CHROMATIC CHARACTERISTIC CLASSES IN ORDINARY GROUP COHOMOLOGY

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Abstract. We study a family of subrings, indexed by the natural numbers, of the mod p cohomology of a finite group G. These subrings are based on a family of v_n-periodic complex oriented cohomology theories and are constructed as rings of generalised characteristic classes. We identify the varieties associated to these subrings in terms of colimits over categories of elementary abelian subgroups of G, naturally interpolating between the work of Quillen on $\mathrm{var}(H^*(BG))$, the variety of the whole cohomology ring, and that of Green and Leary on the variety of the Chern subring, $\mathrm{var}(Ch(G))$. Our subrings give rise to a 'chromatic' (co)filtration, which has both topological and algebraic definitions, of $\mathrm{var}(H^*(BG))$ whose final quotient is the variety $\mathrm{var}(Ch(G))$.

Introduction

This paper develops a structure within the mod p cohomology ring $H^*(BG) = H^*(BG; \mathbb{F}_p)$ of a finite group G which we show to arise both topologically and algebraically. In general, the determination and description of the ring $H^*(BG)$ is a hard problem and we build upon two separate approaches, due respectively to Quillen [18] and to Thomas [22], incorporating also ideas from the chromatic point of view in homotopy theory (in particular [10]) and the theory of coalgebraic rings (Hopf rings) and the homology of infinite loop spaces [8, 11, 12, 19, 23]. We utilise the work of Leary and the first author [9] on categories associated to subrings of $H^*(BG)$, which first hinted at the possibility of a theorem such as our result (0.1) below.

Recall that Quillen showed [18] that the ring $H^*(BG)$ could be described up to F-isomorphism in terms of its elementary abelian subgroup structure. For its most elegant formulation, suppose k is an algebraically closed field of characteristic p and for an $\mathbb{F}_p$ algebra R denote by $\mathrm{var}(R)$ the variety of algebra morphisms from R to k, topologised with the Zariski topology. Quillen's result describes $\mathrm{var}(H^*(BG))$ in terms of a colimit $\mathrm{colim}_{\mathcal{A}} \mathrm{var}(H^*(BV))$ over a certain category $\mathcal{A}$ of elementary abelian subgroups V of G. In particular recall that, for such V, the variety $\mathrm{var}(H^*(BV))$ is isomorphic to $V \underset{\mathbb{F}_p}{\otimes} k$ where V is viewed as an $\mathbb{F}_p$ vector space.

On the other hand, Thomas has shown [22] that a frequently significant as well as computable subring of $H^*(BG)$ is the *Chern subring*, $Ch(G)$, essentially that part of the cohomology ring given most immediately by the complex representation theory of G (specifically, it is the subring generated by all Chern classes of irreducible representations). Leary and the first author [9] have completed this picture by describing $\mathrm{var}(Ch(G))$ as $\mathrm{colim}_{\mathcal{A}^{(1)}} \mathrm{var}(H^*(BV))$, the colimit over a further category $\mathcal{A}^{(1)}$ of elementary abelians. The categories $\mathcal{A}$ and $\mathcal{A}^{(1)}$ have the same

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objects but in general they have different morphisms and the natural surjection $\mathrm{var}(H^*(BG)) \twoheadrightarrow \mathrm{var}(Ch(G))$ is not usually a homeomorphism.

However, Atiyah's theorem [2] linking the (completed) representation theory of G with $K^*(BG)$, the complex K-theory of BG, allows a reinterpretation of $Ch(G)$ in terms of (unstable) maps from BG to the spaces in the Ω-spectrum for K-theory: the homology images of such maps can be identified with Chern classes, the generators of the Chern subring. See section 1 for details. Replacing K-theory by any other representable generalised cohomology theory $E^*(-)$, this construction allows for the definition of 'E-type characteristic classes' in $H^*(BG)$ based on the spaces in the Ω-spectrum for E. These generate a subring, $Ch_E(G)$ say, of $H^*(BG)$. In many instances (including, by the work of Bendersky and the second author [8], those when E is Landweber exact [16]) there are inclusions of rings

$$Ch(G) \subset Ch_E(G) \subset H^*(BG).$$

Our main results concern a family of such subrings, defined by a family of cohomology theories E. Specifically, we are interested in the subrings given by a family of v_n-periodic cohomology theories such as the Johnson-Wilson [14] theories $E(n)$, $n \in \mathbb{N}$. However, to obtain our best results we concentrate on the subrings given by their I_n-adic completions $\widehat{E(n)}$, as given by Baker and Würgler [6], and these subrings we denote $Ch_{\widehat{E(n)}}(G)$. The use of complete theories has the advantage of enabling us to apply the results of Hopkins, Kuhn and Ravenel [10]. In section 3 we define certain categories $\mathcal{A}^{(n)}$ of elementary abelian subgroups of G allowing us to prove

Theorem 0.1. *Let G be a finite group and k an algebraically closed field of characteristic p. Then there is a homeomorphism of varieties*

$$\operatorname*{colim}_{\mathcal{A}^{(n)}} \mathrm{var}(H^*(BV)) \longrightarrow \mathrm{var}(Ch_{\widehat{E(n)}}(G))\,.$$

The categories $\mathcal{A}^{(n)}$ appeared in [9] where, as mentioned, the first case $\mathcal{A}^{(1)}$ was proved to represent the classical Chern subring. It was also suggested that there may be a possible link between the $\mathcal{A}^{(n)}$ and certain v_n-periodic spectra; this theorem provides such a link and supplies a topological construction for subrings of $H^*(BG)$ associated to the $\mathcal{A}^{(n)}$.

The same result almost certainly holds on replacing the spectra $\widehat{E(n)}$ by Morava E-theory or in fact by any Landweber exact spectrum satisfying the main hypotheses of [10]; the recent work of Baker and Lazarev [5] supplies the technical constructions needed for adapting the proof given below for $\widehat{E(n)}$ (the essential element being the analogue of the Baker-Würgler tower [7]). It is less clear that the analagous result for $E(n)$ will hold as these spectra fail the completion hypotheses of [10].

A number of basic results on the $Ch_{\widehat{E(n)}}(G)$ can be proved.

Theorem 0.2.
1. *The varieties $\mathrm{var}(Ch_{\widehat{E(n)}}(G))$ form a (co)filtered space*

$$\mathrm{var}(H^*(BG)) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{var}(Ch_{\widehat{E(n+1)}}(G)) \twoheadrightarrow \mathrm{var}(Ch_{\widehat{E(n)}}(G)) \twoheadrightarrow \cdots \twoheadrightarrow k$$

2. *For each $n \geqslant 1$ there is a finite group G for which $\mathrm{var}(Ch_{\widehat{E(n)}}(G))$ is distinct from $\mathrm{var}(Ch_{\widehat{E(n+1)}}(G))$, and hence $Ch_{\widehat{E(n)}}(G)$ from $Ch_{\widehat{E(n+1)}}(G)$;*
3. *For n at least the p-rank of G, $\mathrm{var}(Ch_{\widehat{E(n)}}(G)) = \mathrm{var}(H^*(BG))$;*

4. $\mathrm{var}(Ch_{\widehat{E(1)}}(G)) = \mathrm{var}(Ch(G))$*, the Chern subring of* $H^*(BG)$*;*
5. *For each* n *there is an inclusion of rings* $Ch(G) \subset Ch_{\widehat{E(n)}}(G)$*.*

Given Theorem 0.1, the statements (1) and (3) follow immediately from the definition of the $\mathcal{A}^{(n)}$ while (2) and (4) follow from results of [9]. A more general version of statement (5) is proved in section 2.

For simplicity we shall assume, at least after section 1, that all our rings R, such as $Ch_E(G)$, are defined as subrings of the even degree part of $H^*(BG)$, and hence are commutative. In fact, as noted in [9], this is not restrictive: if $p \neq 2$ the odd degree elements of $H^*(BG)$ are nilpotent and any homomorphism $H^{\mathrm{even}}(BG) \supset R \xrightarrow{f} k$ extends trivially for any odd degree element; if $p = 2$ such a map would extend uniquely on an odd degree element x to the unique square root of $f(x^2)$.

The rest of the paper is organised as follows. We begin by defining our E-type characteristic classes in section 1, linking this construction to that of the classical Chern subring in the case $E = K$. We also introduce here some of the basic notation and language of coalgebraic algebra [12, 19] in order to discuss the homology of spaces in the Ω-spectrum for E.

In section 2 we recall and extend the main result of [9] and prove that, at least for E Landweber exact, $\mathrm{var}(Ch_E(G))$ has a description as a colimit of $\mathrm{var}(H^*(BV))$ as V runs over some category, $\mathcal{A}_E$ say, of elementary abelian subgroups V of G. Here we also demonstrate that for such E there is an inclusion $Ch(G) \subset Ch_E(G)$.

Most of the rest of the paper is devoted to the proof of the Theorem 0.1, *i.e*, to the identification of the category $\mathcal{A}_E$. In section 3 we introduce the cateogry $\mathcal{A}^{(n)}$ together with an intermediate category $\mathcal{C}_E$. We show that if E both is Landweber exact and satisfies the assumptions of the character isomorphism of [10] then it follows fairly easily that $\mathcal{C}_E = \mathcal{A}^{(n)}$. The harder part is to show that $\mathcal{A}_E = \mathcal{C}_E$ and we prove that this will follow if certain properties of the 'unstable Hurewicz homomorphism'

$$\mathcal{H}\colon E^*(X) = [X, \mathbf{E}_*] \to \mathrm{Hom}(H^*(\mathbf{E}_*), H^*(X))$$

can be established. Our most general result, which for $E = \widehat{E(n)}$ specialises to Theorem 0.1, is stated as Theorem 3.4 and the complete set of hypotheses needed on a spectrum E for its application are set out in (3.2).

Proof that these hypotheses are satisfied by $\widehat{E(n)}$ is given in section 4 which examines in detail the homology of the spaces in its Ω-spectrum, building on Baker and Würgler's description [7] of $\widehat{E(n)}$ as a homotopy limit and Wilson's calculation of the Hopf ring for Morava K-theory [23].

We conclude with some example computations in section 5.

The construction of the subrings $Ch_E(G)$ is firmly based in the world of unstable homotopy through the use of the infinite loop spaces representing E-cohomology. We rely extensively on the techniques of coalgebraic algebra for our proofs and computations; see [12, 19] for introductory material on this field. We also rely on the work of Wilson [23] as the major computational input, but we note one more recent aspect of our work. The results of [10] needed in section 3 demand we work with completed spectra E and consequently with the spaces in their Ω-spectra. These are truly huge spaces whose homologies have only recently become accessible; see [8, 11, 13] for the methods of handling such spaces and for the results we use below.

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1. E-type characteristic classes

In this section we define the class of subrings considered and show that the classical Chern subring arises as a special case. Here and in the next section we can take G to be any compact Lie group; later we shall need to restrict to finite groups. We begin with a very general construction.

Definition 1.1. *Suppose $\{X_i\}$ is a family of spaces and $\mathcal{F}G$ is a set of maps of the form $f\colon BG \to X_i$. Let $Ch_{\mathcal{F}G}$ be the subring of $H^*(BG)$ generated by all elements of the form $f^*(x)$ as f runs over the elements of $\mathcal{F}G$ and x over the homogeneous elements of all the $H^*(X_i)$.*

Example 1.2. Take $\{X_i\}$ to be the set containing just one space, BU. Let $\mathcal{F}$ be the set of maps $BG \to BU$ given by irreducible representations $G \to U(n)$ on embedding $U(n)$ in the infinite unitary group U and taking classifying space. The subring in this instance is the classical Chern subring $Ch(G)$ [22].

Remark 1.3. It is not hard to show that the Chern subring, as given in the last example, can equivalently be defined by taking $\mathcal{F}G$ as the maps given by all representations of G, or even as the maps given by all elements of the representation ring $R(G)$. Clearly enlarging $\mathcal{F}G$ will only if anything enlarge the subring of $H^*(BG)$ it defines, so it suffices to show that elements of $R(G)$ give no new elements than are already in $Ch(G)$ as defined in (1.2). For example, if $\rho_1\colon G \to U(n)$ and $\rho_2\colon G \to U(m)$ are two representations, the H-space structure on BU induced by $U(n) \times U(m) \to U(n+m)$ gives, in cohomology, the commutative diagram

$$\begin{tikzcd} H^*(BU) \otimes H^*(BU) \arrow[r, "B\rho_1^* \otimes B\rho_2^*"] & H^*(BG) \otimes H^*(BG) \arrow[d] \\ H^*(BU) \arrow[u] \arrow[r, "B(\rho_1+\rho_2)^*"] & H^*(BG) \end{tikzcd}$$

sending an element $x \in H^*(BU)$ to $\sum B\rho_1^*(x')\cdot B\rho_2^*(x'')$ in $H^*(BG)$ where $\sum x' \otimes x''$ denotes the image of x in $H^*(BU) \otimes H^*(BU)$. Thus the representation $\rho_1 + \rho_2$ fails to give anything in $H^*(BG)$ that was not already in the subring defined by ρ_1 and ρ_2. The rest of the proof is similar.

To set up notation that will be useful later, recall that $H^*(BU)$ is the polynomial ring on generators $c_1, c_2, \ldots$, the universal Chern classes, where $c_i \in H^{2i}(BU)$. The pull-back $B\rho^*(c_i)$ of a representation $\rho\colon G \to U$ is the called the i^{th} Chern class of ρ, also written $c_i(\rho)$, and is an element of $H^{2i}(BG)$.

For the remainder of this paper we assume $E^*(-)$ is a representable cohomology theory on the category of CW complexes with coefficients E_* concentrated in even dimensions. In particular, there is a representing Ω-spectrum with spaces $\mathbf{E}_r$, $r \in \mathbb{Z}$, and, for all CW complexes X, natural equivalences between $E^r(X)$ and the set of homotopy classes of maps from X to $\mathbf{E}_r$. The following is the main definition of this paper.

Definition 1.4. *Let E be as just described and let G be a compact Lie group. Let $\mathcal{F}G$ be the set of all homotopy classes of maps $BG \to \mathbf{E}_{2r}$, allowing all $r \in \mathbb{Z}$. The E-*Chern subring*, $Ch_E(G)$, is defined as the subring $Ch_{\mathcal{F}G}$ of $H^*(BG)$. Also, define an E-*type characteristic class *of G to be an element $f^*(x) \in H^*(BG)$ for some $f\colon BG \to \mathbf{E}_{2r}$ and some homogeneous $x \in H^*(\mathbf{E}_{2r})$.*

Remark 1.5. Note that it does not matter in this definition whether we take the whole spaces $\mathbf{E}_{2r}$ or just the connected components of their basepoints, usually denoted $\mathbf{E}'_{2r}$. This follows since all components of the $\mathbf{E}_{2r}$ are the same, with $\mathbf{E}_{2r} = \pi_0(\mathbf{E}_{2r}) \times \mathbf{E}'_{2r} = E^{2r} \times \mathbf{E}'_{2r}$.

The following result, building on Remark 1.3, shows that this definition includes that of the classical Chern subring.

Proposition 1.6. *Let G be a compact Lie group. Then the K-Chern subring $Ch_K(G)$ is equal to $Ch(G)$, the classical Chern subring.*

Proof. As any element of the representation ring $R(G)$ gives rise to a homotopy class of maps $BG \to BU$, it is immediate by (1.3) and (1.5) that every element of $Ch(G)$ is also an element of $Ch_K(G)$ since the only even graded spaces in the Ω-spectrum for K-theory are of the homotopy type of $\mathbb{Z} \times BU$. However, not every class of map $BG \to BU$ necessarily arises as the image under $B(-)$ of something from the representation ring. Thus we must check further that $Ch_K(G) \subset Ch(G)$.

Recall from the work of Atiyah and Segal [3] (or Atiyah [2] for the case of finite groups) that the homomorphism $\alpha\colon R(G) \to K^0(BG)$ given by sending the virtual representation ρ to the corresponding map $BG \to \mathbb{Z} \times BU$ is continuous with respect to the I-adic topology on $R(G)$, where I is the augmentation ideal, and the filtration topology on $K^0(BG)$ given by

$$F_m K^0(BG) = \ker(K^0(BG) \to K^0(BG^{m-1}))\,,$$

where BG^{m-1} is the $m-1$ skeleton of BG. Moreover, $K^0(BG)$ is a complete ring and α is an isomorphism after I-adic completion of $R(G)$.

Note that the degree m cohomology of BG is determined by its $(m+1)$-skeleton BG^{m+1}, and so in particular the r^{th} Chern class $c_r(g)$ of a map $g\colon BG \to BU$ is determined by its restriction to $BG^{2r+1} \to BU$. It follows that $c_r(g)$ is zero for all $1 \leqslant r \leqslant n-1$ if $g \in F_{2n}K^0(BG)$.

Now consider a map $f\colon BG \to BU$ in $\mathcal{F}G$. As an element of $K^0(BG)$ it is represented by a Cauchy sequence $\rho_1, \rho_2, \ldots$ in $R(G)$. By the Whitney sum formula, the total Chern class of f satisfies $c(f) = c(\rho_i)c(f - B\rho_i)$. Hence if $f - B\rho_i \in F_{2n}K^0(BG)$, then $c_r(f) = c_r(\rho_i)$ for $1 \leqslant r \leqslant n-1$. Thus, for each $r \geqslant 0$, the sequence $c_r(\rho_1), c_r(\rho_2), \ldots$ in $H^{2r}(BG)$ is eventually constant at $c_r(f) = f^*(c_r)$.

As any homogeneous $x \in H^*(BU)$ can involve only finitely many of the c_r, each element $f^*(x) \in H^*(BG)$ can be realised in the form $B\rho^*(x)$ for some virtual representation $\rho \in R(G)$. This completes the proof. □

Remark 1.7. These definitions of E-type characteristic classes can also be made in the integral or p-local rings $H^*(BG;\mathbb{Z})$ and $H^*(BG;\mathbb{Z}_{(p)})$. If $H^*(\mathbf{E}_{2*};\mathbb{Z})$ is torsion free (as is the case when E is Landweber exact, [8]), then the (mod-p)

E-type characteristic classes are just the mod-p reductions of the integral E-type characteristic classes. In general however this need not be the case. As $H^*(BU;\mathbb{Z})$ is torsion free, the integral analogue of (1.6) holds.

Remark 1.8. The reader may wonder as to the restriction in Definition 1.4 to even graded spaces $\mathbf{E}_{2r}$ in the Ω-spectrum for E. In the case of complex K-theory and finite groups G this restriction is vacuous: there are no non-trivial maps from BG to the odd graded spaces [2]. For more general compact groups Proposition 1.6 would however fail without this hypothesis and so (1.4) would not have been the correct extension of the concept of Chern subring. Below we shall restrict to finite G and Landweber exact E; in light of Kriz's examples such as presented in [15], it is conceivable that even here relaxing Definition 1.4 to include all spaces could enlarge the subring it defines, but the effect remains unclear. The theorems below are proved considering the even spaces alone.

Remark 1.9. There is of course no *a priori* reason why the subring $Ch_E(G)$ defined by (1.4) should lie in $H^{\text{even}}(BG)$, since there may well be odd dimensional elements of $H^*(\mathbf{E}_{2r})$ which pass to non-trivial elements of $H^*(BG)$. However, we shall need from the main result of the next section, Theorem 2.7, and onwards that the spectrum E considered is Landweber exact. We can thus appeal to the work of [13] from which it can easily be deduced that whenever E is Landweber exact $H^*(\mathbf{E}_{2r})$ has no odd dimensional elements.

2. Representations by categories of elementary abelians

Quillen proved in [18] that the variety of $H^*(BG)$ can be built from the elementary abelian p-subgroups using conjugacy information. This result was extended in [9] to cover the varieties of certain large subrings of $H^*(BG)$ as well. The object of this section is to show that for a wide range of theories E, the varieties of the E-Chern subrings $Ch_E(G)$ have a similar description. We start by recalling the definitions and the main result of [9], in a slightly sharpened form.

Definition 2.1. *For a compact Lie group G and homogeneously generated subring R of $H^*(BG)$, define the category $\mathcal{C}(R)$ as that with objects the elementary abelian p-subgroups V of G, and morphisms from W to V to be the set of injective group homomorphisms $f\colon W \to V$ satisfying*

$$f^* \operatorname{Res}_V(x) = \operatorname{Res}_W(x) \tag{2.1}$$

for every homogeneous $x \in R$.

Remark 2.2. This is a variation on the definition given in [9] where the condition on morphisms $f\colon W \to V$ was weakened to satisfying equation (2.1) modulo the nilradical. However, for subrings R that are closed under the action of the Steenrod algebra, the two definitions are equivalent. To see this, suppose that f is a morphism in the "modulo nilradical" version of $\mathcal{C}(R)$ with the property that, for some $x \in R$, the class $w = f^* \operatorname{Res}_V(x) - \operatorname{Res}_W(x)$ is a non-zero nilpotent in $H^*(BW)$. For any non-zero homogeneous element of $H^*(BW)$ there is an operation θ in the mod p Steenrod algebra such that $\theta(w)$ is non-nilpotent: this is immediate for $p = 2$; for p odd, see Lemma 2.6.5 of [21]. Naturality of θ implies that

$$f^* \operatorname{Res}_V(\theta x) - \operatorname{Res}_W(\theta x) = \theta(w)\,.$$

As R is closed under the Steenrod algebra, $\theta(x)$ lies in R, contradicting f being a morphism in $\mathcal{C}(R)$.

Definition 2.3. *Let G be a compact Lie group. A virtual representation ρ of G is called* p-regular *if it has positive virtual dimension and restricts to every elementary abelian p-subgroup as a direct sum of copies of the regular representation. A subring R of $H^*(BG)$ is called* large *if it contains the Chern classes of some p-regular representation.*

Theorem 2.4. [9, 6.1] *Let G be a compact Lie group and R a homogenously generated subring of $H^*(BG)$ which is both large and closed under the Steenrod algebra. Then the natural map $R \to \lim_{V \in \mathcal{C}(R)} H^*(BV)$ induces a homeomorphism*

$$\operatorname*{colim}_{V \in \mathcal{C}(R)} \operatorname{var}(H^*(BV)) \to \operatorname{var}(R)\,. \qquad \square$$

The problem, however, is to identify the category $\mathcal{C}(R)$ for a given R.

Example 2.5. The Quillen category $\mathcal{A} = \mathcal{A}(G)$ is the category with objects the elementary abelian p-subgroups of G, and morphisms generated by inclusion and conjugation. It is shown in [9, 9.2] that $\mathcal{A}$ is the category $\mathcal{C}(H^*(BG))$, thus recovering one of the main results of [18]:

$$\operatorname*{colim}_{V \in \mathcal{A}} \operatorname{var}(H^*(BV)) \cong \operatorname{var}(H^*(BG))\,.$$

Example 2.6. It is shown in [9] that the category $\mathcal{A}^{(1)}$ of elementary abelian p-subgroups of G and morphisms the injective homomorphisms $f\colon W \to V$ which take each element $w \in W$ to a conjugate of itself is the category $\mathcal{C}(Ch(G))$ associated to the Chern subring.

We come to the main result of this section. Recall that the class of Landweber exact spectra [16] includes examples such as BP-theory, complex cobordism, the Johnson-Wilson theories $E(n)$ and their I_n-adic completions $\widehat{E(n)}$ as well as Morava E-theory, complex K-theory, various forms of elliptic cohomology [17] and their completions (in particular the completions $(Ell)\widehat{_{\wp}}$ of Baker [4]).

Theorem 2.7. *Let E be a Landweber exact spectrum and let G be a compact Lie group. Then $Ch_E(G)$ is closed under the Steenrod algebra and is large in the sense of* (2.3)*. Hence there is a category $\mathcal{C}(Ch_E(G))$, which we shall write as $\mathcal{A}_E(G)$ or just $\mathcal{A}_E$, such that*

$$\operatorname{var}(Ch_E(G)) \cong \operatorname*{colim}_{V \in \mathcal{A}_E} \operatorname{var}(H^*(BV))\,.$$

Proof. Closure of $Ch_E(G)$ under the Steenrod algebra is immediate from its definition and it suffices to show that it is also large. This follows from the next result and the fact that the Chern subring $Ch(G)$ is large [9]. $\square$

Proposition 2.8. *For E Landweber exact the homomorphism $(c_1^E)^*\colon H^*(\mathbf{E}_2') \to H^*(BU)$ is surjective for all $n \geqslant 1$. Hence for such E there is an inclusion of subrings $Ch(G) \subset Ch_E(G)$ for all compact Lie groups G and in particular $Ch_E(G)$ contains all Chern classes of complex representations of G.*

Here c_1^E is the first E-theory Chern class: recall that

$$E^*(BU) = E^*[\![c_1^E, c_2^E, \dots]\!] \ \text{ with } \ c_i^E \in E^{2i}(BU)\,;$$

thus $c_1^E\colon BU \to \mathbf{E}_2'$ is an element of $\widetilde{E}^2(BU)$ where, as before, $\mathbf{E}_2'$ means the base point component of $\mathbf{E}_2$.

We need to introduce further notation. Recall that any Landweber exact spectrum E is certainly complex oriented. For any complex oriented spectrum, denote by $x^E\colon \mathbb{C}P^\infty \to \mathbf{E}'_2$ in $\widetilde{E}^2(\mathbb{C}P^\infty)$ the orientation map. The definition of c_1^E is not independent of this and in fact $c_1^E\colon BU \to \mathbf{E}'_2$ restricts to the complex orientation x^E on $\mathbb{C}P^\infty$ [1, Part II, 4.3(ii)]. We assume that the orientation x^K has been chosen compatibly so that the diagram

$$\begin{tikzcd} BU \arrow[rr, "c_1^E"] & & \mathbf{E}'_2 \\ & \mathbb{C}P^\infty \arrow[ul, "x^K"] \arrow[ur, "x^E"'] & \end{tikzcd}$$

commutes.

Take basis elements β_n, $n \in \mathbb{N}$, of $H_*(\mathbb{C}P^\infty)$ with $\beta_n \in H_{2n}(\mathbb{C}P^\infty)$, dual to the basis of $H^*(\mathbb{C}P^\infty)$ given by the powers of the orientation x^H. For a complex oriented theory E define the elements $b_n^E \in H_n(\mathbf{E}'_2)$ by $b_n^E = (x^E)_*(\beta_n)$.

Finally, as $E^r(X)$ is an abelian group for any space X, the space $\mathbf{E}_r$ has an H-space product $\mathbf{E}_r \times \mathbf{E}_r \to \mathbf{E}_r$ which leads, in mod p homology, to a product

$$*\colon H_*(\mathbf{E}_r) \otimes H_*(\mathbf{E}_r) \to H_*(\mathbf{E}_r).$$

Lemma 2.9. *Let E be any complex oriented spectrum and let K denote complex K-theory. Then $(c_1^E)_*\left((b_1^K)^{*r}\right) = (b_1^E)^{*r}$ for all r.*

Proof. The diagram above, in homology, says $(c_1^E)_*(b_1^K) = b_1^E$. The lemma follows by noting that $c_1^E \in E^2(BU)$ is a primitive element in the Hopf algebra $E^*(BU)$, and hence represents an (unstable) additive cohomology operation $K^0(-) \to E^2(-)$. Thus the map c_1^E commutes with $*$-products and

$$(c_1^E)_*\left((b_1^K)^{*r}\right) = \left((c_1^E)_*(b_1^K)\right)^{*r} = (b_1^E)^{*r}$$

for all r. □

Proof of Proposition 2.8. By the work of Bendersky and the second author [8], $H_*(\mathbf{E}_{2r})$ is polynomial (under the $*$-product) for any Landweber exact E and $b_1^E \neq 0$. It follows that $(c_1^E)_*(b_1^{K*r}) = (b_1^E)^{*r}$ is never zero. But $c_r^K \in H^{2r}(BU)$ is dual to $(b_1^K)^{*r} \in H_{2r}(BU)$, with respect to the basis $(b_1^K)^{*r_1} * (b_2^K)^{*r_2} * \cdots * (b_m^K)^{*r_m}$. Hence there is an $\eta_r \in H^{2r}(\mathbf{E}'_2)$ such that $(c_1^E)^*(\eta_r) \equiv c_r^K \mod (c_1^K, \ldots, c_{r-1}^K)$. This proves $(c_1^E)^*$ is surjective.

The Chern subring $Ch(G)$ can be generated by elements of the form $B\rho^*(x)$ for some representations $\rho\colon G \to U$ and homogeneous elements $x \in H^*(BU)$. By the first part of the proposition for any such x there is a homogeneous $y \in H^*(\mathbf{E}'_2)$ with $(c_1^E)^*(y) = x$. Hence $B\rho^*(x) = (c_1^E B\rho)^*(y)$ lies in $Ch_E(G)$. □

Remark 2.10. For the theory $\widehat{E(n)}$, in terms of which our main theorem is stated, we could have appealed to [11, 3.11] instead of [8].

3. The category $\mathcal{A}_E$

In this section, under the restriction to finite groups G, we identify the category $\mathcal{A}_E$ for theories E satisfying certain conditions. The identification of $\mathcal{A}_E$ will depend on a positive natural number n associated to E, and thus we begin by defining a family of categories of elementary abelian subgroups of G, indexed by such n.

Definition 3.1. *For $0 \leqslant n \leqslant \infty$, define $\mathcal{A}^{(n)} = \mathcal{A}^{(n)}(G)$ to be the category with objects the elementary abelian p-subgroups of G and morphisms the injective group homomorphisms $f\colon W \to V$ such that*

$$\forall w_1, \dots, w_n \in W \quad \exists g \in G \quad \forall 1 \leqslant i \leqslant n \quad f(w_i) = g w_i g^{-1}\,.$$

In particular, if t is the p-rank of G, there are equivalences of categories

$$\mathcal{A}^{(t)} = \mathcal{A}^{(t+1)} = \cdots = \mathcal{A}^{(\infty)}$$

and this common category is the Quillen category $\mathcal{A}$. As noted in Example 2.6, it is proved in [9, 7.1] that $\mathcal{A}^{(1)}$ is the category $\mathcal{C}(Ch(G))$ of elementary abelians associated to the Chern subring. Moreover, by [9, 9.2], for each $2 \leqslant n < \infty$ there is a subring R which satisfies the conditions of Theorem 2.4 and has category $\mathcal{C}(R) = \mathcal{A}^{(n)}$. The current paper arose from the desire to find a topological construction of such a ring R.

One way to explain why $\mathcal{A}^{(1)}$ is the right category for $Ch_K(G)$ is via group characters. On the one hand, $K^0(BG)$ is a completed ring of (virtual) characters. On the other hand, the morphisms in $\mathcal{A}^{(1)}$ are the group homomorphisms which preserve the values of characters for G. That is, $f\colon W \to V$ lies in $\mathcal{A}^{(1)}$ if and only if it satisfies the equation

$$f^* \operatorname{Res}_V(\chi) = \operatorname{Res}_W(\chi)$$

for every character χ of G.

Switching attention to $\mathcal{A}^{(n)}$ we recall the work of Hopkins, Kuhn and Ravenel [10]. The morphism $f\colon W \to V$ lies in $\mathcal{A}^{(n)}$ if and only if it satisfies the analagous equation for every *generalised* character (class function) χ of G, *i.e.*, for every function on the set $G_{n,p}$ of commuting n-tuples of elements of G having p-power order which is constant on conjugacy classes and takes values in a certain E^*-algebra $L(E^*)$, which is, roughly speaking, the smallest E^*-algebra which contains all roots of each equation of the form $[p^k](x) = 0$: let $E^*_{cont}(B\mathbb{Z}_p^n) = \operatorname{colim}_r E^*(B(\mathbb{Z}/p^r)^n)$ and S the multiplicatively closed subset generated by Euler classes of continuous homomorphisms $\mathbb{Z}_p^n \to S^1$, then $L(E^*) = S^{-1}E^*_{cont}(B\mathbb{Z}_p^n)$.

For suitable complex oriented E, an element $x \in E^*(BG)$ gives rise to such a class function in essentially the following way: a commuting n-tuple $(g_1, \dots, g_n)$ in G as above can be thought of as a homomorphism $\alpha\colon \mathbb{Z}_p^n \to G$. The value of a generalised character (class function) afforded by x on α is then given by the composite $E^*(BG) \xrightarrow{(B\alpha)^*} E^*_{cont}(B\mathbb{Z}_p^n) \longrightarrow L(E^*)$. Theorem C of [10] asserts that the character map χ_G associating to each $x \in E^*(BG)$ the character it affords induces an isomorphism

$$L(E^*) \otimes_{E^*} E^*(BG) \xrightarrow{\chi_G} Cl_{n,p}(G; L(E^*))\,.$$

Thus $E^*(BG)$ is related to the ring of generalised class functions in the same way as ordinary characters to the K-theory of BG.

Here a suitable theory means a complex oriented ring spectrum E, whose coefficients E^* are a complete, graded, local ring with maximal ideal $\mathfrak{m}$, and residue characteristic $p > 0$, such that the mod $\mathfrak{m}$ reduction of its formal group law has height n and $p^{-1}E^*$ is non-zero. These, together with Landweber exactness condition needed to apply Theorem 2.7, constitute some of our requirements on E.

We need however E to satisfy one further property. Given a space X and any cohomology theory $E^*(-)$ represented by an Ω-spectrum $\mathbf{E}_*$, we have a 'Hurewicz' construction

$$\mathcal{H}_E(X)\colon E^*(X) \longrightarrow \mathrm{Hom}(H^*(\mathbf{E}_*), H^*(X))$$

where Hom denotes, say, morphisms of graded $\mathbb{F}_p$ algebras. The map $\mathcal{H}_E(X)$ is given by sending an element $\alpha \in E^r(X)$, thought of as a homotopy class of maps $\alpha\colon X \to \mathbf{E}_r$, to its corresponding cohomology homomorphism $\alpha^*\colon H^*(\mathbf{E}_r) \to H^*(X)$.

Property 3.2. *Say the theory E satisfies* Property 3.2(n) *if*

(a) *E is Landweber exact and satisfies the hypotheses of* [10]*, namely: the coefficients E^* are a complete, graded, local ring with maximal ideal $\mathfrak{m}$ and residue characteristic $p > 0$, such that $p^{-1}E^*$ is nonzero and the mod $\mathfrak{m}$ reduction of the formal group law has height n;*

(b) *the Hurewicz map $\mathcal{H}_E(X)$ is injective when $X = BV$, the classifying space of an elementary abelian group of finite rank.*

Remark 3.3. In the next section we prove that the Baker-Würgler theory $\widehat{E(n)}$ satisfies Property 3.2(n); it appears that a similar proof works for Morava E theory.

It may well be that the property (b) follows from (a); we know of no examples satisfying (a) which do not also satisfy (b). On the other hand, the Johnson-Wilson (incomplete) theories $E(n)$ are examples of Landweber exact spectra for which property (b) holds, but not (a).

Property (b) is similar to a result of Lannes and Zarati (see, for example, [21, 8.1]) who prove such an injectivity property under the additional assumption that the infinite loop space $\mathbf{E}_r$ concerned has finite type cohomology. However, any theory E satisfying part (a) will be far from having finite type cohomology and we use very different methods in the next section to establish the result for the $\widehat{E(n)}$.

Theorem 3.4. *Let G be a finite group and suppose E satisfies Property* 3.2(n)*. Then there is a homeomorphism of varieties*

$$\mathrm{var}(Ch_E(G)) \to \operatorname*{colim}_{\mathcal{A}^{(n)}} \mathrm{var}(H^*(BV))\,.$$

Theorem 0.1 will of course follow from this and Theorem 4.1.

To prove Theorem 3.4 we introduce a further category $\mathcal{C}_E = \mathcal{C}_E(G)$ of elementary abelians, defined in terms of the theory E.

Definition 3.5. *For a finite group G, let $\mathcal{C}_E = \mathcal{C}_E(G)$ be the category with objects the elementary abelian p-subgroups of G and morphisms the injective group homomorphisms $f\colon W \to V$ such that $f^*\,\mathrm{Res}_V = \mathrm{Res}_W$ holds in E-cohomology:*

$$\begin{tikzcd} E^*(BW) & & E^*(BV) \arrow[ll, "f^*"'] \\ & E^*(BG) \arrow[ul, "\mathrm{Res}_W"] \arrow[ur, "\mathrm{Res}_V"'] & \end{tikzcd}$$

We prove Theorem 3.4 in two stages, identifying respectively $\mathcal{A}^{(n)}$ with $\mathcal{C}_E$, and $\mathcal{C}_E$ with $\mathcal{A}_E$. The former uses part (a) of (3.2); the latter needs part (b).

Proposition 3.6. *Suppose the theory E satisfies part* (a) *of Property* 3.2(n)*. Then $\mathcal{C}_E = \mathcal{A}^{(n)}$ for all finite groups.*

Proof. Firstly recall from Theorem C of [10] that the kernel of the character map consists entirely of p-torsion, as $L(E^*)$ is faithfully flat over $p^{-1}E^*$. So the character map is injective for elementary abelian groups.

Suppose $f\colon W \to V$ is in $\mathcal{A}^{(n)}$ but not in $\mathcal{C}_E$. Then there is an $x \in E^*(BG)$ such that $y := f^* \operatorname{Res}_V(x) - \operatorname{Res}_W(x) \neq 0$. The injectivity of the character map implies that W has a rank (at most) n subgroup S such that $\operatorname{Res}_S(y) \neq 0$: just take S to be the subgroup generated by any commuting n-tuple on which the generalised class function associated to y does not vanish. Now let $T = f(S)$ and $h = f|_S$. Then $h\colon S \to T$ is an isomorphism, and is induced by conjugation by some $g \in G$. Since conjugation by an element of G leaves x fixed, we arrive at

$$\operatorname{Res}_S(y) = h^* \operatorname{Res}_T(x) - \operatorname{Res}_S(x) = \operatorname{Res}_S(g^*x - x) = 0\,,$$

a contradiction.

Now suppose that $f\colon W \to V$ is in $\mathcal{C}_E$ but not in $\mathcal{A}^{(n)}$. Then there is an elementary abelian $S \subset W$ of rank at most n, such that $h = f|_S\colon S \to T = f(S)$ is an isomorphism not induced by conjugation in G. From the definition of $\mathcal{C}_E$ it is clear that h lies in $\mathcal{C}_E$. Let $g_1, \dots, g_m$ be a (minimal) generating set for S, and set $g_{m+1} = \dots = g_n = 1$ if necessary. Then $(g_1, \dots, g_n)$ and $(h(g_1), \dots, h(g_n))$ are two non-conjugate n-tuples in G, and hence are separated by a generalised class function. Surjectivity of the character map gives us a class $x \in E^*(BG)$ with $\operatorname{Res}_S(x) - h^* \operatorname{Res}_T(x) \neq 0$, *i.e.* $\operatorname{Res}_V(x)$ and $f^* \operatorname{Res}_W(x)$ are distinct. □

Remark 3.7. (1) In the definition of $\mathcal{C}_E$, the condition that morphisms be monomorphisms is redundant: suppose $f\colon W \to V$ has kernel K. Then Res_K is trivial on $E^*(BG)$, but this cannot happen unless $K = 1$, as the character isomorphism gives a nontrivial class in the image of restriction.
(2) Instead of $\mathcal{C}_E$ it might seem more appropriate to consider a category $\mathcal{C}'_E$ consisting of all abelian subgroups and group homomorphisms inducing commutative triangles as in (3.5). The respective variant of Proposition 3.6 would still hold, by essentially the same arguments. We have refrained from doing so since our construction ultimately ends up in mod p cohomology, where the difference cannot be seen. This other approach would be relevant were we using p-local or integral cohomology; compare the final remarks in section 17 of Quillen's paper [18].
(3) By construction, every morphism in $\mathcal{C}_E$ is also in $\mathcal{A}_E$. Thus combining Proposition 3.6 with Theorem 2.7 yields the chain of inclusions $\mathcal{A} \subseteq \mathcal{A}^{(n)} \subseteq \mathcal{A}_E \subseteq \mathcal{A}^{(1)}$.

Proposition 3.8. *Suppose the theory E is Landweber exact and satisfies part* (b) *of Property* 3.2(n). *Then $\mathcal{C}_E = \mathcal{A}_E$ for all finite groups.*

Proof. As just noted, it is immediate that $\mathcal{C}_E \subseteq \mathcal{A}_E$ and it is the reverse inclusion we must show. Equivalently, we need to show that the category $\mathcal{C}_E$ does not change upon passing to the subrings of mod p cohomology generated by E-type characteristic classes.

So, assume $f\colon W \to V$ is in $\mathcal{A}_E$ but not in $\mathcal{C}_E$. Consider the diagram

$$\begin{tikzcd}
E^*(BW) \arrow[dd, hook, "\mathcal{H}"'] & & E^*(BV) \arrow[ll, "f^*"'] \arrow[dd, hook, "\mathcal{H}"] \\
& E^*(BG) \arrow[ul] \arrow[ur] \arrow[dd, "\mathcal{H}"'] & \\
\operatorname{Hom}(H^*(\mathbf{E}_*), H^*(BW)) & & \operatorname{Hom}(H^*(\mathbf{E}_*), H^*(BV)) \arrow[ll, "f^*"'] \\
& \operatorname{Hom}(H^*(\mathbf{E}_*), H^*(BG)) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Injectivity of the two outside vertical maps is the assumption that E satisfies part (b) of Property 3.2(n). That f lies in $\mathcal{A}_E$ implies the commutativity of the image of the top triangle in the bottom one; that f does not lie in $\mathcal{C}_E$ means that the top triangle does not commute. Commutativity of the 'sides' of the prism follows from the naturality of the $\mathcal{H}_E(X)$ construction with respect to maps of the space X. A contradiction now follows by chasing round an element $x \in E^*(BG)$ for which $\operatorname{Res}_W(x) - f^* \operatorname{Res}_V(x) \neq 0$. $\square$

4. Injectivity of $\mathcal{H}_{\widehat{E(n)}}(BV)$

The goal of this section is to prove that the I_n-adically complete theory $\widehat{E(n)}$ [6] satisfies Property 3.2(n). Satisfaction of part (a) is well established (we see that it is Landweber exact in [6] and it is noted in [10] that the other properties listed in part (a) also hold). Thus we must demonstrate that the maps

$$\mathcal{H}_{\widehat{E(n)}}(BV)\colon \widehat{E(n)}^*(BV) \to \operatorname{Hom}(H^*(\widehat{\mathbf{E(n)}}_*), H^*(BV))$$

are injective for all finite rank elementary abelian p-groups V.

In fact, we shall prove the following, equivalent result in homology.

Theorem 4.1. *Let V be a finite rank elementary abelian p-group. Then*

$$\mathcal{H}_{\widehat{E(n)}}(BV)\colon \widehat{E(n)}^*(BV) \to \operatorname{Hom}(H_*(BV), H_*(\widehat{\mathbf{E(n)}}_*))$$

defined by $\big(\alpha\colon BV \to \widehat{\mathbf{E(n)}}_r\big) \mapsto \big(\alpha_*\colon H_*(BV) \to H_*(\widehat{\mathbf{E(n)}}_r)\big)$ *is injective. Here* Hom *denotes the morphisms in the category of graded cocommutative* $\mathbb{F}_p$ *coalgebras.*

We start by noting that the map $\mathcal{H}_E(X)$ (in either homology or cohomology, but from now we shall work with the homology variant) satisfies good algebraic properties.

Proposition 4.2. *Suppose that E is a ring spectrum and X is any space. Then* $\operatorname{Hom}(H_*(X), H_*(\mathbf{E}_*))$ *has a natural E_*-algebra structure and*

$$\mathcal{H}_E(X)\colon E^*(X) \longrightarrow \operatorname{Hom}(H_*(X), H_*(\mathbf{E}_*))$$

is a graded E_*-algebra homomorphism.

Proof. This is essentially formal, but it is a good opportunity to introduce the notation and operations from coalgebraic algebra that are needed in the main proof of Theorem 4.1 together with explicit formulæ. For basic references to the algebraic properties and rules of manipulation in coalgebraic algebras (Hopf rings with further structure), see [12, 19].

As each $\mathbf{E}_r$ is an H-space, each $H_*(\mathbf{E}_r)$ is a Hopf algebra with product $*$ as introduced in section 2 and coproduct

$$\psi\colon H_*(\mathbf{E}_r) \to H_*(\mathbf{E}_r) \otimes H_*(\mathbf{E}_r)\,.$$

However, as E is a ring spectrum the graded product in $E^*(X)$ is represented by maps $\mathbf{E}_r \times \mathbf{E}_s \to \mathbf{E}_{r+s}$ giving a further product

$$\circ\colon H_m(\mathbf{E}_r) \otimes H_n(\mathbf{E}_s) \to H_{m+n}(\mathbf{E}_{r+s})\,.$$

An element $e \in E_{-r} = E^r = \pi_0(\mathbf{E}_r)$, thought of as a map from a point into $\mathbf{E}_r$, gives rise to an element $[e] \in H_0(\mathbf{E}_r)$ as the image of $1 \in H_0(\text{point})$. Such an element is grouplike and satisfies $[d]*[e] = [d+e]$ (where defined) and $[d]\circ[e] = [de]$; the subobject of all such elements, written $\mathbb{F}_p[E^*]$, forms a sub-coalgebraic ring and $H_*(\mathbf{E}_*)$ is a coalgebraic algebra over this coalgebraic ring. In fact, $H_0(\mathbf{E}_*) = \mathbb{F}_p[E^*]$ and this should be thought of as the classical group-ring construction, endowed with extra structure. Note that $[0] = b_0$ and is the $*$ unit (and is distinct from 0) and $[1]$ is the $\circ$ unit (and is distinct from 1, which, however, is identical to $[0]$). Note also that $a \circ [0]$ is $[0]$ if $a \in H_0(\mathbf{E}_*)$ but is 0 otherwise.

It is of course entirely formal that the set of coalgebra maps from $H_*(X)$, an $\mathbb{F}_p$ coalgebra, to $H_*(\mathbf{E}_*)$, an algebra in the category of $\mathbb{F}_p$ coalgebras, carries itself an algebra structure. However, it will be useful to identify the operations explicitly. Addition of say $f, g \in \operatorname{Hom}(H_*(X), H_*(\mathbf{E}_*))$ is given by the composite

$$H_*(X) \xrightarrow{\Delta_*} H_*(X) \otimes H_*(X) \xrightarrow{f\otimes g} H_*(\mathbf{E}_*) \otimes H_*(\mathbf{E}_*) \xrightarrow{*} H_*(\mathbf{E}_*)$$

and the product is described similarly using the $\circ$ product. The zero element is given by the composite

$$H_*(X) \to H_*(\text{point}) \to H_*(\mathbf{E}_*)$$

where the second map is that which in $H_0(-)$ sends 1 to $[0]$; the unit is similar, using the map representing $[1]$. Finally, the E_* action is given as follows: if $e \in E^* = E_{-*}$ and $f\colon H_*(X) \to H_*(\mathbf{E}_*)$, then ef is the map $H_*(X) \to H_*(\mathbf{E}_*)$ sending $x \in H_*(X)$ to $[e] \circ f(x)$. It is left to the reader to check that, with these operations, $\mathcal{H}_E(X)$ is an E_*-algebra homomorphism. □

Remark 4.3. It will be useful to note that the construction $\mathcal{H}_E(X)$ is not only natural in the space X (as used in the previous section), but is also natural in the spectrum E. The strategy of the proof of Theorem 4.1 will be to prove the analagous result for $\mathcal{H}_{K(n)}(B\mathbb{Z}/p)$, *i.e.* in Morava K-theory for the rank 1 case, and then deduce (4.1) from naturality in the spectrum via the Baker-Würgler tower [7] linking $K(n)$ and $\widehat{E(n)}$, and the application of an appropriate Künneth theorem.

Recall that the group monomorphism $\mathbb{Z}/p \to S^1$ induces a homomorphism $E^*(\mathbb{C}P^\infty) \to E^*(B\mathbb{Z}/p)$. For a complex oriented theory E we shall just write $x \in E^2(B\mathbb{Z}/p)$ for the image of the complex orientation x^E.

Proposition 4.4. [7, 10, 20] *For $E = K(n)$ or $\widehat{E(n)}$, $E^*(B\mathbb{Z}/p)$ is the free E^* module on basis $\{1, x, \cdots, x^{p^n-1}\}$. Moreover, for both these theories there is a Künneth isomorphism*

$$E^*(B(\mathbb{Z}/p)^r) = E^*(B\mathbb{Z}/p) \underset{E^*}{\otimes} \overset{(r)}{\cdots} \underset{E^*}{\otimes} E^*(B\mathbb{Z}/p).$$

□

As the homomorphism $H_*(B\mathbb{Z}/p) \to H_*(\mathbb{C}P^\infty)$ is an isomorphism in even degrees, we shall extend the notation of section 2 and write $\beta_r \in H_{2r}(B\mathbb{Z}/p)$ for the corresponding elements. Note that the coproduct $\psi\colon H_*(B\mathbb{Z}/p) \to H_*(B\mathbb{Z}/p) \otimes H_*(B\mathbb{Z}/p)$ acts by $\psi(\beta_r) = \sum_{i+j=r} \beta_i \otimes \beta_j$. Note also that $\beta_0 = 1$.

Proposition 4.5. (a) *Suppose $r > 0$. The following formulæ describe some of the action of the map $\mathcal{H}_E(B\mathbb{Z}/p)$ on the class $x^r \in E^{2r}(B\mathbb{Z}/p)$ for a complex oriented theory E.*

$$\begin{array}{llcll} \mathcal{H}_E(B\mathbb{Z}/p)(x^r)\colon & \beta_r & \mapsto & b_1^{\circ r}, & \\ & \beta_t & \mapsto & 0, & \text{if } 0 < t < r, \\ & \beta_0 = 1 & \mapsto & [0] = 1. & \end{array}$$

(b) (*Action on x^0.*) *Suppose $e \in E_*$. Then*

$$\begin{array}{llcll} \mathcal{H}_E(B\mathbb{Z}/p)(e)\colon & \beta_t & \mapsto & 0, & \text{if } 0 < t, \\ & \beta_0 = 1 & \mapsto & [e]. & \end{array}$$

Proof. Given the description of the map $\mathcal{H}_E(X)$ in the proof of Proposition 4.2, these are essentially straightforward calculations in the 'Hopf ring calculus' of [19]. For example, $\mathcal{H}_E(B\mathbb{Z}/p)(x)(\beta_t) = b_t$, by definition of b_t. Then $\mathcal{H}_E(B\mathbb{Z}/p)(x^r)(\beta_t)$ is computed by the composite

$$\beta_t \mapsto \sum_{j_1+\cdots+j_r=t} \beta_{j_1} \otimes \cdots \otimes \beta_{j_r} \mapsto \sum_{j_1+\cdots+j_r=t} b_{j_1} \otimes \cdots \otimes b_{j_r} \mapsto \sum_{j_1+\cdots+j_r=t} b_{j_1} \circ \cdots \circ b_{j_r}.$$

If $t = 0$ there is just one term in the sum – all $j_i = 0$ and $[0] \circ \cdots \circ [0] = [0]$. If $0 < t < r$ then, in every term in the sum, at least one $j_i = 0$ and at least one $j_k > 0$; thus each summand contains the element $b_{j_i} \circ b_{j_k} = [0] \circ b_{j_k} = 0$ and so the whole sum is zero. If $t = r$ then the sum has one term in which all the $j_i = 1$, giving the $b_1^{\circ r}$ of the proposition, and all other summands are zero, as in the previous case. Part (b) follows immediately from the definitions. □

Proposition 4.6. *The morphism*

$$\mathcal{H}_{K(n)}(B\mathbb{Z}/p)\colon K(n)^*(B\mathbb{Z}/p) \longrightarrow \mathrm{Hom}(H_*(B\mathbb{Z}/p), H_*(\mathbf{K(n)}_*))$$

is injective.

Proof. By (4.4) a typical element of $K(n)^*(B\mathbb{Z}/p)$ is of the form $\sum_{i=0}^{p^n-1} c_i x^i$ where $c_i \in K(n)^* = \mathbb{F}_p[v_n, v_n^{-1}]$. In fact, it suffices to consider homogeneous elements, given the construction of $\mathcal{H}$. We shall suppress the unit v_n (and corresponding $[v_n] \in H_*(\mathbf{K(n)}_*)$, as in [23]) for simplicity of notation and so shall assume $c_i \in \mathbb{F}_p$. Then generally homogeneous elements of degree $2i$ are of the simple form $c_i x^i$, $c_i \in \mathbb{F}_p$ and $0 \leqslant i < p^n$; this is not quite true if $i \equiv 0 \bmod 2(p^n - 1)$ in which case the general element is $c_0 + c_{p^n-1} x^{p^n-1}$.

So, for $i \not\equiv 0 \bmod 2(p^n - 1)$, it suffices to show that, for $c_i \neq 0$,

$$\mathcal{H}_{K(n)}(B\mathbb{Z}/p)\left(c_i x^i\right)$$

acts non-trivially on some β_t. By Proposition 4.5 it sends β_i to $[c_i] \circ b_1^{\circ i}$; as b_1 is primitive and $i > 0$, this is just $c_i b_1^{\circ i}$. From [23] we know that all the $b_1^{\circ r} \in H_{2r}(\mathbf{K(n)}_{2r})$ are non-trivial if $r < p^n$.

For $i \equiv 0 \bmod 2(p^n - 1)$ we must consider the general element $c_0 + c_{p^n-1}x^{p^n-1}$. If $c_{p^n-1} = 0$ then (4.5)(b) shows $\mathcal{H}_{K(n)}(B\mathbb{Z}/p)(c_0)$ is non-zero on β_0. Otherwise, assuming $c_{p^n-1} \neq 0$,

$$\begin{aligned} \mathcal{H}_{K(n)}(B\mathbb{Z}/p)(c_0 + c_{p^n-1}x^{p^n-1})\colon \beta_{p^n-1} &\mapsto [c_0] * ([c_{p^n-1}] \circ b_1^{\circ p^n-1}) \\ &= [c_0] * (c_{p^n-1} b_1^{\circ p^n-1}) . \end{aligned}$$

As $[c_0]$ is a $*$ unit (with $*$ inverse $[-c_0]$), this last expression is non-zero and the proof is complete. $\square$

It is interesting to note that in $H_*(\mathbf{K(n)}_*)$ although $b_1^{\circ p^n-1} \neq 0$, one more $\circ$ power of b_1 (or even one more suspension) kills this element. In this sense the above proof only 'just' works.

We now recall the tower of spectra defined in [7] (and implicitly in [6]). This is a tower

$$\cdots \longrightarrow E(n)/I_n^{k+1} \longrightarrow E(n)/I_n^k \longrightarrow \cdots \longrightarrow K(n) = E(n)/I_n^1$$

where the spectrum $E(n)/I_n^k$ has homotopy $E(n)_*/I_n^k$. The homotopy limit of this tower is the Baker-Würgler spectrum $\widehat{E(n)}$. Unstably, (*i.e.*, passing to Ω spectra), this corresponds to a tower of fibrations of the relevant spaces; the fibre of the map

$$(\mathbf{E(n)/I_n^{k+1}})_* \longrightarrow (\mathbf{E(n)/I_n^k})_*$$

is a product of copies of $\mathbf{K(n)}_*$ indexed by a basis of I_n^{k+1}/I_n^k, *i.e.*, by monomials in the v_l, $0 \leqslant l < n$ (using the convention of putting $v_0 = p$) of degree $k+1$.

This tower of spectra gives rise to Baker and Würgler's $K(n)$ Bockstein spectral sequence [7]. An example of this sequence is that for the space $B\mathbb{Z}/p$ in which the E_2-page is just

$$\widehat{E(n)}_* \underset{K(n)_*}{\otimes} K(n)^*(B\mathbb{Z}/p).$$

This is entirely in even dimensions and the sequence, converging to $(\widehat{E(n)})^*(B\mathbb{Z}/p)$, collapses; *cf.* Proposition 4.4. We shall prove the rank 1 case of Theorem 4.1 by examining the Hurewicz image of this spectral sequence.

Theorem 4.7. *The morphism*

$$\mathcal{H}_{\widehat{E(n)}}(B\mathbb{Z}/p)\colon \widehat{E(n)}^*(B\mathbb{Z}/p) \longrightarrow \mathrm{Hom}(H_*(B\mathbb{Z}/p), H_*(\widehat{\mathbf{E(n)}}_*))$$

is injective.

Proof. Let $0 \neq \alpha \in (\widehat{E(n)})^s(B\mathbb{Z}/p)$ and consider it as a map $B\mathbb{Z}/p \to \widehat{\mathbf{E(n)}}_s$. Either there is some integer $k \geqslant 1$ for which composition of α with the maps in the Baker-Würgler tower gives an essential map $\tilde{\alpha}\colon B\mathbb{Z}/p \to (\mathbf{E(n)/I_n^{k+1}})_s$ but a null map to $(\mathbf{E(n)/I_n^k})_s$, or else α maps to a non-zero element of $K(n)^s(B\mathbb{Z}/p)$ (at the bottom of the tower).

In the former case, the map $\tilde{\alpha}$ lifts to an essential map to the fibre of the map $(\mathbf{E(n)/I_n^{k+1}})_s \longrightarrow (\mathbf{E(n)/I_n^k})_s$, a product of spaces from the Ω spectrum for $K(n)$. Thus in both this or the second case, α gives rise to a non-trivial map

$$a\colon B\mathbb{Z}/p \longrightarrow \prod_i \mathbf{K(n)}_{r_i}$$

where the product is finite, indexed by the monomials, $\mathbf{w}_i$ say, in the v_l of degree k. (Note that the r_i will generally differ from the original s, their value depending on the dimension of the $\mathbf{w}_i$ concerned.)

Consider the commutative diagram

$$\begin{tikzcd} H_*(B\mathbb{Z}/p) \arrow[r] \arrow[d, "a_*"'] & H_*(\widehat{\mathbf{E}(\mathbf{n})}_s) \arrow[d] \\ H_*(\prod_i \mathbf{K}(\mathbf{n})_{r_i}) \arrow[r] & H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^{\mathbf{k+1}})_s)\,. \end{tikzcd}$$

The top map is $\mathcal{H}_{\widehat{E(n)}}(B\mathbb{Z}/p)(\alpha)$, the map we wish to show to be non-trivial. This will follow by finding an element $\beta_r \in H_*(B\mathbb{Z}/p)$ which passes to something non-zero in $H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^{\mathbf{k+1}})_s)$ and we do this by examining the action of a_* (via 4.6) and the bottom horizontal map which is given by the inclusion of the fibre $\iota\colon \prod_i \mathbf{K}(\mathbf{n})_{r_i} \to (\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^{\mathbf{k+1}})_s$.

First examine the map $a\colon B\mathbb{Z}/p \to \prod_i \mathbf{K}(\mathbf{n})_{r_i}$. Keeping track of the components and the monomials $\mathbf{w}_i$ they correspond to, we can write this map as a tuple $(\dots, \mathbf{w}_i c_i x^{t_i}, \dots)$ where, as before, we assume $c_i \in \mathbb{F}_p$ and $0 \leqslant t_i < p^n$ (and, strictly speaking, in the dimension congruent to 0 mod $2(p^n-1)$, components may be of the form $\mathbf{w}_i(c_{0,i} + c_{p^n-1,i}x^{p^n-1})$). The composite with the inclusion, in homology,

$$H_*(B\mathbb{Z}/p) \xrightarrow{a_*} H_*(\prod \mathbf{K}(\mathbf{n})_{r_i}) \longrightarrow H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^{\mathbf{k+1}})_s)$$

sends an element β_r to

$$\sum {}_i^*[\mathbf{w}_i] \circ \iota_* \mathcal{H}_{K(n)}(B\mathbb{Z}/p)\left(c_i x^{t_i}\right)(\beta_i')$$

where the sum is over all the terms in the iterated coproduct of β_r, and where we write $\psi(\beta_r) = \sum \dots \otimes \beta_i' \otimes \cdots$. (Again, reading the longer expression $c_{0,i} + c_{p^n-1,i}x^{p^n-1}$ in the displayed formula where necessary.)

If all the powers of x in this expression are zero, so that we are just dealing with a 'constant' term, then the result is easy – taking β_0 in the top left hand of the commutative square we map to $\sum_i [c_i \mathbf{w}_i] \neq 0 \in H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^{\mathbf{k+1}})_*)$, where the $c_i \in \mathbb{F}_p$ (not all zero) are the coefficients of x^0 in each factor in the above expression for a.

So suppose r is the smallest positive power of x which appears in the expression for a above. By (4.6), the image of β_r in the bottom right of the commutative square is then (up to a $*$ multiple of $*$ invertible elements $[c_{0,j}]$)

$$\sum_i [\mathbf{w}_i] \circ c_i b_1^{\circ r} = \left(\sum_i c_i [\mathbf{w}_i]\right) \circ b_1^{\circ r}$$

where the sum is now over only some of the indexing elements i (namely those for which $t_i = r$).

It suffices now to show that expressions of the form $(\sum_i c_i[\mathbf{w}_i]) \circ b_1^{\circ r}$ are non-zero in $H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^{\mathbf{k+1}})_*)$. This follows from the following lemma, thus completing the present proof. □

Lemma 4.8. *Let $\mathbf{w}_j$ be a set of monomials in the v_l, $0 < l < n$, of degree k and of some fixed homotopy dimension and suppose $c_j \in \mathbb{F}_p$, not all zero. Then $(\sum_i c_j[\mathbf{w}_j]) \circ b_1^{\circ q}$ are non-zero in $H_*((\mathbf{E(n)}/\mathbf{I_n^{k+1}})_*)$ for all $0 \leqslant q < p^n$.*

Proof. As $\circ$ product with b_1 represents (double) suspension, one way to prove this would be to observe that $(\sum_i c_j[\mathbf{w}_j])$ was a non-zero element of $H_0((\mathbf{E(n)}/\mathbf{I_n^{k+1}})_*)$ and that this suspends to a non-zero element $(\sum_i c_j\mathbf{w}_j)$ of $H_*(E(n)/I_n^{k+1})$. Then every intermediate $(\sum_i c_j[\mathbf{w}_j]) \circ b_1^{\circ q}$ must be non-zero as well. Although the statement about $(\sum_i c_j[\mathbf{w}_j]) \in H_0((\mathbf{E(n)}/\mathbf{I_n^{k+1}})_*)$ is true, it is not true that this suspends to a non-zero element of the stable gadget as, indeed, $H_*(E(n)/I_n^{k+1}) = 0$. However, this proof would work if H was replaced with $K(n)$ and the strategy of proof will be to deduce the H result from that for $K(n)$ by arguing with the Atiyah-Hirzebruch spectral sequence.

First check the statements for $K(n)$. It is a basic fact on the homology of Ω spectra that $(\sum_i c_j[\mathbf{w}_j]) \neq 0 \in K(n)_0((\mathbf{E(n)}/\mathbf{I_n^{k+1}})_*)$ as this is just the group-ring [19]. To see the stable result it suffices to check that the right unit

$$E(n)_*/I_n^{k+1} \longrightarrow K(n)_*(E(n)/I_n^{k+1})$$

is an inclusion on sums of monomials of degree k. However, the cofibration of spectra

$$\bigvee \Sigma^\bullet K(n) \longrightarrow E(n)/I_n^{k+1} \longrightarrow E(n)/I_n^k$$

gives rise to a long exact sequence in Morava K-theory. The elements in question map to 0 in $K(n)_*(E(n)/I_n^k)$ but lift non-trivially in $K(n)_*(\bigvee \Sigma^\bullet K(n))$.

Recall that in $H_*(\mathbf{K(n)}_*)$ the element b_1 suspends by iterated $\circ$ product with itself to the elements $b_1^{\circ r} \neq 0 \in H_{2r}(\mathbf{K(n)}_{2r})$. These are non-zero until we get to $b_1^{\circ p^n-1}$ (non-zero) suspending to $b_1^{\circ p^n-1} \circ e_1 = 0 \in H_{2p^n-1}(\mathbf{K(n)}_{2p^n-1})$. Here we write $e_1 \in H_1(\mathbf{K(n)}_1)$ for the single suspension element following the notation of [19]. However, in $K(n)_*(\mathbf{K(n)}_*)$ we have $b_1^{\circ p^n-1} \circ e_1 = v_n e_1$ and suspension continues indefinitely, ultimately reaching the stable element given by the image of 1 under the right unit in $K(n)_*(K(n))$.

Now consider the map of Atiyah-Hirzebruch spectral sequences

$$\begin{array}{ccc}
K(n)_* \underset{\mathbb{F}_p}{\otimes} H_*(\prod \mathbf{K(n)}_{r_i}) & \Longrightarrow & K(n)_*(\prod \mathbf{K(n)}_{r_i}) \\
\downarrow & & \downarrow \\
K(n)_* \underset{\mathbb{F}_p}{\otimes} H_*((\mathbf{E(n)}/\mathbf{I_n^{k+1}})_s) & \Longrightarrow & K(n)_*((\mathbf{E(n)}/\mathbf{I_n^{k+1}})_s).
\end{array}$$

Take an element $\left(\sum_j c_j[\mathbf{w}_j]\right) \circ b_1^{\circ q} \in H_*(\prod \mathbf{K(n)}_{r_i})$ with $0 \leqslant q < p^n$. We know it is a permanent (and non-trivial) cycle in the top spectral sequence [23] and that regarded as an element of $K(n)_*(\prod \mathbf{K(n)}_{r_i})$ it maps non-trivially to $K(n)_*((\mathbf{E(n)}/\mathbf{I_n^{k+1}})_*)$. The only way the version of this element in $H_*(-)$ could map to zero in $H_*((\mathbf{E(n)}/\mathbf{I_n^{k+1}})_*)$ would be if the $K(n)_*(-)$ version dropped Atiyah-Hirzebruch filtration in mapping to $K(n)_*((\mathbf{E(n)}/\mathbf{I_n^{k+1}})_*)$. As $\dim([v] \circ b_1^{\circ q}) \leqslant 2p^n - 2$ this cannot happen, for dimensional reasons. □

Theorem 4.1 now follows from (4.4) and (4.7) together with the observation that $H_*(B(\mathbb{Z}/p)^r)$ also satisfies a Künneth isomorphism. □

Remark 4.9. The method of proof used for Theorem 4.1 can be adapted to establish analagous results for other theories. One of the key elements of our proof is the use of the Baker-Würgler tower, and the recent work of Baker and Lazarev [5] now allows such towers to be built in quite general circumstances. In particular, it would seem that height n Morava E-theory also satisfies Property 3.2(n).

Baker's study [4] of the homotopy type of elliptic spectra shows that if $\wp$ is any suitable prime ideal of Ell_*, the complete spectrum $(Ell)\widehat{_\wp}$ splits as a wedge of suspensions of $\widehat{E(2)}$ (see [4] for precise details of the spectra Ell and ideals $\wp$ considered). Theorem 4.1 thus shows that all these complete spectra also satisfy Property 3.2(2) and thus, in some sense, identifies the subring of 'elliptic characteristic classes' in the mod p cohomology of a finite group (even though we are still not sure what an 'elliptic object' actually is).

5. Examples

We finish by sketching some of the calculations for the chromatic subrings $Ch_{\widehat{E(n)}}(G)$ for the simplest non-trivial example, that of G the alternating group A_4. We take the prime p to be 2.

As the 2-rank of A_4 is two, the category $\mathcal{A}^{(2)}$ is the Quillen category $\mathcal{A}$. The skeleton of $\mathcal{A}$ may be represented as

$$\begin{array}{lccc} \textit{Automorphism group} & 1 & 1 & C_3 \\ & \bullet & \text{———} & \bullet \\ \textit{Rank of elem. abelian} & 1 & & 2 \end{array}$$

Here, the nodes represent isomorphism classes of elementary abelians, labelled by 2-rank and automorphism group. The edges represent equivalence classes of morphisms under conjugacy, the label denoting the stabilizer in the automorphism group of the target group. Thus here there are $3 = |C_3 : 1|$ morphisms from a rank 1 to a rank 2 elementary abelian.

The skeleton of the category $\mathcal{A}^{(1)}$ is

$$\begin{array}{ccc} 1 & C_2 & S_3 \\ \bullet & \text{———} & \bullet \\ 1 & & 2 \end{array}$$

Thus $\mathcal{A}$ is strictly contained in $\mathcal{A}^{(1)}$ and so, by Theorem 0.1, the subring $Ch_{\widehat{E(1)}}(A_4)$ is strictly contained in $Ch_{\widehat{E(2)}}(A_4)$. The following calculations demonstrate an element in the latter not in the former as well as showing explicitly the equivalence $\operatorname{var}(Ch_{\widehat{E(2)}}(A_4)) = \operatorname{var}(H^*(BA_4))$.

Let V be the Sylow 2-subgroup of A_4. The Weyl group of V in A_4 is the cyclic group C_3, permuting the non-trivial elements transitively. As V is abelian, $H^*(BA_4)$ is the ring of $N_{A_4}(V)$-invariants in $H^*(BV)$. Let x, y be the basis for V^* dual to the basis $(1\,2)(3\,4)$, $(1\,3)(2\,4)$ for V. Then we have

$$H^*(BA_4) \cong \mathbb{F}_2[x,y]^{C_3} \quad |x| = |y| = 1$$

where the C_3-action is $x \mapsto y \mapsto x + y$. Over $\mathbb{F}_4$ we can diagonalize this action and so calculate the invariants: $H^*(BA_4)$ is generated by D_1, D_0 and η, where

$$D_1 = x^2 + xy + y^2 \quad D_0 = x^2y + xy^2 \quad \eta = x^3 + x^2y + y^3\,.$$

Observe that D_1 and D_0 are Dickson invariants, and that η is the orbit sum of x^2y. The natural permutation representation π of A_4 has Chern classes $c_2(\pi) = D_1^2$ and $c_3(\pi) = D_0^2$. These generate the Chern subring $Ch(A_4) = Ch_{\widehat{E(1)}}(A_4)$, and are $\widehat{E(2)}$-type classes as well by Proposition 2.8.

Now $\widehat{E(2)}^*(BV) \cong \widehat{E(2)}^*[[w,z]]/([2]_F(w),[2]_F(z))$ where $[2]_F(-)$ denotes the 2-series of the formal group law for $\widehat{E(2)}$. Set $\theta = \mathrm{Tr}_{BV}^{BA_4}(w^2z)$. By the Mackey formula, we have

$$\mathrm{Res}_{BV}\,\theta = w^2z + z^2(w +_F z) + (w +_F z)^2w \colon BV \to \widehat{\mathbf{E(2)}}_6\,.$$

Write $\beta(s)$ for the generating function of the $\beta_i \in H_{2i}(B\mathbb{Z}/2)$, that is $\beta(s)$ is the formal power series $\sum_{i\geqslant 0}\beta_i s^i \in H_*(B\mathbb{Z}/2)[\![s]\!]$, and similarly regard $\beta(s)\otimes\beta(t)$ as the corresponding generating function in $H_*(BV) = H_*(B\mathbb{Z}/2)\underset{\mathbb{F}_2}{\otimes} H_*(B\mathbb{Z}/2)$. Likewise, write $b(s)$ for the generating function of the b_i in $H_*(\widehat{\mathbf{E(2)}}_2)$. The standard Ravenel-Wilson relation [19, 3.6] in $H_*(\mathbf{E}_*)$ for any complex oriented theory E now allows us to compute

$$\begin{aligned}(\mathrm{Res}_{BV}\,\theta)_*&(\beta(s)\otimes\beta(t))\\ &= \big(b(s)^{\circ 2}\circ b(t)\big) * \big(b(t)^{\circ 2}\circ b(s+t)\big) * \big(b(s+t)^{\circ 2}\circ b(s)\big)\,.\end{aligned}$$

This reduces to

$$b_0^{\circ 3} + \bar{b}(s)^{\circ 2}\circ\bar{b}(t) + \bar{b}(t)^{\circ 2}\circ\bar{b}(s+t) + \bar{b}(s+t)^{\circ 2}\circ\bar{b}(s) \quad \text{mod indecomposables}$$

where $\bar{b}(t)$ is defined as $b(t) - b_0 = \sum_{r\geqslant 1} b_r t^r$ (recall that $b_r \circ b_0 = 0$ for $r > 0$ and that b_0 is the $*$ unit).

The (s,t)-degree 3 part of this expression is

$$b_1^{\circ 3}(s^2t + t^2(s+t) + (s+t)^2s) = b_1^{\circ 3}(s^3 + s^2t + t^3)\,.$$

Note that $b_1^{\circ 3} \in H_6(\widehat{\mathbf{E(2)}}_6)$ represents a non-zero indecomposable: for example, it maps to the corresponding element in $H_6(\mathbf{K(2)}_6)$ through the Baker-Würgler tower, and this represents a non-zero indecomposable here [23]. By duality, there is a class $\gamma \in H^6(\widehat{\mathbf{E(2)}}_6)$ which kills all decomposables in H_6 and sends $b_1^{\circ 3}$ to 1. Then the $\widehat{E(2)}$-type characteristic class $\theta^*(\gamma)$ restricts to $H^*(BV)$ as η^2. Hence $Ch_{\widehat{E(2)}}(A_4)$ contains $Ch_{\widehat{E(1)}}(A_4)$ as a subring, and the inclusion of $Ch_{\widehat{E(2)}}(A_4)$ in $H^*(BA_4)$ is an inseparable isogeny.

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