COMMUTATIVITY PROPERTIES OF QUINN SPECTRA

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ABSTRACT. We give a simple sufficient condition for Quinn's "bordism-type" spectra to be weakly equivalent to commutative symmetric ring spectra. We also show that the symmetric signature is (up to weak equivalence) a monoidal transformation between symmetric monoidal functors, which implies that the Sullivan-Ranicki orientation of topological bundles is represented by a ring map between commutative symmetric ring spectra. In the course of proving these statements we give a new description of symmetric L theory which may be of independent interest.

1. INTRODUCTION

In [Qui95], Frank Quinn gave a general machine for constructing spectra from "bordism-type theories." In our paper [LM] we gave axioms for a structure we call an *ad theory* and showed that when these axioms are satisfied (as they are for all of the standard examples) the Quinn machine can be improved to give a symmetric spectrum **M**. We also showed that when the ad theory is multiplicative (that is, when its "target category" is graded monoidal) the symmetric spectrum **M** is a symmetric ring spectrum. Finally, we showed that there are monoidal functors to the category of symmetric spectra which represent Poincaré bordism over $B\pi$ (considered as a functor of π) and symmetric L-theory (considered as a functor of a ring R with involution).

In this paper we consider commutativity properties. The reader does not need to have a thorough knowledge of our previous paper; the only sections that are relevant are 3, 6, 7, 9, 10, 17, 18 and 19, and only the definitions and the statements of the results (not the proofs) are used in the present paper.

A "commutative ad theory" is (essentially) an ad theory whose target category is graded symmetric monoidal (the precise definition is given in Section 3). Our first main result is

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Theorem 1.1. *Let* $\mathbf{M}$ *be the symmetric ring spectrum associated to a commutative ad theory. There is a commutative symmetric ring spectrum* $\mathbf{M}^{\mathrm{comm}}$ *which is weakly equivalent in the category of symmetric ring spectra to* $\mathbf{M}$.

The proof gives a specific chain of weak equivalences (of length 2) between $\mathbf{M}$ and $\mathbf{M}^{\mathrm{comm}}$.

In particular, Theorem 1.1 shows that the L-theory spectrum of a commutative ring can be realized as a commutative symmetric ring spectrum.[1]

For the ad theory $\mathrm{ad}_{\mathrm{STop}}$ of oriented topological bordism ([LM, Section 6]), we showed in [LM, Section 17 and Appendix B] that the underlying spectrum of $\mathbf{M}_{\mathrm{STop}}$ is weakly equivalent to the usual Thom spectrum $M\mathrm{STop}$. It is well-known that $M\mathrm{STop}$ is a commutative symmetric ring spectrum, and we have

Theorem 1.2. $(\mathbf{M}_{\mathrm{STop}})^{\mathrm{comm}}$ *and* $M\mathrm{STop}$ *are weakly equivalent in the category of commutative symmetric ring spectra.*

The proof gives a specific chain of weak equivalences between them.

We also prove a multiplicative property of the symmetric signature. The symmetric signature is a basic tool in surgery theory. In its simplest form, it assigns to an oriented Poincaré complex X an element of the symmetric L-theory of $\pi_1(X)$; this element determines the surgery obstruction up to 8-torsion. Ranicki proved that the symmetric signature of a Cartesian product is the product of the symmetric signatures ([Ran80b, Proposition 8.1(i)]). The symmetric signature gives a map of spectra from Poincaré bordism to L-theory ([KMM, Proposition 7.10]), and we showed in [LM] that it gives a map of symmetric spectra. In order to investigate the multiplicativity of this map, we give a new (but equivalent) description of the L-spectrum, using "relaxed" algebraic Poincaré complexes (the relation between these and the usual algebraic Poincaré complexes is similar to the relation between Γ-spaces and E_∞ spaces). For a ring with involution R there is an ad theory $\mathrm{ad}_{\mathrm{rel}}^R$, and the associated spectrum $\mathbf{M}_{\mathrm{rel}}^R$ is equivalent to the usual L-spectrum. The symmetric signature gives a map $\mathrm{Sig}_{\mathrm{rel}}$ from the Poincaré bordism spectrum (which we denote by $\mathbf{M}_{e,*,1}$; see [LM, Section 7]) to $\mathbf{M}_{\mathrm{rel}}^R$. We prove that this map is weakly equivalent to a ring map between commutative symmetric ring spectra:

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[1]Lurie [Lur] has explained another way to prove that the L-theory spectrum of a commutative ring can be realized as a commutative symmetric ring spectrum. The method used in [Lur] does not include other examples we consider such as Poincaré bordism and (in [BLM]) Witt bordism.

Theorem 1.3. *There are symmetric ring spectra* **A** *and* **B***, commutative symmetric ring spectra* **C** *and* **D***, and a commutative diagram*

$$
\begin{array}{ccccc}
\mathbf{M}_{e,*,1} & \longleftarrow & \mathbf{A} & \longrightarrow & \mathbf{C} \\
\Big\downarrow{\scriptstyle \mathrm{Sig}_{\mathrm{rel}}} & & \Big\downarrow & & \Big\downarrow \\
\mathbf{M}^{\mathbb{Z}}_{\mathrm{rel}} & \longleftarrow & \mathbf{B} & \longrightarrow & \mathbf{D}
\end{array}
$$

in which the horizontal arrows and the right vertical arrow are ring maps and the horizontal arrows are weak equivalences.

In fact **C** is $(\mathbf{M}_{e,*,1})^{\mathrm{comm}}$, and **D** is weakly equivalent to $(\mathbf{M}^{\mathbb{Z}}_{\mathrm{rel}})^{\mathrm{comm}}$ in the category of commutative symmetric ring spectra (see Remark 17.3).

As far as we are aware, there is no previous result in the literature showing multiplicativity of the symmetric signature at the spectrum level.

In Section 18 we prove a stronger statement, that up to weak equivalence the symmetric signature is a monoidal transformation between symmetric monoidal functors. In [BLM] we will prove the analogous statement about the symmetric signature for Witt bordism, using the methods of the present paper.

Remark 1.4. The *Sullivan-Ranicki orientation* for topological bundles ([Sul05], [MM79], [Ran92, Remark 16.3], [KMM, Section 13.5]) is the following composite in the homotopy category of spectra

$$
M\mathrm{STop} \simeq \mathbf{Q}_{\mathrm{STop}} \xrightarrow{\;\mathrm{Sig}\;} L^{\mathbb{Z}},
$$

where $\mathbf{Q}_{\mathrm{STop}}$ denotes the Quinn spectrum of oriented topological bordism (which was shown to be equivalent to $M\mathrm{STop}$ in [LM, Appendix B]). Combining Theorems 1.3 and 1.2 shows that the Sullivan-Ranicki orientation is represented by a ring map of commutative symmetric ring spectra.

Here is an outline of the paper. In Sections 2 and 3 we give the definition of commutative ad theory. The proof of Theorem 1.1 occupies Sections 4–10. We begin in Sections 4 and 5 by giving a multisemisimplicial analogue $\Sigma\mathcal{S}_{\mathrm{ss}}$ of the category of symmetric spectra. We observe that an ad theory gives rise to an object **R** of $\Sigma\mathcal{S}_{\mathrm{ss}}$ whose realization is the symmetric spectrum **M** mentioned above. Section 6 explains the key idea of the proof, which is to interpolate between the various permutations of the multiplication map by allowing a different order of multiplication for each cell. In Sections 7 and 8 we use this idea to create a monad in the category $\Sigma\mathcal{S}_{\mathrm{ss}}$ which acts on **R**, and in Section

10 (after a brief technical interlude in Section 9) we use a standard
rectification argument (as in [May72]) to convert $\mathbf{R}$ with this action to
a strictly commutative object of $\Sigma \mathcal{S}_{ss}$; passage to geometric realization
gives $\mathbf{M}^{comm}$. Next we turn to the proof of Theorem 1.3. In Sections
11–13 we introduce the relaxed symmetric Poincaré ad theory and the
corresponding version of the symmetric signature. In Sections 14–16
we create a monad in the category $\Sigma \mathcal{S}_{ss} \times \Sigma \mathcal{S}_{ss}$ which acts on the pair
$(\mathbf{R}_{e,*,1}, \mathbf{R}_{rel}^{\mathbb{Z}})$, and in Section 17 we adapt the argument of Section 10 to
prove Theorem 1.3. Section 18 gives the statement of the stronger ver-
sion of Theorem 1.3 mentioned above, and Section 19 gives the proof.
Appendix A gives the proof of Theorem 1.2.

2. Some redefinitions

One of the ingredients in the definition of ad theory in [LM] is the
target "$\mathbb{Z}$-graded category" $\mathcal{A}$ (see [LM, Definitions 3.3 and 3.10]). For
the purposes of that paper, there was no reason to allow morphisms in
$\mathcal{A}$ between objects of the same dimension (except for identity maps).
For the present paper, we do need such morphisms (see Definition 3.1
below and the proof of Theorem 1.1). We therefore begin by giving
modified versions of some of the definitions of [LM].

Definition 2.1. (cf. Definition 3.3 of [LM]) A $\mathbb{Z}$-*graded category* is a
small category $\mathcal{A}$ with involution i, together with involution-preserving
functors $d : \mathcal{A} \longrightarrow \mathbb{Z}$ (called the *dimension function*) and $\emptyset : \mathbb{Z} \longrightarrow \mathcal{A}$
such that
   (a) $d\emptyset$ is equal to the identity functor, and
   (b) if $d(a) > d(b)$ then there are no morphisms from a to b.

The definition of a strict monoidal structure on a $\mathbb{Z}$-graded category
([LM, Definition 18.1]) needs no change, provided that one uses the
new definition of $\mathbb{Z}$-graded category.
   Next we explain how to modify the specific examples of target cate-
gories in [LM] by adding morphisms which preserve dimension.

For the category $\mathcal{A}_{\mathrm{STop}}$ ([LM, Example 3.5]) the morphisms between objects of the same dimension are the maps of degree 1.

For the category $\mathcal{A}_{\pi,Z,w}$ ([LM, Definition 7.3]) the morphisms between objects (X, f, ξ) and (X', f', ξ') of the same dimension are the maps $g : X \to X'$ such that $f' \circ g = f$ and $g_*(\xi) = \xi'$.

We do not need the analogous modification for the category $\mathcal{A}^R$ ([LM, Definition 9.5]) because we will be using the version in Section 11.

3. COMMUTATIVE AD THEORIES

Definition 3.1. Let $\mathcal{A}$ be a $\mathbb{Z}$-graded category. A *permutative structure* on $\mathcal{A}$ is a strict monoidal structure $(\boxtimes, \varepsilon)$ ([LM, Definition 18.1]) together with a natural isomorphism

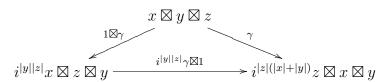$$\gamma_{x,y} : x \boxtimes y \to i^{|x||y|}y \boxtimes x$$

such that
- (a) $i\gamma_{x,y} = \gamma_{ix,y} = \gamma_{x,iy}$,
- (b) each of the maps $\gamma_{\emptyset,y}$, $\gamma_{x,\emptyset}$ is the identity map of $\emptyset$,
- (c) the composite

$$x \boxtimes y \xrightarrow{\gamma_{x,y}} i^{|x||y|}y \boxtimes x \xrightarrow{i^{|x||y|}(\gamma_{y,x})} x \boxtimes y$$

is the identity.
- (d) $\gamma_{x,\varepsilon}$ is the identity, and
- (e) the diagram

$$
\begin{array}{ccc}
& x \boxtimes y \boxtimes z & \\
{}^{1\boxtimes\gamma}\swarrow & & \searrow{}^{\gamma} \\
i^{|y||z|}x \boxtimes z \boxtimes y \xrightarrow{\quad i^{|y||z|}\gamma\boxtimes 1 \quad} & & i^{|z|(|x|+|y|)}z \boxtimes x \boxtimes y
\end{array}
$$

commutes.

Remark 3.2. The analogue of the coherence theorem for symmetric monoidal categories [ML63] holds in this context with essentially the same proof.

Definition 3.3. A *commutative ad theory* is a multiplicative ad theory [LM, Definitions 3.10 and 18.4], with the extra property that every pre K-ad which is isomorphic to a K-ad is a K-ad, together with a permutative structure on the target category $\mathcal{A}$.

Examples are ad_C when C is a commutative DGA (see [LM, Example 3.12]), $\mathrm{ad}_{\mathrm{STop}}$ (see [LM, Section 6]), $\mathrm{ad}_{e,*,1}$ (see [LM, Section 7]),

$\text{ad}_{\text{STopFun}}$ (see [LM, end of Section 8]), $\text{ad}_{\text{rel}}^{R}$ when R is commutative (see Section 11 below), and $\text{ad}_{\text{IP},e,*}$ (see [BLM]).

Remark 3.4. The extra property in Definition 3.3 is used in the proof of Lemma 6.3 below.

For later use we record some notation for iterated products.

Definition 3.5. (i) For a permutation $\eta \in \Sigma_j$, let $\epsilon(\eta)$ denote 0 if η is even and 1 if η is odd.

(ii) Let $\mathcal{A}$ be a $\mathbb{Z}$-graded category with a permutative structure. Let $\eta \in \Sigma_j$. Define a functor

$$\eta_{\bigstar} : \mathcal{A}^{\times j} \to \mathcal{A}$$

(where $\mathcal{A}^{\times j}$ is the j-fold Cartesian product) by

$$\eta_{\bigstar}(x_1, \ldots, x_j) = i^{\epsilon(\bar{\eta})}(x_{\eta^{-1}(1)} \boxtimes \cdots \boxtimes x_{\eta^{-1}(j)})$$

where $\bar{\eta}$ is the block permutation that takes blocks $\mathbf{b}_1$, $\ldots$, $\mathbf{b}_j$ of size $|x_1|$, $\ldots$, $|x_j|$ into the order $\mathbf{b}_{\eta^{-1}(1)}$, $\ldots$, $\mathbf{b}_{\eta^{-1}(j)}$.

Remark 3.6. Note that, by Remark 3.2, $\eta_{\bigstar}(x_1, \ldots, x_j)$ is canonically isomorphic to $x_1 \boxtimes \cdots \boxtimes x_j$.

4. Multisemisimplicial symmetric spectra

We now turn to the proof of Theorem 1.1.

In this section we define a category $\Sigma\mathcal{S}_{\text{ss}}$ (the ss stands for "semisimplicial") which is a multisemisimplicial version of the category $\Sigma\mathcal{S}$ of symmetric spectra. The motivation for the definition is that the sequence R_k in [LM, Definition 17.2] should give an object of $\Sigma\mathcal{S}_{\text{ss}}$.

Recall that we write Δ_{inj} for the category whose objects are the sets $\{0, \ldots, n\}$ and whose morphisms are the monotonically increasing injections.

A based k-fold *multisemisimplicial set* is a contravariant functor from the Cartesian product $(\Delta_{\text{inj}})^{\times k}$ to the category Set_* of based sets. In particular, a based 0-fold multisemisimplicial set is just a based set.

Next note that given a category $\mathcal{C}$ with a left action of a group G one can define a category $G \ltimes \mathcal{C}$ whose objects are those of $\mathcal{C}$ and whose morphisms are pairs (α, f) with $\alpha \in G$ and f a morphism of $\mathcal{C}$; the domain of (α, f) is the domain of f and the target is α applied to the target of f. Composition is defined by

$$(\alpha, f) \circ (\beta, g) = (\alpha\beta, \beta^{-1}(f) \circ g)$$

Remark 4.1. (i) $\mathcal{C}$ is imbedded in $G \ltimes \mathcal{C}$ by taking the morphism f of $\mathcal{C}$ to the morphism (e, f) of $G \ltimes \mathcal{C}$, where e is the identity element of G.

(ii) The morphism (α, f) is the composite $(\alpha, \mathrm{id}) \circ (e, f)$.

Definition 4.2. Let Σ_k act on $(\Delta_{\mathrm{inj}}^{\mathrm{op}})^{\times k}$ by permuting the factors (when $k = 0$, Σ_0 is the trivial group). For each subgroup H of Σ_k let $H\mathrm{ss}_k$ be the category of functors from $H \ltimes (\Delta_{\mathrm{inj}}^{\mathrm{op}})^{\times k}$ to Set_*.

By Remark 4.1, an object of $H\mathrm{ss}_k$ can be thought of as a based k-fold multisemisimplicial set with a left "action" of H in which H also acts on the multidegrees.

Definition 4.3. (i) A multisemisimplicial symmetric sequence $\mathbf{X}$ is a sequence X_k, $k \geq 0$, such that X_k is an object of $\Sigma_k\mathrm{ss}_k$.

(ii) A *morphism* of multisemisimplicial symmetric sequences from $\mathbf{X}$ to $\mathbf{Y}$ is a sequence of morphisms $f_k : X_k \to Y_k$ in $\Sigma_k\mathrm{ss}_k$.

The category of multisemisimplicial symmetric sequences will be denoted by ss^Σ.

For our next definition, recall [LM, Definitions 17.2 and 17.3].

Definition 4.4. (i) For each $k \geq 0$ extend the object R_k of ss_k to an object of $\Sigma_k\mathrm{ss}_k$ by letting

$$(\alpha, \mathrm{id})_*(F) = i^{\epsilon(\alpha)} \circ F \circ \alpha_\#$$

(where $\alpha \in \Sigma_k$ and $F \in \mathrm{ad}^k(\Delta^{\mathbf{n}})$).

(ii) Let $\mathbf{R}$ denote the object of ss^Σ whose k-th term is R_k.

Next we assemble the ingredients needed to define a symmetric monoidal structure on ss^Σ.

Definition 4.5. Given $A \in \Sigma_k\mathrm{ss}_k$ and $B \in \Sigma_l\mathrm{ss}_l$, define $A \wedge B \in (\Sigma_k \times \Sigma_l)\mathrm{ss}_{k+l}$ by

$$(A \wedge B)_{\mathbf{m},\mathbf{n}} = A_{\mathbf{m}} \wedge B_{\mathbf{n}}$$

(where $\mathbf{m}$ is a k-fold multi-index and $\mathbf{n}$ is an l-fold multi-index).

Definition 4.6. Given $H \subset G \subset \Sigma_k$, define a functor

$$I_H^G : H\mathrm{ss}_k \to G\mathrm{ss}_k$$

by letting $I_H^G A$ be the left Kan extension of A along $H \ltimes (\Delta_{\mathrm{inj}}^{\mathrm{op}})^{\times k} \to G \ltimes (\Delta_{\mathrm{inj}}^{\mathrm{op}})^{\times k}$.

Remark 4.7. For later use we give an explicit description of $I_H^G A$. For each multi-index $\mathbf{n}$, we have

$$(I_H^G A)_{\mathbf{n}} = \left(\bigvee_{\alpha \in G} A_{\alpha^{-1}(\mathbf{n})} \right) / H$$

where the action of H is defined as follows: if $\beta \in H$ and x is an element in the α-summand then β takes x to the element $(\beta, \mathrm{id})_*(x)$ in the $\alpha\beta^{-1}$-summand.

Notation 4.8. We denote the equivalence class of an element x in the α-summand of $I_H^G A$ by $[\alpha, x]$; note that $[\alpha, x] = (\alpha, \mathrm{id})_*[e, x]$.

Definition 4.9. Given $\mathbf{X}, \mathbf{Y} \in \mathrm{ss}^\Sigma$, define $\mathbf{X} \otimes \mathbf{Y} \in \mathrm{ss}^\Sigma$ by

$$(\mathbf{X} \otimes \mathbf{Y})_k = \bigvee_{j_1+j_2=k} I_{\Sigma_{j_1} \times \Sigma_{j_2}}^{\Sigma_k} \left(\mathbf{X}_{j_1} \wedge \mathbf{Y}_{j_2} \right).$$

The proof that $\otimes$ is a symmetric monoidal product is essentially the same as the corresponding proof in [HSS00, Section 2.1]. The symmetry map

$$\tau : \mathbf{X} \otimes \mathbf{Y} \to \mathbf{Y} \otimes \mathbf{X}$$

is given by

(4.1) $$\tau([\alpha, x \wedge y]) = [\alpha\beta, y \wedge x]$$

where $x \in (X_k)_{\mathbf{m}}$, $y \in (Y_l)_{\mathbf{n}}$, and β is the permutation of $\{1, \ldots, k+l\}$ which switches the first k and the last l elements.

Next we give the definition of the category $\Sigma\mathcal{S}_{\mathrm{ss}}$ and its symmetric monoidal product.

Let S^1 denote the based semisimplicial set that consists of the base point together with a 1-simplex. We can extend the k-fold multi-semisimplicial set $(S^1)^{\wedge k}$ to an object of $\Sigma_k \mathrm{ss}_k$ by defining

$$(\alpha, \mathrm{id})_*(x_1 \wedge \cdots \wedge x_k) = x_{\alpha^{-1}(1)} \wedge \cdots \wedge x_{\alpha^{-1}(k)}.$$

(where the x_i are simplices of S^1). We write S^k for this object and $\mathbf{S}$ for the object of ss^Σ whose k-th term is S^k. It is easy to check that $\mathbf{S}$ is a commutative monoid in ss^Σ.

Definition 4.10. $\Sigma\mathcal{S}_{\mathrm{ss}}$ is the category of modules over $\mathbf{S}$.

Remark 4.11. One can give a more explicit version of this definition: an object of $\Sigma\mathcal{S}_{\mathrm{ss}}$ consists of an object $\mathbf{X}$ of ss^Σ together with suspension maps

$$\omega : S^1 \wedge X_k \to X_{k+1}$$

for each k, such that the iterates of the ω's satisfy appropriate equivariance conditions.

Example 4.12. The object $\mathbf{R}$ of Definition 4.4 can be given suspension maps as follows: with the notation of [LM, Definition 17.4(i)], define

$$\omega : S^1 \wedge R_k \to R_{k+1}$$

by
$$\omega(s \wedge F) = \lambda^*(F)$$
(where s is the 1-simplex of S^1 and $F \in \mathrm{ad}^k(\Delta^{\mathbf{n}})$). The resulting object of $\Sigma\mathcal{S}_{\mathrm{ss}}$ will also be denoted $\mathbf{R}$.

Definition 4.13. (cf. [HSS00, Definition 2.2.3]) For $\mathbf{X}, \mathbf{Y} \in \Sigma\mathcal{S}_{\mathrm{ss}}$, define the *smash product* $\mathbf{X} \wedge \mathbf{Y}$ to be the coequalizer of the diagram
$$\mathbf{X} \otimes \mathbf{S} \otimes \mathbf{Y} \rightrightarrows \mathbf{X} \otimes \mathbf{Y}$$
where the right action of $\mathbf{S}$ on $\mathbf{X}$ is the composite
$$\mathbf{X} \otimes \mathbf{S} \to \mathbf{S} \otimes \mathbf{X} \to \mathbf{X}.$$

The proof that $\wedge$ is a symmetric monoidal product is essentially the same as the corresponding proof in [HSS00, Section 2.2].

5. Geometric realization

Let G be a subgroup of Σ_k. By Remark 4.1(i), an object of $G\mathrm{ss}_k$ has an underlying k-fold multisemisimplicial set.

Definition 5.1. The *geometric realization* $|A|$ of an object $A \in G\mathrm{ss}_k$ is the geometric realization of its underlying k-fold multisemisimplicial set.

Definition 5.2. (i) A map in $G\mathrm{ss}_k$ is a *weak equivalence* if it induces a weak equivalence of realizations.

(ii) A map $\mathbf{X} \to \mathbf{Y}$ in ss^Σ or in $\Sigma\mathcal{S}_{\mathrm{ss}}$ is a *weak equivalence* if each map $X_k \to Y_k$ is a weak equivalence.

Proposition 5.3. *For $A \in \Sigma_k\mathrm{ss}_k$, the following formula gives a natural left Σ_k action on $|A|$:*
$$\alpha([u_1, \ldots, u_k, a]) = [u_{\alpha^{-1}(1)}, \ldots, u_{\alpha^{-1}(k)}, (\alpha, \mathrm{id})_*(a)];$$
here $(u_1, \ldots, u_k) \in \Delta^{\mathbf{n}}$, $a \in A_{\mathbf{n}}$, and $[u_1, \ldots, u_k, a]$ denotes the class of $(u_1, \ldots, u_k, a)$ in $|A|$. □

Proposition 5.4. *For $H \subset G \subset \Sigma_k$ and $A \in H\mathrm{ss}_k$ there is a natural isomorphism of based G-spaces*
$$|I_H^G A| \cong G_+ \wedge_H |A|.$$

Proof. The proof is easy, using Remark 4.7. □

Corollary 5.5. *Geometric realization induces a symmetric monoidal functor from $\Sigma\mathcal{S}_{\mathrm{ss}}$ to the category of symmetric spectra $\Sigma\mathcal{S}$; in particular, the realization of a (commutative) monoid in $\Sigma\mathcal{S}_{\mathrm{ss}}$ is a (commutative) monoid in $\Sigma\mathcal{S}$.* □

6. A family of multiplication maps

In this section we begin the proof of Theorem 1.1. From now until the end of Section 10 we fix a $\mathbb{Z}$-graded permutative category $\mathcal{A}$ and a commutative ad theory with values in $\mathcal{A}$. Let $\mathbf{R}$ be the object of $\Sigma \mathcal{S}_{\mathrm{ss}}$ constructed from this ad theory as in Example 4.12.

Let $\mathbf{M}$ be the symmetric ring spectrum associated to the ad theory ([LM, Proposition 17.5 and Theorem 18.5]). By definition, $M_k = |R_k|$. The multiplication of $\mathbf{M}$ is induced by the collection of maps

$$\mu : (R_k)_{\mathbf{m}} \wedge (R_l)_{\mathbf{n}} \to (R_{k+l})_{\mathbf{m},\mathbf{n}}$$

defined by

(6.1) $\mu(F \wedge G)(\sigma_1 \times \sigma_2, o_1 \times o_2) = i^{l \dim \sigma_1} F(\sigma_1, o_1) \boxtimes G(\sigma_2, o_2)$

(this is well-defined because, by [LM, Definition 18.1(b)], reversing the orientations o_1 and o_2 does not change the right-hand side). These maps give $\mathbf{R}$ the structure of a monoid in $\Sigma \mathcal{S}_{\mathrm{ss}}$ (the proof is is essentially the same as for [LM, Theorem 18.5]).

In general, even though the ad theory is commutative, $\mathbf{R}$ is not a commutative monoid (this would require the product in the target category $\mathcal{A}$ to be *strictly* graded commutative). Instead we have the following. Recall Definition 3.5 and Notation 4.8.

Lemma 6.1. *Let $m : \mathbf{R} \wedge \mathbf{R} \to \mathbf{R}$ be the product and let $\eta \in \Sigma_j$. Then the composite*

$$m_\eta : \mathbf{R}^{\wedge j} \xrightarrow{\eta} \mathbf{R}^{\wedge j} \xrightarrow{m} \mathbf{R}$$

is determined by the formula

$$m_\eta([e, F_1 \wedge \cdots \wedge F_j])(\sigma_1 \times \cdots \times \sigma_j, o_1 \times \cdots \times o_j)$$
$$= i^{\epsilon(\zeta)} \eta_\bigstar(F_1(\sigma_1, o_1), \ldots, F_j(\sigma_j, o_j)),$$

where e is the identity element of the relevant symmetric group and ζ is the block permutation that takes blocks $\mathbf{b}_1, \ldots, \mathbf{b}_j, \mathbf{c}_1, \ldots, \mathbf{c}_j$ of size $\deg F_1, \ldots, \deg F_j, \dim \sigma_1, \ldots, \dim \sigma_j$ into the order $\mathbf{b}_1, \mathbf{c}_1, \ldots, \mathbf{b}_j, \mathbf{c}_j$.

Proof. It suffices to prove this when η is a transposition, and in this case the proof is an easy calculation using Equations (4.1) and (6.1) and [LM, Definition 17.3]. $\square$

The key idea in the proof of Theorem 1.1 is that there is a family of operations which can be used to interpolate between the various m_η. To construct this family, we allow a different permutation of the factors for each cell of $\Delta^{\mathbf{n}_1} \times \cdots \times \Delta^{\mathbf{n}_j}$, as explained in our next definition.

We begin by defining the operations for pre-ads (see [LM, Definitions 3.8(i) and 3.10(ii)]).

Definition 6.2. (i) Given a ball complex K, let $U(K)$ denote the set of all cells of K.

(ii) Let $k_1, \ldots, k_j$ be non-negative integers and let $\mathbf{n}_i$ be a k_i-fold multi-index for $1 \leq i \leq j$. For any map

$$a : U(\Delta^{\mathbf{n}_1} \times \cdots \times \Delta^{\mathbf{n}_j}) \to \Sigma_j$$

define

$$a_* : \mathrm{pre}^{k_1}(\Delta^{\mathbf{n}_1}) \times \cdots \times \mathrm{pre}^{k_j}(\Delta^{\mathbf{n}_j}) \to \mathrm{pre}^{k_1 + \cdots + k_j}(\Delta^{(\mathbf{n}_1, \ldots, \mathbf{n}_j)})$$

by

$$
\begin{aligned}
a_*(F_1, \ldots, &F_j)(\sigma_1 \times \cdots \times \sigma_j, o_1 \times \cdots \times o_j)) \\
&= i^{\epsilon(\zeta)}(a(\sigma_1 \times \cdots \times \sigma_j))_\star (F_1(\sigma_1, o_1), \ldots, F_j(\sigma_j, o_j)),
\end{aligned}
$$

where ζ is the block permutation that takes blocks $\mathbf{b}_1, \ldots, \mathbf{b}_j, \mathbf{c}_1, \ldots, \mathbf{c}_j$ of size $k_1, \ldots, k_j, \dim \sigma_1, \ldots, \dim \sigma_j$ into the order $\mathbf{b}_1, \mathbf{c}_1, \ldots, \mathbf{b}_j, \mathbf{c}_j$.

Lemma 6.3. *If $F_i \in \mathrm{ad}^{k_i}(\Delta^{\mathbf{n}_i})$ for $1 \leq i \leq j$ then $a_*(F_1, \ldots, F_j) \in \mathrm{ad}^{k_1 + \cdots + k_j}(\Delta^{(\mathbf{n}_1, \ldots, \mathbf{n}_j)})$.*

Proof. This is immediate from the extra property in Definition 3.3, Remark 3.6, and [LM, Definition 18.4(b)]. $\qquad\square$

Recalling that $(R_k)_{\mathbf{n}} = \mathrm{ad}^k(\Delta^{\mathbf{n}})$ with basepoint at the trivial ad (see [LM, Definitions 3.8(ii), 3.10(b) and 18.1(c)]), we have now constructed an operation

$$a_* : (R_{k_1})_{\mathbf{n}_1} \wedge \cdots \wedge (R_{k_j})_{\mathbf{n}_j} \to (R_{k_1 + \cdots + k_j})_{\mathbf{n}_1, \ldots, \mathbf{n}_j}$$

for each

$$a : U(\Delta^{\mathbf{n}_1} \times \cdots \times \Delta^{\mathbf{n}_j}) \to \Sigma_j.$$

For later use, we give the relation between a_* and the suspension map $\omega : S^1 \wedge R_k \to R_{k+1}$.

Definition 6.4. For a ball complex K, let

$$\Pi : U(\Delta^1 \times K) \to U(K)$$

be the map which takes $\sigma \times \tau$ to τ, where σ is a simplex of Δ^1 and τ is a simplex of K.

Lemma 6.5. *Let s be the 1-simplex of S^1, let $F_i \in (R_{k_i})_{\mathbf{n}_i}$ for $1 \leq i \leq j$, and let $a : U(\Delta^{\mathbf{n}_1} \times \cdots \times \Delta^{\mathbf{n}_j}) \to \Sigma_j$. Then*

$$\omega(s \wedge a_*(F_1 \wedge \cdots \wedge F_j)) = (a \circ \Pi)_*(\omega(s \wedge F_1) \wedge \cdots \wedge F_j).$$

Proof. This follows from Lemma 6.1 (because the permuted multiplication commutes with suspension). It can also be proved by a straightforward calculation using Definitions 3.5 and 4.12 and [LM, Definitions 17.4 and 3.7(ii)].   □

In the remainder of this section, we show that the action of the operations a_* can be described in a way that begins to resemble the action of an operad; this resemblance will be developed further in the next two sections.

Definition 6.6. (i) For $j, k \geq 0$ define an object $\mathcal{O}(j)_k$ of $\Sigma_k \mathrm{ss}_k$ by

$$(\mathcal{O}(j)_k)_{\mathbf{n}} = \mathrm{Map}(U(\Delta^{\mathbf{n}}), \Sigma_j)_+$$

(where the $+$ denotes a disjoint basepoint); the morphisms in $(\Delta_{\mathrm{inj}}^{\mathrm{op}})^{\times k}$ act in the evident way, and the morphisms of the form (α, id) with $\alpha \in \Sigma_k$ act by permuting the factors in $\Delta^{\mathbf{n}}$.

(ii) For $j \geq 0$ define $\mathcal{O}(j)$ to be the object of ss^{Σ} with k-th term $\mathcal{O}(j)_k$.

Definition 6.7. (i) For $A, B \in \Sigma_k \mathrm{ss}_k$, define the *degreewise smash product*

$$A \overline{\wedge} B \in \Sigma_k \mathrm{ss}_k$$

by

$$(A \overline{\wedge} B)_{\mathbf{n}} = A_{\mathbf{n}} \wedge B_{\mathbf{n}},$$

with the diagonal action of Σ_k.

(ii) For $\mathbf{X}, \mathbf{Y} \in \mathrm{ss}^{\Sigma}$, define $\mathbf{X} \overline{\wedge} \mathbf{Y} \in \mathrm{ss}^{\Sigma}$ by

$$(\mathbf{X} \overline{\wedge} \mathbf{Y})_k = X_k \overline{\wedge} Y_k.$$

Remark 6.8. The difference between the degreewise smash product $A \overline{\wedge} B$ and the previously defined smash product $A \wedge B$ is that the former is only defined when A and B are k-fold multisemisimplicial sets for the same k, and the result is again a k-fold multisemisimplicial set, whereas $A \wedge B$ is defined when A is k-fold and B is l-fold, and the result is $(k + l)$-fold.

Our next definition assembles the operations a_* for a given j into a single map.

Definition 6.9. Let $j \geq 0$. Define a map

$$\phi_j : \mathcal{O}(j) \overline{\wedge} \mathbf{R}^{\otimes j} \to \mathbf{R}$$

in ss^{Σ} by the formulas

$$\phi_j(a \wedge [e, F_1 \wedge \cdots \wedge F_j]) = a_*(F_1 \wedge \cdots \wedge F_j)$$

(where e denotes the identity element of the relevant symmetric group) and

$$\phi_j(a \wedge [\alpha, F_1 \wedge \cdots \wedge F_j]) = (\alpha, \mathrm{id})_* \phi_j((\alpha^{-1}, \mathrm{id})_* a \wedge [e, F_1 \wedge \cdots \wedge F_j]).$$

Lemma 6.10. *The map ϕ_j induces a map*

$$\psi_j : \mathcal{O}(j) \,\overline{\wedge}\, \mathbf{R}^{\wedge j} \to \mathbf{R}$$

in ss^{Σ}.

Proof. This is a straightforward calculation using Example 4.12, Definition 4.13, and [LM, Definition 17.3 and Lemma 18.7]. $\qquad\square$

7. A MONAD IN ss^{Σ}

In the next section we will show that there is a monad $\mathbb{P}$ in $\Sigma \mathcal{S}_{\mathrm{ss}}$ with the property that the maps ψ_j constructed in Lemma 6.10 give an action of $\mathbb{P}$ on $\mathbf{R}$. As preparation, in this section we prove the analogous result in ss^{Σ}; that is, we show that there is a monad $\mathbb{O}$ in ss^{Σ} for which the maps ϕ_j of Definition 6.9 give an action of $\mathbb{O}$ on $\mathbf{R}$.

First we observe that the collection of objects $\mathcal{O}(j)$ has a composition map analogous to that of an operad. Recall that May defines an operad $\mathcal{M}$ in the category of sets with $\mathcal{M}(j) = \Sigma_j$ ([May72, Definition 3.1(i)]). Let $\gamma_{\mathcal{M}}$ denote the composition operation in $\mathcal{M}$. Also recall Definition 4.9 and Notation 4.8.

Definition 7.1. Given $j_1, \ldots, j_i \geq 0$ define a map

$$\gamma : \mathcal{O}(i) \,\overline{\wedge}\, \big(\mathcal{O}(j_1) \otimes \cdots \otimes O(j_i)\big) \to \mathcal{O}(j_1 + \cdots + j_i)$$

in ss^{Σ} by the formulas

$$\gamma(a \wedge [e, b_1 \wedge \ldots \wedge b_i])(\sigma_1 \times \cdots \times \sigma_i) = \gamma_{\mathcal{M}}(a(\sigma_1 \times \cdots \times \sigma_i), b_1(\sigma_1), \ldots, b_i(\sigma_i))$$

(where e is the identity element of the relevant symmetric group) and

$$\gamma(a \wedge [\alpha, b_1 \wedge \ldots \wedge b_i]) = (\alpha, \mathrm{id})_* \gamma((\alpha^{-1}, \mathrm{id})_* a \wedge [e, b_1 \wedge \ldots \wedge b_i]).$$

In order to formulate the associativity property of γ, we note that for $\mathbf{X}_1, \ldots, \mathbf{X}_i, \mathbf{Y}_1, \ldots, \mathbf{Y}_i \in \mathrm{ss}^{\Sigma}$ there is a natural map

$$\chi : (\mathbf{X}_1 \,\overline{\wedge}\, \mathbf{Y}_1) \otimes \cdots \otimes (\mathbf{X}_i \,\overline{\wedge}\, \mathbf{Y}_i) \to (\mathbf{X}_1 \otimes \cdots \otimes \mathbf{X}_i) \,\overline{\wedge}\, (\mathbf{Y}_1 \otimes \cdots \otimes \mathbf{Y}_i)$$

given by

$$\chi([\alpha, x_1 \wedge y_1 \wedge \cdots \wedge x_i \wedge y_i]) = [\alpha, x_1 \wedge \cdots \wedge x_i] \wedge [\alpha, y_1 \wedge \cdots \wedge y_i].$$

Lemma 7.2. *The operation γ has the following associativity property:
the composite*

$$\mathcal{O}(i)\overline{\wedge}\Big(\big(\mathcal{O}(j_1)\overline{\wedge}(\mathcal{O}(l_{11})\otimes\cdots\otimes\mathcal{O}(l_{1j_1}))\big)\otimes\cdots\otimes\big(\mathcal{O}(j_i)\overline{\wedge}(\mathcal{O}(l_{i1})\otimes\cdots\otimes\mathcal{O}(l_{ij_i}))\big)\Big)$$

$$\xrightarrow{1\overline{\wedge}(\gamma\otimes\cdots\gamma)} \mathcal{O}(i)\,\overline{\wedge}\,\big(\mathcal{O}(l_{11}+\cdots+l_{1j_1})\otimes\cdots\otimes\mathcal{O}(l_{i1}+\cdots+l_{ij_i})\big)$$

$$\xrightarrow{\gamma} \mathcal{O}(l_{11}+\cdots+l_{ij_i})$$

is the same as the composite

$$\mathcal{O}(i)\overline{\wedge}\Big(\big(\mathcal{O}(j_1)\overline{\wedge}(\mathcal{O}(l_{11})\otimes\cdots\otimes\mathcal{O}(l_{1j_1}))\big)\otimes\cdots\otimes\big(\mathcal{O}(j_i)\overline{\wedge}(\mathcal{O}(l_{i1})\otimes\cdots\otimes\mathcal{O}(l_{ij_i}))\big)\Big)$$

$$\xrightarrow{1\overline{\wedge}\chi} \mathcal{O}(i)\,\overline{\wedge}\,\big(\mathcal{O}(j_1)\otimes\cdots\otimes\mathcal{O}(j_i)\big)\,\overline{\wedge}\,\big(\mathcal{O}(l_{11})\otimes\cdots\otimes\mathcal{O}(l_{i,j_i})\big)$$

$$\xrightarrow{\gamma\overline{\wedge}1} \mathcal{O}(j_1+\cdots+j_i)\,\overline{\wedge}\,\big(\mathcal{O}(l_{11})\otimes\cdots\otimes\mathcal{O}(l_{i,j_i})\big)$$

$$\xrightarrow{\gamma} \mathcal{O}(l_{11}+\cdots+l_{ij_i}).$$

$$\square$$

To formulate the unital property of γ we first need to consider the unit object for the operation $\overline{\wedge}$.

Definition 7.3. Let $\overline{S}_k$ be the object of $\Sigma_k \mathrm{ss}_k$ which has a copy of S^0 in each multidegree (with each morphism of $\Sigma_k \ltimes (\Delta_{\mathrm{inj}}^{\mathrm{op}})^{\times k}$ acting as the identity of S^0), and let $\overline{\mathbf{S}}$ be the object of ss^Σ with k-th term $\overline{S}_k$.

Remark 7.4. (i) $\overline{\mathbf{S}}\,\overline{\wedge}\,\mathbf{X} \cong \mathbf{X}$ for any $\mathbf{X} \in \mathrm{ss}^\Sigma$.
 (ii) $\mathcal{O}(0)$ and $\mathcal{O}(1)$ are both equal to $\overline{\mathbf{S}}$.
 (iii) $\overline{\mathbf{S}}$ is a commutative monoid in ss^Σ with multiplication

$$m : \overline{\mathbf{S}} \otimes \overline{\mathbf{S}} \to \overline{\mathbf{S}}$$

given by

$$m([\alpha, s_1 \wedge s_2]) = t,$$

where s_1 and s_2 are any nontrivial simplices and t is the nontrivial simplex in the relevant multidegree.

Lemma 7.5. *The operation γ has the following unital property: the diagrams*

$$
\begin{array}{ccc}
\mathcal{O}(j)\,\overline{\wedge}\,\overline{\mathbf{S}}^{\otimes j} & \xrightarrow{\ =\ } & \mathcal{O}(j)\,\overline{\wedge}\,\mathcal{O}(1)^{\otimes j} \\[2pt]
\scriptstyle{1\overline{\wedge}m}\downarrow & & \downarrow\scriptstyle{\gamma} \\[2pt]
\mathcal{O}(j)\,\overline{\wedge}\,\overline{\mathbf{S}} & \xrightarrow{\ \cong\ } & \mathcal{O}(j)
\end{array}
$$

and

$$\overline{\mathbf{S}} \,\overline{\wedge}\, \mathcal{O}(j) \xrightarrow{\ \cong\ } \mathcal{O}(j)$$

$$\mathcal{O}(1) \,\overline{\wedge}\, \mathcal{O}(j)$$

with the left vertical map labeled $=$ and the diagonal map labeled γ.

commute. $\square$

To complete the analogy between γ and the composition map of an operad, we need an equivariance property.

Definition 7.6. Define a right action of Σ_j on $\mathcal{O}(j)$ by

$$(a\alpha)(\sigma) = a(\sigma) \cdot \alpha,$$

where $a \in \mathrm{Map}(U(\Delta^{\mathbf{n}}), \Sigma_j)_+$, $\sigma \in U(\Delta^{\mathbf{n}})$, and $\cdot$ is multiplication in Σ_j.

Lemma 7.7. *(i) The following diagram commutes for all $\alpha \in \Sigma_i$.*

$$
\begin{array}{ccc}
\mathcal{O}(i) \,\overline{\wedge}\, \big(\mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_i)\big) & \longrightarrow & \mathcal{O}(i) \,\overline{\wedge}\, \big(\mathcal{O}(j_{\alpha^{-1}(1)}) \otimes \cdots \otimes \mathcal{O}(j_{\alpha^{-1}(i)})\big) \\
\Big\downarrow {\scriptstyle \alpha\overline{\wedge}(\beta_1 \otimes \cdots \otimes \beta_i)} & & \Big\downarrow {\scriptstyle \gamma} \\
& & \mathcal{O}(j_1 + \cdots + j_i) \\
& & \Big\downarrow {\scriptstyle \gamma_M(\alpha, \beta_1, \ldots, \beta_i)} \\
\mathcal{O}(i) \,\overline{\wedge}\, \big(\mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_i)\big) & \xrightarrow{\ \gamma\ } & \mathcal{O}(j_1 + \cdots + j_i)
\end{array}
$$

$\square$

Now we use the data defined so far to construct a monad in the category ss^{Σ}.

Definition 7.8. (i) For $\mathbf{X} \in \mathrm{ss}^{\Sigma}$, give $\mathcal{O}(j) \,\overline{\wedge}\, \mathbf{X}^{\otimes j}$ the diagonal right Σ_j action.

(ii) Define a functor $\mathbb{O} : \mathrm{ss}^{\Sigma} \to \mathrm{ss}^{\Sigma}$ by

$$\mathbb{O}(\mathbf{X}) = \bigvee_{j \geq 0} \big(\mathcal{O}(j) \,\overline{\wedge}\, \mathbf{X}^{\otimes j}\big)/\Sigma_j.$$

(iii) Define a natural transformation

$$\iota : \mathbf{X} \to \mathbb{O}\mathbf{X}$$

to be the composite

$$\mathbf{X} \xrightarrow{\cong} \overline{\mathbf{S}} \,\overline{\wedge}\, \mathbf{X} = \mathcal{O}(1) \,\overline{\wedge}\, \mathbf{X} \hookrightarrow \mathbb{O}(\mathbf{X}).$$

(iv) Define

$$\mu : \mathbb{O}\mathbb{O}\mathbf{X} \to \mathbb{O}\mathbf{X}$$

to be the natural transformation induced by the maps

$$\mathcal{O}(i) \overline{\wedge} \left(\left(\mathcal{O}(j_1) \overline{\wedge} \mathbf{X}^{\otimes j_1} \right) \otimes \cdots \otimes \left(\mathcal{O}(j_i) \overline{\wedge} \mathbf{X}^{\otimes j_i} \right) \right)$$

$$\xrightarrow{1 \overline{\wedge} \chi} \left(\mathcal{O}(i) \overline{\wedge} \left(\mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_i) \right) \right) \overline{\wedge} \mathbf{X}^{\otimes(j_1 + \cdots + j_i)}$$

$$\xrightarrow{\gamma \overline{\wedge} 1} \mathcal{O}(j_1 + \cdots + j_i) \overline{\wedge} \mathbf{X}^{\otimes(j_1 + \cdots + j_i)}.$$

Proposition 7.9. *The transformations μ and ι define a monad structure on $\mathbb{O}$.*

Proof. This is immediate from Lemmas 7.2 and 7.5. $\square$

We conclude this section by giving the action of $\mathbb{O}$ on $\mathbf{R}$. Observe that the map

$$\phi_j : \mathcal{O}(j) \overline{\wedge} \mathbf{R}^{\otimes j} \to \mathbf{R}$$

of Definition 6.9 induces a map

(7.1) $\left(\mathcal{O}(j) \overline{\wedge} \mathbf{R}^{\otimes j} \right)/\Sigma_j \to \mathbf{R}.$

Definition 7.10. Define

$$\nu : \mathbb{O}\mathbf{R} \to \mathbf{R}$$

to be the map whose restriction to $\left(\mathcal{O}(j) \overline{\wedge} \mathbf{X}^{\otimes j} \right)/\Sigma_j$ is the map (7.1).

Proposition 7.11. *ν is an action of $\mathbb{O}$ on $\mathbf{R}$.*

Proof. We need to show that the diagrams

$$\mathbf{R} \xrightarrow{\iota} \mathbb{O}\mathbf{R}$$
$$\underset{=}{\searrow} \quad \downarrow \nu$$
$$\mathbf{R}$$

and

$$\mathbb{O}\mathbb{O}\mathbf{R} \xrightarrow{\mu} \mathbb{O}\mathcal{R}$$
$$\mathbb{O}\nu \downarrow \qquad \qquad \nu \downarrow$$
$$\mathbb{O}\mathcal{R} \xrightarrow{\nu} \mathbf{R}$$

commute. The first is obvious and for the second it suffices to check that the composite

$$\mathcal{O}(i) \overline{\wedge} \left(\left(\mathcal{O}(j_1) \overline{\wedge} \mathbf{R}^{\otimes j_1} \right) \otimes \cdots \otimes \left(\mathcal{O}(j_i) \overline{\wedge} \mathbf{R}^{\otimes j_i} \right) \right)$$

$$\xrightarrow{1 \overline{\wedge} \chi} \mathcal{O}(i) \overline{\wedge} \left(\mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_i) \right) \overline{\wedge} \mathbf{R}^{\otimes(j_1 + \cdots + j_i)}$$

$$\xrightarrow{\gamma \overline{\wedge} 1} \mathcal{O}(j_1 + \cdots + j_i) \overline{\wedge} \mathbf{R}^{\otimes(j_1 + \cdots + j_i)} \xrightarrow{\phi} \mathbf{R}$$

is the same as the composite

$$\mathcal{O}(i)\overline{\wedge}\Big(\big(\mathcal{O}(j_1)\overline{\wedge}\mathbf{R}^{\otimes j_1}\big)\otimes\cdots\otimes\big(\mathcal{O}(j_i)\overline{\wedge}\mathbf{R}^{\otimes j_i}\big)\Big) \xrightarrow{1\overline{\wedge}(\phi\otimes\cdots\phi)} \mathcal{O}(i)\overline{\wedge}\mathbf{R}^{\otimes i} \xrightarrow{\phi} \mathbf{R}.$$

$\square$

8. A monad in $\Sigma\mathcal{S}_{\mathrm{ss}}$

We begin by giving $\mathcal{O}(j)\overline{\wedge}\mathbf{X}^{\wedge j}$ the structure of a multisemisimplicial symmetric spectrum when $\mathbf{X} \in \mathrm{ss}^{\Sigma}$. The definition is motivated by Lemma 6.5. Recall Definition 6.4.

Definition 8.1. Let $j, k \geq 0$. Let s be the 1-simplex of S^1. Define

$$\omega : S^1 \wedge (\mathcal{O}(j)\,\overline{\wedge}\,\mathbf{X}^{\wedge j})_k \to (\mathcal{O}(j)\,\overline{\wedge}\,\mathbf{X}^{\wedge j})_{k+1}$$

as follows: for $a \in (\mathcal{O}(j)_k)_{\mathbf{n}}$ and $x \in ((X^{\wedge j})_k)_{\mathbf{n}}$, let

$$\omega(s \wedge (a \wedge x)) = (a \circ \Pi) \wedge \omega(s \wedge x).$$

Definition 8.2. (i) For $\mathbf{X} \in \Sigma\mathcal{S}_{\mathrm{ss}}$, give $\mathcal{O}(j)\,\overline{\wedge}\,\mathbf{X}^{\wedge j}$ the diagonal right Σ_j action.
 (ii) Define a functor $\mathbb{P} : \Sigma\mathcal{S}_{\mathrm{ss}} \to \Sigma\mathcal{S}_{\mathrm{ss}}$ by

$$\mathbb{P}(\mathbf{X}) = \bigvee_{j\geq 0}\big(\mathcal{O}(j)\,\overline{\wedge}\,\mathbf{X}^{\wedge j}\big)/\Sigma_j.$$

To give $\mathbb{P}$ a monad structure we need

Lemma 8.3. *The composite in Definition 7.8(iv) induces a map*

$$\mathcal{O}(i)\,\overline{\wedge}\,\Big(\big(\mathcal{O}(j_1)\,\overline{\wedge}\,\mathbf{X}^{\wedge j_1}\big)\wedge\cdots\wedge\big(\mathcal{O}(j_i)\,\overline{\wedge}\,\mathbf{X}^{\wedge j_i}\big)\Big)$$
$$\to \mathcal{O}(j_1+\cdots+j_i)\,\overline{\wedge}\,\mathbf{X}^{\wedge(j_1+\cdots+j_i)}.$$

in $\Sigma\mathcal{S}_{\mathrm{ss}}$. $\square$

Definition 8.4. (i) Define a natural transformation

$$\iota : \mathbf{X} \to \mathbb{P}\mathbf{X}$$

to be the composite

$$\mathbf{X} \xrightarrow{\cong} \overline{\mathbf{S}}\,\overline{\wedge}\,\mathbf{X} = \mathcal{O}(1)\,\overline{\wedge}\,\mathbf{X} \hookrightarrow \mathbb{P}(\mathbf{X}).$$

 (ii) Define

$$\mu : \mathbb{P}\mathbb{P}\mathbf{X} \to \mathbb{P}\mathbf{X}$$

to be the natural transformation induced by the maps constructed in Lemma 8.3.

Proposition 8.5. *The transformations μ and ι define a monad structure on $\mathbb{P}$.*

Proof. This follows from Proposition 7.9 by passage to quotients. □

Next we give the action of $\mathbb{P}$ on $\mathbf{R}$. By Definition 8.1 and Lemma 6.5, the map

$$\psi_j : \mathcal{O}(j)\,\overline{\wedge}\,\mathbf{R}^{\wedge j} \to \mathbf{R}$$

of Lemma 6.10 is a map in $\Sigma\mathcal{S}_{\mathrm{ss}}$. It induces a map

(8.1) $$\big(\mathcal{O}(j)\,\overline{\wedge}\,\mathbf{R}^{\wedge j}\big)/\Sigma_j \to \mathbf{R}.$$

in $\Sigma\mathcal{S}_{\mathrm{ss}}$.

Definition 8.6. Define

$$\nu : \mathbb{P}\mathbf{R} \to \mathbf{R}$$

to be the map whose restriction to $\big(\mathcal{O}(j)\,\overline{\wedge}\,\mathbf{X}^{\wedge j}\big)/\Sigma_j$ is the map (8.1).

Proposition 8.7. *ν is an action of $\mathbb{P}$ on $\mathbf{R}$.*

Proof. This follows from Proposition 7.11 by passage to quotients. □

For use in Section 10 we record a lemma.

Lemma 8.8. *(i) There is a functor Υ from $\mathbb{P}$ algebras to monoids in $\Sigma\mathcal{S}_{\mathrm{ss}}$ (with respect to $\wedge$) which is the identity on objects.*
(ii) The geometric realization of $\Upsilon(\mathbf{R})$ is the symmetric ring spectrum $\mathbf{M}$ of [LM, Theorem 18.5].

Proof. Part (i). Let $\mathbb{A}$ be the monad

$$\mathbb{A}(\mathbf{X}) = \bigvee_{j\geq 0} \mathbf{X}^{\wedge j}.$$

Then a monoid in $\Sigma\mathcal{S}_{\mathrm{ss}}$ is the same thing as an $\mathbb{A}$-algebra, so it suffices to give a map of monads from $\mathbb{A}$ to $\mathbb{P}$.

For each $j, k \geq 0$ and each k-fold multi-index $\mathbf{n}$, define an element

$$e_{j,k,\mathbf{n}} \in (\mathcal{O}(j)_k)_{\mathbf{n}}$$

to be the constant function $U(\Delta^{\mathbf{n}}) \to \Sigma_j$ whose value is the identity element of Σ_j. Next define a map

$$\overline{\mathbf{S}} \to \mathcal{O}(j)$$

by taking the nontrivial simplex of $(\overline{\mathbf{S}}_k)_{\mathbf{n}}$ to $e_{j,k,\mathbf{n}}$.

Now the composite

$$\mathbb{A}(\mathbf{X}) = \bigvee_{j\geq 0} \mathbf{X}^{\wedge j} \cong \bigvee_{j\geq 0} \overline{\mathbf{S}}\,\overline{\wedge}\,\mathbf{X}^{\wedge j} \to \bigvee_{j\geq 0}\big(\mathcal{O}(j)\,\overline{\wedge}\,\mathbf{X}^{\wedge j}\big)/\Sigma_j = \mathbb{P}(\mathbf{X})$$

is a map of monads.

Part (ii) is an easy consequence of the definitions. □

9. Degreewise smash product and geometric realization

For the proof of Theorem 1.1 we need to know the relation between $\overline{\wedge}$ and geometric realization.

There is a natural map

$$\kappa : |A \overline{\wedge} B| \to |A| \wedge |B|$$

defined by

$$\kappa([u, x \wedge y]) = [u, x] \wedge [u, y].$$

The analogous map for multisimplicial sets is a homeomorphism, but the situation for multisemisimplicial sets is more delicate.

Definition 9.1. A multisemisimplicial set *has compatible degeneracies* if it is in the image of the forgetful functor from multisimplicial sets to multisemisimplicial sets.

Example 9.2. (i) One can define compatible degeneracies on $\mathcal{O}(j)_k$ for each $j, k \geq 0$ by using the codegeneracy maps between the $\Delta^{\mathbf{n}}$.

(ii) If $\mathbf{X} \in \Sigma \mathcal{S}_{\mathrm{ss}}$ and X_k has compatible degeneracies for all k then each $\left(\mathcal{O}(j) \overline{\wedge} \mathbf{X}^{\wedge j}\right)_k$ has compatible degeneracies.

Proposition 9.3. *If the underlying multisemisimplicial sets of A and B have compatible degeneracies then κ is a weak equivalence.*

Proof. Let $\tilde{A}$ and $\tilde{B}$ be multisimplicial sets whose underlying multisemisimplicial sets are A and B. Then the underlying multisemisimplicial set of the degreewise smash product $\tilde{A} \overline{\wedge} \tilde{B}$ is $A \overline{\wedge} B$. Consider the following commutative diagram, where $| \; |$ in the bottom row denotes realization of multisimplicial sets, $\tilde{\kappa}$ is defined analogously to κ, and the vertical maps collapse the degeneracies:

$$
\begin{array}{ccc}
|A \overline{\wedge} B| & \xrightarrow{\kappa} & |A| \wedge |B| \\
\downarrow & & \downarrow \\
|\tilde{A} \overline{\wedge} \tilde{B}| & \xrightarrow{\tilde{\kappa}} & |\tilde{A}| \wedge |\tilde{B}|
\end{array}
$$

The map $\tilde{\kappa}$ is a homeomorphism, and the vertical arrows are weak equivalences by the multisimplicial analogue of [Seg74, Lemma A.5], so κ is a weak equivalence. $\square$

Next we give a sufficient condition for a multisemisimplicial set to have compatible degeneracies. Let D^n denote the semisimplicial set consisting of the nondegenerate simplices of the standard simplicial n-simplex. For a multi-index $\mathbf{n}$, let $D^{\mathbf{n}}$ denote the k-fold multisemisimplicial set

$$D^{n_1} \times \cdots \times D^{n_k}.$$

Definition 9.4. (i) A *horn* in $D^{\mathbf{n}}$ is a subcomplex E which contains all elements of $D^{\mathbf{n}}$ except for the top-dimensional element and one of its faces.

(ii) A k-fold multisemisimplicial set A satisfies the *multi-Kan condition* if every map from a horn in $D^{\mathbf{n}}$ to A extends to a map $D^{\mathbf{n}} \to A$.

The following result is proved in [McC].

Proposition 9.5. *If A satisfies the multi-Kan condition then it has compatible degeneracies.*

Our next result is proved in the same way as [LM, Lemma 15.12] and does not require the ad theory to be commutative.

Proposition 9.6. *For each k, R_k satisfies the multi-Kan condition.*

10. Rectification

In this section we complete the proof of Theorem 1.1.

First we consider a monad in $\Sigma \mathcal{S}_{\mathrm{ss}}$ which is simpler than $\mathbb{P}$.

Definition 10.1. (i) Define $\mathbb{P}'(\mathbf{X})$ to be $\bigvee_{j \geq 0} \mathbf{X}^{\wedge j}/\Sigma_j$.

(ii) For each $j \geq 0$, let

$$\xi_j : \mathcal{O}(j) \to \overline{\mathbf{S}}$$

be the map which takes each nontrivial simplex of $\mathcal{O}(j)_k$ to the nontrivial simplex of $\overline{S}_k$ in the same multidegree. Define a natural transformation

$$\Xi : \mathbb{P} \to \mathbb{P}'$$

to be the wedge of the composites

$$\big(\mathcal{O}(j) \,\overline{\wedge}\, \mathbf{X}^{\wedge j}\big)/\Sigma_j \xrightarrow{\xi_j \overline{\wedge} 1} \big(\overline{\mathbf{S}} \,\overline{\wedge}\, \mathbf{X}^{\wedge j}\big)/\Sigma_j \xrightarrow{\cong} \mathbf{X}^{\wedge j}/\Sigma_j.$$

Proposition 10.2. (i) *An algebra over $\mathbb{P}'$ is the same thing as a commutative monoid in $\Sigma \mathcal{S}_{\mathrm{ss}}$.*

(ii) *Ξ is a map of monads.*

(iii) *Suppose that each X_k has compatible degeneracies (see Definition 9.1). Let $\mathbb{P}^q$ denote the q-th iterate of $\mathbb{P}$. Then each map*

$$\Xi : \mathbb{P}^q(\mathbf{X}) \to \mathbb{P}'\mathbb{P}^{q-1}(\mathbf{X})$$

is a weak equivalence.

Parts (i) and (ii) are immediate from the definitions. Part (iii) will be proved at the end of this section.

Proof of Theorem 1.1. We apply the monadic bar construction ([May72, Construction 9.6]) to obtain simplicial objects $B_\bullet(\mathbb{P}, \mathbb{P}, \mathbf{R})$ and $B_\bullet(\mathbb{P}', \mathbb{P}, \mathbf{R})$ in $\Sigma\mathcal{S}_{\mathrm{ss}}$. We write $\mathbf{R}_\bullet$ for the constant simplicial object which is $\mathbf{R}$ in each simplicial degree. There are maps of simplicial $\mathbb{P}$-algebras

$$(10.1) \qquad \mathbf{R}_\bullet \xleftarrow{\varepsilon} B_\bullet(\mathbb{P}, \mathbb{P}, \mathbf{R}) \xrightarrow{\Xi_\bullet} B_\bullet(\mathbb{P}', \mathbb{P}, \mathbf{R}),$$

where ε is induced by the action of $\mathbb{P}$ on $\mathbf{R}$ (see [May72, Lemma 9.2(ii)]). The map ε is a homotopy equivalence of simplicial objects ([May72, Proposition 9.8]) and the map $\Xi_\bullet$ is a weak equivalence in each simplicial degree by Propositions 9.5, 9.6 and 10.2(iii). $B_\bullet(\mathbb{P}', \mathbb{P}, \mathbf{R})$ is a simplicial algebra over $\mathbb{P}'$, which by Proposition 10.2(i) is the same thing as a simplicial commutative monoid in $\Sigma\mathcal{S}_{\mathrm{ss}}$. Moreover, by Lemma 8.8(i), $\mathbf{R}_\bullet$ and $B_\bullet(\mathbb{P}, \mathbb{P}, \mathbf{R})$ are simplicial monoids, and ε and $\Xi_\bullet$ are maps of simplicial monoids.

The objects of the diagram (10.1) are simplicial objects in $\Sigma\mathcal{S}_{\mathrm{ss}}$. We obtain a diagram

$$(10.2) \qquad |\mathbf{R}_\bullet| \xleftarrow{|\varepsilon|} |B_\bullet(\mathbb{P}, \mathbb{P}, \mathbf{R})| \xrightarrow{|\Xi_\bullet|} |B_\bullet(\mathbb{P}', \mathbb{P}, \mathbf{R})|$$

of simplicial objects in $\Sigma\mathcal{S}$ (the category of symmetric spectra) by applying the geometric realization functor $\Sigma\mathcal{S}_{\mathrm{ss}} \to \Sigma\mathcal{S}$ to the diagram (10.1) in each simplicial degree. The map $|\varepsilon|$ is a homotopy equivalence of simplicial objects and the map $|\Xi_\bullet|$ is a weak equivalence in each simplicial degree. The object $|B_\bullet(\mathbb{P}', \mathbb{P}, \mathbf{R})|$ is a simplicial commutative symmetric ring spectrum, the objects $|\mathbf{R}_\bullet|$ and $|B_\bullet(\mathbb{P}, \mathbb{P}, \mathbf{R})|$ are simplicial symmetric ring spectra, and the maps $|\varepsilon|$ and $|\Xi_\bullet|$ are maps of simplicial symmetric ring spectra.

Finally, we apply geometric realization to the diagram (10.2). We define $\mathbf{M}^{\mathrm{comm}}$ to be $||B_\bullet(\mathbb{P}', \mathbb{P}, \mathbf{R})||$. Now we have a diagram

$$(10.3) \quad \mathbf{M} = |\mathbf{R}| \xleftarrow{||\varepsilon||} ||B_\bullet(\mathbb{P}, \mathbb{P}, \mathbf{R})|| \xrightarrow{||\Xi_\bullet||} ||B_\bullet(\mathbb{P}', \mathbb{P}, \mathbf{R})|| = \mathbf{M}^{\mathrm{comm}}$$

in $\Sigma\mathcal{S}$. The map $||\varepsilon||$ is a homotopy equivalence (cf. [May72, Corollary 11.9]) and the map $||\Xi_\bullet||$ is a weak equivalence by [Ree, Theorem E]. $\mathbf{M}^{\mathrm{comm}}$ is a commutative symmetric ring spectrum, $\mathbf{M}$ is the symmetric ring spectrum of [LM, Theorem 18.5] (by Lemma 8.8(ii)), $||B_\bullet(\mathbb{P}, \mathbb{P}, \mathbf{R})||$ is a symmetric ring spectrum, and $||\varepsilon||$ and $||\Xi_\bullet||$ are maps of symmetric ring spectra. $\qquad\square$

We conclude this section with the proof of part (iii) of Proposition 10.2. First we need a lemma (which for later use we state in more generality than we immediately need). Recall that a preorder is a set with a reflexive and transitive relation $\leq$. Examples are Σ_j, with

every element $\leq$ every other, and $U(K)$ (see Definition 6.2(i)), with $\leq$ induced by inclusions of cells.

Lemma 10.3. *Let P be a preorder with an element which is $\geq$ all other elements, and let $k \geq 0$. Define a k-fold multisemisimplicial set A by*

$$A_{\mathbf{n}} = \mathrm{Map}_{\mathrm{preorder}}(U(\Delta^{\mathbf{n}}), P).$$

Then

(i) *A has compatible degeneracies, and*
(ii) *A is weakly equivalent to a point.*

Proof. For (i), we can give A compatible degeneracies by using the codegeneracy maps between the $\Delta^{\mathbf{n}}$.

Part (ii). Let $\tilde{A}$ be a multisimplicial set whose underlying multi-semisimiplicial set is A. Let $d\tilde{A}$ be its diagonal. The multisimplicial analogue of [Seg74, Lemma A.5] implies that $|A|$ is weakly equivalent to $|\tilde{A}|$, and it is well-known that the latter is homeomorphic to $|d\tilde{A}|$. It therefore suffices to show that the simplicial set $d\tilde{A}$ is weakly equivalent to a point.

Let Δ^n_{simp} denote the standard simplicial n simplex and let $\partial\Delta^n_{\mathrm{simp}}$ denote its boundary. Then it suffices by [GJ99, Theorem I.11.2] to show that every map from $\partial\Delta^n_{\mathrm{simp}}$ to $d\tilde{A}$ extends to Δ^n_{simp}.

Let D^n (respectively ∂D^n) be the semisimplicial set consisting of the nondegenerate simplices of Δ^n_{simp} (resp., $\partial\Delta^n_{\mathrm{simp}}$). Since Δ^n_{simp} (resp., $\partial\Delta^n_{\mathrm{simp}}$) is the free simplicial set generated by D^n (respectively ∂D^n), it suffices to show that every semisimplicial map from ∂D^n to $d\tilde{A}$ extends to D^n, and this is obvious from the definition of A. $\square$

Note that if P is Σ_j with the preorder described above then A_+ is $\tilde{\mathcal{O}}(j)_k$.

Proof of 10.2(iii). We begin with the case $q = 1$, so we want to show that the map $\Xi : \mathbb{P}(\mathbf{X}) \to \mathbb{P}'(\mathbf{X})$ is a weak equivalence. It suffices to show that the map

$$(\mathcal{O}(j) \overline{\wedge} \mathbf{X}^{\wedge j})/\Sigma_j \to \mathbf{X}^{\wedge j}/\Sigma_j$$

is a weak equivalence for each j. Equation (4.1) shows that the Σ_j actions are free away from the basepoint, so it suffices to show that each map

$$\mathcal{O}(j) \overline{\wedge} \mathbf{X}^{\wedge j} \to \mathbf{X}^{\wedge j}$$

is a weak equivalence. Now the object $\left(\mathcal{O}(j) \overline{\wedge} \mathbf{X}^{\wedge j}\right)_k$ is

$$\bigvee_{k_1+\cdots+k_j=k} \mathcal{O}(j)_k \overline{\wedge} I^{\Sigma_k}_{\Sigma_{k_1} \times \cdots \times \Sigma_{k_j}}\left(\mathbf{X}_{k_1} \wedge \cdots \wedge \mathbf{X}_{k_j}\right)$$

and we have

$$\mathcal{O}(j)_k \overline{\wedge} I^{\Sigma_k}_{\Sigma_{k_1} \times \cdots \times \Sigma_{k_j}} \big(\mathbf{X}_{k_1} \wedge \cdots \wedge \mathbf{X}_{k_j} \big) \cong I^{\Sigma_k}_{\Sigma_{k_1} \times \cdots \times \Sigma_{k_j}} \Big(\mathcal{O}(j)_k \overline{\wedge} \big(\mathbf{X}_{k_1} \wedge \cdots \wedge \mathbf{X}_{k_j} \big) \Big),$$

so it suffices by Proposition 5.4 to show that each map

$$\mathcal{O}(j)_k \overline{\wedge} (\mathbf{X}_{k_1} \wedge \cdots \wedge \mathbf{X}_{k_j}) \to \mathbf{X}_{k_1} \wedge \cdots \wedge \mathbf{X}_{k_j}$$

is a weak equivalence, and this follows from Example 9.2, Proposition 9.3, and Lemma 10.3.

The general case follows from the case $q = 1$ and Example 9.2(ii). $\square$

11. Relaxed symmetric Poincaré complexes

For a commutative ring R with the trivial involution, we would like to apply Theorem 1.1 to obtain a commutative model for the symmetric L-spectrum of R. However, the ad theory ad^R defined in Section 9 of [LM], with the product defined in [LM, Definition 9.12], is not commutative. The difficulty is that this product is defined using a noncommutative coproduct

$$\Delta : W \to W \otimes W.$$

In this section and the next we give an equivalent ad theory which is commutative.

Fix a ring R with involution. Recall the definition of the category $\mathcal{D}$ from [LM, Definition 9.2(v)].

Definition 11.1. A *relaxed quasi-symmetric complex of dimension n* is a quadruple (C, D, β, φ), where C is an object of $\mathcal{D}$, D is an object of $\mathcal{D}$ with a $\mathbb{Z}/2$ action, β is a quasi-isomorphism $C^t \otimes_R C \to D$ which is also a $\mathbb{Z}/2$ equivariant map, and φ is an element of $D_n^{\mathbb{Z}/2}$.

Example 11.2. (i) If (C, φ) is a quasi-symmetric complex as defined in [LM, Definition 9.3], then the quadruple $(C, (C^t \otimes_R C)^W, \beta, \varphi)$ is a relaxed quasi-symmetric complex, where $\beta : C^t \otimes_R C \to (C^t \otimes_R C)^W$ is induced by the augmentation $W \to \mathbb{Z}$.

(ii) Relaxed quasi-symmetric complexes arise naturally from the construction of the symmetric signature of a Witt space given in [FM]; see [BLM].

Definition 11.3. We define a category $\mathcal{A}^R_{\mathrm{rel}}$ (the rel stands for relaxed) as follows. The objects of $\mathcal{A}^R_{\mathrm{rel}}$ are the relaxed quasi-symmetric complexes. A morphism $(C, D, \beta, \varphi) \to (C', D', \beta', \varphi')$ is a pair $(f : C \to C', g : D \to D')$, where f and g are R-linear chain maps, g is $\mathbb{Z}/2$ equivariant, $g\beta = \beta'(f \otimes f)$, and (if $\dim \varphi = \dim \varphi'$) $g_*(\varphi) = \varphi'$.

$\mathcal{A}^R_{\mathrm{rel}}$ is a balanced ([LM, Definition 5.1]) $\mathbb{Z}$-graded category, where i takes (C, D, β, φ) to $(C, D, \beta, -\varphi)$ and $\emptyset_n$ is the n-dimensional object for which C and D are zero in all degrees.

Remark 11.4. The construction of Example 11.2(i) gives a morphism

$$\mathcal{A}^R \to \mathcal{A}^R_{\mathrm{rel}}$$

of $\mathbb{Z}$-graded categories.

Next we must say what the K-ads with values in $\mathcal{A}^R_{\mathrm{rel}}$ are. We need some preliminary definitions and a lemma. For a balanced pre K-ad F we will use the notation

$$F(\sigma, o) = (C_\sigma, D_\sigma, \beta_\sigma, \varphi_{\sigma, o}).$$

Recall [LM, Definition 9.7].

Definition 11.5. A balanced pre K-ad F is *well-behaved* if C and D are well-behaved.

Next recall [LM, Definition 12.2].

Lemma 11.6. *Let F be a well-behaved pre K-ad. Then*
  (i) $C^t \otimes_R C$ *is well-behaved, and*
  (ii) *the map*

$$\beta_* : H_*((C^t \otimes_R C)_\sigma, (C^t \otimes_R C)_{\partial\sigma}) \to H_*(D_\sigma, D_{\partial\sigma})$$

is an isomorphism.

Proof. Part (i) follows from the fact that $C_{\partial\sigma} \to C_\sigma$ is the inclusion of a direct summand for each σ (see [LM, Definition 9.7(b) and 9.6(ii)]).

For part (ii), first observe that the fact that $C^t \otimes_R C$ and D are well-behaved implies that they are Reedy cofibrant ([Hir03, Definition 15.3.3(2)]). The colim that defines $(C^t \otimes_R C)_{\partial\sigma}$ is a hocolim by [Hir03, Theorem 19.9.1(1) and Proposition 15.10.2(2)], and similarly for $D_{\partial\sigma}$, so the map

$$\beta_* : H_*((C^t \otimes_R C)_{\partial\sigma}) \to H_*(D_{\partial\sigma})$$

is an isomorphism by [Hir03, Theorem 19.4.2(1)], and this implies the lemma. $\square$

Recall [LM, Example 3.12].

Definition 11.7. A balanced pre K-ad F is *closed* if, for each σ, the map

$$\mathrm{cl}(\sigma) \to D_\sigma$$

which takes $\langle \tau, o \rangle$ to $\varphi_{\tau, o}$ is a chain map.

Note that if F is closed then $\varphi_{\sigma,o}$ represents an element $[\varphi_{\sigma,o}] \in H_*(D_\sigma, D_{\partial\sigma})$.

Definition 11.8. A pre K-ad F is a K-ad if
  (a) it is balanced, well-behaved and closed, and
  (b) for each σ the slant product with $\beta_*^{-1}([\varphi_{\sigma,o}])$ is an isomorphism
$$H^*(\mathrm{Hom}_R(C_\sigma, R)) \to H_{\dim\sigma - \deg F - *}(C_\sigma/C_{\partial\sigma}).$$

We write $\mathrm{ad}_{\mathrm{rel}}^R$ for the set of K ads with values in $\mathcal{A}_{\mathrm{rel}}^R$.

Remark 11.9. The morphism of Remark 11.4 takes ads to ads.

Theorem 11.10. $\mathrm{ad}_{\mathrm{rel}}^R$ *is an ad theory.*

Remark 11.11. When R is commutative with the trivial involution, the proof will show that $\mathrm{ad}_{\mathrm{rel}}^R$ is a commutative ad theory.

For the proof of Theorem 11.10 we need a product operation on ads.

Definition 11.12. (i) For $i = 1, 2$, let R_i be a ring with involution and let $(C^i, D^i, \beta^i, \varphi^i)$ be an object of $\mathcal{A}_{\mathrm{rel}}^R$. Define
$$(C^1, D^1, \beta^1, \varphi^1) \otimes (C^2, D^2, \beta^2, \varphi^2)$$
to be the following object of $\mathcal{A}_{\mathrm{rel}}^{R_1 \otimes R_2}$:
$$(C^1 \otimes C^2, D^1 \otimes D^2, \gamma, \varphi^1 \otimes \varphi^2),$$
where γ is the composite
$$(C^1 \otimes C^2)^t \otimes_{R_1 \otimes R_2} (C^1 \otimes C^2) \cong ((C^1)^t \otimes_{R_1} C^1) \otimes ((C^2)^t \otimes_{R_2} C^2)$$
$$\xrightarrow{\beta^1 \otimes \beta^2} D^1 \otimes D^2.$$

(ii) For $i = 1, 2$, suppose given a ball complex K_i and a pre K_i-ad F_i of degree k_i with values in $\mathcal{A}_{\mathrm{rel}}^{R_i}$. Define a pre $(K_1 \times K_2)$-ad $F_1 \otimes F_2$ with values in $\mathcal{A}_{\mathrm{rel}}^{R_1 \otimes R_2}$ by
$$(F_1 \otimes F_2)(\sigma \times \tau, o_1 \times o_2) = i^{k_2 \dim\sigma} F_1(\sigma, o_1) \otimes F_2(\tau, o_2).$$

Lemma 11.13. *For $i = 1, 2$, suppose given a ball complex K_i and a K_i-ad F_i with values in $\mathcal{A}_{\mathrm{rel}}^{R_i}$. Then $F_1 \otimes F_2$ is a $(K_1 \times K_2)$-ad.* $\square$

Proof of 11.10. We only need to verify parts (f) and (g) of [LM, Definition 3.10].

For part (f), let F be a K'-ad and let
$$F(\sigma, o) = (C_\sigma, D_\sigma, \beta_\sigma, \varphi_{\sigma,o}).$$
We need to define a K-ad E which agrees with F on each residual subcomplex of K. As in the proof of [LM, Theorem 6.5], we may

assume by induction that K is a ball complex structure for the n disk with one n cell τ, and that K' is a subdivision of K which agrees with K on the boundary. We only need to define E on the top cell τ of K. We define $E(\tau, o)$ to be $(C_\tau, D_\tau, \beta_\tau, \varphi_{\tau,o})$, where

- $C_\tau = \operatorname{colim}_{\sigma \in K'} C_\sigma$,
- $D_\tau = \operatorname{colim}_{\sigma \in K'} D_\sigma$,
- $\beta = \operatorname{colim}_{\sigma \in K'} \beta_\sigma$, and
- $\varphi_{\tau,o} = \sum \varphi_{\sigma,o'}$, where (σ, o') runs through the n-dimensional cells of K' with orientations induced by o.

The fact that E satisfies part (a) of Definition 11.8 is a consequence of [LM, Proposition A.1(ii)]. We will deduce the isomorphism in part (b) of Definition 11.8 from [LM, Proposition 12.4], and for this we need some facts from [LM, Section 12].

First recall that for a well-behaved functor

$$B : \mathcal{C}ell^\flat(K') \to \mathcal{D},$$

we write

$$\operatorname{Nat}(\mathrm{cl}, B)$$

for the chain complex of natural transformations of graded abelian groups; the differential is given by

$$\partial(\nu) = \partial \circ \nu - (-1)^{|\nu|} \nu \circ \partial.$$

Recall [LM, Definition 12.3] and also the map Φ defined just before the statement of [LM, Lemma 12.6]. Consider the diagram

$$
\begin{array}{ccc}
H_*(\operatorname{Nat}(\mathrm{cl}, D)) & \xrightarrow{\ \ \Phi\ \ } & H_{*+n}(D_\tau, D_{\partial\tau}) \\
\beta \big\uparrow & & \beta \big\uparrow \\
H_*(\operatorname{Nat}(\mathrm{cl}, C^t \otimes_R C)) & \xrightarrow{\ \Phi\ } & H_{*+n}((C^t \otimes_R C)_\tau, (C^t \otimes_R C)_{\partial\tau}).
\end{array}
$$

The horizontal maps are isomorphisms by [LM, Lemma 12.6], and the right-hand vertical map is an isomorphism by the proof of Lemma 11.6(ii). Hence the left-hand vertical map is an isomorphism.

The collection $\{\varphi_{\sigma,o}\}$ gives a cycle ν in $\operatorname{Nat}(\mathrm{cl}, D)$. Let $\mu \in \operatorname{Nat}(\mathrm{cl}, C^t \otimes_R C)$ be a representative for $\beta^{-1}([\nu])$. Now fix an orientation o for τ. Let $\psi \in C_\tau^t \otimes_R C_\tau$ be $\sum \mu(\langle \sigma, o' \rangle)$, where (σ, o') runs through the n-dimensional cells of K' with orientations induced by o. Then ψ is a representative of $\beta^{-1}([\varphi_{\tau,o}])$, so it suffices to show that the cap product with ψ is an isomorphism $H^*(\operatorname{Hom}_R(C_\tau, R)) \to H_{n-\deg F-*}(C_\tau/C_{\partial\tau})$, and this follows from [LM, Proposition 12.4].

It remains to verify part (g) of [LM, Definition 3.10]. Let $0, 1, \iota$ denote the three cells of the unit interval I, with their standard orientations. As in the proof of [LM, Theorem 9.11], it suffices to construct a relaxed symmetric Poincaré I-ad H over $\mathbb{Z}$ which takes both 0 and 1 to the object $(\mathbb{Z}, \mathbb{Z}, \gamma, 1)$, where γ is the isomorphism $\mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$. The proof of [LM, Theorem 9.11] gives a symmetric Poincaré I-ad G with $G(0) = G(1) = (\mathbb{Z}, \epsilon)$, where $\epsilon : W \to \mathbb{Z} \otimes \mathbb{Z}$ is the composite of the augmentation with γ^{-1}. Let us denote the object $G(\iota)$ by (C, φ). Applying Remark 11.9 to G gives a relaxed symmetric Poincaré I-ad G' with $G'(\iota) = (C, (C \otimes C)^W, \beta, \varphi)$, where β is induced by the augmentation. Let e_0 (resp., e_1) be the inclusion $0 \hookrightarrow \iota$ (resp., $1 \hookrightarrow \iota$) and for $i = 0, 1$ let $g_i = G(e_i) : \mathbb{Z} \to C$. Then

$$\partial \varphi = (g_1 \otimes g_1) \circ \epsilon - (g_0 \otimes g_0) \circ \epsilon$$

because G is closed. We can therefore construct the required I-ad H from G' by replacing $G'(0)$ and $G'(1)$ by $(\mathbb{Z}, \mathbb{Z}, \gamma, 1)$. $\square$

12. Equivalence of the spectra associated to ad^R and $\mathrm{ad}^R_{\mathrm{rel}}$

By Remark 11.9, the morphism

$$\mathcal{A}^R \to \mathcal{A}^R_{\mathrm{rel}}$$

of Remark 11.4 induces a map of spectra

$$\mathbf{Q}^R \to \mathbf{Q}^R_{\mathrm{rel}}$$

(see [LM, Section 15]) and a map of symmetric spectra

$$\mathbf{M}^R \to \mathbf{M}^R_{\mathrm{rel}}$$

(see [LM, Section 17]).

Theorem 12.1. *The maps*

$$\mathbf{Q}^R \to \mathbf{Q}^R_{\mathrm{rel}}$$

and

$$\mathbf{M}^R \to \mathbf{M}^R_{\mathrm{rel}}.$$

are weak equivalences.

Remark 12.2. The method that will be used to prove Theorem 1.3 can be used to show that $\mathbf{M}^R \to \mathbf{M}^R_{\mathrm{rel}}$ is weakly equivalent to a map of symmetric ring spectra.

Recall [LM, Definitions 4.1 and 4.2]. By [LM, Theorem 16.1, Remark 14.2(i) and Corollary 17.9(iii)], Theorem 12.1 follows from

Proposition 12.3. *The map of bordism groups*

$$\Omega_*^R \to (\Omega_{\mathrm{rel}}^R)_*$$

is an isomorphism.

The proof of Proposition 12.3 will occupy the rest of this section. The following lemma proves surjectivity. As usual, for a chain complex A with a $\mathbb{Z}/2$ action, we write $A^{h\mathbb{Z}/2}$ for $(A^W)^{\mathbb{Z}/2}$. The augmentation induces a map $A^{\mathbb{Z}/2} \to A^{h\mathbb{Z}/2}$.

Lemma 12.4. *Let*

$$(C, D, \beta, \varphi)$$

be a relaxed symmetric Poincaré $*$-ad and let

$$\psi \in (C^t \otimes_R C)^{h\mathbb{Z}/2}$$

represent the image of φ under the map

$$H_*(D^{\mathbb{Z}/2}) \to H_*(D^{h\mathbb{Z}/2}) \overset{\cong}{\leftarrow} H_*((C^t \otimes_R C)^{h\mathbb{Z}/2})$$

(where the isomorphism is induced by β). Then (C, ψ) is a symmetric Poincaré $*$-ad, and (C, D, β, φ) is bordant to

$$(C, (C^t \otimes_R C)^W, \gamma, \psi),$$

where γ is induced by the augmentation.

For the proof we need another lemma.

Lemma 12.5. *Let (C, D, β, φ) be a relaxed symmetric Poincaré $*$-ad.*
  (i) *If $\psi \in D^{\mathbb{Z}/2}$ is any representative for the homology class $[\varphi] \in H_*(D^{\mathbb{Z}/2})$ then (C, D, β, ψ) is bordant to (C, D, β, φ).*
  (ii) *If*

$$(f, g) : (C, D, \beta, \varphi) \to (C, D', \beta', \varphi')$$

is a map of $*$-ads for which $f : C \to C$ is the identity map then (C, D, β, φ) and $(C, D', \beta', \varphi')$ are bordant.

The proof of Lemma 12.5 is deferred to the end of the section.

Proof of Lemma 12.4. The fact that (C, ψ) satisfies [LM, Definition 9.9] (only part (b) is relevant) is immediate from Definition 11.8(b).

To see that (C, D, β, φ) and $(C, (C^t \otimes_R C)^W, \gamma, \psi)$ are bordant, let δ denote the composite

$$C^t \otimes_R C \overset{\beta}{\to} D \to D^W,$$

let $\omega \in D^{h\mathbb{Z}/2}$ be the image of φ, and let $\omega' \in D^{h\mathbb{Z}/2}$ be the image of ψ under the map $(C \otimes C)^{h\mathbb{Z}/2} \to D^{h\mathbb{Z}/2}$ induced by β. Part (ii) of Lemma 12.5 shows that (C, D, β, φ) and (C, D^W, δ, ω) are bordant,

and also that $(C, (C^t \otimes_R C)^W, \gamma, \psi)$ and $(C, D^W, \delta, \omega')$ are bordant. But $[\omega] = [\omega']$ in $H_*(D^{h\mathbb{Z}/2})$, so the result follows from part (i) of Lemma 12.5. $\qquad\square$

Next we show that the map in Proposition 12.3 is 1-1. So let (C_0, φ_0) and (C_1, φ_1) be symmetric Poincaré $*$-ads and let F be a relaxed symmetric Poincaré bordism between them. Let $0, 1, \iota$ denote the three cells of the unit interval I, with their standard orientations. Denote the object $F(\iota)$ by (C, D, β, φ). It suffices to show that there is a symmetric Poincaré I-ad G with

(12.1) $G(0) = (C_0, \varphi_0), \quad G(1) = (C_1, \varphi_1), \quad G(\iota) = (C, \chi)$

for an element χ which we will now construct.

φ represents an element

$$[\varphi] \in H_*(D^{\mathbb{Z}/2}, (C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2}).$$

The map $D \to D^W$ induced by the augmentation gives a map

$$H_*(D^{\mathbb{Z}/2}, (C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2}).$$
$$\to H_*(D^{h\mathbb{Z}/2}, ((C_0^t \otimes_R C_0)^W)^{h\mathbb{Z}/2} \oplus ((C_1^t \otimes_R C_1)^W)^{h\mathbb{Z}/2});$$

let x be the image of φ under this map. The map $\beta : C^t \otimes_R C \to D$ gives an isomorphism

$$(\beta^{h\mathbb{Z}/2})_* : H_*((C^t \otimes_R C)^{h\mathbb{Z}/2}, (C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2}).$$
$$\to H_*(D^{h\mathbb{Z}/2}, ((C_0^t \otimes_R C_0)^W)^{h\mathbb{Z}/2} \oplus ((C_1^t \otimes_R C_1)^W)^{h\mathbb{Z}/2});$$

let $y = (\beta^{h\mathbb{Z}/2})_*^{-1}(x)$.

Lemma 12.6. *The image of y under the boundary map*

$$H_*((C^t \otimes_R C)^{h\mathbb{Z}/2}, (C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2})$$
$$\xrightarrow{\partial} H_{*-1}((C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2})$$

is $-[\varphi_0] + [\varphi_1]$.

Before proving this we conclude the proof of Proposition 12.3. The lemma implies that there is a representative χ of y with

(12.2) $\partial \chi = -\varphi_0 + \varphi_1.$

It suffices to show that, with this choice of χ, the symmetric Poincaré pre I-ad G given by equation (12.1) is an ad. Equation (12.2) says that

G is closed, and part (b) of [LM, Definition 9.9] follows from Definition 11.8(b) and the fact that the image of $[\chi]$ under the map

$$H_*((C^t \otimes_R C)^{h\mathbb{Z}/2}, (C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2})$$
$$\to H_*((C^t \otimes_R C)^W, (C_0^t \otimes_R C_0)^W \oplus (C_1^t \otimes_R C_1)^W)$$
$$\stackrel{\cong}{\Longleftarrow} H_*(C^t \otimes_R C, (C_0^t \otimes_R C_0) \oplus (C_1^t \otimes_R C_1))$$

is the same as the image of $[\varphi]$ under the map

$$H_*(D^{\mathbb{Z}/2}, (C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2})$$
$$\to H_*(D, (C_0^t \otimes_R C_0)^W \oplus (C_1^t \otimes_R C_1)^W)$$
$$\xrightarrow{\beta_*^{-1}} H_*(C^t \otimes_R C, (C_0^t \otimes_R C_0) \oplus (C_1^t \otimes_R C_1)).$$
$\square$

Proof of Lemma 12.6. We know that the image of $[\varphi]$ under the boundary map

$$H_*(D^{\mathbb{Z}/2}, (C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2})$$
$$\xrightarrow{\partial} H_{*-1}((C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2})$$

is $-[\varphi_0] + [\varphi_1]$, so it suffices to show that for $i = 0, 1$ the maps

$$(C_i^t \otimes_R C_i)^{h\mathbb{Z}/2} \to ((C_i^t \otimes_R C_i)^W)^{h\mathbb{Z}/2}$$

induced by $D \to D^W$ and by β give the same map in homology. If we think of these as maps

$$a_i, b_i : ((C_i^t \otimes_R C_i)^W)^{\mathbb{Z}/2} \to ((C_i^t \otimes_R C_i)^{W \otimes W})^{\mathbb{Z}/2}$$

(with diagonal $\mathbb{Z}/2$ action on $W \otimes W$) then a_i and b_i are induced by the maps

$$e_1, e_2 : W \otimes W \to W$$

given by the augmentations on the two factors. Now the $\mathbb{Z}/2$ equivariant map

$$\Delta : W \to W \otimes W$$

of [Ran80a, page 175] has the property that $e_1 \circ \Delta$ and $e_2 \circ \Delta$ are both the identity map, so if

$$d : ((C_i^t \otimes_R C_i)^{W \otimes W})^{\mathbb{Z}/2} \to ((C_i^t \otimes_R C_i)^W)^{\mathbb{Z}/2}$$

is the map induced by Δ then $d \circ a_i$ and $d \circ b_i$ are both the identity map. But Δ is a $\mathbb{Z}/2$ chain homotopy equivalence, so d is a homology isomorphism and it follows that a_i and b_i induce the same map in homology as required. $\square$

It remains to prove Lemma 12.5. Let F be the cylinder of (C, D, β, φ) ([LM, Definition 3.10(g)]). Then $F(0)$ and $F(1)$ are both (C, D, β, φ). Write

$$F(\iota) = (C_\iota, D_\iota, \beta_\iota, \varphi_\iota)$$

and let

$$(h, k) : (C, D, \beta, \varphi) \to (C_\iota, D_\iota, \beta_\iota, \varphi_\iota)$$

be the map $F(1) \to F(\iota)$.

For part (i), the hypothesis gives an element $\rho \in D^{\mathbb{Z}/2}$ with $\partial \rho = \psi - \varphi$. Let $\rho' \in D_\iota^{\mathbb{Z}/2}$ be the image of ρ under $k : D \to D_\iota$. Define an I-ad G by

$$G(0) = (C, D, \beta, \varphi), \quad G(1) = (C, D, \beta, \psi), \quad G(\iota) = (C_\iota, D_\iota, \beta_\iota, \varphi_\iota + \rho').$$

Then G is the desired bordism.

For part (ii), the idea is to construct a suitable mapping cylinder. Let E be the pushout of the diagram

$$
\begin{array}{ccc}
D & \xrightarrow{k} & D_\iota \\
{\scriptstyle g}\downarrow & & \\
D' & &
\end{array}
$$

Let χ be the image of φ_ι in E and let γ be the composite

$$C_\iota^t \otimes_R C_\iota \to D_\iota \to E.$$

Define an I-ad H by

$$H(0) = (C, D, \beta, \varphi), \quad H(1) = (C, D', \beta', \varphi'), \quad H(\iota) = (C_\iota, E, \gamma, \chi).$$

Then H is the desired bordism. $\qquad\square$

13. The symmetric signature revisited

Fix a group π, a simply-connected free π-space Z, and a homomorphism $w : \pi \to \{\pm 1\}$, and recall the symmetric spectrum $\mathbf{M}_{\pi, Z, w}$ ([LM, Sections 7 and 17]) which represents w-twisted Poincaré bordism over Z/π.

Let R denote the group ring $\mathbb{Z}[\pi]$ with the w-twisted involution ([Ran80b, page 196]).

In [LM, Section 10] we constructed a morphism of ad theories

$$\mathrm{Sig} : \mathrm{ad}_{\pi, Z, w} \to \mathrm{ad}^R.$$

We begin this section by constructing a morphism of ad theories

$$\mathrm{Sig}_{\mathrm{rel}} : \mathrm{ad}_{\pi, Z, w} \to \mathrm{ad}_{\mathrm{rel}}^R.$$

Let (X, f, ξ) be an object of $\mathcal{A}_{\pi, Z, w}$ ([LM, Definition 7.3]).

In the special case where π is the trivial group and Z is a point, the definition is easy:

$$\mathrm{Sig}_{\mathrm{rel}}(X, f, \xi) = (S_*X, S_*(X \times X), \beta, \varphi),$$

where β is the cross product $S_*X \otimes S_*X \xrightarrow{\times} S_*(X \times X)$ and φ is the image of ξ under the diagonal map.

The definition in the general case is similar. Recall that we write $\widetilde{X}$ for the pullback of Z along f and $\mathbb{Z}^w$ for $\mathbb{Z}$ with the right R action determined by w. Also recall [LM, Definition 7.1].

Definition 13.1. (i) Give $S_*(\widetilde{X})$ the left R module structure determined by the action of π on $\widetilde{X}$ and give $S_*(\widetilde{X} \times \widetilde{X})$ and $S_*(\widetilde{X}) \otimes S_*(\widetilde{X})$ the left R module structures determined by the diagonal actions of π.

(ii) Define

$$\mathrm{Sig}_{\mathrm{rel}}(X, f, \xi) = (S_*(\widetilde{X}), \mathbb{Z}^w \otimes_R S_*(\widetilde{X} \times \widetilde{X}), \beta, \varphi),$$

where β is the composite

$$S_*(\widetilde{X})^t \otimes_R S_*(\widetilde{X}) \cong \mathbb{Z}^w \otimes_R (S_*(\widetilde{X}) \otimes S_*(\widetilde{X})) \xrightarrow{1 \otimes \times} \mathbb{Z}^w \otimes_R S_*(\widetilde{X} \times \widetilde{X})$$

and φ is the image of ξ under the map

$$S_*(X, \mathbb{Z}^f) = \mathbb{Z}^w \otimes_R S_*(\widetilde{X}) \to \mathbb{Z}^w \otimes_R S_*(\widetilde{X} \times \widetilde{X})$$

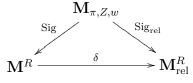(where the unmarked arrow is induced by the diagonal map).

Remark 13.2. (i) For set-theoretic reasons one should modify this definition as in [LM, Section 10]; we leave this to the reader.

(ii) $\mathrm{Sig}_{\mathrm{rel}}$ takes ads to ads, because the composite of the cross product with the Alexander-Whitney map is naturally chain homotopic to the map induced by the diagonal.

Next we compare Sig to $\mathrm{Sig}_{\mathrm{rel}}$.

Let us denote by δ both the map $\mathcal{A}^R \to \mathcal{A}^R_{\mathrm{rel}}$ of Remark 11.4 and the map $\mathbf{M}^R \to \mathbf{M}^R_{\mathrm{rel}}$ which it induces.

Proposition 13.3. *The diagram*

$$\begin{array}{ccc}
 & \mathbf{M}_{\pi, Z, w} & \\
{}^{\mathrm{Sig}}\swarrow & & \searrow^{\mathrm{Sig}_{\mathrm{rel}}} \\
\mathbf{M}^R & \xrightarrow{\ \ \delta\ \ } & \mathbf{M}^R_{\mathrm{rel}}
\end{array}$$

commutes in the homotopy category of symmetric spectra.

The rest of this section is devoted to the proof of Proposition 13.3. The basic idea is similar to the proof of Theorem 1.1.

First observe that the extended Eilenberg-Zilber map

$$W \otimes S_*(Y \times Z) \to S_*(Y) \otimes S_*(Z)$$

([FM, proof of Proposition 5.8]) gives a map

$$S_*(Y \times Z) \to ((S_*(Y) \otimes S_*(Z))^W,$$

and this gives a natural transformation

$$\nu : \mathrm{Sig}_{\mathrm{rel}} \to \delta \circ \mathrm{Sig}.$$

Definition 13.4. Let P be the poset whose two elements are the functors $\mathrm{Sig}_{\mathrm{rel}}$ and $\delta \circ \mathrm{Sig}$, with $\mathrm{Sig}_{\mathrm{rel}} \leq \delta \circ \mathrm{Sig}$.

Recall Definition 6.2(i), and note that $U(K)$ has a poset structure given by inclusions of cells. Our next definition is analogous to Definition 6.2(ii).

Definition 13.5. Let $k \geq 0$, let $\mathbf{n}$ be a k-fold multi-index, and let $F \in \mathrm{pre}^k_{\pi,Z,w}(\Delta^{\mathbf{n}})$. Let

$$b : U(\Delta^{\mathbf{n}}) \to P$$

be a map of posets.

(i) For an object (σ, o) of $\mathcal{C}\mathit{ell}(\Delta^{\mathbf{n}})$ define the object $b_*(F)(\sigma, o)$ of $\mathcal{A}^R_{\mathrm{rel}}$ to be $b(\sigma)(F(\sigma, o))$.

(ii) For a morphism $f : (\sigma, o) \to (\sigma', o')$ of $\mathcal{C}\mathit{ell}(\Delta^{\mathbf{n}})$ define the morphism

$$b_*(F)(f) : b_*(F)(\sigma, o) \to b_*(F)(\sigma', o')$$

to be

$$\begin{cases} \mathrm{Sig}_{\mathrm{rel}}(f) & \text{if } b(\sigma', o') = \mathrm{Sig}_{\mathrm{rel}}, \\ \mathrm{Sig}(f) & \text{if } b(\sigma, o) = \mathrm{Sig}, \\ \nu \circ \mathrm{Sig}_{\mathrm{rel}}(f) & \text{otherwise.} \end{cases}$$

(iii) Let

$$b_* : \mathrm{pre}^k_{\pi,Z,w}(\Delta^{\mathbf{n}}) \to (\mathrm{pre}^R_{\mathrm{rel}})^k(\Delta^{\mathbf{n}})$$

be the map defined by (i) and (ii).

Lemma 13.6. b_* *takes ads to ads.*

Proof. This follows from the fact that ν is a quasi-isomorphism. $\square$

Recall that we write $\mathbf{R}$ for the object of $\Sigma\mathcal{S}_{\mathrm{ss}}$ associated to an ad theory (Example 4.12). Then b_* gives a map

$$((\mathbf{R}_{\pi,Z,w})_k)_{\mathbf{n}} \to ((\mathbf{R}^R_{\mathrm{rel}})_k)_{\mathbf{n}}.$$

Definition 13.7. (i) For $k \geq 0$ define an object $\mathcal{P}_k$ of $\Sigma_k\mathrm{ss}_k$ by

$$(\mathcal{P}_k)_\mathbf{n} = \mathrm{Map}_\mathrm{posets}(U(\Delta^\mathbf{n}), P)_+$$

(where the $+$ denotes a disjoint basepoint); the morphisms in $(\Delta_\mathrm{inj}^\mathrm{op})^{\times k}$ act in the evident way, and the morphisms of the form (α, id) with $\alpha \in \Sigma_k$ act by permuting the factors in $\Delta^\mathbf{n}$.

(ii) Define $\mathcal{P}$ to be the object of ss^Σ with k-th term $\mathcal{P}_k$.

Next we give $\mathcal{P} \overline{\wedge} \mathbf{R}_{\pi,Z,w}$ the structure of a multisemisimplicial symmetric spectrum (cf. Definition 8.1). Recall Definition 6.4 and let s be the 1-simplex of S^1. Define

$$\omega : S^1 \wedge (\mathcal{P} \overline{\wedge} \mathbf{R}_{\pi,Z,w})_k \to (\mathcal{P} \overline{\wedge} \mathbf{R}_{\pi,Z,w})$$

as follows: for $b \in (\mathcal{P}_k)_\mathbf{n}$ and $x \in ((\mathbf{R}_{\pi,Z,w})_k)_\mathbf{n}$, let

$$\omega(s \wedge (b \wedge x)) = (b \circ \Pi) \wedge \omega(s \wedge x).$$

It follows from the definitions that we obtain a map

$$\beta : \mathcal{P} \overline{\wedge} \mathbf{R}_{\pi,Z,w} \to \mathbf{R}_\mathrm{rel}^R$$

in $\Sigma\mathcal{S}_\mathrm{ss}$ by

$$\beta(b \wedge F) = b_*(F).$$

For each k and $\mathbf{n}$, define elements $c_{k,\mathbf{n}}, d_{k,\mathbf{n}} \in (\mathcal{P}_k)_\mathbf{n}$ to be the constant functions $U(\Delta^\mathbf{n}) \to P$ whose values are respectively $\mathrm{Sig}_\mathrm{rel}$ and $\delta \circ \mathrm{Sig}$. Then define maps

$$c, d : \overline{\mathbf{S}} \to \mathcal{P}$$

in ss^Σ by taking the nontrivial simplex of $(\overline{\mathbf{S}}_k)_\mathbf{n}$ to $c_{k,\mathbf{n}}$, resp., $d_{k,\mathbf{n}}$. Finally, define maps

$$\mathbf{c}, \mathbf{d} : \mathbf{R}_{\pi,Z,w} \to \mathcal{P} \overline{\wedge} \mathbf{R}_{\pi,Z,w}$$

in $\Sigma\mathcal{S}_\mathrm{ss}$ by letting $\mathbf{c}$ be the composite

$$\mathbf{R}_{\pi,Z,w} \cong \overline{\mathbf{S}} \wedge \mathbf{R}_{\pi,Z,w} \xrightarrow{c \wedge 1} \mathcal{P} \overline{\wedge} \mathbf{R}_{\pi,Z,w}$$

and similarly for $\mathbf{d}$.

Now $\beta \circ \mathbf{c}$ is the map $\mathrm{Sig}_\mathrm{rel}$ and $\beta \circ \mathbf{d}$ is the map $\delta \circ \mathrm{Sig}$, so to complete the proof of Proposition 13.3 it suffices to show:

Lemma 13.8. $\mathbf{c}$ *and* $\mathbf{d}$ *are homotopic in* $\Sigma\mathcal{S}_\mathrm{ss}$.

Proof of Lemma 13.8. For each $k \geq 0$ and each $\mathbf{n}$ let $e_{k,\mathbf{n}} : (\mathcal{P}_k)_\mathbf{n} \to S^0$ be the map which takes every simplex except the basepoint to the nontrivial element of S^0, and let

$$e : \mathcal{P} \to \overline{\mathbf{S}}$$

be the map given by the $e_{k,\mathbf{n}}$. Let

$$\mathbf{e} : \mathcal{P} \,\overline{\wedge}\, \mathbf{R}_{\pi,Z,w} \to \mathbf{R}_{\pi,Z,w}$$

be the composite

$$\mathcal{P} \,\overline{\wedge}\, \mathbf{R}_{\pi,Z,w} \xrightarrow{e \wedge 1} \overline{\mathbf{S}} \,\overline{\wedge}\, \mathbf{R}_{\pi,Z,w} \cong \mathbf{R}_{\pi,Z,w}.$$

Then $\mathbf{e} \circ \mathbf{c}$ and $\mathbf{e} \circ \mathbf{d}$ are both equal to the identity. But $\mathbf{e}$ is a weak equivalence by Proposition 9.3 and Lemma 10.3, and the result follows.
$\square$

14. Background for the proof of Theorem 1.3

Notation 14.1. In order to distinguish the product in $\mathcal{A}_{e,*,1}$ from the Cartesian product of categories, we will denote the former by $\boxtimes$ from now on.

We now turn to the proof of Theorem 1.3, which will follow the general outline of the proof of Theorem 1.1. The key ingredient in that proof was the action of the monad $\mathbb{P}$ on $\mathbf{R}$. That action was constructed from the family of operations given in Definition 6.2(ii), and this family in turn was constructed from the family of functors $\eta_\star$ given in Definition 3.5(ii). For our present purpose we need the functors $\eta_\star$ and also a family of functors

$$\mathbf{d}_\blacksquare : \mathcal{A}_1 \times \cdots \times \mathcal{A}_j \to \mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}},$$

where each $\mathcal{A}_i$ is equal to $\mathcal{A}_{e,*,1}$ or $\mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}$; these will be built from the symmetric monoidal structures of $\mathcal{A}_{e,*,1}$ and $\mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}$ and the functor

$$\mathrm{Sig}_{\mathrm{rel}} : \mathcal{A}_{e,*,1} \to \mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}.$$

It is convenient to represent this situation by a function r from $\{1, \ldots, j\}$ to a two element set $\{u, v\}$, with $\mathcal{A}_i = \mathcal{A}_{e,*,1}$ if $r(i) = u$ and $\mathcal{A}_i = \mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}$ if $r(i) = v$.

Example 14.2. A typical example is the functor

$$\mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}} \times (\mathcal{A}_{e,*,1})^{\times 5} \times \mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}} \to \mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}$$

which takes $(x_1, \ldots, x_7)$ to

$$i^\epsilon \mathrm{Sig}_{\mathrm{rel}}(x_4 \boxtimes x_3) \otimes x_7 \otimes \mathrm{Sig}_{\mathrm{rel}}(x_6 \boxtimes x_2 \boxtimes x_5) \otimes x_1,$$

where i^ϵ is the sign that arises from permuting $(x_1, \ldots, x_7)$ into the order $(x_4, x_3, x_7, x_6, x_2, x_5, x_1)$. In Definition 14.4(iv) we will represent such a functor by a surjection h which keeps track of which inputs go to which output factors and a permutation η which keeps track of the

order in which the inputs to each $\mathrm{Sig}_{\mathrm{rel}}$ factor are multiplied. In the present example h is the surjection $\{1, \ldots, 7\} \to \{1, 2, 3, 4\}$ with

$$h^{-1}(1) = \{3, 4\}, h^{-1}(2) = 7, h^{-1}(3) = \{2, 5, 6\}, h^{-1}(4) = 1$$

and η is the permutation $(256)(34)$.

In order to get the signs right we need a preliminary definition.

Definition 14.3. (i) For totally ordered sets $S_1, \ldots, S_m$, define

$$\coprod_{i=1}^{n} S_i$$

to be the disjoint union with the order relation given as follows: $s < t$ if either $s \in S_i$ and $t \in S_j$ with $i < j$, or $s, t \in S_i$ with $s < t$ in the order of S_i.

(ii) For a surjection

$$h : \{1, \ldots, j\} \to \{1, \ldots, m\}$$

define $\theta(h)$ to be the permutation

$$\{1, \ldots, j\} \cong h^{-1}(1) \coprod \cdots \coprod h^{-1}(m) \cong \{1, \ldots, j\};$$

here the first map restricts to the identity on each $h^{-1}(i)$ and the second is the unique ordered bijection.

In Example 14.2. $\theta(h)$ takes $1, \ldots, 7$ respectively to $7, 4, 1, 2, 5, 6, 3$.

Definition 14.4. Let $j \geq 0$ and let $r : \{1, \ldots, j\} \to \{u, v\}$. Let $\mathcal{A}_i$ denote $\mathcal{A}_{e, *, 1}$ if $r(i) = u$ and $\mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}$ if $r(i) = v$.

(i) Let $1 \leq m \leq j$. A surjection

$$h : \{1, \ldots, j\} \to \{1, \ldots, m\}$$

is *adapted* to r if r is constant on each set $h^{-1}(i)$ and h is monic on $r^{-1}(v)$.

(ii) Given a surjection

$$h : \{1, \ldots, j\} \to \{1, \ldots, m\}$$

which is adapted to r, define

$$h_{\blacklozenge} : \mathcal{A}_1 \times \cdots \times \mathcal{A}_j \to (\mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}})^{\times m}$$

by

$$h_{\blacklozenge}(x_1, \ldots, x_j) = (i^{\epsilon} y_1, \ldots, y_m),$$

where i^ϵ is the sign that arises from putting the objects $x_1, \ldots, x_j$ into the order $x_{\theta(h)^{-1}(1)}, \ldots, x_{\theta(h)^{-1}(j)}$ and

$$
y_i = \begin{cases} \mathrm{Sig}_{\mathrm{rel}}(\boxtimes_{l \in h^{-1}(i)} x_l) & \text{if } h^{-1}(i) \subset r^{-1}(u), \\ x_{h^{-1}(i)} & \text{if } h^{-1}(i) \in r^{-1}(v). \end{cases}
$$

(iii) A *datum* of type r is a pair

$$
(h, \eta),
$$

where h is a surjection which is adapted to r and η is an element of Σ_j with the property that $h \circ \eta = h$.

(iv) Given a datum

$$
\mathbf{d} = (h, \eta),
$$

of type r, define

$$
\mathbf{d}_\blacksquare : \mathcal{A}_1 \times \cdots \times \mathcal{A}_j \to \mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}
$$

to be the composite

$$
\mathcal{A}_1 \times \cdots \times \mathcal{A}_j \xrightarrow{\eta} \mathcal{A}_{\eta^{-1}(1)} \times \cdots \times \mathcal{A}_{\eta^{-1}(j)} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_j \xrightarrow{h_\blacklozenge} (\mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}})^{\times m} \xrightarrow{\otimes} \mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}},
$$

where η permutes the factors with the usual sign.

We also need natural transformations between the functors $\mathbf{d}_\blacksquare$. First observe that the cross product gives a natural transformation from the functor

$$
(\mathcal{A}_{e,*,1})^{\times l} \xrightarrow{\mathrm{Sig}_{\mathrm{rel}}^{\times l}} (\mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}})^{\times l} \xrightarrow{\otimes} \mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}
$$

to the functor

$$
(\mathcal{A}_{e,*,1})^{\times l} \xrightarrow{\boxtimes} \mathcal{A}_{e,*,1} \xrightarrow{\mathrm{Sig}_{\mathrm{rel}}} \mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}.
$$

Combining this with the symmetric monoidal structures of $\mathcal{A}_{e,*,1}$ and $\mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}$, we obtain a natural transformation $\mathbf{d}_\blacksquare \to \mathbf{d}'_\blacksquare$ whenever $\mathbf{d} \leq \mathbf{d}'$, as defined in:

Definition 14.5. For data of type r, define

$$
(h, \eta) \leq (h', \eta')
$$

if each set $\eta^{-1}(h^{-1}(i))$ is contained in some set $\eta'^{-1}(h'^{-1}(l))$.

Our next definition is analogous to Definition 6.6 (the presence of the letter v in the symbols $P_{r;v}$ and $\mathcal{O}(r; v)$ will be explained in a moment).

Definition 14.6. Let $r : \{1 \ldots, j\} \to \{u, v\}$.

(i) Let $P_{r;v}$ be the preorder whose elements are the data of type r, with the order relation given by Definition 14.5.

(ii) Define an object $\mathcal{O}(r; v)_k$ of $\Sigma_k \mathrm{ss}_k$ by

$$
(\mathcal{O}(r; v)_k)_\mathbf{n} = \mathrm{Map}_{\mathrm{preorder}}(U(\Delta^\mathbf{n}), P_{r;v})_+
$$

(where the $+$ denotes a disjoint basepoint); the morphisms in $(\Delta_{\mathrm{inj}}^{\mathrm{op}})^{\times k}$ act in the evident way, and the morphisms of the form (α, id) with $\alpha \in \Sigma_k$ act by permuting the factors in $\Delta^{\mathbf{n}}$.

(iii) Define $\mathcal{O}(r; v)$ to be the object of ss^{Σ} with k-th term $\mathcal{O}(r; v)_k$.

Remark 14.7. (i) Given $r : \{1 \ldots, j\} \to \{u, v\}$, let $m = |r^{-1}(v)| + 1$, let

$$h : \{1, \ldots, j\} \to \{1, \ldots, m\}$$

be any surjection which is adapted to r, and let e be the identity element of Σ_j. Then the datum (h, e) is $\geq$ every element in $P_{r;v}$.

(ii) Lemma 10.3 shows that each of the objects $\mathcal{O}(r; v)_k$ has compatible degeneracies and is weakly equivalent to a point.

We also need a preorder corresponding to the family of functors

$$\eta_{\bigstar} : (\mathcal{A}_{e,*,1})^{\times j} \to \mathcal{A}_{e,*,1} :$$

Notation 14.8. Let $r_u(j)$ (resp., $r_v(j)$) denote the constant function $\{1, \ldots, j\} \to \{u, v\}$ with value u (resp., v).

Definition 14.9. Let $r : \{1 \ldots, j\} \to \{u, v\}$.

(i) If $r = r_u(j)$, let $P_{r;u}$ be the set Σ_j with the preorder in which every element is $\leq$ every other, and let $\mathcal{O}(r; u)_k$ be the object $\mathcal{O}(j)_k$ of Definition 6.6.

(ii) Otherwise let $P_{r;u}$ be the empty set and let $\mathcal{O}(r; u)_k$ be the multisemisimplicial set with a point in every multidegree.

(iii) In either case, let $\mathcal{O}(r; u)$ be the object of ss^{Σ} with k-th term $\mathcal{O}(r; u)_k$.

In the next section we will use the objects $\mathcal{O}(r; v)$ and $\mathcal{O}(r; u)$ to construct a monad. In preparation for that, we show that the collection of preorders $P_{r;v}$ and $P_{r;u}$ has suitable composition maps. Specifically, we show that it is a colored operad (also called a multicategory) in the category of preorders.

We refer the reader to [EM06, Section2] for the definition of multicategory; we will mostly follow the notation and terminology given there. In our case there are two objects u and v, and we think of a function $r : \{1, \ldots, j\} \to \{u, v\}$ as a sequence of u's and v's.

Remark 14.10. Let $r : \{1, \ldots, j\} \to \{u, v\}$ and let $\mathcal{A}_i$ denote $\mathcal{A}_{e,*,1}$ if $r(i) = u$ and $\mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}$ if $r(i) = v$. Let us write $\mathcal{A}_r$ for the category $\mathcal{A}_1 \times \cdots \times \mathcal{A}_j$ and $\mathcal{A}_{r;u}$ (resp., $\mathcal{A}_{r;v}$) for the category of functors $\mathcal{A}_r \to \mathcal{A}_{e,*,1}$ (resp., $\mathcal{A}_r \to \mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}$). Then the objects $\mathcal{A}_{r;u}$ and $\mathcal{A}_{r;v}$ form a multicategory (with the obvious Σ_j actions and composition maps). Moreover, Definitions 3.5(ii) and 14.4(iv) give imbeddings

$$\Phi_{r;u} : P_{r;u} \to \mathcal{A}_{r;u}$$

and
$$\Phi_{r;v} : P_{r;v} \to \mathcal{A}_{r;v}.$$
The following definitions are chosen so that these imbeddings preserve the Σ_j actions and the composition operations.

We define the right Σ_j action on the collection of j-morphisms as follows. Let $\alpha \in \Sigma_j$ and $r : \{1, \ldots, j\} \to \{u, v\}$. Define r^α to be the composite
$$\{1, \ldots, j\} \xrightarrow{\alpha} \{1, \ldots, j\} \xrightarrow{r} \{u, v\}.$$
If $r = r_u(j)$ then $r^\alpha = r$ and the map
$$\alpha : P_{r;u} \to P_{r^\alpha;u}$$
is the right action of Σ_j on itself. If $(h, \eta) \in P_{r,v}$ define $(h, \eta)\alpha \in P_{r^\alpha;v}$ to be $(h \circ \bar{\alpha}, \bar{\alpha}^{-1} \circ \eta \circ \alpha)$, where $\bar{\alpha} \in \Sigma_j$ is the permutation whose restriction to each $\alpha^{-1}h^{-1}(i)$ is the order-preserving bijection to $h^{-1}(i)$.

We define the composition operation as follows. If the composition involves only $P_{r;u}$'s then it is the composition in the operad $\mathcal{M}$ of [May72, Definition 3.1(i)]. Otherwise let $i, j_1, \ldots, j_i \geq 0$, let $r : \{1, \ldots, i\} \to \{u, v\}$, and for $1 \leq l \leq i$ let $r_l : \{1, \ldots, j_l\} \to \{u, v\}$; assume that if $r(l) = u$ then r_l is $r_u(j_l)$. Let $(h, \eta) \in P_{r;v}$ and for $1 \leq l \leq i$ let $x_i \in P_{r_l;r(l)}$. If $r(l) = v$ then x_l has the form (h_l, η_l), otherwise x_l is an element $\eta_l \in \Sigma_{j_l}$ and we write h_l for the map $\{1, \ldots, j_l\} \to \{1\}$. Define the composition operation Γ by

(14.1) $$\Gamma((h, \eta), x_1, \ldots, x_l) = (H, \theta),$$

where H is the multivariable composite $h \circ (h_1, \ldots, h_i)$ and θ is the composite $\gamma_{\mathcal{M}}(\eta, \eta_1, \ldots, \eta_i)$ in the operad $\mathcal{M}$ of [May72, Definition 3.1(i)].

Proposition 14.11. *With these definitions, the collection of preorders $P_{r;u}$ and $P_{r;v}$ is a multicategory.*

Proof. This is immediate from Remark 14.10. $\square$

15. A MONAD IN $\mathrm{ss}^\Sigma \times \mathrm{ss}^\Sigma$

In this section we construct a monad in $\mathrm{ss}^\Sigma \times \mathrm{ss}^\Sigma$ which acts on the pair $(\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}})$.

Definition 15.1. Let $j \geq 0$ and let $\mathbf{X}, \mathbf{Y} \in \mathrm{ss}^\Sigma$.
   (i) For $\alpha \in \Sigma_j$ and $r : \{1, \ldots, j\} \to \{u, v\}$, define
$$\tilde{\alpha} : \mathcal{O}(r; v) \to \mathcal{O}(r^\alpha; v)$$
by
$$(\tilde{\alpha}(a))(\sigma) = (a(\sigma))\alpha$$
where $a \in \mathrm{Map}_{\mathrm{preorder}}(U(\Delta^{\mathbf{n}}), P_{r;v})_+$ and $\sigma \in U(\Delta^{\mathbf{n}})$.

(ii) Define
$$(\mathbf{X}, \mathbf{Y})^{\otimes r} = \mathbf{Z}_1 \otimes \cdots \otimes \mathbf{Z}_j,$$
where $\mathbf{Z}_i$ denotes $\mathbf{X}$ if $r(i) = u$ and $\mathbf{Y}$ if $r(i) = v$.

(iii) For $\alpha \in \Sigma_j$ define
$$\bar{\alpha} : \bigvee_r \mathcal{O}(r; v) \overline{\wedge} (\mathbf{X}, \mathbf{Y})^{\otimes r} \to \bigvee_r \mathcal{O}(r; v) \overline{\wedge} (\mathbf{X}, \mathbf{Y})^{\otimes r}$$
to be the map which takes the r-summand to the r^α-summand by means of the map
$$\mathcal{O}(r; v) \overline{\wedge} (\mathbf{X}, \mathbf{Y})^{\otimes r} \xrightarrow{\tilde{\alpha}\overline{\wedge}\alpha} \mathcal{O}(r^\alpha; v) \overline{\wedge} (\mathbf{X}, \mathbf{Y})^{\otimes r^\alpha}.$$

Note that the maps $\bar{\alpha}$ give $\bigvee_r \mathcal{O}(r; v) \overline{\wedge} (\mathbf{X}, \mathbf{Y})^{\otimes r}$ a right Σ_j action. Recall Notation 14.8.

Definition 15.2. (i) Define a functor $\mathbb{O} : \mathrm{ss}^\Sigma \times \mathrm{ss}^\Sigma \to \mathrm{ss}^\Sigma \times \mathrm{ss}^\Sigma$ by
$$\mathbb{O}(\mathbf{X}, \mathbf{Y}) = (\mathbb{O}_1(\mathbf{X}), \mathbb{O}_2(\mathbf{X}, \mathbf{Y})),$$
where
$$\mathbb{O}_1(\mathbf{X}) = \bigvee_{j \geq 0} \left(\mathcal{O}(r_u(j); u) \overline{\wedge} \mathbf{X}^{\otimes j} \right) / \Sigma_j$$
and
$$\mathbb{O}_2(\mathbf{X}, \mathbf{Y}) = \bigvee_{j \geq 0} \bigvee_r \left(\mathcal{O}(r; v) \overline{\wedge} (\mathbf{X}, \mathbf{Y})^{\otimes r} \right) / \Sigma_j.$$

(ii) Define a natural transformation
$$\iota : (\mathbf{X}, \mathbf{Y}) \to \mathbb{O}(\mathbf{X}, \mathbf{Y})$$
to be (ι_1, ι_2), where ι_1 is the composite
$$\mathbf{X} \xrightarrow{\cong} \overline{\mathbf{S}} \overline{\wedge} \mathbf{X} = \mathcal{O}(r_u(1); u) \overline{\wedge} \mathbf{X} \hookrightarrow \mathbb{O}(\mathbf{X})$$
and ι_2 is the composite
$$\mathbf{Y} \xrightarrow{\cong} \overline{\mathbf{S}} \overline{\wedge} \mathbf{Y} = \mathcal{O}(r_v(1); v) \overline{\wedge} \mathbf{Y} \hookrightarrow \mathbb{O}(\mathbf{Y}).$$

For the structure map $\mu : \mathbb{O}\mathbb{O} \to \mathbb{O}$ we need a composition operation for the collection of objects $\mathcal{O}(r; u)$ and $\mathcal{O}(r; v)$. Recall Definition 14.3(i) and the map Γ defined in Equation (14.1).

Definition 15.3. Let $i, j_1, \ldots, j_i \geq 0$, let $r : \{1, \ldots, i\} \to \{u, v\}$, and for $1 \leq l \leq i$ let $r_l : \{1, \ldots, j_l\} \to \{u, v\}$; assume that if $r(l) = u$ then r_l is $r_u(j_l)$. Let
$$R : \{1, \ldots, \sum j_l\} \to \{u, v\}$$

be the composite

$$\{1, \ldots, \sum j_l\} \cong \coprod_{l=1}^{i} \{1, \ldots, j_l\} \to \{u, v\},$$

where the first map is the unique order-preserving bijection and the second restricts on each $\{1, \ldots, j_l\}$ to r_l. Define a map

$$\gamma : \mathcal{O}(r; v) \overline{\wedge} \left(\mathcal{O}(r_1; r(1)) \otimes \cdots \otimes O(r_i; r(i)) \right) \to \mathcal{O}(R; v)$$

in ss^{Σ} by the formulas

$$\gamma(a \wedge [e, b_1 \wedge \ldots \wedge b_i])(\sigma_1 \times \cdots \times \sigma_i) = \Gamma(a(\sigma_1 \times \cdots \times \sigma_i), b_1(\sigma_1), \ldots, b_i(\sigma_i))$$

(where e is the identity element of the relevant symmetric group) and

$$\gamma(a \wedge [\alpha, b_1 \wedge \ldots \wedge b_i]) = (\alpha, \mathrm{id})_* \gamma((\alpha^{-1}, \mathrm{id})_* a \wedge [e, b_1 \wedge \ldots \wedge b_i]).$$

This operation satisfies the analogues of Lemmas 7.2, 7.5, and 7.7. Now we can define

$$\mu : \mathbb{O}\mathbb{O} \to \mathbb{O}$$

to be (μ_1, μ_2), where μ_1 is given by Definition 7.8(iv) and μ_2 is defined in a similar way using Definition 15.3.

Proposition 15.4. *The transformations μ and ι define a monad structure on $\mathbb{O}$.* $\qquad\square$

We conclude this section by giving the action of $\mathbb{O}$ on the pair $(\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}})$. Recall Remark 14.10.

Definition 15.5. Let $k_1, \ldots, k_j$ be non-negative integers and let $\mathbf{n}_i$ be a k_i-fold multi-index for $1 \le i \le j$. Let $r : \{1, \ldots, j\} \to \{u, v\}$, and for $1 \le i \le j$ let pre_i denote $\mathrm{pre}_{e,*,1}$ if $r(i) = u$ and $\mathrm{pre}_{\mathrm{rel}}^{\mathbb{Z}}$ if $r(i) = v$. For any map of preorders

$$a : U(\Delta^{\mathbf{n}_1} \times \cdots \times \Delta^{\mathbf{n}_j}) \to P_{r;v}$$

define

$$a_* : \mathrm{pre}_1^{k_1}(\Delta^{\mathbf{n}_1}) \times \cdots \times \mathrm{pre}_j^{k_j}(\Delta^{\mathbf{n}_j}) \to (\mathrm{pre}_{\mathrm{rel}}^{\mathbb{Z}})^{k_1 + \cdots + k_j}(\Delta^{(\mathbf{n}_1, \ldots, \mathbf{n}_j)})$$

by

$$a_*(F_1, \ldots, F_j)(\sigma_1 \times \cdots \times \sigma_j, o_1 \times \cdots \times o_j)$$
$$= i^{\epsilon(\zeta)} \Phi_{r;v}(a(\sigma_1 \times \cdots \times \sigma_j))(F_1(\sigma_1, o_1), \ldots, F_j(\sigma_j, o_j)),$$

where ζ is the block permutation that takes blocks $\mathbf{b}_1, \ldots, \mathbf{b}_j, \mathbf{c}_1, \ldots, \mathbf{c}_j$ of size $k_1, \ldots, k_j, \dim \sigma_1, \ldots, \dim \sigma_j$ into the order $\mathbf{b}_1, \mathbf{c}_1, \ldots, \mathbf{b}_j, \mathbf{c}_j$.

Lemma 15.6. *If $F_i \in \mathrm{ad}_i^{k_i}(\Delta^{\mathbf{n}_i})$ for $1 \leq i \leq j$ then $a_*(F_1, \ldots, F_j) \in (\mathrm{ad}_{\mathrm{rel}}^{\mathbb{Z}})^{k_1+\cdots+k_j}(\Delta^{(\mathbf{n}_1,\ldots,\mathbf{n}_j)})$.*

Proof. This is a straightforward consequence of the fact that the natural transformation from the functor

$$(\mathcal{A}_{e,*,1})^{\times l} \xrightarrow{\mathrm{Sig}_{\mathrm{rel}}^{\times l}} (\mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}})^{\times l} \xrightarrow{\otimes} \mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}$$

to the functor

$$(\mathcal{A}_{e,*,1})^{\times l} \xrightarrow{\boxtimes} \mathcal{A}_{e,*,1} \xrightarrow{\mathrm{Sig}_{\mathrm{rel}}} \mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}$$

given by the cross product is a quasi-isomorphism. $\square$

Definition 15.7. Let $j \geq 0$ and let $r : \{1, \ldots, j\} \to \{u, v\}$. Define a map

$$\phi_r : \mathcal{O}(r; v) \overline{\wedge} (\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}})^{\otimes r} \to \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}}$$

in ss^{Σ} by the formulas

$$\phi_r(a \wedge [e, F_1 \wedge \cdots \wedge F_j]) = a_*(F_1 \wedge \cdots \wedge F_j)$$

(where e denotes the identity element of the relevant symmetric group) and

$$\phi_r(a \wedge [\alpha, F_1 \wedge \cdots \wedge F_j]) = (\alpha, \mathrm{id})_* \phi_r((\alpha^{-1}, \mathrm{id})_* a \wedge [e, F_1 \wedge \cdots \wedge F_j]).$$

Next observe that the maps ϕ_r induce a map

$$(15.1) \qquad \left(\bigvee_r \left(\mathcal{O}(r; v) \overline{\wedge} (\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}})^{\otimes r}\right)\right)/\Sigma_j \to \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}}$$

for each $j \geq 0$. We define

$$\nu : \mathbb{O}(\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}}) \to (\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}})$$

to be the pair (ν_1, ν_2), where ν_1 is given by Definition 7.10 and ν_2 is given by the maps (15.1).

Proposition 15.8. *ν is an action of $\mathbb{O}$ on $(\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}})$.*

Proof. This is a straightforward consequence of Remark 14.10. $\square$

16. A MONAD IN $\Sigma\mathcal{S}_{\mathrm{ss}} \times \Sigma\mathcal{S}_{\mathrm{ss}}$

First we give $\mathcal{O}(r; v) \overline{\wedge} (\mathbf{X}, \mathbf{Y})^{\wedge r}$ the structure of a multisemisimplicial symmetric spectrum when $\mathbf{X}, \mathbf{Y} \in \mathrm{ss}^{\Sigma}$. The definition is analogous to Definition 8.1. Recall Definition 6.4.

Definition 16.1. Let $j, k \geq 0$. Let s be the 1-simplex of S^1. Define

$$\omega : S^1 \wedge (\mathcal{O}(r;v) \,\overline{\wedge}\, (\mathbf{X}, \mathbf{Y})^{\wedge r})_k \rightarrow (\mathcal{O}(r;v) \,\overline{\wedge}\, (\mathbf{X}, \mathbf{Y})^{\wedge r})_{k+1}$$

as follows: for $a \in (\mathcal{O}(r;v)_k)_{\mathbf{n}}$ and $x \in (((\mathbf{X}, \mathbf{Y})^{\wedge r})_k)_{\mathbf{n}}$, let

$$\omega(s \wedge (a \wedge x)) = (a \circ \Pi) \wedge \omega(s \wedge x).$$

Definition 16.2. Define a functor $\mathbb{P} : \mathrm{ss}^\Sigma \times \mathrm{ss}^\Sigma \rightarrow \mathrm{ss}^\Sigma \times \mathrm{ss}^\Sigma$ by

$$\mathbb{P}(\mathbf{X}, \mathbf{Y}) = (\mathbb{P}_1(\mathbf{X}), \mathbb{P}_2(\mathbf{X}, \mathbf{Y})),$$

where

$$\mathbb{P}_1(\mathbf{X}) = \bigvee_{j \geq 0} \left(\mathcal{O}(r_u(j); u) \,\overline{\wedge}\, \mathbf{X}^{\wedge j} \right)/\Sigma_j$$

and

$$\mathbb{P}_2(\mathbf{X}, \mathbf{Y}) = \bigvee_{j \geq 0} \left(\bigvee_r \left(\mathcal{O}(r;v) \,\overline{\wedge}\, (\mathbf{X}, \mathbf{Y})^{\wedge r} \right) \right)/\Sigma_j.$$

The proof that $\mathbb{P}$ inherits a monad structure and an action on $(\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}})$ is the same as the corresponding proof in Section 8.

For use in the next section we record a lemma. Let $\mathcal{C}$ be the category whose objects are triples $(\mathbf{X}, \mathbf{Y}, f)$, where $\mathbf{X}$ and $\mathbf{Y}$ are monoids in $\Sigma \mathcal{S}_{\mathrm{ss}}$ and f is a map $\mathbf{X} \rightarrow \mathbf{Y}$ in $\Sigma \mathcal{S}_{\mathrm{ss}}$ which is not required to be a monoid map; the morphisms are commutative diagrams

$$\begin{array}{ccc} \mathbf{X} & \longrightarrow & \mathbf{X}' \\ {\scriptstyle f} \downarrow & & \downarrow {\scriptstyle f'} \\ \mathbf{Y} & \longrightarrow & \mathbf{Y}', \end{array}$$

where the horizontal arrows are monoid maps.

Lemma 16.3. (i) *There is a functor* Υ *from* $\mathbb{P}$ *algebras to* $\mathcal{C}$ *which takes* $(\mathbf{X}, \mathbf{Y})$ *to a map* $\mathbf{X} \rightarrow \mathbf{Y}$; *in particular,* $\mathbf{X}$ *and* $\mathbf{Y}$ *have natural monoid structures.*

(ii) $\Upsilon(\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}})$ *is the map*

$$\mathrm{Sig}_{\mathrm{rel}} : \mathbf{R}_{e,*,1} \rightarrow \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}},$$

where $\mathbf{R}_{e,*,1}$ *and* $\mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}}$ *have the monoid structures given by Lemma 8.8(i).*

Proof. Recall Notation 14.8, and let e denote the identity element of Σ_j.

Part (i). The map $f : \mathbf{X} \rightarrow \mathbf{Y}$ is the composite

$$\mathbf{X} \cong \overline{\mathbf{S}} \,\overline{\wedge}\, \mathbf{X} = \mathcal{O}(r_u(1); v) \,\overline{\wedge}\, \mathbf{X} \hookrightarrow \mathbb{P}_2(\mathbf{X}, \mathbf{Y}) \rightarrow \mathbf{Y}.$$

The monoid structure on $\mathbf{X}$ is given by Lemma 8.8; it remains to give the monoid structure on $\mathbf{Y}$. It suffices to give an action on $\mathbf{Y}$ of

the monad $\mathbb{A}$ defined in the proof of Lemma 8.8, and for this in turn it suffices to give a suitable natural transformation $\mathbb{A} \to \mathbb{P}_2$.

For each $j \geq 0$ let h_0 be the identity map of $\{1, \ldots, j\}$. Then h_0 is adapted to $r_v(j)$, so we obtain an element $(h_0, e) \in P_{r_v(j);v}$.

For each $j, k \geq 0$ and each k-fold multi-index $\mathbf{n}$, define an element

$$b_{j,k,\mathbf{n}} \in (\mathcal{O}(r_v(j); v)_k)_{\mathbf{n}}$$

to be the constant function $U(\Delta^{\mathbf{n}}) \to P_{r_v(j);v}$ whose value is (h_0, e). Next define a map

$$\overline{\mathbf{S}} \to \mathcal{O}(r_v(j); v)$$

by taking the nontrivial simplex of $(\overline{\mathbf{S}}_k)_{\mathbf{n}}$ to $b_{j,k,\mathbf{n}}$.

Now the composite

$$\mathbb{A}(\mathbf{Y}) = \bigvee_{j \geq 0} \mathbf{Y}^{\wedge j} \cong \bigvee_{j \geq 0} \overline{\mathbf{S}} \,\overline{\wedge}\, \mathbf{Y}^{\wedge j}$$

$$\to \bigvee_{j \geq 0} \left(\bigvee_r \left(\mathcal{O}(r; v) \,\overline{\wedge}\, (\mathbf{X}, \mathbf{Y})^{\wedge r} \right) \right) / \Sigma_j = \mathbb{P}_2(\mathbf{X}, \mathbf{Y})$$

is the desired map.

Part (ii) is an easy consequence of the definitions. $\qquad\square$

17. Rectification

In this section we prove Theorem 1.3. The argument is analogous to that in Section 10.

First we consider a monad in $\Sigma\mathcal{S}_{\mathrm{ss}} \times \Sigma\mathcal{S}_{\mathrm{ss}}$ which is simpler than $\mathbb{P}$.

Definition 17.1. (i) Define $\mathbb{P}'(\mathbf{X}, \mathbf{Y})$ to be

$$\left(\bigvee_{j \geq 0} \mathbf{X}^{\wedge j}/\Sigma_j, \ \bigvee_{j \geq 0} \left(\bigvee_r (\mathbf{X}, \mathbf{Y})^{\wedge r} \right)/\Sigma_j \right).$$

(ii) For each $j \geq 0$ and each $r : \{1, \ldots, j\} \to \{u, v\}$, let

$$\xi_j : \mathcal{O}(r_u(j); u) \to \overline{\mathbf{S}}$$

and

$$\zeta_r : \mathcal{O}(r; v) \to \overline{\mathbf{S}}$$

be the maps which take each nontrivial simplex of the k-th object to the nontrivial simplex of $\overline{S}_k$ in the same multidegree. Define a natural transformation

$$\Xi : \mathbb{P} \to \mathbb{P}'$$

to be the pair (Ξ_1, Ξ_2), where Ξ_1 is the wedge of the composites

$$\left(\mathcal{O}(r_u(j); u) \,\overline{\wedge}\, \mathbf{X}^{\wedge j} \right)/\Sigma_j \xrightarrow{\xi_j \overline{\wedge} 1} \left(\overline{\mathbf{S}} \,\overline{\wedge}\, \mathbf{X}^{\wedge j} \right)/\Sigma_j \xrightarrow{\cong} \mathbf{X}^{\wedge j}/\Sigma_j.$$

and Ξ_2 is the wedge of the composites

$$\left(\bigvee_r \mathcal{O}(r;v) \,\overline{\wedge}\, (\mathbf{X},\mathbf{Y})^{\wedge r}\right)/\Sigma_j \xrightarrow{\vee \zeta_r \overline{\wedge} 1} \left(\bigvee_r \overline{\mathbf{S}} \,\overline{\wedge}\, (\mathbf{X},\mathbf{Y})^{\wedge r}\right)/\Sigma_j$$

$$\xrightarrow{\cong} \left(\bigvee_r (\mathbf{X},\mathbf{Y})^{\wedge r}\right)/\Sigma_j.$$

Proposition 17.2. (i) *An algebra over $\mathbb{P}'$ is the same thing as a pair of commutative monoids $(\mathbf{X},\mathbf{Y})$ in $\Sigma\mathcal{S}_{\mathrm{ss}}$ together with a monoid map $\mathbf{X} \to \mathbf{Y}$.*

(ii) *Ξ is a map of monads.*

(iii) *Suppose that each X_k and each Y_k has compatible degeneracies (see Definition 9.1). Let $\mathbb{P}^q$ denote the q-th iterate of $\mathbb{P}$. Then each map*

$$\Xi : \mathbb{P}^q(\mathbf{X},\mathbf{Y}) \to \mathbb{P}'\mathbb{P}^{q-1}(\mathbf{X},\mathbf{Y})$$

is a weak equivalence.

Proof. Part (i). Let $(\mathbf{X},\mathbf{Y})$ be an algebra over $\mathbb{P}'$. The fact that $\mathbf{X}$ and $\mathbf{Y}$ are commutative monoids is immediate from the definitions. The map $f : \mathbf{X} \to \mathbf{Y}$ is constructed as in the proof of Lemma 16.3(i).

To show that f is a monoid map, we first observe that there are two inclusions of $\mathbf{X}^{\wedge j}$ into $\mathbb{P}'_2\mathbb{P}'(\mathbf{X},\mathbf{Y})$. Let i_1 be the composite

$$\mathbf{X}^{\wedge j} \hookrightarrow \mathbb{P}'_1(\mathbf{X}) \hookrightarrow \mathbb{P}'_2\mathbb{P}'(\mathbf{X},\mathbf{Y}),$$

where the second arrow is the inclusion of the summand indexed by $j = 1, r = r_u(1)$. Let i_2 be the composite

$$\mathbf{X}^{\wedge j} \hookrightarrow \mathbb{P}'_2(\mathbf{X},\mathbf{Y})^{\wedge j} \hookrightarrow \mathbb{P}'_2\mathbb{P}'(\mathbf{X},\mathbf{Y}),$$

where the first arrow is the j-fold smash of the inclusion of the $r_u(1)$ summand, and the second arrow is the inclusion of the $r_v(j)$ summand.
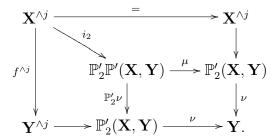
Consider the commutative diagram

$$\begin{array}{ccc}
\mathbf{X}^{\wedge j} & \xrightarrow{\quad = \quad} & \mathbf{X}^{\wedge j} \\
\downarrow & \searrow^{i_1} & \downarrow \\
 & \mathbb{P}'_2\mathbb{P}'(\mathbf{X},\mathbf{Y}) \xrightarrow{\mu} \mathbb{P}'_2(\mathbf{X},\mathbf{Y}) & \\
 & \downarrow^{\mathbb{P}'_2\nu} & \downarrow^{\nu} \\
\mathbf{X} & \to \mathbb{P}'_2(\mathbf{X},\mathbf{Y}) \xrightarrow{\nu} & \mathbf{Y}.
\end{array}$$

Let H denote the composite of the right-hand vertical arrows. Then the diagram shows that the composite

$$(17.1) \qquad\qquad \mathbf{X}^{\wedge j} \to \mathbf{X} \xrightarrow{f} \mathbf{Y}$$

is H.

Next consider the commutative diagram

$$
\begin{array}{ccc}
\mathbf{X}^{\wedge j} & \xrightarrow{\quad = \quad} & \mathbf{X}^{\wedge j} \\
\end{array}
$$

This diagram shows that the composite

(17.2) $$\mathbf{X}^{\wedge j} \xrightarrow{f^{\wedge j}} \mathbf{Y}^{\wedge j} \to \mathbf{Y}$$

is also H. Therefore the composites (17.1) and (17.2) are equal as reqired.

Part (ii) is immediate from the definitions, and the proof of part (iii) is the same as for Proposition 10.2(iii) (but using Remark 14.7(ii)). □

Proof of Theorem 1.3. The proof follows the outline of the proof of Theorem 1.1 (given in Section 10); we refer the reader to that proof for some of the details. We have a diagram of simplicial $\mathbb{P}$-algebras

$$(\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}})_{\bullet} \xleftarrow{\varepsilon} B_{\bullet}(\mathbb{P}, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}})) \xrightarrow{\Xi_{\bullet}} B_{\bullet}(\mathbb{P}', \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}})).$$

By Lemma 16.3, this gives a diagram
(17.3)

$$
\begin{array}{ccc}
(\mathbf{R}_{e,*,1})_{\bullet} & \xleftarrow{\varepsilon} B_{\bullet}(\mathbb{P}_1, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}})) & \xrightarrow{(\Xi_1)_{\bullet}} B_{\bullet}(\mathbb{P}'_1, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}})) \\
\mathrm{Sig}_{\mathrm{rel}} \downarrow & \downarrow & \downarrow \\
(\mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}})_{\bullet} & \xleftarrow{\varepsilon} B_{\bullet}(\mathbb{P}_2, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}})) & \xrightarrow{(\Xi_2)_{\bullet}} B_{\bullet}(\mathbb{P}'_2, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}})),
\end{array}
$$

in which all objects are simplicial monoids and all horizontal arrows are monoid maps. By Proposition 17.2(i), the right column is a simplicial monoid map between simplicial commutative monoids. Moreover, each map ε is a homotopy equivalence of simplicial objects, and (using Proposition 17.2(iii)) $(\Xi_1)_{\bullet}$ and $(\Xi_2)_{\bullet}$ are weak equivalences in each simplicial degree.

The objects of the diagram (17.3) are simplicial objects in $\Sigma\mathcal{S}_{\mathrm{ss}}$. We obtain a diagram

(17.4)

$$
\begin{array}{ccccc}
|(\mathbf{R}_{e,*,1})_\bullet| & \xleftarrow{\;|\varepsilon|\;} & |B_\bullet(\mathbb{P}_1, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}}))| & \xrightarrow{\;|(\Xi_1)_\bullet|\;} & |B_\bullet(\mathbb{P}_1', \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}}))| \\
{\scriptstyle |\mathrm{Sig}_{\mathrm{rel}}|}\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
|(\mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}})_\bullet| & \xleftarrow{\;|\varepsilon|\;} & |B_\bullet(\mathbb{P}_2, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}}))| & \xrightarrow{\;|(\Xi_2)_\bullet|\;} & |B_\bullet(\mathbb{P}_2', \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}}))|
\end{array}
$$

of simplicial objects in $\Sigma\mathcal{S}$ (the category of symmetric spectra) by applying the geometric realization functor $\Sigma\mathcal{S}_{\mathrm{ss}} \to \Sigma\mathcal{S}$ to the diagram (17.3) in each simplicial degree. All objects are simplicial monoids and all horizontal arrows are monoid maps, and the right column is a simplicial monoid map between simplicial commutative monoids. The maps $|\varepsilon|$ are homotopy equivalences of simplicial objects and the maps $|(\Xi_1) \bullet |$ and $|(\Xi_1) \bullet |$ are weak equivalences in each simplicial degree.

Finally, we apply geometric realization to the diagram (17.4). We define $\mathbf{A}$ to be $||B_\bullet(\mathbb{P}_1, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}}))||$, $\mathbf{B}$ to be $||B_\bullet(\mathbb{P}_2, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}}))||$, $\mathbf{C}$ to be $||B_\bullet(\mathbb{P}_1', \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}}))||$, and $\mathbf{D}$ to be $||B_\bullet(\mathbb{P}_2', \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}_{\mathrm{rel}}^{\mathbb{Z}}))||$. This gives the diagram of Theorem 1.3. $\qquad\square$

Remark 17.3. The symmetric ring spectrum $\mathbf{C}$ is the same as the symmetric ring spectrum $\mathbf{M}_{e,*,1}^{\mathrm{comm}}$ given by Theorem 1.1. There is a ring map

$$(\mathbf{M}_{\mathrm{rel}}^{\mathbb{Z}})^{\mathrm{comm}} \to \mathbf{D}$$

which is a weak equivalence (because there is a commutative diagram whose first row is Diagram (10.3) and whose second row is the second row of the diagram in Theorem 1.3).

18. Improved versions of geometric and symmetric Poincaré bordism.

In order to state our next theorem we need some background.

Let $\Sigma\mathcal{S}$ denote the category of symmetric spectra.

Recall (from [LM, Definition 13.2(i) and the second paragraph of Section 19]) the strict monoidal category $\mathcal{T}$ whose objects are the triples (π, Z, w), where π is a group, Z is a simply-connected free π-space, and w is a homomorphism $\pi \to \{\pm 1\}$. There is a monoidal functor

$$\mathbf{M}_{\mathrm{geom}} : \mathcal{T} \to \Sigma\mathcal{S}$$

which takes (π, Z, w) to $\mathbf{M}_{\pi,Z,w}$ ([LM, Definition 19.1 and Theorem 19.2]).

Let $\mathcal{R}$ be the category of rings with involution. There is a functor

$$\mathbf{M}_{\text{sym}} : \mathcal{R} \to \Sigma\mathcal{S}$$

which takes R to $\mathbf{M}_{\text{rel}}^R$ (we used the symbol $\mathbf{M}_{\text{sym}}$ for a different but related functor in [LM, Section 19]). The proof of ([LM, Theorem 19.2]) shows that $\mathbf{M}_{\text{sym}}$ is monoidal.

There is a functor $\rho : \mathcal{T} \to \mathcal{R}$ which takes (π, Z, w) to $\mathbb{Z}[\pi]$ with the w-twisted involution ([LM, Definition 13.2(ii)]). In Section 13 we constructed a natural transformation

$$\text{Sig}_{\text{rel}} : \mathbf{M}_{\text{geom}} \to \mathbf{M}_{\text{sym}} \circ \rho.$$

Now Sig_{rel} is not a monoidal transformation, and $\mathbf{M}_{\text{geom}}$ and $\mathbf{M}_{\text{sym}}$ are not symmetric monoidal functors (we recall the definitions of monoidal transformation and symmetric monoidal functor below). Our next result shows that there is a monoidal transformation between symmetric monoidal functors which is weakly equivalent to Sig_{rel}.

Theorem 18.1. *There are symmetric monoidal functors* $\mathbf{P} : \mathcal{T} \to \Sigma\mathcal{S}$, $\mathbf{L}_{\text{sym}} : \mathcal{R} \to \Sigma\mathcal{S}$, *and a monoidal natural transformation* $\text{Sig} : \mathbf{P} \to \mathbf{L}_{\text{sym}} \circ \rho$ *such that*

(i) $\mathbf{P}$ *is weakly equivalent as a monoidal functor to* $\mathbf{M}_{\text{geom}}$; *specifically, there is a monoidal functor* $\mathbf{A} : \mathcal{T} \to \Sigma\mathcal{S}$ *and monoidal weak equivalences*

$$\mathbf{M}_{\text{geom}} \leftarrow \mathbf{A} \to \mathbf{P}.$$

(ii) $\mathbf{L}_{\text{sym}}$ *is weakly equivalent as a monoidal functor to* $\mathbf{M}_{\text{sym}}$; *specifically, there is a monoidal functor* $\mathbf{B} : \mathcal{R} \to \Sigma\mathcal{S}$ *and monoidal weak equivalences*

$$\mathbf{M}_{\text{sym}} \leftarrow \mathbf{B} \to \mathbf{L}_{\text{sym}}.$$

(iii) *The natural transformations* $\text{Sig} : \mathbf{P} \to \mathbf{L}_{\text{sym}} \circ \rho$ *and* $\text{Sig}_{\text{rel}} : \mathbf{M}_{\text{geom}} \to \mathbf{M}_{\text{sym}} \circ \rho$ *are weakly equivalent; specifically, there is a natural transformation* $\mathbf{A} \to \mathbf{B} \circ \rho$ *which makes the following diagram strictly commute*

$$
\begin{array}{ccccc}
\mathbf{M}_{\text{geom}} & \longleftarrow & \mathbf{A} & \longrightarrow & \mathbf{P} \\
{\scriptstyle \text{Sig}_{\text{rel}}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \text{Sig}} \\
\mathbf{M}_{\text{sym}} \circ \rho & \longleftarrow & \mathbf{B} \circ \rho & \longrightarrow & \mathbf{L}_{\text{sym}} \circ \rho.
\end{array}
$$

Remark 18.2. (i) It is perhaps worth mentioning that there is no such thing as a *symmetric* monoidal transformation, just as there is no such thing as a commutative homomorphism between commutative rings.

(ii) Theorem 18.1 implies that $\mathbf{L}_{\text{sym}}(R)$ is a strictly commutative symmetric ring spectrum when R is commutative. Also, $\mathbf{P}(e, *, 1)$ is a strictly commutative symmetric ring spectrum and $\text{Sig} : \mathbf{P}(e, *, 1) \to$

$\mathbf{L}_{\mathrm{sym}}(\mathbb{Z})$ is a map of symmetric ring spectra. This is compatible with Theorem 1.3: there is a commutative diagram

$$
\begin{array}{ccc}
\mathbf{C} & \longrightarrow & \mathbf{P}(e,*,1) \\
\downarrow & & {\scriptstyle \mathrm{Sig}} \downarrow \\
\mathbf{D} & \longrightarrow & \mathbf{L}_{\mathrm{sym}}(\mathbb{Z})
\end{array}
$$

in which the horizontal arrows are ring maps, and they are weak equivalences by the argument given in Remark 17.3.

(iii) The fact that Sig is a monoidal functor is a spectrum-level version of Ranicki's multiplicativity formula for the symmetric signature ([Ran80b, Proposition 8.1(i)]). It seems likely that his multiplicativity formula for the surgery obstruction ([Ran80b, Proposition 8.1(ii)]) can also be given a spectrum-level interpretation.

We recall the definitions of symmetric monoidal functor and monoidal transformation. The theorem says that $\mathbf{L}_{\mathrm{sym}}$ (and similarly $\mathbf{P}$) is a monoidal functor with the additional property that the diagram

$$
\begin{array}{ccc}
\mathbf{L}_{\mathrm{sym}}(R) \wedge \mathbf{L}_{\mathrm{sym}}(S) & \longrightarrow & \mathbf{L}_{\mathrm{sym}}(R \otimes S) \\
\downarrow & & \downarrow \\
\mathbf{L}_{\mathrm{sym}}(S) \wedge \mathbf{L}_{\mathrm{sym}}(R) & \longrightarrow & \mathbf{L}_{\mathrm{sym}}(S \otimes R)
\end{array}
$$

strictly commutes. Moreover, Sig has the property that the diagrams

$$
\begin{array}{ccc}
 & \mathbf{S} & \\
 \swarrow & & \searrow \\
\mathbf{P}(e,*,1) & \xrightarrow{\ \mathrm{Sig}\ } & \mathbf{L}_{\mathrm{sym}}(\mathbb{Z})
\end{array}
$$

and

$$
\begin{array}{ccc}
\mathbf{P}(\pi,Z,w) \wedge \mathbf{P}(\pi',Z',w') & \xrightarrow{\mathrm{Sig}\wedge\mathrm{Sig}} & \mathbf{L}_{\mathrm{sym}}(\mathbb{Z}[\pi]^w) \wedge \mathbf{L}_{\mathrm{sym}}(\mathbb{Z}[\pi']^{w'}) \\
\downarrow & & \downarrow \\
 & & \mathbf{L}_{\mathrm{sym}}(\mathbb{Z}[\pi]^w \otimes \mathbb{Z}[\pi']^{w'}) \\
\downarrow & & \downarrow \\
\mathbf{P}(\pi \times \pi', Z \times Z', w \cdot w') & \xrightarrow{\ \mathrm{Sig}\ } & \mathbf{L}_{\mathrm{sym}}(\mathbb{Z}[\pi \times \pi']^{w \cdot w'})
\end{array}
$$

strictly commute.

19. Proof of Theorem 18.1

The proof is a modification of the proof of Theorem 1.3; the main difference is that we need more elaborate notation.

Notation 19.1. (i) For an object x of $\mathcal{T}$ or $\mathcal{R}$, write $\mathcal{A}_x$ for the corresponding $\mathbb{Z}$-graded category and $\mathbf{R}_x$ for the associated object of $\Sigma\mathcal{S}_{\mathrm{ss}}$.

(ii) Given a j-tuple $(x_1, \ldots, x_j)$, where each x_i is an object of $\mathcal{T}$ or $\mathcal{R}$, write

$$[x_1, \ldots, x_j]$$

for $y_1 \otimes \cdots \otimes y_j$, where y_i is x_i if x_i is an object of $\mathcal{R}$ and $\rho(x_i)$ if x_i is an object of $\mathcal{T}$.

(iii) Given a j-tuple $(f_1, \ldots, f_j)$, where each f_i is a morphism in $\mathcal{T}$ or $\mathcal{R}$, write

$$[f_1, \ldots, f_j]$$

for $g_1 \otimes \cdots \otimes g_j$, where g_i is f_i if f_i is a morphism in $\mathcal{R}$ and $\rho(f_i)$ if f_i is a morphism in $\mathcal{T}$.

The reader should see Proposition 19.11(i) below for motivation for the following definitions.

Definition 19.2. (i) Let y be an object of $\mathcal{T}$. An *entity* of type $(r_u(j), y)$ is a $j + 1$-tuple $(x_1, \ldots, x_j, f)$, where f is a morphism in $\mathcal{T}$ from $x_1 \boxtimes \cdots \boxtimes x_j$ to y.

(ii) Let $\mathcal{E}_{r_u(j),y}$ denote the set of entities of type $(r_u(j), y)$.

(iii) Let z be an object of $\mathcal{R}$ and let $r : \{1, \ldots, j\} \to \{u, v\}$ be a function. An *entity* of type (r, z) is a $j + 1$-tuple $(x_1, \ldots, x_j, f)$, where each x_i is an object of $\mathcal{T}$ or $\mathcal{R}$ and f is a morphism in $\mathcal{R}$ from $[x_1, \ldots, x_j]$ to z.

(iv) Let $\mathcal{E}_{r,z}$ denote the set of entities of type (r, z).

Notation 19.3. (i) Let $\mathfrak{S}$ denote the union of the set of objects of $\mathcal{T}$ and the set of objects of $\mathcal{R}$.

(ii) Let $\Pi\Sigma\mathcal{S}_{\mathrm{ss}}$ be the infinite product of copies of $\Sigma\mathcal{S}_{\mathrm{ss}}$, indexed over $\mathfrak{S}$.

We will define a monad in $\Pi\Sigma\mathcal{S}_{\mathrm{ss}}$.

First we need to define the relevant right Σ_j actions. Recall Definition 15.1(i).

Definition 19.4. Let $\{\mathbf{X}_x\}_{x \in \mathfrak{S}}$ be an object of $\Pi\Sigma\mathcal{S}_{\mathrm{ss}}$ and let $j \geq 0$.

(i) Given an object y of $\mathcal{T}$ and $\alpha \in \Sigma_j$, define a map $\bar{\alpha}$ from

$$\bigvee_{(x_1, \ldots, x_j, f) \in \mathcal{E}_{r_u(j),y}} \mathcal{O}(r_u(j); u) \,\overline{\wedge}\, (\mathbf{X}_{x_1} \wedge \cdots \wedge \mathbf{X}_{x_j})$$

to itself to be the map which takes the summand indexed by $(x_1, \ldots, x_j, f)$ to the summand indexed by $(x_{\alpha(1)}, \ldots, x_{\alpha(1)}, f \circ \alpha)$ by means of the map

$$\mathcal{O}(r_u(j); u) \overline{\wedge} (\mathbf{X}_{x_1} \wedge \cdots \wedge \mathbf{X}_{x_j}) \xrightarrow{\alpha \overline{\wedge} \alpha} \mathcal{O}(r_u(j); u) \overline{\wedge} (\mathbf{X}_{x_{\alpha(1)}} \wedge \cdots \wedge \mathbf{X}_{x_{\alpha(j)}}).$$

(ii) Given an object z of $\mathcal{R}$ and $\alpha \in \Sigma_j$, define a map $\bar{\alpha}$ from

$$\bigvee_r \bigvee_{(x_1, \ldots, x_j, f) \in \mathcal{E}_{(r,z)}} \mathcal{O}(r; v) \overline{\wedge} (\mathbf{X}_{x_1} \wedge \cdots \wedge \mathbf{X}_{x_j})$$

to itself to be the map which takes the summand indexed by $(x_1, \ldots, x_j, f) \in \mathcal{E}_{(r,z)}$ to the summand indexed by $(x_{\alpha(1)}, \ldots, x_{\alpha(1)}, f \circ \alpha) \in \mathcal{E}_{(r^\alpha, z)}$ by means of the map

$$\mathcal{O}(r; v) \overline{\wedge} (\mathbf{X}_{x_1} \wedge \cdots \wedge \mathbf{X}_{x_j}) \xrightarrow{\bar{\alpha} \overline{\wedge} \alpha} \mathcal{O}(r^\alpha; v) \overline{\wedge} (\mathbf{X}_{x_{\alpha(1)}} \wedge \cdots \wedge \mathbf{X}_{x_{\alpha(j)}}).$$

Note that this definition gives right Σ_j actions on the objects mentioned.

Definition 19.5. Let $\{\mathbf{X}_x\}_{x \in \mathfrak{S}}$ be an object of $\Pi\Sigma\mathcal{S}_{ss}$.
(i) Given an object y of $\mathcal{T}$, define

$$\mathbb{P}_y(\{\mathbf{X}_x\}_{x \in \mathfrak{S}})$$

to be

$$\bigvee_{j \geq 0} \bigvee_{((x_1, \ldots, x_j, f) \in \mathcal{E}_{r_u(j), y}} \mathcal{O}(r_u(j); u) \overline{\wedge} (\mathbf{X}_{x_1} \wedge \cdots \wedge \mathbf{X}_{x_j}) \Big) / \Sigma_j.$$

(ii) Given an object z of $\mathcal{R}$, define

$$\mathbb{P}_z(\{\mathbf{X}_x\}_{x \in \mathfrak{S}})$$

to be

$$\bigvee_{j \geq 0} \Big(\bigvee_r \bigvee_{(x_1, \ldots, x_j, f) \in \mathcal{E}_{(r,z)}} \mathcal{O}(r; v) \overline{\wedge} (\mathbf{X}_{x_1} \wedge \cdots \wedge \mathbf{X}_{x_j}) \Big) / \Sigma_j.$$

(iii) Define $\mathbb{P} : \Pi\Sigma\mathcal{S}_{ss} \to \Pi\Sigma\mathcal{S}_{ss}$ to be the functor whose projection on the y factor (where y is an object of $\mathcal{T}$) is $\mathbb{P}_y$ and whose projection on the z factor (where z is an object of $\mathcal{R}$) is $\mathbb{P}_z$.

Definition 19.6. Let $\{\mathbf{X}_x\}_{x \in \mathfrak{S}}$ be an object of $\Pi\Sigma\mathcal{S}_{ss}$.
(i) For an object y of $\mathcal{T}$, define

$$\iota_y : \mathbf{X}_y \to \mathbb{P}_y(\{\mathbf{X}_x\}_{x \in \mathfrak{S}})$$

to be the composite

$$\mathbf{X}_y \cong \overline{\mathbf{S}} \overline{\wedge} \mathbf{X}_y = \mathcal{O}(r_u(1); u) \overline{\wedge} \mathbf{X}_y \hookrightarrow \mathbb{P}_y(\{\mathbf{X}_x\}_{x \in \mathfrak{S}}),$$

where the last map is the inclusion of the summand corresponding to the entity (y, id).

(ii) For an object z of $\mathcal{R}$, define

$$\iota_z : \mathbf{X}_z \to \mathbb{P}_z(\{\mathbf{X}_x\}_{x \in \mathfrak{S}})$$

to be the composite

$$\mathbf{X}_z \cong \overline{\mathbf{S}} \,\overline{\wedge}\, \mathbf{X}_z = \mathcal{O}(r_v(1); v) \,\overline{\wedge}\, \mathbf{X}_y \hookrightarrow \mathbb{P}_z(\{\mathbf{X}_x\}_{x \in \mathfrak{S}}),$$

where the last map is the inclusion of the summand corresponding to the entity (z, id).

(iii) Define

$$\iota : \{\mathbf{X}_x\}_{x \in \mathfrak{S}} \to \mathbb{P}(\{\mathbf{X}_x\}_{x \in \mathfrak{S}})$$

to be the map whose projection on the y factor (where y is an object of $\mathcal{T}$) is ι_y and whose projection on the z factor (where z is an object of $\mathcal{R}$) is ι_z.

In order to define the structure map $\mu : \mathbb{P}\mathbb{P} \to \mathbb{P}$ we need a composition operation on entities. For part (ii) we use Notation 19.1(iii) and the notation of Definition 15.3.

Definition 19.7. Let $i \geq 0$, and for each l with $1 \leq l \leq i$ let $j_l \geq 0$.
(i) Let y be an object of $\mathcal{T}$ and let

$$\mathbf{e} = (x_1, \ldots, x_i, f) \in \mathcal{E}_{r_u(i), y}.$$

For each l with $1 \leq l \leq i$ let

$$\mathbf{e}_l = (x_1^{(l)}, \ldots, x_{j_l}^{(l)}, f^{(l)}) \in \mathcal{E}_{r_u(j_l), x_l}.$$

Define

$$\mathbf{e} \circ (\mathbf{e}_1, \ldots, \mathbf{e}_i) \in \mathcal{E}_{r_u(j_1 + \cdots + j_i), y}$$

to be

$$(x_1^{(1)}, \ldots, x_{j_i}^{(i)}, g),$$

where g is the composite

$$x_1^{(1)} \boxtimes \cdots \boxtimes x_{j_i}^{(i)} \xrightarrow{f^{(1)} \boxtimes \cdots \boxtimes f^{(i)}} x_1 \boxtimes \cdots \boxtimes x_i \xrightarrow{f} y.$$

(ii) Let z be an object of $\mathcal{R}$, let $r : \{1, \ldots, i\} \to \{u, v\}$, and let

$$\mathbf{e} = (x_1, \ldots, x_i, f) \in \mathcal{E}_{r, z}.$$

For each l with $1 \leq l \leq i$ let $r_l : \{1, \ldots, j_l\} \to \{u, v\}$; assume that if $r(l) = u$ then r_l is $r_u(j_l)$. Let

$$\mathbf{e}_l = (x_1^{(l)}, \ldots, x_{j_l}^{(l)}, f^{(l)}) \in \mathcal{E}_{r_l, x_l}.$$

Define

$$\mathbf{e} \circ (\mathbf{e}_1, \ldots, \mathbf{e}_i) \in \mathcal{E}_{R, z}$$

to be

$$(x_1^{(1)}, \ldots, x_{j_i}^{(i)}, g),$$

where g is the composite

$$[x_1^{(1)}, \ldots, x_{j_i}^{(i)}] \xrightarrow{[f^{(1)}, \ldots, f^{(i)}]} [x_1, \ldots, x_i] \xrightarrow{f} z.$$

Now we can define $\mu : \mathbb{PP} \to \mathbb{P}$. We begin with the projection on the y-factor,

$$\mu_y : \mathbb{P}_y \mathbb{P} \to \mathbb{P}_y,$$

where y is an object of $\mathcal{T}$. A collection of entities $\mathbf{e}, \mathbf{e}_1, \ldots, \mathbf{e}_i$ as in Definition 19.7(i) determines a summand

$$\mathcal{O}(r_u(i); u) \overline{\wedge} \left(\left(\mathcal{O}(r_u(j_1); u) \overline{\wedge} (\mathbf{X}_{x_1^{(1)}} \wedge \cdots \wedge \mathbf{X}_{x_{j_1}^{(1)}}) \right) \wedge \cdots \right)$$

in $\mathbb{P}_y \mathbb{P}(\{\mathbf{X}_x\}_{x \in \mathfrak{S}})$. We define the restriction of μ_y to this summand to be the map to the summand of $\mathbb{P}_y(\{\mathbf{X}_x\}_{x \in \mathfrak{S}})$ indexed by $\mathbf{e} \circ (\mathbf{e}_1, \ldots, \mathbf{e}_i)$ which is induced (after passage to quotients) by the composite

$$\mathcal{O}(r_u(i); u) \overline{\wedge} \left(\left(\mathcal{O}(r_u(j_1); u) \overline{\wedge} (\mathbf{X}_{x_1^{(1)}} \wedge \cdots \wedge \mathbf{X}_{x_{j_1}^{(1)}}) \right) \wedge \cdots \right)$$

$$\to \left(\mathcal{O}(r_u(i); u) \overline{\wedge} \left(\mathcal{O}(r_u(j_1); u) \otimes \cdots \right) \right) \overline{\wedge} (\mathbf{X}_{x_1^{(1)}} \otimes \cdots \otimes \mathbf{X}_{x_{j_i}^{(i)}})$$

$$\xrightarrow{\gamma \overline{\wedge} 1} \mathcal{O}(r_u(j_1 + \cdots + j_i); u) \overline{\wedge} (\mathbf{X}_{x_1^{(1)}} \otimes \cdots \otimes \mathbf{X}_{x_{j_i}^{(i)}}).$$

The projection of μ on the z factor (where z is an object of $\mathcal{R}$) is defined similarly (using Definition 15.3).

Next we give the action of $\mathbb{P}$ on the object $\{\mathbf{R}_x\}_{x \in \mathfrak{S}}$. Let y be an object of $\mathcal{T}$ and let $(x_1, \ldots, x_j, f)$ be an entity of type $(r_u(j), y)$. A slight modification of Definition 6.9 gives a map

$$\mathcal{O}(r_u(j); u) \overline{\wedge} (\mathbf{R}_{x_1} \wedge \cdots \wedge \mathbf{R}_{x_j}) \to \mathbf{R}_{x_1 \boxtimes \cdots \boxtimes x_j},$$

and composing with the map induced by f gives a map

(19.1) $$\mathcal{O}(r_u(j); u) \overline{\wedge} (\mathbf{R}_{x_1} \wedge \cdots \wedge \mathbf{R}_{x_j}) \to \mathbf{R}_y.$$

We define

$$\nu_y : \mathbb{P}_y(\{\mathbf{R}_x\}_{x \in \mathfrak{S}}) \to \mathbf{R}_y$$

to be the map whose restriction to the summand indexed by $(x_1, \ldots, x_j, f)$ is the map (19.1). We define

$$\nu_z : \mathbb{P}_z(\{\mathbf{R}_x\}_{x \in \mathfrak{S}}) \to \mathbf{R}_z$$

similarly when z is an object of $\mathcal{R}$ (using a slight modification of Definition 15.7), and we define

$$\nu : \mathbb{P}(\{\mathbf{R}_x\}_{x \in \mathfrak{S}}) \to \{\mathbf{R}_x\}_{x \in \mathfrak{S}}$$

to be the map with projections ν_y and ν_z.

Lemma 19.8. ν *is an action of* $\mathbb{P}$ *on* $\{\mathbf{R}_x\}_{x\in\mathfrak{S}}$. $\hfill\square$

Now we need the analogue of Lemma 16.3. Let $\mathcal{C}$ be the category whose objects are triples $(\mathbf{F}, \mathbf{G}, \mathbf{t})$, where $\mathbf{F}$ is a monoidal functor $\mathcal{T} \to \Sigma\mathcal{S}_{\mathrm{ss}}$, $\mathbf{G}$ is a monoidal functor $\mathcal{R} \to \Sigma\mathcal{S}_{\mathrm{ss}}$, and $\mathbf{t}$ is a natural transformation $\mathbf{F} \to \mathbf{G} \circ \rho$ which is not required to be a monoidal transformation; the morphisms are commutative diagrams

$$
\begin{array}{ccc}
\mathbf{F} & \longrightarrow & \mathbf{F}' \\
{\scriptstyle\mathbf{t}}\downarrow & & \downarrow{\scriptstyle\mathbf{t}'} \\
\mathbf{G}\circ\rho & \longrightarrow & \mathbf{G}'\circ\rho,
\end{array}
$$

where the horizontal arrows are monoidal transformations.

Let us write $\mathbf{R}_{\mathrm{geom}}$ (resp., $\mathbf{R}_{\mathrm{sym}}$) for the functor $\mathcal{T} \to \Sigma\mathcal{S}_{\mathrm{ss}}$ (resp., $\mathcal{R} \to \Sigma\mathcal{S}_{\mathrm{ss}}$) which takes x to $\mathbf{R}_x$.

Lemma 19.9. (i) *There is a functor* Υ *from* $\mathbb{P}$ *algebras to* $\mathcal{C}$ *which takes* $\{\mathbf{X}_x\}_{x\in\mathfrak{S}}$ *to a triple* $(\mathbf{F}, \mathbf{G}, \mathbf{t})$ *with* $\mathbf{F}(y) = \mathbf{X}_y$ *and* $\mathbf{G}(z) = \mathbf{X}_z$.
(ii) $\Upsilon(\{\mathbf{R}_x\}_{x\in\mathfrak{S}})$ *is the triple*

$$(\mathbf{R}_{\mathrm{geom}}, \mathbf{R}_{\mathrm{sym}}, \mathrm{Sig}_{\mathrm{rel}})$$

Proof. Part (i). Let $\{\mathbf{X}_x\}_{x\in\mathfrak{S}}$ be a $\mathbb{P}$ algebra. Define a functor

$$\mathbf{F} : \mathcal{T} \to \Sigma\mathcal{S}_{\mathrm{ss}}$$

on objects by $\mathbf{F}(y) = \mathbf{X}_y$ and on morphisms by letting $\mathbf{F}(f : y \to y')$ be the composite

$$\mathbf{X}_y \cong \mathcal{O}(r_u(1); u) \,\overline{\wedge}\, \mathbf{X}_y \hookrightarrow \mathbb{P}_{y'}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) \xrightarrow{\nu_{y'}} \mathbf{X}_{y'},$$

where the unlabeled arrow is the inclusion of the summand indexed by the entity (y, f). We define

$$\mathbf{G} : \mathcal{R} \to \Sigma\mathcal{S}_{\mathrm{ss}}$$

similarly. The proof that $\mathbf{F}$ and $\mathbf{G}$ are monoidal functors is similar to the argument, in the proof in Lemma 16.3(i), that $\mathbf{X}$ and $\mathbf{Y}$ are monoids.

It remains to give the natural transformation

$$\mathbf{t} : \mathbf{F} \to \mathbf{G} \circ \rho.$$

For an object y of $\mathcal{T}$, let $\mathbf{t}_y$ be the composite

$$\mathbf{X}_y \cong \mathcal{O}(r_u(1); v) \,\overline{\wedge}\, \mathbf{X}_y \hookrightarrow \mathbb{P}_{\rho(y)}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) \xrightarrow{\nu_{\rho(y)}} \mathbf{X}_{\rho(y)},$$

where the unlabeled arrow is the inclusion of the summand indexed by the entity $(y, \mathrm{id}) \in \mathcal{E}_{r_u(1), \rho(y)}$. To show that $\mathbf{t}$ is a natural transformation, let $f : y \to y'$ be a morphism in $\mathcal{T}$, and let $z = \rho(y)$, $z' = \rho(y')$. Let i_1 be the composite

$$\mathbf{X}_y \cong \mathcal{O}(r_u(1); u)\overline{\wedge}\mathbf{X}_y \hookrightarrow \mathbb{P}_{y'}(\{\mathbf{X}_x\}_{x \in \mathfrak{S}}) \cong \mathcal{O}(r_u(1); v)\overline{\wedge}\mathbb{P}_{y'}(\{\mathbf{X}_x\}_{x \in \mathfrak{S}})$$
$$\hookrightarrow \mathbb{P}_{z'}\mathbb{P}(\{\mathbf{X}_x\}_{x \in \mathfrak{S}}),$$

where the first arrow is the inclusion of the summand indexed by the entity (y, f) and the second is the inclusion of the summand indexed by (y', id). Let j_1 be the composite
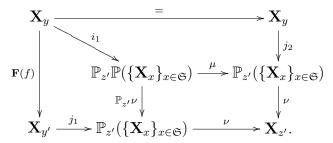
$$\mathbf{X}_{y'} \cong \mathcal{O}(r_u(1); v)\overline{\wedge}\mathbf{X}_{y'} \hookrightarrow \mathbb{P}_{z'}(\{\mathbf{X}_x\}_{x \in \mathfrak{S}}),$$

where the inclusion is indexed by (y', id), and let j_2 be the composite

$$\mathbf{X}_y \cong \mathcal{O}(r_u(1); v)\overline{\wedge}\mathbf{X}_y \hookrightarrow \mathbb{P}_{z'}(\{\mathbf{X}_x\}_{x \in \mathfrak{S}}),$$

where the inclusion is indexed by $(y, \rho(f))$.

Consider the commutative diagram

Let H denote the composite of the right-hand vertical arrows. Then the diagram shows that the composite

(19.2) $$\mathbf{X}_y \xrightarrow{\mathbf{F}(f)} \mathbf{X}_{y'} \xrightarrow{\mathbf{t}} \mathbf{X}_{z'}$$

is H.

Let i_2 be the composite

$$\mathbf{X}_y \cong \mathcal{O}(r_u(1); v)\overline{\wedge}\mathbf{X}_y \hookrightarrow \mathbb{P}_z(\{\mathbf{X}_x\}_{x \in \mathfrak{S}}) \cong \mathcal{O}(r_v(1); v)\overline{\wedge}\mathbb{P}_z(\{\mathbf{X}_x\}_{x \in \mathfrak{S}})$$
$$\hookrightarrow \mathbb{P}_{z'}\mathbb{P}(\{\mathbf{X}_x\}_{x \in \mathfrak{S}}),$$

where the first inclusion is indexed by (y, id) and the second is indexed by $(z, \rho(f))$. Let j_2 be as above and let j_3 be the composite

$$\mathbf{X}_z \cong \mathcal{O}(r_v(1); v)\overline{\wedge}\mathbf{X}_z \hookrightarrow \mathbb{P}_z(\{\mathbf{X}_x\}_{x \in \mathfrak{S}}),$$

where the inclusion is indexed by $(z, \rho(f))$.

Consider the commutative diagram

$$
\begin{array}{ccc}
\mathbf{X}_y & \xrightarrow{\quad = \quad} & \mathbf{X}_y \\
\end{array}
$$

$$
\mathbf{X}_y \xrightarrow{i_1} \mathbb{P}_{z'}\mathbb{P}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) \xrightarrow{\mu} \mathbb{P}_{z'}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) \xrightarrow{j_2} \mathbf{X}_y
$$

$$
\mathbf{X}_z \xrightarrow{j_3} \mathbb{P}_{z'}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) \xrightarrow{\nu} \mathbf{X}_{z'}.
$$

This diagram shows that the composite

(19.3) $$\mathbf{X}_y \xrightarrow{\mathbf{t}} \mathbf{X}_z \xrightarrow{\mathbf{G}(f)} \mathbf{X}_{z'}$$

is also H, so the composites (19.2) and (19.3) are equal as reqired.

Part (ii) is an easy consequence of the definitions. $\square$

Finally, we have the analogues of Definition 17.1 and Proposition 17.2.

Definition 19.10. Let $\{\mathbf{X}_x\}_{x\in\mathfrak{S}}$ be an object of $\Pi\Sigma\mathcal{S}_{\mathrm{ss}}$.
(i) Given an object y of $\mathcal{T}$, define

$$\mathbb{P}'_y(\{\mathbf{X}_x\}_{x\in\mathfrak{S}})$$

to be

$$\bigvee_{j\geq 0}\Big(\bigvee_{((x_1,\ldots,x_j,f)\in\mathcal{E}_{r_u(j),y}} \mathbf{X}_{x_1}\wedge\cdots\wedge\mathbf{X}_{x_j}\Big)/\Sigma_j.$$

(ii) Given an object z of $\mathcal{R}$, define

$$\mathbb{P}'_z(\{\mathbf{X}_x\}_{x\in\mathfrak{S}})$$

to be

$$\bigvee_{j\geq 0}\Big(\bigvee_r\bigvee_{(x_1,\ldots,x_j,f)\in\mathcal{E}_{(r,z)}} \mathbf{X}_{x_1}\wedge\cdots\wedge\mathbf{X}_{x_j}\Big)/\Sigma_j.$$

(iii) Define $\mathbb{P}' : \Pi\Sigma\mathcal{S}_{\mathrm{ss}} \to \Pi\Sigma\mathcal{S}_{\mathrm{ss}}$ to be the functor whose projection on the y factor (where y is an object of $\mathcal{T}$) is $\mathbb{P}'_y$ and whose projection on the z factor (where z is an object of $\mathcal{R}$) is $\mathbb{P}'_z$.

A routine modification of Definition 17.1(ii) gives a natural transformation

$$\Xi : \mathbb{P} \to \mathbb{P}'.$$

Proposition 19.11. (i) *An algebra over $\mathbb{P}'$ is the same thing as a pair of symmetric monoidal functors $\mathbf{F}$ and $\mathbf{G}$ with a monoidal transformation $\mathbf{F} \to \mathbf{G} \circ \rho$.*
(ii) *Ξ is a map of monads.*

(iii) *Suppose that each $(X_x)_k$ has compatible degeneracies (see Definition 9.1). Let $\mathbb{P}^q$ denote the q-th iterate of $\mathbb{P}$. Then each map*

$$\Xi : \mathbb{P}^q(\{\mathbf{X}_x\}_{x \in \mathfrak{S}}) \to \mathbb{P}'\mathbb{P}^{q-1}(\{\mathbf{X}_x\}_{x \in \mathfrak{S}})$$

is a weak equivalence.

Proof. Part (i). Let $\{\mathbf{X}_x\}_{x \in \mathfrak{S}}$ be an algebra over $\mathbb{P}'$. The fact that $\mathbf{F}$ and $\mathbf{G}$ are symmetric monoidal functors is an easy consequence of the definitions. The natural transformation $\mathbf{t} : \mathbf{F} \to \mathbf{G} \circ \rho$ is constructed as in the proof of Lemma 19.9(i). The proof that $\mathbf{t}$ is monoidal is similar to the proofs of Proposition 17.2(i) and Lemma 19.9(i), using the maps

$$i_1 : \mathbf{X}_{y_1} \wedge \cdots \wedge \mathbf{X}_{y_j} \hookrightarrow \mathbb{P}'_{y_1 \boxtimes \cdots \boxtimes y_j}(\{\mathbf{X}_x\}_{x \in \mathfrak{S}}) \hookrightarrow \mathbb{P}'_{\rho(y_1 \boxtimes \cdots \boxtimes y_j)}\mathbb{P}'(\{\mathbf{X}_x\}_{x \in \mathfrak{S}}),$$

(where the first inclusion is indexed by $(y_1, \ldots, y_j, \mathrm{id})$ and the second by $(y_1 \boxtimes \cdots \boxtimes y_j, \mathrm{id})$), and

$$i_2 : \mathbf{X}_{y_1} \wedge \cdots \wedge \mathbf{X}_{y_j} \hookrightarrow \mathbb{P}'_{\rho(y_1)}(\{\mathbf{X}_x\}_{x \in \mathfrak{S}}) \wedge \cdots \wedge \mathbb{P}'_{\rho(y_j)}(\{\mathbf{X}_x\}_{x \in \mathfrak{S}})$$
$$\hookrightarrow \mathbb{P}'_{\rho(y_1 \boxtimes \cdots \boxtimes y_j)}\mathbb{P}'(\{\mathbf{X}_x\}_{x \in \mathfrak{S}}),$$

(where the first map is the smash product of the inclusions indexed by (y_i, id) and the second is indexed by $(\rho(y_1), \ldots, \rho(y_j), \rho(y_1) \otimes \cdots \otimes \rho(y_j) \to \rho(y_1 \boxtimes \cdots \boxtimes y_j)))$.

Part (ii) is immediate from the definitions, and the proof of part (iii) is the same as for Proposition 10.2(iii) (but using Remark 14.7(ii)). $\square$

Now the proof of Theorem 18.1 is the same as the proof of Theorem 1.3 given in Section 17, with only the notation changed.

Appendix A. Proof of Theorem 1.2

It is well known that Thom spectra are commutative symmetric ring spectra (see for example [Sch09]; we recall this below). In this appendix we show that the Thom spectrum $M\mathrm{STop}$ is weakly equivalent, in the category of commutative symmetric ring spectra, to the commutative symmetric ring spectrum $(\mathbf{M}_{\mathrm{STop}})^{\mathrm{comm}}$ given by Theorem 1.1.

Our first task is to construct the following chain of weak equivalences in the category of symmetric spectra.

$$(\mathrm{A.1}) \qquad \mathbf{M}_{\mathrm{STop}} \xleftarrow{f_1} \mathbf{Y} \xrightarrow{f_2} \mathbf{X} \xrightarrow{f_3} M\mathrm{STop}$$

First recall that $M\mathrm{STop}$ has as k-th space the Thom space $T(\mathrm{STop}(k))$. The Σ_k action on $T(\mathrm{STop}(k))$ is induced by the conjugation action on $\mathrm{STop}(k)$.

For the construction of $\mathbf{X}$ we need some facts about multisimplicial sets. Given a space Z and $k \geq 1$, let $S_\bullet^{k\text{-multi}}(Z)$ be the k-fold multi-simplicial set whose simplices in multidegree $\mathbf{n}$ are the maps $\Delta^\mathbf{n} \to Z$. There is a natural map

$$|S_\bullet^{k-\text{multi}}(Z)| \to Z$$

(where $|\ |$ denotes realization of the underlying multisimplicial set) which is a weak equivalence by [And] and the multisimplicial analogue of [Seg74, Lemma A.5]. If Z is a based space, there are natural maps

$$\lambda : \Sigma |S_\bullet^{k-\text{multi}}(Z)| \to |S^1 \wedge S_\bullet^{k-\text{multi}}(Z)|$$

and

$$\kappa : S^1 \wedge S_\bullet^{k-\text{multi}}(Z) \to S_\bullet^{(k+1)-\text{multi}}(\Sigma Z)$$

defined as follows. Given $t \in [0,1]$, $u \in \Delta^\mathbf{n}$, and $g : \Delta^\mathbf{n} \to Z$, let $\bar{t}$ denote the image of t under the oriented affine homeomorphism $[0,1] \to \Delta^1$, and define

$$\lambda(t \wedge [u,g]) = [(\bar{t},u), s \wedge g],$$

where s is the nontrivial simplex of S^1. Define

$$\kappa(s \wedge g)(\bar{t},u) = t \wedge g(u).$$

Then the diagram

$$\text{(A.2)} \qquad \begin{array}{ccc} \Sigma |S_\bullet^{k-\text{multi}}(Z)| & \xrightarrow{\lambda} & |S^1 \wedge S_\bullet^{k-\text{multi}}(Z)| \\ \downarrow & & {\scriptstyle |\kappa|} \downarrow \\ \Sigma X & \longleftarrow & |S_\bullet^{(k+1)-\text{multi}}(\Sigma Z)| \end{array}$$

commutes.

Now let $X_k = |S_\bullet^{k-\text{multi}}(T(\text{STop}(k)))|$. We define the Σ_k action on X_k as follows. For $\alpha \in \Sigma_k$ and $g : \Delta^\mathbf{n} \to T(\text{STop}(k)))$, let $\alpha(\mathbf{n}) = (n_{\alpha^{-1}(1)}, \ldots, n_{\alpha^{-1}(k)})$ and let $\alpha(g)$ be the composite

$$\Delta^{\alpha(\mathbf{n})} \xrightarrow{\alpha^{-1}} \Delta^\mathbf{n} \xrightarrow{g} T(\text{STop}(k)) \xrightarrow{\alpha} T(\text{STop}(k)).$$

This makes $S_\bullet^{k-\text{multi}}(T(\text{STop}(k)))$ an object of $\Sigma_k \text{ss}_k$, and now Proposition 5.3 gives the Σ_k action on X_k. Next define the structure map

$$\Sigma X_k \to X_{k+1}$$

to be the composite

$$\Sigma |S_\bullet^{k-\text{multi}}(T(\text{STop}(k)))| \xrightarrow{\lambda} |S^1 \wedge S_\bullet^{k-\text{multi}}(T(\text{STop}(k)))|$$

$$\xrightarrow{|\kappa|} |S_\bullet^{(k+1)-\text{multi}}(\Sigma T(\text{STop}(k)))| \to |S_\bullet^{(k+1)-\text{multi}}(T(\text{STop}(k+1)))|,$$

where the last map is induced by the structure map of MSTop. Let $\mathbf{X}$ be the symmetric spectrum consisting of the spaces X_k with these structure maps. Define $f_3 : \mathbf{X} \to M$STop to be the sequence of weak equivalences

$$|S^{k-\mathrm{multi}}_\bullet(T(\mathrm{STop}(k)))| \to T(\mathrm{STop}(k)).$$

The commutativity of diagram (A.2) shows that f_3 is a map of symmetric spectra.

Next let $S^{k-\mathrm{multi},\pitchfork}_\bullet(T(\mathrm{STop}(k)))$ be the sub-multisemisimplicial set of $S^{k-\mathrm{multi}}_\bullet(T(\mathrm{STop}(k)))$ consisting of maps whose restrictions to each face of $\Delta^{\mathbf{n}}$ are transverse to the zero section. Let $\mathbf{Y}$ be the subspectrum of $\mathbf{X}$ with k-th space $|S^{k-\mathrm{multi},\pitchfork}_\bullet(T(\mathrm{STop}(k)))|$, and let $f_2 : \mathbf{Y} \to \mathbf{X}$ be the inclusion.

Lemma A.1. f_2 is a weak equivalence.

Proof. Since $S^{k-\mathrm{multi},\pitchfork}_\bullet(T(\mathrm{STop}(k)))$ and $S^{k-\mathrm{multi}}_\bullet(T(\mathrm{STop}(k)))$ satisfy the multi-Kan condition, they have compatible degeneracies by Proposition 9.5. It therefore suffices to show that the inclusion $S^{k-\mathrm{multi},\pitchfork}_\bullet(T(\mathrm{STop}(k))) \subset S^{k-\mathrm{multi}}_\bullet(T(\mathrm{STop}(k)))$ induces a weak equivalence on the diagonal semi-simplicial sets, and this follows from [FQ90, Section 9.6] and the definition of homotopy groups ([May92, Definition 3.6]). □

It remains to construct f_1. Let $S \subset T(\mathrm{STop}(k))$ be the zero section. First we observe that, if $g : \Delta^{\mathbf{n}} \to T(\mathrm{STop}(k))$ is a map whose restriction to each face is transverse to S, we obtain an element $F \in \mathrm{ad}_{\mathrm{STop}}(\Delta^{\mathbf{n}})$ by letting $F(\sigma, o)$ be $g^{-1}(S) \cap \sigma$ with the orientation determined by o. This construction gives a map

$$S^{k-\mathrm{multi},\pitchfork}_\bullet(T(\mathrm{STop}(k))) \to (\mathbf{R}_{\mathrm{STop}})_k,$$

but unfortunately it doesn't commute with the Σ_k actions, as the reader can verify. To fix this, given g as above and an orientation o of $\Delta^{\mathbf{n}}$, let $\mathbf{m}$ be the numbers $n_1, \ldots, n_k$ written in increasing order, let

$$\beta : \Delta^{\mathbf{m}} \to \Delta^{\mathbf{n}}$$

be the map which permutes the factors without changing the order of factors of equal dimension, and give $\Delta^{\mathbf{m}}$ the orientation o' determined by o and β. Now define

$$\mathrm{Inv}(g, o) = \beta^{-1} g^{-1}(S)$$

with the orientation determined by o'. We obtain an element $F \in \mathrm{ad}_{\mathrm{STop}}(\Delta^{\mathbf{n}})$ by letting $F(\sigma, o) = \mathrm{Inv}(g|_\sigma, o)$. This gives a map

$$S^{k-\mathrm{multi},\pitchfork}_\bullet(T(\mathrm{STop}(k))) \to (\mathbf{R}_{\mathrm{STop}})_k$$

in $\Sigma_k\mathrm{ss}_k$, and applying geometric realization gives a Σ_k equivariant map $Y_k \to (\mathbf{M}_{\mathrm{STop}})_k$; we let f_1 be the sequence of these maps.

Lemma A.2. f_1 *is a weak equivalence.*

Proof. For a k-fold multisemisimplicial set A, let A' be the semisimplicial set whose n-th set is $A_{0,\dots,0,n}$. There is an evident map

$$\phi : |A'| \to |A|.$$

If A is $S_\bullet^{k-\mathrm{multi}}(Z)$, then A' is $S_\bullet(Z)$, and if A is R_k then A' is the semisimplicial set P_k of [LM, Definition 15.4(i)], with realization $(\mathbf{Q}_{\mathrm{STop}})_k$ ([LM, Definitions 15.4(ii) and 15.8]). Now we have a commutative diagram

$$
\begin{array}{ccccccc}
(\mathbf{M}_{\mathrm{STop}})_k & \xleftarrow{\;f_1\;} & Y_k & \xrightarrow{\;f_2\;} & X_k & \xrightarrow{\;f_3\;} & T(\mathrm{STop}(k)) \\
\big\uparrow{\scriptstyle\phi_1} & & \big\uparrow{\scriptstyle\phi_2} & & & & \big\uparrow{\scriptstyle =} \\
(\mathbf{Q}_{\mathrm{STop}})_k & \xleftarrow{\;g_1\;} & |S_\bullet^\pitchfork(T(\mathrm{STop}(k)))| & \xrightarrow{\;g_2\;} & |S_\bullet(T(\mathrm{STop}(k)))| & \xrightarrow{\;g_3\;} & T(\mathrm{STop}(k))
\end{array}
$$

Here g_3 is the usual weak equivalence, and g_2 is a weak equivalence by [FQ90, Section 9.6] and the definition of homotopy groups ([May92, Definition 3.6]), so ϕ_2 is a weak equivalence. g_1 was shown to be a weak equivalence in [LM, Appendix B], and ϕ_1 was shown to be a weak equivalence in [LM, Section 15], so f_1 is a weak equivalence as required. $\qquad\square$

This completes the construction of diagram (A.1).

Next we recall that $M\mathrm{STop}$ is a commutative symmetric ring spectrum with product

$$T(\mathrm{STop}(k)) \wedge T(\mathrm{STop}(l)) \to T(\mathrm{STop}(k+l)).$$

$\mathbf{X}$ is also a commutative symmetric ring spectrum, with the product

$$|S_\bullet^{k-\mathrm{multi}}(T(\mathrm{STop}(k)))| \wedge |S_\bullet^{k-\mathrm{multi}}(T(\mathrm{STop}(l)))|$$
$$\to |S_\bullet^{(k+l)-\mathrm{multi}}(T(\mathrm{STop}(k))\wedge T(\mathrm{STop}(l)))| \to |S_\bullet^{(k+l)-\mathrm{multi}}(T(\mathrm{STop}(k+l)))|,$$

and $\mathbf{Y}$ is a commutative symmetric ring spectrum with the product it inherits from $\mathbf{X}$. The maps f_2 and f_3 are maps of symmetric ring spectra, so to complete the proof of Theorem 1.2, it suffices to show

Lemma A.3. $(\mathbf{M}_{\mathrm{STop}})^{\mathrm{comm}}$ *and* $\mathbf{Y}$ *are isomorphic in the homotopy category of commutative symmetric ring spectra.*

The remainder of this appendix is devoted to the proof of Lemma A.3.

Let $\mathbf{W}$ denote the multisemisimplicial spectrum whose k-th object is $S_{\bullet}^{k-\mathrm{multi},\mathrm{fh}}(T(\mathrm{STop}(k)))$, so that $|\mathbf{W}| = \mathbf{Y}$. We begin by showing that the monad $\mathbb{P}$ of Definition 16.2 acts on the pair $(\mathbf{W}, \mathbf{R}_{\mathrm{STop}})$.

Let us define a $\mathbb{Z}$-graded category $\mathcal{B}$ as follows. The objects of $\mathcal{B}$ are pairs
$$(g : \Delta^{\mathbf{n}} \to T(\mathrm{STop}(k)), o),$$
where both $\mathbf{n}$ and k are allowed to vary and o is an orientation of $\Delta^{\mathbf{n}}$; the grading is given by $d(g, o) = \dim(\Delta^{\mathbf{n}}) - k$. The morphisms are commutative diagrams

$$
\begin{array}{ccc}
\Delta^{\mathbf{n}} & \xrightarrow{\;g\;} & T(\mathrm{STop}(k)) \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \alpha} \\
\Delta^{\mathbf{n}'} & \xrightarrow{\;g'\;} & T(\mathrm{STop}(k))
\end{array}
$$

in which ϕ is a composite of coface maps and permutations of the factors and α is a permutation; we require ϕ to be orientation preserving if the dimensions are equal. $\mathcal{B}$ is a symmetric monoidal $\mathbb{Z}$-graded category with product $\square$, where
$$(g, o) \,\square\, (g', o')$$
is the pair consisting of the composite
$$\Delta^{\mathbf{n}} \times \Delta^{\mathbf{n}'} \xrightarrow{\;g \times g'\;} T(\mathrm{STop}(k)) \times T(\mathrm{STop}(k')) \to T(\mathrm{STop}(k + k'))$$
and the orientation $o \times o'$. The symmetry isomorphism γ is

$$
\begin{array}{ccc}
\Delta^{\mathbf{n}} \times \Delta^{\mathbf{n}'} \xrightarrow{\;g \times g'\;} T(\mathrm{STop}(k)) \times T(\mathrm{STop}(k')) & \longrightarrow & T(\mathrm{STop}(k + k')) \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \alpha} \\
\Delta^{\mathbf{n}'} \times \Delta^{\mathbf{n}} \xrightarrow{\;g' \times g\;} T(\mathrm{STop}(k')) \times T(\mathrm{STop}(k)) & \longrightarrow & T(\mathrm{STop}(k' + k))
\end{array}
$$

where ϕ and α are the evident permutations.

In the construction of Definition 14.4, if we replace $\mathcal{A}_{e,*,1}$ by $\mathcal{B}$, $\mathcal{A}_{\mathrm{rel}}^{\mathbb{Z}}$ by $\mathcal{A}_{\mathrm{STop}}$, $\boxtimes$ by $\square$ and $\mathrm{Sig}_{\mathrm{rel}}$ by Inv we obtain a functor
$$\mathbf{d}_{\blacksquare} : \mathcal{A}_1 \times \cdots \times \mathcal{A}_j \to \mathcal{A}_{\mathrm{STop}},$$
for each datum $\mathbf{d}$, where $\mathcal{A}_i$ denotes $\mathcal{B}$ if $r(i) = u$ and $\mathcal{A}_{\mathrm{STop}}$ if $r(i) = v$.

Next we have the analogue of Definition 15.5.

Definition A.4. Let $k_1, \ldots, k_j$ be non-negative integers and let $\mathbf{n}_i$ be a k_i-fold multi-index for $1 \le i \le j$. Let $r : \{1, \ldots, j\} \to \{u, v\}$, and for $1 \le i \le j$ let $\mathbf{Z}_i$ denote $\mathbf{W}$ if $r(i) = u$ and $\mathbf{R}_{\mathrm{STop}}$ if $r(i) = v$. Then for each map of preorders
$$a : U(\Delta^{\mathbf{n}_1} \times \cdots \times \Delta^{\mathbf{n}_j}) \to P_{r;v}$$

we define

$$a_* : ((\mathbf{Z}_1)_{k_1})_{\mathbf{n}_1} \times \cdots \times ((\mathbf{Z}_j)_{k_j})_{\mathbf{n}_j} \to ((\mathbf{R}_{\mathrm{STop}})_{k_1 + \cdots + k_j})_{(\mathbf{n}_1, \ldots, \mathbf{n}_j)}$$

by

$$a_*(z_1, \ldots, z_j)(\sigma_1 \times \cdots \times \sigma_j, o_1 \times \cdots \times o_j)$$
$$= i^{\epsilon(\zeta)} a(\sigma_1 \times \cdots \times \sigma_j)_\blacksquare (z_1(\sigma_1, o_1), \ldots, z_j(\sigma_j, o_j)),$$

where

- if $r(i) = u$ then $z_i(\sigma_i, o_i))$ denotes $(z_i|_{\sigma_i}, o_i)$, and
- ζ is the block permutation that takes blocks $\mathbf{b}_1$, ..., $\mathbf{b}_j$, $\mathbf{c}_1$, ..., $\mathbf{c}_j$ of size k_1, ..., k_j, $\dim \sigma_1$, ..., $\dim \sigma_j$ into the order $\mathbf{b}_1$, $\mathbf{c}_1$, ..., $\mathbf{b}_j$, $\mathbf{c}_j$.

As in Section 16, this definition leads to a map

$$(\mathrm{A.3}) \qquad\qquad \mathbb{P}_2(\mathbf{W}, \mathbf{R}_{\mathrm{STop}}) \to \mathbf{R}_{\mathrm{STop}}.$$

Since $\mathbf{W}$ is a commutative multisemisimplicial symmetric ring spectrum, we have a map

$$(\mathrm{A.4}) \qquad\qquad \mathbb{P}_1(\mathbf{W}) \xrightarrow{\Xi_1} \bigvee_{j \geq 0} \mathbf{W}^{\wedge j}/\Sigma_j \to \mathbf{W},$$

where Ξ_1 is given in Definition 17.1(ii).

The maps (A.3) and (A.4) give the required action of $\mathbb{P}$ on $(\mathbf{W}, \mathbf{R}_{\mathrm{STop}})$. Now the proof of Theorem 1.3 (given in Section 17) gives a map of commutative symmetric ring spectra

$$|B_\bullet(\mathbb{P}'_1, \mathbb{P}, (\mathbf{W}, \mathbf{R}_{\mathrm{STop}}))| \to |B_\bullet(\mathbb{P}'_2, \mathbb{P}, (\mathbf{W}, \mathbf{R}_{\mathrm{STop}}))|$$

which is a weak equivalence by Lemma A.1. As in Remark 17.3, there is a weak equivalence of commutative symmetric ring spectra

$$(\mathbf{M}_{\mathrm{STop}})^{\mathrm{comm}} \to |B_\bullet(\mathbb{P}'_2, \mathbb{P}, (\mathbf{W}, \mathbf{R}_{\mathrm{STop}}))|.$$

To complete the proof, we observe that there is a weak equivalence of commutative symmetric ring spectra

$$|B_\bullet(\mathbb{P}'_1, \mathbb{P}, (\mathbf{W}, \mathbf{R}_{\mathrm{STop}}))| = |B_\bullet(\mathbb{P}'_1, \mathbb{P}_1, \mathbf{W})| \to |B_\bullet(\mathbb{P}'_1, \mathbb{P}'_1, \mathbf{W})| \to |\mathbf{W}| = \mathbf{Y},$$

where the first arrow is a weak equivalence by Proposition 17.2(iii) and the second by [May72, Proposition 9.8 and Corollary 11.9].

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