.$$

□

Before proving 1.10 we need two more lemmas.

LEMMA 1.18. *For all s, t we have*

$$a_{st} = \frac{\binom{s+t}{s}}{\epsilon(s+t)} d_{s+t} \in Q_{2(s+t)}(C_2(E)).$$

PROOF. The third relation of 1.14 reads modulo $C_2(E)_+^2$

$$\binom{n}{n-t} a_{mn} = \binom{s}{m} a_{st}$$

for all $m+n = s+t,\ m \leq s$. We conclude

$$\frac{\binom{s+t}{s}}{\epsilon(s+t)} d_{s+t} = \sum_{m+n=s+t} n_{s+t}^m \frac{\binom{s+t}{s}}{\epsilon(s+t)} a_{mn} = a_{st} \sum_{m+n=s+t} n_{s+t}^m \frac{\binom{s+t}{m}}{\epsilon(s+t)} = a_{st}$$

□

LEMMA 1.19. (i) *For all $s \geq 0$ and prime numbers p the map*

$$Q_{2p^s} \iota_* : Q_{2p^s}(H_*(BSU; \mathbb{Z}_p)) \longrightarrow Q_{2p^s}(H_*(BU; \mathbb{Z}_p))$$

is null.

(ii) *Let ρ_t denote the Poincaré series of a graded vector space. Then we have*

$$\rho_t(Q(H_*(BSU; \mathbb{Z}_p)) = (1-t^2)^{-1} - t^2.$$

PROOF. (i) For a Hopf algebra A let us write $P(A)$ for the group of primitives. It is enough to show the dual statement that the map

$$P_{2p^s} \iota^* : P_{2p^s}(H^*(BU; \mathbb{Z}_p)) \longrightarrow P_{2p^s}(H^*(BSU; \mathbb{Z}_p))$$

vanishes. The p^s-power of the first Chern class generates the source since

$$c_1^{p^s}(\xi \oplus \eta) = c_1^{p^s}(\xi) + c_1^{p^s}(\eta)$$

and the dual is one dimensional. This class obviously vanishes in $H^*(BSU; \mathbb{Z}_p)$.

(ii) As in [**Sin68**] 1.4 and 1.5 one sees

$$\begin{aligned}\rho_t(Q(H_*(BSU;\mathbb{Z}_p))) &= \rho_t(P(H_*(BSU;\mathbb{Z}_p))) = \rho_t(Q(H^*(BSU;\mathbb{Z}_p))) \\ &= \rho_t(Q(\mathbb{Z}_p[c_2,c_3,\dots])) = (1-t^2)^{-1} - t^2\end{aligned}$$

□

PROOF OF 1.10. The proof will fall into several steps: First consider the case when E is rational ordinary homology. Then by 1.17 the composite

$$\mathbb{Q}[d_2,d_3,\dots] \longrightarrow H_*(BSU,\mathbb{Q}) \xrightarrow{\iota_*} H_*(BU,\mathbb{Q}) \longrightarrow \mathbb{Q}[b_1',b_2',\dots]/b_1'$$

is a surjection and consequently is an isomorphism. Thus there cannot be any relation between the monomials in the d_i's and the statement follows from the homotopy equivalence

$$\iota + g : BSU \times \mathbb{C}P^\infty \cong BU.$$

by counting dimensions in each degree.

Next observe that the class d_k must generate $Q_{2k}(H_*(BSU;\mathbb{Z}_p))$: by 1.19(i) this vector space is one dimensional for any prime p. Pick a generator e of the latter. Then a multiple n of e coincides with d_k. If k is not a prime power the integer n is invertible since the element d_k is sent to the generator b_k under the map to BU. For prime power degrees the integer n again can not be a multiple of p since else e is mapped to b_k which contradicts 1.19 (ii). In particular, we have shown that the canonical map

$$\mathbb{Z}_p[d_2,d_3,\dots] \longrightarrow H_*(BSU;\mathbb{Z}_p)$$

is a surjection which in turn means that it is an isomorphism. The theorem now holds for integral singular homology.

Next let E be complex bordism MU. Since the Atiyah Hirzebruch spectral sequence collapses we may choose an isomorphism of π_*MU-modules

$$MU_*BSU \cong E_\infty = E_2 = H_*(BSU;\pi_*MU).$$

It is enough to show that a monomial in the d_k's reduces to the corresponding monomial on the 0-line of the E_2-term $H_*(BSU;\mathbb{Z})$ since then the canonical map

$$\pi_*MU[d_2,d_3,\dots] \xrightarrow{\cong} MU_*BSU$$

is an isomorphism. This follows from the fact that the map induced from the complex orientation from MU_*BSU to $H_*(BSU;\mathbb{Z})$ respects the d_k's and is the projection onto the 0 line of the spectral sequence.

Finally, for arbitrary complex oriented E we may simply tensor the isomorphism

$$\pi_*MU[d_2, d_3, \dots] \xrightarrow{\cong} MU_*BSU$$

with π_*E and the result follows from the universal coefficients spectral sequence [**Ada69**]. □

Bibliography

[ABP66] D.W. Anderson, Edgar H.jun. Brown, and Franklin P. Peterson, *Spin cobordism*, Bull. Am. Math. Soc. **72** (1966), 256–260.

[Ada69] J.F. Adams, *Lectures on generalised cohomology*, Category Theory Homology Theory Appl., Proc. Conf. Seattle Res. Center Battelle Mem. Inst. 1968, 3, 1-138, 1969.

[Ada74] J.F. Adams, *Stable homotopy and generalised homology.*, Chicago Lectures in Mathematics. Chicago - London: The University of Chicago Press. X, 373, 1974.

[AL70] A.O.L. Atkin and J. Lehner, *Hecke operators on* $\Gamma_0(m)$, Math. Ann. **185** (1970), 134–160.

[And64] D.W. Anderson, *The real K-theory of classifying spaces*, Proc. Natl. Acad. Sci. USA **51** (1964), 634–636 (English).

[And83] D.W. Anderson, *There are no phantom cohomology operations in K-theory.*, Pac. J. Math. **107** (1983), 279–306.

[BMMS86] R.R. Bruner, J.P. May, J.E. McClure, and M. Steinberger, H_∞ *ring spectra and their applications.*, Lecture Notes in Mathematics, 1176. Berlin etc.: Springer-Verlag. VII, 388 p., 1986.

[Bot69] Raoul Bott, *Lectures on* $K(X)$, New York-Amsterdam: W.A. Benjamin, Inc. X, 203 p., 1969.

[Bou79] A.K. Bousfield, *The localization of spectra with respect to homology.*, Topology **18** (1979), 257–281.

[Bou96a] A.K. Bousfield, *On* λ*-rings and the* K*-theory of infinite loop spaces.*, K-Theory **10** (1996), no. No.1, 1–30.

[Bou96b] A.K. Bousfield, *On* p*-adic* λ*-rings and the* K*-theory of* H*-spaces.*, Math. Z. **223** (1996), no. 3, 483–519.

[Bou99] A.K. Bousfield, *The* K*-theory localizations and* v_1*-periodic homotopy groups of H-spaces.*, Topology **38** (1999), no. 6, 1239–1264.

[BW85] Michael Barr and Charles Wells, *Toposes, triples and theories.*, Grundlehren der Mathematischen Wissenschaften, 278. New York etc.: Springer-Verlag. XIII, 345, 1985.

[EKMM97] A.D. Elmendorf, I. Kriz, M.A. Mandell, and J.P. May, *Rings, modules, and algebras in stable homotopy theory. With an appendix by M. Cole.*, Mathematical Surveys and Monographs. 47. Providence, RI: American Mathematical Society. xi, 249 p. [ISBN 0-8218-0638-6/hbk], 1997.

[Fra92] Jens Franke, *On the construction of elliptic cohomology.*, Math. Nachr. **158** (1992), 43–65.

[Gou88] Fernando Q. Gouvea, *Arithmetic of p-adic modular forms.*, Lecture Notes in Mathematics, 1304. Berlin etc.: Springer-Verlag. VIII, 121 p., 1988.

[HAS98] M.J. Hopkins, M. Ando, and N.P. Strickland, *Elliptic spectra, the Witten genus and the theorem of the cube*, preprint, 1998.

[HAS99] M.J. Hopkins, M. Ando, and N.P. Strickland, *Elliptic homology of* $BO\langle 8\rangle$ *in positive charachteristic*, preprint, 1999.

[HBJ92] Friedrich Hirzebruch, Thomas Berger, and Rainer Jung, *Manifolds and modular forms. Transl. by Peter S. Landweber.*, Aspects of Mathematics. E 20. Wiesbaden: Friedr. Vieweg. xi, 211 p., 1992.

[HH92] Michael J. Hopkins and Mark A. Hovey, *Spin cobordism determines real K-theory.*, Math. Z. **210** (1992), no. 2, 181–196.

[Hod72] Luke Hodgkin, *The K-theory of some well-known spaces. I:* QS^0*.*, Topology **11** (1972), 371–375.

[Hop95] Michael J. Hopkins, *Topological modular forms, the Witten genus, and the theorem of the cube.*, Chatterji, S. D. (ed.), Proceedings of the international congress of mathematicians, ICM '94, August 3-11, 1994, Zuerich, Switzerland. Vol. I. Basel: Birkhaeuser. 554-565. [ISBN 3-7643-5153-5/hbk], 1995.

[Hop98a] M.J. Hopkins, *Communications with the author*, 1998, Oberwolfach.

[Hop98b] M.J. Hopkins, $K(1)$*-local* E_∞ *ring spectra*, unpublished notes, 1998.

[Hov95] Mark Hovey, *Bousfield localization functors and Hopkins' chromatic splitting conjecture.*, Cenkl, Mila (ed.) et al., The Cech centennial. A conference on homotopy theory dedicated to Eduard Cech on the occasion of his 100th birthday, June 22-26, 1993, Northeastern University, Boston, MA, USA. Providence, RI: American Mathematical Society. Contemp. Math. 181, 225-250 [ISBN 0-8218-0296-8/pbk], 1995.

[HS75] L. Hodgkin and V. Snaith, *Topics in K-theory. L. Hodgkin: The equivariant Kuenneth theorem in K- theory. V. Snaith: Dyer-Lashof operations in K-theory.*, Lecture Notes in Mathematics. 496. Berlin - Heidelberg - New York: Springer-Verlag. 294 p., 1975.

[HSS00] Mark Hovey, Brooke Shipley, and Jeff Smith, *Symmetric spectra.*, J. Am. Math. Soc. **13** (2000), no. 1, 149–208.

[Joa01] Michael Joachim, *A symmetric ring spectrum representing* KO*-theory.*, Topology **40** (2001), 299–308.

[Kat75] Nicholas M. Katz, *Higher congruences between modular forms.*, Ann. Math., II. Ser. **101** (1975), 332–367.

[Kat77] Nicholas M. Katz, *The Eisenstein measure and p-adic interpolation.*, Am. J. Math. **99** (1977), 238–311.

[KM63] Michel A. Kervaire and John W. Milnor, *Groups of homotopy spheres. I.*, Ann. of Math. II. Ser. **77** (1963), 504–537.

[KS93] Matthias Kreck and Stephan Stolz, $\mathbf{HP}^2$*-bundles and elliptic homology.*, Acta Math. **171** (1993), no. 2, 231–261.

[Lau93] Gerd Laures, *Parametrisierte Spektren*, Forschergruppe: Topologie und nicht kommutative Geometrie **Nr. 69** (1993), 1–35.

[Lau99] Gerd Laures, *The topological q-expansion principle.*, Topology **38** (1999), no. 2, 387–425.

[Lau00] Gerd Laures, *Characteristic numbers from 2-structures on formal groups.*, to appear, 2000.

[LMS86] L.G.jun. Lewis, J.P. May, and M. Steinberger, *Equivariant stable homotopy theory. With contributions by J. E. McClure.*, Lecture Notes in Mathematics, 1213. Berlin etc.: Springer-Verlag. IX, 538 p., 1986.

[LRS95] Peter S. Landweber, Douglas C. Ravenel, and Robert E. Stong, *Periodic cohomology theories defined by elliptic curves.*, Cenkl, Mila (ed.) et al., The Cech centennial. A conference on homotopy theory dedicated to Eduard Cech on the occasion of his 100th birthday, June 22-26, 1993, Northeastern University, Boston, MA, USA. Providence, RI: American Mathematical Society. Contemp. Math. 181, 317-337. [ISBN 0-8218-0296-8/pbk; ISSN 0271-4132], 1995.

[May77] J.Peter May, *E_∞ ring spaces and E_∞ ring spectra. With contributions by Frank Quinn, Nigel Ray, and Jorgen Tornehave.*, Lecture Notes in Mathematics. 577. Berlin - Heidelberg - New York: Springer-Verlag. 268 p., 1977.

[Mei79] W. Meier, *Complex and real K-theory and localization.*, J. Pure Appl. Algebra **14** (1979), 59–71.

[Pen82] David J. Pengelley, *The homotopy type of MSU.*, Am. J. Math. **104** (1982), 1101–1123.

[Pri75] Stewart Priddy, *Dyer-Lashof operations for the classifying spaces of certain matrix groups.*, Q. J. Math., Oxf. II. Ser. **26** (1975), 179–193.

[Qui71] D. Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Adv. Math. **7** (1971), 29–56.

[Rav84] Douglas C. Ravenel, *Localization with respect to certain periodic homology theories.*, Am. J. Math. **106** (1984), 351–414.

[Rav86] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres.*, Pure and Applied Mathematics, Vol. 121. Orlando etc.: Academic Press, Inc. (Harcourt Brace Jovanovich, Publishers). XIX, 413 p., 1986.

[Sch84] W.H. Schikhof, *Ultrametric calculus. An introduction to p-adic analysis.*, Cambridge Studies in Advanced Mathematics, 4. Cambridge etc.: Cambridge University Press. XI, 306 p., 1984.

[Seg88] Graeme Segal, *Elliptic cohomology [after Landweber-Stong, Ochanine, Witten, and others].*, Semin. Bourbaki, 40eme Annee, Vol. 1987/88, Exp. No.695, Asterisque 161-162, 187-201. [ISSN 0303-1179], 1988.

[Sil86] Joseph H. Silverman, *The arithmetic of elliptic curves.*, Graduate Texts in Mathematics, 106. New York etc.: Springer-Verlag. XII, 400 p., 1986.

[Sil98] Joseph H. Silverman, *Advanced topics in the arithmetic of elliptic curves.*, Graduate Texts in Mathematics. 151. New York, NY: Springer-Verlag. xiii, 525 p., 1998.

[Sin68] W.M. Singer, *Connective fiberings over BU and U*, Topology **7** (1968), 271–303.

[Sto94] Stephan Stolz, *Splitting certain MSpin-module spectra.*, Topology **33** (1994), no. 1, 159–180.

[tDKP70] Tammo tom Dieck, K.H. Kamps, and Dieter Puppe, *Homotopietheorie*, Lecture Notes in Mathematics. 157. Berlin-Heidelberg-New York: Springer- Verlag. VI, 265 S., 1970.

[Wit88] Edward Witten, *The index of the Dirac operator in loop space.*, Elliptic curves and modular forms in algebraic topology, Proc. Conf., Princeton/NJ 1986, Lect. Notes Math. 1326, 161-181, 1988.