Characteristic numbers from 2-cocycles on formal groups

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Abstract. We give explicit polynomial generators for the homology rings of BSU and $BSpin$ for complex oriented theories. Using these we are able to provide an alternative proof of the result of Hopkins, Ando and Strickland for symmetric 2-cocycles on formal group laws.

1. Introduction and statement of results

Hopkins, Ando and Strickland have recently shown (see [**AHS98**][**Hop95**] [**HMM98**]) that for any complex oriented theory E the ring E_*BSU carries the universal symmetric 2-cocycle on the formal group of E. In this paper we give an alternative proof of their result which is based on the following choice of polynomial generators for E_*BSU: Let L be the canonical line bundle over $\mathbb{C}P^\infty$ and let $\beta_i \in E_{2i}\mathbb{C}P^\infty$ be dual to $c_1(L)^i$. Let

$$f : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow BSU$$

be the map which classifies the product $(1-L_1)(1-L_2)$. For each natural number k and $1 \leq i \leq k-1$ choose integers n_k^i such that

$$(1) \qquad \sum_{i=1}^{k-1} n_k^i \binom{k}{i} \quad = \quad \text{g.c.d.}\{\binom{k}{1}, \ldots, \binom{k}{k-1}\}.$$

Then our first result is

Theorem 1.1. *Define elements*

$$(2) \qquad d_k \quad = \quad \sum_{i=1}^{k-1} n_k^i f_*(\beta_i \otimes \beta_{k-i}) \in E_{2k}BSU.$$

Then for any complex oriented E we have

$$(3) \qquad E_*BSU \quad \cong \quad \pi_*E[d_2, d_3, d_4, \ldots].$$

It must be emphasized that the conceptual basis and the proof of the above theorem owes many ideas to the work of [**AHS98**]. However, the present approach is more elementary and does not use the language of schemes. We also show how the generators relate to the map from E_*BSp.

Next we investigate the homology ring of $BSpin$ for mod 2 K-theory. Our main result is

Theorem 1.2. *Let ω be the canonical quaternian line bundle over $\mathbb{H}P^\infty$ and let $z_k \in K_*(\mathbb{H}P^\infty; \mathbb{F}_2)$ be dual to $c_2(\omega)^j$. Setting $d'_{2k} = d_{2k} + z_k \in K_*(BSpin; \mathbb{F}_2)$ for all k we have*

$$K_0(BSpin; \mathbb{F}_2) \quad \cong \quad \mathbb{F}_2[d_{2k} | k \neq 2^s] \otimes \mathbb{F}_2[d'_4, d'_8, d'_{16}, \ldots].$$

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Moreover, each z_k *is decomposable in* $K_0(BSpin;\mathbb{F}_2)$*.*

As a consequence, we are able to give a new proof of the result of [**HAS99**] that the ring $K_*(BSpin;\mathbb{F}/2)$ carries the universal real symmetric 2-cocycle for the multiplicative formal group.

2. The homology of $\mathbb{C}P^\infty$ and binomial coefficients of formal group laws

Recall from [**Ada74**] that for any complex oriented ring theory E we are given a class $x \in \tilde{E}^2\mathbb{C}P^\infty$ such that

$$E^*\mathbb{C}P^\infty \cong \pi^*E[\![x]\!].$$

The H-space structure of $BS^1 \cong \mathbb{C}P^\infty$ induces a comultiplication

$$\mu^* : E^*\mathbb{C}P^\infty \longrightarrow E^*\mathbb{C}P^\infty \hat{\otimes} E^*\mathbb{C}P^\infty;\ x \mapsto x +_F y$$

and a ring structure map

$$E_*\mathbb{C}P^\infty \otimes E_*\mathbb{C}P^\infty \longrightarrow E_*\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\mu_*} E_*\mathbb{C}P^\infty.$$

Let $\beta_0 = 1, \beta_1, \beta_2, \ldots$ be the additive basis of $E_*\mathbb{C}P^\infty$ dual to $x^0, x^1, x^2, \ldots$.

DEFINITION 2.1. *The binomial coefficients of the formal group law* F

$$\binom{k}{i,j}_F \in \pi_{2(i+j-k)}E$$

are defined by the equation

$$(x +_F y)^k = \sum_{i,j} \binom{k}{i,j}_F x^i y^j.$$

With this notation we easily see

LEMMA 2.2.

$$\beta_i\beta_j = \sum_{k=0}^{i+j} \binom{k}{i,j}_F \beta_k.$$

EXAMPLE 2.3. Let E be integral singular homology. Then F is the additive formal group law $\hat{G}_a$ and

$$\binom{k}{i,j}_{\hat{G}_a} = \begin{cases} \binom{k}{i} & \text{if } i+j=k \\ 0 & \text{else} \end{cases}.$$

Hence $H\mathbb{Z}_*\mathbb{C}P^\infty$ is the divided power algebra $\Gamma[\beta_1]$.

Next let E be K-theory with its standard orientation $F = \hat{G}_m$. Then

$$(x +_{\hat{G}_m} y)^k = (x + y - v^{-1}xy)^k = \sum_{s=0}^{k}\sum_{t=0}^{s} \binom{k}{s}\binom{s}{t}(-v)^{s-k}x^{k-s+t}y^{k-t}$$

and hence

$$\begin{aligned}\binom{k}{i,j}_{\hat{G}_m} &= \binom{k}{2k-i-j}\binom{2k-i-j}{k-j}(-v)^{k-i-j} \\ &= \frac{k!}{(i+j-k)!\,(k-j)!\,(k-i)!}(-v)^{k-i-j}\end{aligned}$$

Finally, let E be complex bordism MU. The coefficients of the universal formal group law FGL are the Milnor hypersurfaces $H_{i,j}$ of type $(1,1)$ in $\mathbb{C}P^i \times \mathbb{C}P^j$ and hence

$$\binom{k}{i,j}_{FGL} = \sum_{\substack{i_1\cdots+i_k=i \\ j_1+\cdots j_k=j}} \prod_{l=1}^{k} H_{i_l,j_l}.$$

We close this section with an observation which will help finding the binomial coefficients in the situation of positive characteristic. Let F be a formal group law with coefficients in an $\mathbb{F}_p$-algebra. Then we define

$$x +_{pF} y = \sum_{i,j} \binom{1}{i,j}_F^p x^i y^j .$$

Since $x^p +_{pF} y^p = (x +_F y)^p$ we obtain a new formal group law pF, i.e. it satisfies the associativity condition. The group law pF carries the name Frobenius of F. Be aware that the grading has changed

$$\left| \binom{k}{i,j}_{pF} \right| = 2p(i+j-k)$$

EXAMPLE 2.4. Let H_n be the Honda formal group law of height n. Then the $[p]$-series of pH_n read

$$[p](x^p) = x^p +_{pH_n} \dots +_{pH_n} x^p = (x +_{H_n} \cdots +_{H_n} x)^p = (v_n x^{p^n})^p = v_n^p x^{p^{n+1}}.$$

Hence pH_n again is the Honda group law of height n over the ring $\mathbb{F}_p[v_n^{\pm p}]$.

LEMMA 2.5. *Let $k = \sum_{i=0}^n k_i p^i$ be the p-adic expansion of k. Then mod p*

$$\binom{k}{i,j}_F = \sum_{\substack{i_0 p^0 + \cdots + i_n p^n = i \\ j_0 p^0 + \cdots j_n p^n = j}} \prod_{s=0}^n \binom{k_s}{i_s, j_s}_{p^s F}$$

In particular

$$\binom{pk}{pi, pj}_F = \binom{k}{i,j}_{pF}.$$

PROOF.

$$\begin{aligned} (x +_F y)^k &= \prod_{s=0}^n (x^{p^s} +_{p^s F} y^{p^s})^{k_s} = \prod_{s=0}^n \sum_{i_s, j_s} \binom{k_s}{i_s, j_s}_{p^s F} x^{i_s p^s} y^{j_s p^s} \\ &= \sum_{\substack{i_0 p^0 + \cdots + i_n p^n = i \\ j_0 p^0 + \cdots + j_n p^n = j}} \prod_{s=0}^n \binom{k_s}{i_s, j_s}_{p^s F} x^i y^j \end{aligned}$$

□

3. The homology ring of BSU and symmetric 2-cocycles on formal groups

In the following let $f : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow BSU$ be the map which classifies $(1-L_1)(1-L_2)$. Even though f is not a map of H-spaces we may use it to produce interesting classes in $E_* BSU$. Let $a_{ij} \in E_{2(i+j)} BSU$ be the image of $\beta_i \otimes \beta_j$ under the induced map f_*.

LEMMA 3.1. *The following relations hold for all i, j, k*

$$a_{0,0} = 1 \quad ; \quad a_{0i} = a_{i0} = 0 \text{ for all } i \neq 0 \tag{4}$$

$$a_{ij} = a_{ji} \tag{5}$$

$$\sum_{l,s,t} \binom{l}{s,t}_F a_{j-s,k-t} a_{il} = \sum_{l,s,t} \binom{l}{s,t}_F a_{lk} a_{i-s,j-t} \tag{6}$$

PROOF. The first relation is obvious. Let τ be the self map of $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ which switches the two factors. Then the second relation immedately follows from the fact that $f\tau$ is homotopic to f. To do the last consider the two maps g, h from $(\mathbb{C}P^\infty)^{\times 3}$ to $(\mathbb{C}P^\infty)^{\times 4}$ given by

$$\begin{aligned} g(x,y,z) &= (y,z,x,y\,z) \\ h(x,y,z) &= (x\,y,z,x,y) \end{aligned}$$

Their effect on our generators is

$$g_*(\beta_i \otimes \beta_j \otimes \beta_k) = \sum_{l,s,t} \binom{l}{s,t}_F \beta_{j-s} \otimes \beta_{k-t} \otimes \beta_i \otimes \beta_l \tag{7}$$

$$h_*(\beta_i \otimes \beta_j \otimes \beta_k) = \sum_{l,s,t} \binom{l}{s,t}_F \beta_l \otimes \beta_k \otimes \beta_{i-s} \otimes \beta_{j-t} \tag{8}$$

This can be verified by pairing the left hand side with the cohomological monomials in the the x_i's.The maps g and h become homotopic in BSU when composed with $\mu(f \times f)$ since

$$\begin{aligned} &(\mu(f \times f)g)^* \xi_{univ} = (1-L_2)(1-L_3) + (1-L_1)(1-L_2L_3) \\ &= \ (1-L_1L_2)(1-L_3) + (1-L_1)(1-L_2) = (\mu(f \times f)h)^* \xi_{univ} \end{aligned}$$

The desired relation now follows from the above by applying $\mu(f \times f)_*$ to the right hand side of (7) and (8). ☐

There is another way to look at the classes a_{ij} and the relations of 3.1. First note that $E \wedge BSU_+$ is itself a complex oriented ring theory with

$$x_{E \wedge BSU_+} = (1 \wedge \eta)_* x_E$$

In abuse of the notation we will simply denote this orientation by x in the following. Hence, we may view

$$(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+ \xrightarrow{f_+} BSU_+ \xrightarrow{\eta \wedge 1} E \wedge BSU_+$$

as a power series

$$f(x,y) = 1 + \sum_{i,j \geq 1} b_{ij} x^i y^j \in (E \wedge BSU_+)^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

for some $b_{ij} \in E_{2(i+j)}BSU$. Of course, we have

$$\begin{aligned} b_{ij} &= \sum_{k,l} b_{ij} (1 \wedge \eta)_* \left\langle \beta_i \otimes \beta_j, x^k y^l \right\rangle \\ &= \left\langle (1 \wedge \eta)_* \beta_i \otimes (1 \wedge \eta)_* \beta_j, 1 + \sum_{k,l} b_{kl} x^k y^l \right\rangle \\ &= \langle (1 \wedge \eta)_* (\beta_i \otimes \beta_j), f^*(\eta \wedge 1) \rangle = (\mu f (1 \wedge \eta))_* (\beta_i \otimes \beta_j) = a_{ij}. \end{aligned}$$

The power series f is a symmetric 2-cocycle or 2-structure on the fomal group law F in the sense of [**AHS98**]3.1.[**HAS99**]1.2: This means that the relations

$$f(x,0) = f(0,y) = 1 \tag{9}$$

$$f(x,y) = f(y,x) \tag{10}$$

$$f(y,z) f(x, y +_F z) = f(x +_F y, z) f(x,y). \tag{11}$$

hold. In fact, one easily checks that these are equivalent to 3.1.

REMARK 3.2. There is another way to look at symmetric 2-cocycles: any such f defines a commutative central extension

$$G_m \longrightarrow E \longrightarrow F$$

of F by the multiplicative formal group. Here, E is the product $F \times G_m$ and the group structure is given by the formula

$$(a, \lambda) \cdot (b, \mu) = (a + b, f(a, b)\lambda\mu).$$

The equation 11 is then equivalent to the associativity of the multiplication. These objects have been extensively studied; for example Lazard's symmetric 2-cocycle lemma classifies central extensions of the additive formal group by itself

$$G_a \longrightarrow E \longrightarrow G_a.$$

DEFINITION 3.3. *For any complex oriented E let $C_2(E)$ be the graded ring freely generated by symbols a_{ij}'s subject to the relations of 3.1. We write α for the canonical map from $C_2(E)$ to E_*BSU.*

Here, we denoted the generators of $C_2(E)$ and E_*BSU by the same letters to simplify the notation. We will see below that the map α is an isomorphism. Equivalently, given any 2-cocycle f' on the formal group F over an π_*E-algebra S then there is a unique algebra homomorphism

$$\varphi : E_*BSU \longrightarrow S$$

with $\varphi f = f'$. Hence, E_*BSU carries the universal symmetric 2-cocycle on F which is the result of [**AHS98**].

Consider the canonical map of H-spaces $\iota : BSU \longrightarrow BU$. Writing g for the map from $\mathbb{C}P^\infty$ to BU which classifies $1-L$ we see from the homotopy equivalence

$$\iota + g : BSU \times \mathbb{C}P^\infty \longrightarrow BU$$

that ι is an inclusion in homology. It is well known [**Ada74**] that E_*BU is a polynomial algebra with generators $b_i = g_*\beta_i$. Alternatively, let g' be the map which classifies $L-1$. Then the classes $g'\beta_i = b'_i$ again give polynomial generators of E_*BU. Since the map $\mu_{BU}(g \times g')\Delta$ is null the generators b_i and b'_i are related by the equation

$$\sum_{i=0}^{\infty} b'_i x^i = (\sum_{i=0}^{\infty} b_i x^i)^{-1}.$$

PROPOSITION 3.4. *We have the formula*

$$\iota_* a_{ij} = \sum_{\substack{s=0,\dots,i;\ t=0,\dots,j \\ k=0,\dots,s+t}} \binom{k}{s,t}_F b'_k b_{i-s} b_{j-t}$$

In particular, modulo decomposables in $\tilde{E}_*BU$

$$\iota_* a_{ij} = \sum_{k=0}^{i+j} \binom{k}{i,j}_F b'_k.$$

PROOF. Decompose f by writing

$$\begin{aligned} f^*\xi_{univ} &= 1 - L_1 - L_2 + L_1L_2 = (L_1L_2 - 1) + (1 - L_1) + (1 - L_2) \\ &= (g'\mu_{\mathbb{C}P^\infty} + g\,p_1 + g\,p_2)^*\xi_{univ} \end{aligned}$$

and calculate

$$\begin{aligned}
\iota_* a_{ij} &= \iota_* f_*(\beta_i \otimes \beta_j) \\
&= \mu_*^{BU}(g' \mu_*^{\mathbb{C}P^\infty} \otimes g\, p_{1*} \otimes g\, p_{2*}) \sum_{\substack{i_1+i_2+i_3=i \\ j_1+j_2+j_3=j}} \beta_{i_1} \otimes \beta_{j_1} \otimes \beta_{i_2} \otimes \beta_{j_2} \otimes \beta_{i_3} \otimes \beta_{j_3} \\
&= \sum_{\substack{i_1+i_2=i \\ j_1+j_3=j}} \sum_{k=0}^{i_1+j_1} \binom{k}{i_1, j_2}_F b'_k b_{i_2} b_{j_3}
\end{aligned}$$

□

Now choose n_k^i and d_k as in 1.1 and set

$$\epsilon(k) \overset{def}{=} \sum_{i=1}^{k-1} n_k^i \binom{k}{i} = \text{g.c.d.} \left\{ \binom{k}{1}, \ldots, \binom{k}{k-1} \right\} = \begin{cases} p & \text{for } k = p^s \\ 1 & \text{else} \end{cases} .$$

For a graded ring R we write R_+ for the elements in positive degrees and

$$Q(R) = R/R_+^2.$$

Corollary 3.5. *In $Q(E_*BU)$ we have $\iota_* d_k = \epsilon(k) b'_k$.*

Proof. Using the identity

$$\binom{s+t}{s, t}_F = \binom{s+t}{s}$$

we compute with the proposition

$$\iota_* d_k = \sum_{i+j=k} n_k^i \iota_* a_{ij} = \sum_{i=1}^{k-1} n_k^i \binom{k}{i} b'_k = \epsilon(k) b'_k.$$

□

Before proving 1.1 we need two more lemmas.

Lemma 3.6. *For all s, t we have*

$$a_{st} = \frac{\binom{s+t}{s}}{\epsilon(s+t)} d_{s+t} \in Q_{2(s+t)}(C_2(E)).$$

Proof. The third relation of 3.1 reads modulo $C_2(E)_+^2$

$$\binom{n}{n-t} a_{mn} = \binom{s}{m} a_{st}$$

for all $m + n = s + t,\ m \leq s$. We conclude

$$\begin{aligned}
\frac{\binom{s+t}{s}}{\epsilon(s+t)} d_{s+t} &= \sum_{m+n=s+t} n_{s+t}^m \frac{\binom{s+t}{s}}{\epsilon(s+t)} a_{mn} \\
&= a_{st} \sum_{m+n=s+t} n_{s+t}^m \frac{\binom{s+t}{m}}{\epsilon(s+t)} = a_{st}
\end{aligned}$$

□

Lemma 3.7. (i) *For all $s \geq 0$ and prime numbers p the map*

$$Q_{2p^s} \iota_* : Q_{2p^s}(H_*(BSU; \mathbb{F}_p)) \longrightarrow Q_{2p^s}(H_*(BU; \mathbb{F}_p))$$

is null.

(ii) *Let ρ_t denote the Poincaré series of a graded vector space. Then we have*

$$\rho_t(Q(H_*(BSU; \mathbb{F}_p)) = (1-t^2)^{-1} - t^2.$$

PROOF. (i) For a Hopf algebra A let us write $P(A)$ for the group of primitives. It is enough to show the dual statement that the map

$$P_{2p^s}\iota^* : P_{2p^s}(H^*(BU;\mathbb{F}_p)) \longrightarrow P_{2p^s}(H^*(BSU;\mathbb{F}_p))$$

vanishes. The p^s-power of the first Chern class generates the source since

$$c_1^{p^s}(\xi \oplus \eta) = c_1^{p^s}(\xi) + c_1^{p^s}(\eta)$$

and the dual is one dimensional. This class obviously vanishes in $H^*(BSU;\mathbb{F}_p)$.

(ii) As in [**Sin67**] 1.4 and 1.5 one sees

$$\begin{aligned}\rho_t(Q(H_*(BSU;\mathbb{F}_p))) &= \rho_t(P(H_*(BSU;\mathbb{F}_p))) = \rho_t(Q(H^*(BSU;\mathbb{F}_p))) \\ &= \rho_t(Q(\mathbb{F}_p[c_2, c_3, \dots])) = (1-t^2)^{-1} - t^2\end{aligned}$$

□

PROOF OF 1.1. The proof will fall into several steps: First consider the case when E is rational ordinary homology. Then by 3.5 the composite

$$\mathbb{Q}[d_2, d_3, \dots] \longrightarrow H_*(BSU,\mathbb{Q}) \xrightarrow{\iota_*} H_*(BU,\mathbb{Q}) \longrightarrow \mathbb{Q}[b_1', b_2', \dots]/b_1'$$

is a surjection and consequently is an isomorphism. Thus there cannot be any relation between the monomials in the d_i's and the statement follows from the homotopy equivalence

$$\iota + g : BSU \times \mathbb{C}P^\infty \cong BU.$$

by counting dimensions in each degree.

Next observe that the class d_k must generate $Q_{2k}(H_*(BSU;\mathbb{F}_p))$ for all $k \geq 2$: by 3.7(ii) this vector space is one dimensional for any prime p. Pick a generator e of the latter. Then a multiple n of e coincides with d_k. If k is not a prime power the integer n is invertible since the element d_k is sent to the generator b_k under the map to BU. For prime power degrees the integer n again can not be a multiple of p since else a multiple of e is mapped to b_k by 3.5 which contradicts 3.7 (i). In particular, we have shown that the canonical map

$$\mathbb{F}_p[d_2, d_3, \dots] \longrightarrow H_*(BSU;\mathbb{F}_p)$$

is a surjection which in turn means that it is an isomorphism. The theorem now holds for integral singular homology.

Next let E be complex bordism MU. Since the Atiyah Hirzebruch spectral sequence collapses we may choose an isomorphism of π_*MU-modules

$$MU_*BSU \cong E_\infty = E_2 = H_*(BSU;\pi_*MU).$$

It is enough to show that a monomial in the d_k's reduces to the corresponding monomial on the 0-line of the E_2-term $H_*(BSU;\mathbb{Z})$ since then the canonical map

$$\pi_*MU[d_2, d_3, \dots] \xrightarrow{\cong} MU_*BSU$$

is an isomorphism. This follows from the fact that the map induced from the complex orientation from MU_*BSU to $H_*(BSU;\mathbb{Z})$ respects the d_k's and is the projection onto the 0 line of the spectral sequence.

Finally, for arbitrary complex oriented E we may simply tensor the isomorphism

$$\pi_*MU[d_2, d_3, \dots] \xrightarrow{\cong} MU_*BSU$$

with π_*E and the result follows from the universal coefficients spectral sequence [**Ada69**]. □

COROLLARY 3.8. *The map $\alpha : C_2(E) \longrightarrow E_*BSU$ is an isomorphism.*

PROOF. Consider the obvious map $\varphi : \pi_* E[d_2, d_3, \dots] \longrightarrow C_2(E)$. Since its composite with α is an isomorphism φ must be injective. Moreover, 3.6 tells us that the a_{ij} can be written as polynomials in the d_i's. Consequently φ is an isomorphism and so is α. □

It is hard to give an explicit formula for the n_k^i. However, in the situation of positive characteristic we are better off.

LEMMA 3.9. (i) *It is possible to choose $n^i_{p^s} = 0 \bmod p$ for all i not equal to p^{s-1}.*

(ii) *If n is not a prime power it is possible to choose $n^i_n = 0 \bmod p$ for all i not equal to $p^{\nu_p(n)}$. Here, $\nu_p(n)$ is the exponent of p in the prime decomposition of n.*

PROOF. $p^s!/(p^s - p^{s-1})!$ is once more divisible than $p^{s-1}!$. Hence $\binom{p^s}{p^{s-1}}/p$ is not divisible by p and we find $a, b \in \mathbb{Z}$ such that

$$a\binom{p^s}{1} + b\frac{\binom{p^s}{p^{s-1}}}{p} = 1.$$

Hence we take $n^1_{p^s} = pa$ and $n^{p^{s-1}}_{p^s} = b$.

A similar argument works for the second statement: Since $\binom{n}{p^{\nu_p(n)}}$ is not divisible by p we find natural numbers $a_0, a_1, \dots, a_{n-1}$ such that

$$a_0\binom{n}{p^{\nu_p(n)}} + \sum_{i=1,\dots,n-1;\ i\neq p^{\nu_p(n)}} a_i p\binom{n}{i} = 1$$

which gives the result. □

4. The homology ring of BSp

In this section we are going to determine the canonical map from BSp to BSU in E-homology for complex oriented theories. The calculations will prove useful in things to come.

The (trivial) fibration $det : U(n) \longrightarrow S^1$ with fibre $SU(n)$ allows an identification of $E^*BSU(n)$ with $E^*BU(n) = \pi^*E[\![c_1, \cdots, c_n]\!]$ divided by the ideal generated by the first E-Chern class c_1 of the determinant bundle. The determinant restricted to the standard maximal torus of $U(n)$ is just the multiplication map. Hence in formal Chern roots we compute

$$c_1(det) = x_1 +_F \dots +_F x_n.$$

In particular, for the E-cohomology of $\mathbb{H}P^\infty \cong BSU(2)$ we have

$$c_1(det) = x_1 +_F x_2 = c_1(\omega) + \text{terms of higher order}$$

Here, ω is the canonical quaternian line bundle over $\mathbb{H}P^\infty$. This gives the

PROPOSITION 4.1. $E^*\mathbb{H}P^\infty \cong \pi^*E[\![c_2(\omega)]\!]$.

Observe that in general $c_1(\omega)$ does not vanish: for K-theory we get

$$x_1 +_{\hat{G}_m} x_2 = x_1 + x_2 - x_1x_2 = c_1(\omega) - c_2(\omega)$$

and the first two Chern classes hence coincide.

Let $z_i \in E_{4i}\mathbb{H}P^\infty$ be dual to $c_2(\omega)^i$. Abusing the notation denote the image of z_i under the canonical map

$$E_*\mathbb{H}P^\infty = E_*BSp(1) \longrightarrow E_*BSp.$$

by the same letter. Then one easily verifies

PROPOSITION 4.2. $E_*BSp \cong \pi_*E[z_1, z_2, \dots]$.

We next consider the standard fibration $p : \mathbb{C}P^{2k+1} \longrightarrow \mathbb{H}P^k$ with fibre $\mathbb{C}P^1$. The quaternian line bundle ω splits on the total space

$$p^*\omega \cong L \oplus jL.$$

Here, L is the canonical complex line bundle. Moreover, since i anti commutes with j we see that

$$jL \cong \bar{L}.$$

When passing to infinity the fibration fits into the commutative diagram

$$\begin{array}{ccc} \mathbb{H}P^\infty & \xrightarrow{\quad h \quad} & BSU \\ {\scriptstyle p}\uparrow & & \uparrow{\scriptstyle (1-L_1)(1-L_2)} \\ \mathbb{C}P^\infty & \xrightarrow{(1\wedge\bar{1})\Delta} & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \end{array}$$

because

$$(1 \wedge \bar{1})^* \Delta^* (1 - L_1)(1 - L_2) = 1 - L - \bar{L} + L\bar{L} = (1 - L) + (1 - \bar{L}).$$

Hence, in homology the map

$$E_*\mathbb{C}P^\infty \longrightarrow E_*\mathbb{H}P^\infty \longrightarrow E_*BSU$$

sends β_k to the kth coefficient of the power series

$$f(x, -_F x) = \sum_{i,j} a_{ij} x^i (-_F x)^j.$$

Here, $f(x, y)$ is the universal symmetric 2-cocycle on the formal group law F. It is not hard to see that p is a surjection in homology. Thus we should be able to lift each z_k to $E_*\mathbb{C}P^\infty$and compute its image from there. In fact, we have the following nice formula:

Theorem 4.3. *The map $p : \mathbb{C}P_+^\infty \longrightarrow \mathbb{H}P_+^\infty \wedge E$ is given by the power series*

$$p(x) = \sum_{j=0}^{\infty} z_j x^j (-_F x)^j.$$

As a consequence, the map h is determined by the equality of power series

$$\sum_{i,j} a_{ij} x^i (-_F x)^j = f(x, -_F x) = hp(x) = \sum_j z_j x^j (-_F x)^j.$$

Proof. It remains to compute the image of β_i under p_*:

$$\left\langle p_*\beta_i, c_2^j \right\rangle \quad = \quad \left\langle \beta_i, p^* c_2^j \right\rangle = \left\langle \beta_i, c_2(L + \bar{L})^j \right\rangle = \left\langle \beta_i, (x(-_F x))^j \right\rangle.$$

Hence, β_i is sent to $\sum_j \left\langle \beta_i, x^j (-_F x)^j \right\rangle z_j$ and the claim follows. □

Let us see how this formula works for K-theory. Setting $y = -_F x$ the left hand side becomes the symmetric polynomial

$$\sum_{i,j} a_{ij} x^i y^j \quad = \quad \sum_i a_{ii} (xy)^i + \sum_{i<j} a_{ij} (x^i y^j + x^j y^i) \tag{12}$$

Let Q_k be the Newton polynomial expressing the power sum in terms of the elementary symmetric functions e_1, e_2. That is,

$$t_1^k + t_2^k = Q_k(e_1, e_2)$$

and set $q_k(a) = Q_k(a, a)$. Then since

$$x + (-_F x) = c_1(\omega) = c_2(\omega) = x(-_F x)$$

we have

$$x^k + y^k = q_k(c_2(\omega)).$$

The polynomials q_k satisfy the Newton identities

$$q_k = a(q_{k-1} - q_{k-2}) \text{ for all } k > 2,\ q_1 = a,\ q_2 = a^2 - 2a$$

and a simple induction shows

$$q_k = \sum_s (-1)^{k+s}(\binom{s-1}{k-s-1} + \binom{s}{k-s})a^s.$$

Hence equation (12) reads

$$\begin{aligned} \sum_{i,j} a_{ij}x^i y^j &= \sum_k a_{kk}c_2^k + \sum_{i<j} a_{ij}c_2^i q_{j-i}(c_2) \\ &= \sum_r (a_{rr} + \sum_{i<j}(-1)^{j-i}(\binom{r-i-1}{j-r-1} + \binom{r-i}{j-r})a_{ij})c_2^r \end{aligned}$$

We have shown

COROLLARY 4.4. *The map $h_* : K_*BSp \longrightarrow K_*BSU$ is given by the formula*

$$z_r \mapsto a_{rr} + \sum_{i<j}(-1)^{j-i}(\binom{r-i-1}{j-r-1} + \binom{r-i}{j-r})a_{ij}$$

5. The K-homology ring of $BSpin$ and real symmetric 2-cocycles

In this section we are going to prove 1.2. First we need

THEOREM 5.1 ([**Sna75**]8.4 8.11). (i) *The canonical map*

$$K_*(BSpin; \mathbb{F}_2) \longrightarrow K_*(BSO; \mathbb{F}_2)$$

is an algebra isomorphism.

(ii) *The composite of*

$$K_*(pt; \mathbb{F}_2)[b_2, b_4, b_6, \dots] \longrightarrow K_*(BU; \mathbb{F}_2) \xrightarrow{\rho_*} K_*(BSO; \mathbb{F}_2)$$

is an algebra isomorphism. Moreover, each $\rho_* b_{2k+1}$ lies in the image of $K_*(pt; \mathbb{F}_2)[b_2, b_4, \dots, b_{2k}]$.

LEMMA 5.2. *The composite*

$$K_*(\mathbb{H}P^\infty; \mathbb{F}_2) \xrightarrow{h} K_*(BSU; \mathbb{F}_2) \xrightarrow{\iota_*} K_*(BSO; \mathbb{F}_2)$$

sends z_j to b_j^2 modulo the ideal generated by $b_1^2, b_2^2, \dots, b_{j-1}^2$. Hence z_j is decomposable in $K_*(BSpin; \mathbb{F}_2)$.

PROOF. Since the diagram

$$\begin{array}{ccc} \mathbb{C}P^\infty & \xrightarrow{\bar{\imath}} & \mathbb{C}P^\infty \\ \downarrow & & \downarrow \\ BU & \longrightarrow & BSO \end{array}$$

commutes we get

$$g(x) = g(-_{\hat{G}_m} x) \in K_*(BSO, \mathbb{F}_2)[\![x]\!].$$

Hence using 3.4 we compute

$$f(x, -_{\hat{G}_m} x) = g'(x + (-_{\hat{G}_m} x))g(x)g(-_{\hat{G}_m} x) = g(x)^2 = \sum_i b_i^2 x^{2i}$$

and with 5.2

$$f(x, -_{\hat{G}_m} x) = \sum_i z_i x^i (-_{\hat{G}_m} x)^i.$$

Hence the assertion follows by induction since $x^j(-_{\hat{G}_m} x)^j = x^{2j} + o(x^{2j+1})$. The last claim is a consequence of 5.1(ii). □

DEFINITION 5.3. *For any natural number i let 1_i be the set of indices of the 1-digits of i in its binary decomposition. Declare a new product $n \star m$ of natural numbers n, m by*

$$1_{n\star m} = 1_n \cup 1_m.$$

EXAMPLE 5.4. Let $i = \sum_j i_j 2^j$ be the 2-adic expansion of i. Then by its definition $\nu_2(i)$ is the minimum of the set 1_i and hence

$$i = (i - 2^{\nu_2(i)}) \star 2^{\nu_2(i)}.$$

The importance of the $\star$-product comes from the

LEMMA 5.5. *For the multipicative formal group law we have modulo 2*

$$\binom{k}{i,j} = 1 \text{ iff } k = i \star j.$$

PROOF. Since

$$\binom{1}{1,1} = \binom{1}{0,1} = \binom{1}{1,0} = \binom{0}{0,0} = 1$$

and

$$\binom{0}{1,0} = \binom{0}{0,1} = \binom{0}{1,1} = \binom{1}{0,0} = 0$$

the lemma holds for $k = 0, 1$. Hence for arbitrary k we see with 2.5 that the binomial coefficient is non zero iff $k_s = i_s \star j_s$ for all s and the assertion follows. □

It is interesting to note

COROLLARY 5.6. *Modulo decomposables we have*

$$a_{ij} = \begin{cases} d_{2^{s+1}} & \text{for } i = j = 2^s \\ d_{i\star j} & \text{else} \end{cases}.$$

PROOF. Modulo decomposables (6) gives

$$a_{i,j\star k} = \sum_l \binom{l}{j,k} a_{il} = \sum_l \binom{l}{i,j} a_{lk} = a_{i\star j,k}.$$

In particular if $i * j \neq 2^s$

$$a_{ij} = a_{2^{\nu_2(i\star j)},i\star j} = a_{2^{\nu_2(i\star j)},i\star j - 2^{\nu_2(i\star j)}} = d_{i\star j}.$$

Here we used 3.9. □

PROOF OF 1.2. We may assume that we have chosen the n_i^k as in 3.9. By 5.1 it remains to show that the map

$$\mathbb{F}_2[d_k | k \neq 2^s] \otimes \mathbb{F}_2[d'_4, d'_8, \dots] \longrightarrow \mathbb{F}_2[b_2, b_4, \dots];\ d_k \mapsto \iota_*(d_k),\ d'_{2k} \mapsto \iota_*(d_{2k} + h_* z_k)$$

is an isomorphism. By 3.4 and 5.5 we have

$$\iota_* d_{2^{r+1}} = \iota_* a_{2^r,2^r} = \sum_{s,t=0}^{2^r} b'_{s\star t} b_{2^r-s} b_{2^r-t} = \sum_{s=0}^{2^r} b'_s b^2_{2^r-s}$$

and hence with 5.2

$$\iota_* d'_{2^{r+1}} = b'_{2^r} \bmod (b_2, b_4, \dots, b_{2^r-2}).$$

The claim follows since a similar relation holds for the other d_k's by 3.5:

$$\iota_* d_k = \iota_* a_{2^{\nu_2(k)}, k-2^{\nu_2(k)}} = b'_k \bmod (b_1 b_2, \dots, b_{k-1}) \text{ for } k \neq 2^s.$$

□

Since the composite

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{f} BSU \xrightarrow{V-\bar{V}} BSpin$$

is null we have for any complex oriented E the relation

$$f(x,y) \quad = \quad f(-_F x, -_F y) \in E_* BSpin[\![x,y]\!] \tag{13}$$

Example 5.7. Explicitly, the relations (13) read for $F = \hat{G}_m$

$$\sum_{s,t} \binom{i}{s-1} \binom{j}{t-1} a_{st} = a_{ij} \text{ for all } i,j$$

as one checks easily.

Recall from [**HAS99**] the

Definition 5.8. *For any complex oriented E let $C_2^r(E)$ be the ring which carries the universal real symmetric 2-cocycles on F. That is, $C_2^r(E)$ is the quotient of the graded ring $C_2(E)$ by the relations implied by the equation 13. We write β for the canonical map from $C_2^r(E)$ to $E_* BSpin$.*

We are going to show that β is an isomorphism for mod 2 K-theory. Note that this statement is wrong for mod 2 singular homology. However, for general E we have

Lemma 5.9. *The map*

$$(\iota + g)_* : E_* BSU \otimes_{\pi_* E} E_* \mathbb{C}P^\infty \longrightarrow E_* BSU \times \mathbb{C}P^\infty \longrightarrow E_* BU$$

is an isomorphism of $E_* BSU$-modules.

Proof. The diagram

$$\begin{array}{ccccc} BSU \times BSU \times \mathbb{C}P^\infty & \xrightarrow{\iota \times \iota \times g} & BU \times BU \times BU & \xrightarrow{1 \times \mu} & BU \times BU \\ \big\downarrow{\scriptstyle \mu \times 1} & & \big\downarrow{\scriptstyle \mu \times 1} & & \big\downarrow{\scriptstyle \mu} \\ BSU \times \mathbb{C}P^\infty & \xrightarrow{\iota \times g} & BU \times BU & \xrightarrow{\mu} & BU \end{array}$$

commutes. □

A complex 1-structure on F simply is a power series $g(x)$ with leading term 1. The universal ring $C_1(E)$ of these objects can be identified with $E_* BU$ in the obvious way. A real 1-structure is such a power series g which satisfies the real relation

$$g(x) = g(-_F x).$$

Let us write $C_1^r(E)$ for the universal ring of real 1-structures. That is $C_1^r(E)$ is $C_1(E)$ subject to the real relation. It is clear that the map $\iota : C_2 \longrightarrow C_1$ for which

$$\iota(f(x,y)) = g(x +_F y) g(x)^{-1} g(y)^{-1}$$

induces a map on the real universal rings which we denote with the same letter. For mod 2 K-theory we have

Proposition 5.10. *The obvious map from C_1^r to $K_*(BSO, \mathbb{F}_2)$ is an isomorphism.*

PROOF. This is an immediate consequence from 5.1 as the composite

$$\mathbb{F}_2[b_2, b_4, \dots] \longrightarrow C_1(K\mathbb{F}_2) \longrightarrow C_1^r(K\mathbb{F}_2)$$

is easily checked to be surjective with the real relations. □

There is a ring inbetween C_1^r and C_2^r which will be useful in the sequel: let $T(x)$ be the power series $g(x)g(-_{\hat{G}_m}x)^{-1}$ and let $C_1^{r'}$ be the quotient ring of C_1 subject to the relation generated by the set I' consisting of the coefficients of

$$T(x +_{\hat{G}_m} y) = T(x)T(y). \tag{14}$$

Then we have

LEMMA 5.11. (i) *The canonical map $\iota' : C_2^r \longrightarrow C_1^{r'}$ is an injection.*
(ii) *T is an even power series.*

PROOF. For (i) observe that we have

$$\iota' \frac{f(x+y)}{f(-x,-y)} = \frac{g(x+y)g(x)g(y)}{g(-x-y)g(-x)g(-y)} = \frac{T(x+y)}{T(x)T(y)}$$

when ommiting the formal addition from the notation. Hence it suffices to check that the ideal IC_2 generated by $f(x,y)f(-x,-y)^{-1}$ is the intersection of the ideal $I'C_1$ with C_2. By 5.9 there exists a retraction homomorphism

$$\rho : C_1 \cong K_*(BU; \mathbb{F}_2) \longrightarrow K_*(BSU; \mathbb{F}_2) \cong C_2$$

of C_2-modules. Hence any

$$a = \sum_k i_k s_k \in C_2$$

with $s_k \in C_1^{r'}$ and $i_k \in I'$ satisfies

$$a = \rho(a) = \sum_k i_k \, \rho(s_k) \in IC_2$$

and the first part of the lemma follows.

For the second we have with $T(x) = \sum t_i x^i$ and 5.5

$$T(x+y) = \sum_{i,j,k} t_i \binom{i}{j,k} x^j y^k = \sum_{j,k} t_{j\star k} x^j y^k$$

and hence with $T(x+y) = T(x)T(y)$ for each odd n

$$t_n = t_{1\star n} = t_1 t_n = 0$$

since $t_1 = 0$. □

LEMMA 5.12. *The power series $S(x) = f(x,-x)f(x,x)^{-1}$ with coefficients in C_2^r satisfies the relation*

$$f(x^2, y^2) = \frac{S(x)S(y)}{S(x+y)}.$$

PROOF. The cocylce relation (11) gives

$$\begin{aligned} \frac{S(x)S(y)}{S(x+y)} &= \frac{f(x+y,x+y)f(x,-x)f(y,-y)f(-x,-y)}{f(-x,x+y)f(y,-y)f(x,x)f(y,y)} \\ &= \frac{f(x+y,x+y)f(x,y)f(-x,-y)}{f(x,x)f(y,y)} \end{aligned}$$

and

$$\begin{aligned} f(x^2,y^2) &= \frac{f(x^2+y,y)f(x^2,y)}{f(y,y)} = \frac{f(x,y)f(x+y,x+y)f(x,y)f(x,x+y)}{f(y,y)f(x+y,x)f(x,x)} \\ &= \frac{f(x,y)f(x+y,x+y)f(x,y)}{f(y,y)f(x,x)} \end{aligned}$$

Hence the claim follows from the real relation $f(x,y) = f(-x,-y)$. □

Corollary 5.13. *$\beta : C_2^r(K\mathbb{F}_2) \longrightarrow K_*(BSpin;\mathbb{F}_2)$ is an isomorphism.*

Proof. The composite of

$$\pi_*K\mathbb{F}_2[d_k | k \neq 2^s] \otimes_{\pi_*K\mathbb{F}_2} \pi_*K\mathbb{F}_2[d_4', d_8', d_{16}', \dots] \longrightarrow C_2^r(K\mathbb{F}_2)$$

with β is an isomorphism. Hence, β must be surjective. It remains to check that the map ι from C_2^r to C_1^r is an injection. First we claim that the power series $S(x)$ of 5.12 is even. Since $f(x,x)$ is even we only need to investigate $f(x,-x)$. Using the injection ι' of 5.11 we get

$$\iota' f(x,-x) = g(x)g(-x)^{-1} = T(x)g(-x)^{-2}.$$

Since $T(x)$ was even by 5.11(ii) the assertion follows. Next define the ring homomorphism κ from C_1^r to C_2^r by demanding

$$\kappa g(x^2) = S(x)^{-1}.$$

Then we see with 5.12

$$\kappa\iota f(x^2,y^2) = \frac{\kappa g(x^2+y^2)}{\kappa(g(x^2))\kappa(g(y^2))} = \frac{S(x)S(y)}{S(x+y)} = f(x^2,y^2).$$

Thus the universal property of C_2^r shows that we have constructed a left inverse to the map ι. □

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