Technical appendix for "Asymptotic results for multivariate local Whittle estimation with applications"

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The detailed proofs for the results in the article "Asymptotic results for multivariate local Whittle estimation with applications" are presented. Some additional comments on the article are also provided, as well as some other details that go beyond what is solely necessary to justify the results in the article. Thereby, we adopt the notation of the article and refer to its labels by adding square brackets. For example, Theorem [2.1] here refers to Theorem 2.1 in the article.

Sections 1 and 2 below provide respectively the technical details for Sections [II] and [III] in the article. We refrain from giving more details for Section [IV], since the article already provides a proof of Theorem [4.1] with sufficient details. Section 3 gives additional numerical results for Section [V] in the article.

1 Details for Section [II]

We provide here details for the proofs of the results in Section [II] of the article. Section 1.1 contains the proof of Theorem [2.1], the asymptotic normality result for the local Whittle estimator. The Sections 1.2, 1.3 and 1.4 shed additional light on Remarks [2.1], [2.2] and [2.3].

For the readers' convenience, we first recall some notation from the article. We consider the negative log-likelihood

$$\ell(D,G) = \frac{1}{m}\sum_{j=1}^{m}(\log|\lambda_j^{-D}G\lambda_j^{-D}| + \operatorname{tr}(I_X(\lambda_j)\lambda_j^{D}G^{-1}\lambda_j^{D})), \tag{1.1}$$

where $|\cdot|$ and $\operatorname{tr}(\cdot)$ denote the determinant and the trace of a matrix,

$$I_X(\lambda) = \frac{1}{2\pi N}\Big(\sum_{n=1}^{N}X_n e^{-in\lambda}\Big)\Big(\sum_{n=1}^{N}X_n e^{in\lambda}\Big)'$$

is the periodogram for sample size N and m is the number of Fourier frequencies $\lambda_j = 2\pi j/N$ used in estimation. In order to define the parameter vector of interest, we introduce the matrix $\widetilde{G} = (\widetilde{g}_{kl})_{k,l=1,\dots,p}$, $\widetilde{g}_{kl} = g_{kk}\mathbb{1}_{\{k=l\}} + r_{1,kl}\mathbb{1}_{\{k>l\}} + r_{2,kl}\mathbb{1}_{\{k<l\}}$, where $g_{kl} = r_{1,kl} + ir_{2,kl}$, and the vector $\widetilde{D} = (d_1,\dots,d_p)'$. Then, the parameter vector can be written as

$$\theta = \begin{pmatrix}\operatorname{vec}(\widetilde{G}) \\ \widetilde{D}\end{pmatrix}. \tag{1.2}$$

The respective pairs of matrices $\widetilde{G}$, G and $\widetilde{D}$, D can be related as

$$\operatorname{vec}(\widetilde{G}) = L_p \operatorname{vec}(G), \ \ \operatorname{vec}(\widetilde{D}) = E_{p,p^2} \operatorname{vec}(D). \tag{1.3}$$

The matrix $L_p \in \mathbb{C}^{p\times p}$ is defined as

$$L_p = \frac{1}{2}(J_{p^2} I_{p^2} + J_{p^2}^* K_p), \tag{1.4}$$

where $*$ denotes the Hermitian conjugate and $J_{p^2} = \operatorname{diag}(\operatorname{vec}(\widetilde{J}))$ with $\widetilde{J} = (\mathbb{1}_{\{k\le l\}} + i\mathbb{1}_{\{k>l\}})_{k,l=1,\dots,p}$. The matrix K_p denotes the commutation matrix, which transforms $\operatorname{vec}(M)$ into $\operatorname{vec}(M')$ for a square matrix M; see Magnus and Neudecker (2007) for more details on these kinds of operations. The matrix E_{p,p^2} is defined as

$$E_{p,p^2} = (e_1, \mathbf{0}_{p\times p}, e_2, \mathbf{0}_{p\times p}, \dots, e_p), \tag{1.5}$$

where e_i denotes the ith standard basis vector of $\mathbb{R}^p$ and $\mathbf{0}_{p\times p}$ a $p \times p$-matrix with all entries equal to zero. Its Moore-Penrose inverse is $E^+_{p,p^2} = E'_{p,p^2}(E_{p,p^2}E'_{p,p^2})^{-1} = E'_{p,p^2}$.

1.1 Proof of Theorem [2.1]

The asymptotic normality result can be derived as in Baek et al. (2019) and Robinson (2008) who considered the case $p = 2$. We focus on calculating the information matrix for all model parameters for arbitrary dimension p. The negative log-likelihood $\ell(\theta) = \ell(D, G)$ of the model is given in (1.1). The information matrix is a 2×2 block matrix

$$I(\theta) = \begin{pmatrix} M_G & M_{G,D} \\ M^*_{G,D} & M_D \end{pmatrix} \tag{1.6}$$

with

$$M_{G,D} = \mathrm{E}(\mathbf{D}_{\widetilde{G}}(\mathbf{D}_{\widetilde{D}}\,\ell)), \ \ M_G = M_{G,G}, \ \ M_D = M_{D,D},$$

where $\mathbf{D}_V$ denotes the derivative matrix with respect to a vector V. As proven in Lemma 1.1 below, the resulting blocks of (1.6) can be expressed as

$$M_G = (L_p^{-1})^*((G^{-1})' \otimes G^{-1})L_p^{-1}, \tag{1.7}$$

$$M_{G,D} = -T_1(L_p^{-1})^*(G^{-1} \oplus (G^{-1})')E'_{p,p^2}, \tag{1.8}$$

$$M_D = T_2 E_{p,p^2}(G \oplus G')(G^{-1} \oplus (G^{-1})')E'_{p,p^2}, \tag{1.9}$$

where $\oplus$ denotes the Kronecker sum defined as $A \oplus B = (I_p \otimes A) + (B \otimes I_p)$, L_p, E_{p,p^2} are as in (1.4) and (1.5), and

$$T_1 = \frac{1}{m}\sum_{j=1}^{m} \log\lambda_j, \ \ T_2 = \frac{1}{m}\sum_{j=1}^{m}(\log\lambda_j)^2. \tag{1.10}$$

The block structure of the information matrix (1.6) leads to the inverse

$$I(\theta)^{-1} = \begin{pmatrix} L_p(G' \otimes G)L_p^* + \frac{T_1^2}{T_2 - T_1^2} T_2 O_{G,D} & \frac{T_1}{T_2 - T_1^2} T_2 L_p(G \oplus G')E'_{p,p^2} M_D^{-1} \\ \frac{T_1}{T_2 - T_1^2} T_2 M_D^{-1} E_{p,p^2}(G \oplus G')L_p^* & \frac{1}{T_2 - T_1^2} T_2 M_D^{-1} \end{pmatrix} \tag{1.11}$$

with

$$O_{G,D} = L_p(G \oplus G')E'_{p,p^2} M_D^{-1} E_{p,p^2}(G \oplus G')L_p^*;$$

see Lemma 1.2 below. By Lemma 1.3 below, the inverse of the information matrix (1.11) can be simplified to

$$I(\theta)^{-1} = \begin{pmatrix} M_G^{-1} + \frac{T_1^2}{T_2-T_1^2} L_p R Z_G^{-1} R^* L_p^* & \frac{T_1}{T_2-T_1^2} L_p R Z_G^{-1} \\ \frac{T_1}{T_2-T_1^2} Z_G^{-1} R^* L_p^* & \frac{1}{T_2-T_1^2} Z_G^{-1} \end{pmatrix}, \tag{1.12}$$

where $Z_G = 2(G \odot G^{-1} + I_p)$, $\odot$ denotes the Hadamard product and the matrix R is defined as

$$R = (\mathrm{vec}(G) \odot \mathrm{vec}(Y_1), \ldots, \mathrm{vec}(G) \odot \mathrm{vec}(Y_p)) \tag{1.13}$$

with $Y_i = (\mathbb{1}_{\{i=k\}} + \mathbb{1}_{\{i=l\}})_{k,l=1,\ldots,p}$. The asymptotic orders of T_1 and T_2 in (1.10) are $T_1 \sim \log(m/N)$, $T_2 \sim (\log(m/N))^2$ and $T_2 - T_1^2 = 1 + o(1)$ and lead to the limiting covariance matrix

$$C = \begin{pmatrix} L_p R Z_G^{-1} R^* L_p^* & L_p R Z_G^{-1} \\ Z_G^{-1} R^* L_p^* & Z_G^{-1} \end{pmatrix},$$

as stated in the article. □

Lemma 1.1. *The information matrix* (1.6) *is given by* (1.7)*,* (1.8) *and* (1.9)*.*

Proof: We first give the resulting negative score function and second derivative matrices, and take the expected values to conclude (1.6) for the information matrix. Set

$$A^{(k)} = \frac{1}{m} \sum_{j=1}^{m} \lambda_j^D I_X(\lambda_j) \lambda_j^D (\log(\lambda_j))^k, \;\; k = 0, 1, 2. \tag{1.14}$$

Taking the expected value of $A^{(k)}$ and assuming $\mathrm{E}\, I_X(\lambda_j) = f(\lambda_j) = \lambda_j^{-D} G \lambda_j^{-D}$ for simplicity yields

$$\mathrm{E}\, A^{(0)} = G, \;\; \mathrm{E}\, A^{(1)} = T_1 G, \;\; \mathrm{E}\, A^{(2)} = T_2 G \tag{1.15}$$

with T_1, T_2 as in (1.10).

For the negative score function $\mathbf{D}_\theta\, \ell(D, G)$ with respect to θ in (1.2), we calculate $\mathbf{D}_{\widetilde{G}}\, \ell(D, G)$ and $\mathbf{D}_{\widetilde{D}}\, \ell(D, G)$ as

$$\mathbf{D}_{\widetilde{G}}\, \ell = ((L_p^{-1})^* \mathrm{vec}(G^{-1}))^* - ((L_p^{-1})^* \mathrm{vec}(G^{-1} A^{(0)} G^{-1}))^*, \tag{1.16}$$

$$\mathbf{D}_{\widetilde{D}}\, \ell = -2T_1 \mathbf{1}_p' + (E_{p,p^2} \mathrm{vec}(G^{-1} A^{(1)}))^* + (E_{p,p^2} \mathrm{vec}(A^{(1)} G^{-1}))^*, \tag{1.17}$$

where $\mathbf{1}_p = (1, \ldots, 1)'$; details are given below. To get the information matrix, we need to calculate the second derivative matrices, which leads to

$$\begin{aligned} \mathbf{D}_{\widetilde{G}}(\mathbf{D}_{\widetilde{G}}\, \ell) &= -(L_p^{-1})^* ((G^{-1})' \otimes G^{-1}) L_p^{-1} \\ &\quad + (L_p^{-1})^* \Big(((G^{-1} A^{(0)} G^{-1})' \otimes G^{-1}) + ((G^{-1})' \otimes G^{-1} A^{(0)} G^{-1}) \Big) L_p^{-1}, \end{aligned} \tag{1.18}$$

$$\mathbf{D}_{\widetilde{D}}(\mathbf{D}_{\widetilde{G}}\, \ell) = -(L_p^{-1})^* (((A^{(1)} G^{-1})' \otimes G^{-1}) + ((G^{-1})' \otimes G^{-1} A^{(1)})) E_{p,p^2}'. \tag{1.19}$$

$$\begin{aligned} \mathbf{D}_{\widetilde{D}}(\mathbf{D}_{\widetilde{D}}\, \ell) &= E_{p,p^2} \Big(((A^{(2)})' \otimes G^{-1}) + (I_p \otimes (A^{(2)} G^{-1})') \\ &\quad + ((A^{(2)} G^{-1})' \otimes I_p) + ((G^{-1})' \otimes A^{(2)} G^{-1}) \Big) E_{p,p^2}', \end{aligned} \tag{1.20}$$

with details given below. Taking the expected value of the second derivatives (1.18), (1.19) and (1.20) and using (1.15) gives the information matrix (1.6) with blocks (1.7), (1.8) and (1.9).

To derive the derivatives (1.16) and (1.17) note the following relations for some differentials for non-singular matrices X,

$$
\begin{aligned}
\mathbf{d}\,|X| &= |X|\operatorname{tr}(X^{-1}\,\mathbf{d}\,X), && (1.21)\\
\mathbf{d}\,X^{-1} &= -X^{-1}(\mathbf{d}\,X)X^{-1}, && (1.22)\\
\mathbf{d}\,X^{j} &= \sum_{k=1}^{j} X^{k-1}(\mathbf{d}\,X)X^{j-k}, \ \ j=1,2,\ldots. && (1.23)
\end{aligned}
$$

For (1.21)–(1.22) and (1.23) see p. 202 and p. 208 in Magnus and Neudecker (2007), respectively. The Taylor expansion of the exponential function and the formula for the differential of a power function (1.23) combined together give

$$
\begin{aligned}
\mathbf{d}\,\lambda^{D} = \sum_{j=0}^{\infty}\frac{(\log(\lambda))^{j}}{j!}\sum_{k=1}^{j} D^{k-1}(\mathbf{d}\,D)D^{j-k} &= \sum_{j=1}^{\infty}\frac{(\log(\lambda))^{j}}{j!}\sum_{k=1}^{j} D^{j-1}\,\mathbf{d}\,D\\
&= \log(\lambda)\sum_{j=1}^{\infty}\frac{(\log(\lambda))^{j-1}}{(j-1)!}D^{j-1}\,\mathbf{d}\,D \qquad (1.24)\\
&= \log(\lambda)\lambda^{D}\,\mathbf{d}\,D,
\end{aligned}
$$

where we used commutativity of diagonal matrices. For more insights about differentials and derivatives of matrix-valued functions, we refer the reader to Chapter 9 in Magnus and Neudecker (2007).

The derivative with respect to $\widetilde{G}$ in (1.16) is a consequence of the differentials

$$
\begin{aligned}
\mathbf{d}\,\frac{1}{m}\sum_{j=1}^{m}\log|\lambda_j^{-D}G\lambda_j^{-D}| &= \frac{1}{|G|}|G|\operatorname{tr}(G^{-1}\,\mathbf{d}\,G) = (\operatorname{vec}(G^{-1}))^{*}\operatorname{vec}(\mathbf{d}\,G)\\
&= (\operatorname{vec}(G^{-1}))^{*}L_p^{-1}\operatorname{vec}(\mathbf{d}\,\widetilde{G})\\
&= ((L_p^{-1})^{*}\operatorname{vec}(G^{-1}))^{*}\operatorname{vec}(\mathbf{d}\,\widetilde{G}),\\
\mathbf{d}\,\frac{1}{m}\sum_{j=1}^{m}\operatorname{tr}(I_X(\lambda_j)\lambda_j^{D}G^{-1}\lambda_j^{D}) &= \mathbf{d}\operatorname{tr}(A^{(0)}G^{-1})\\
&= \operatorname{tr}(G^{-1}A^{(0)}(-1)G^{-1}(\mathbf{d}\,G))\\
&= -(\operatorname{vec}(G^{-1}A^{(0)}G^{-1}))^{*}L_p^{-1}\operatorname{vec}(\mathbf{d}\,\widetilde{G})\\
&= -((L_p^{-1})^{*}\operatorname{vec}(G^{-1}A^{(0)}G^{-1}))^{*}\operatorname{vec}(\mathbf{d}\,\widetilde{G}),
\end{aligned}
$$

where (1.21), (1.22) and (1.3) are used.

The derivative with respect to $\widetilde{D}$ in (1.17) is a consequence of the differentials

$$
\begin{aligned}
\mathbf{d}\,\frac{1}{m}\sum_{j=1}^{m}\log|\lambda_j^{-D}G\lambda_j^{-D}| &= \frac{1}{m}\sum_{j=1}^{m}2\operatorname{tr}(\lambda_j^{D}\,\mathbf{d}\,\lambda_j^{-D})\\
&= -2\frac{1}{m}\sum_{j=1}^{m}\log(\lambda_j)(\operatorname{vec}(I_p))^{*}E'_{p,p^2}\operatorname{vec}(\mathbf{d}\,\widetilde{D})\\
&= -2\frac{1}{m}\sum_{j=1}^{m}\log(\lambda_j)\mathbf{1}'_p\operatorname{vec}(\mathbf{d}\,\widetilde{D}) = -2T_1\mathbf{1}'_p\operatorname{vec}(\mathbf{d}\,\widetilde{D}),
\end{aligned}
$$

$$\mathbf{d}\,\frac{1}{m}\sum_{j=1}^{m}\operatorname{tr}(I_X(\lambda_j)\lambda_j^D G^{-1}\lambda_j^D) = \frac{1}{m}\sum_{j=1}^{m}(\operatorname{tr}(I_X(\lambda_j)(\mathbf{d}\,\lambda_j^D)G^{-1}\lambda_j^D) + \operatorname{tr}(I_X(\lambda_j)\lambda_j^D G^{-1}(\mathbf{d}\,\lambda_j^D)))$$
$$= \Big((E_{p,p^2}\operatorname{vec}(G^{-1}A^{(1)}))^* + (E_{p,p^2}\operatorname{vec}(A^{(1)}G^{-1}))^*\Big)\operatorname{vec}(\mathbf{d}\,\widetilde{D}), \quad (1.25)$$

where we used (1.21) and (1.24). The equation (1.25) is a consequence of

$$\begin{aligned}\operatorname{tr}(I_X(\lambda)(\mathbf{d}\,\lambda^D)G^{-1}\lambda^D) &= \log(\lambda)\operatorname{tr}(G^{-1}\lambda^D I_X(\lambda)\lambda^D\,\mathbf{d}\,D)\\ &= \log(\lambda)(\operatorname{vec}(G^{-1}\lambda^D I_X(\lambda)\lambda^D))^* E'_{p,p^2}\operatorname{vec}(\mathbf{d}\,\widetilde{D})\\ &= \log(\lambda)(E_{p,p^2}\operatorname{vec}(G^{-1}\lambda^D I_X(\lambda)\lambda^D))^*\operatorname{vec}(\mathbf{d}\,\widetilde{D}).\end{aligned}$$

These calculations also used (1.24) and Theorem 2 in Magnus and Neudecker (2007), p. 35.

We continue with the second derivative matrix of $\ell(D, G)$. The relation (1.18) for $\mathbf{D}_{\widetilde{G}}(\mathbf{D}_{\widetilde{G}}\,\ell)$ is a consequence of (1.17) and the derivatives

$$\begin{aligned}\mathbf{d}(L_p^{-1})^*\operatorname{vec}(G^{-1}) &= (L_p^{-1})^*\operatorname{vec}(\mathbf{d}\,G^{-1})\\ &= -(L_p^{-1})^*\operatorname{vec}(G^{-1}(\mathbf{d}\,G)G^{-1})\\ &= -(L_p^{-1})^*((G^{-1})'\otimes G^{-1})L_p^{-1}\operatorname{vec}(\mathbf{d}\,\widetilde{G}),\\ -\,\mathbf{d}(L_p^{-1})^*\operatorname{vec}(G^{-1}A^{(0)}G^{-1}) &= -(L_p^{-1})^*\operatorname{vec}((\mathbf{d}\,G^{-1})A^{(0)}G^{-1} + G^{-1}A^{(0)}(\mathbf{d}\,G^{-1}))\\ &= -(L_p^{-1})^*\operatorname{vec}(-G^{-1}(\mathbf{d}\,G)G^{-1}A^{(0)}G^{-1} + G^{-1}A^{(0)}(-1)G^{-1}(\mathbf{d}\,G)G^{-1})\\ &= (L_p^{-1})^*((G^{-1}A^{(0)}G^{-1})'\otimes G^{-1})\operatorname{vec}(\mathbf{d}\,G)\\ &\quad + ((G^{-1})'\otimes G^{-1}A^{(0)}G^{-1})\operatorname{vec}(\mathbf{d}\,G)\\ &= (L_p^{-1})^*\Big(((G^{-1}A^{(0)}G^{-1})'\otimes G^{-1})\\ &\quad + ((G^{-1})'\otimes G^{-1}A^{(0)}G^{-1})\Big)L_p^{-1}\operatorname{vec}(\mathbf{d}\,\widetilde{G}).\end{aligned}$$

The second derivative $\mathbf{D}_{\widetilde{D}}(\mathbf{D}_{\widetilde{G}}\,\ell)$ in (1.19) follows from the differentials (with respect to $\widehat{D}$)

$$\begin{aligned}\mathbf{d}(L_p^{-1})^*\operatorname{vec}(G^{-1}) &= 0,\\ -\,\mathbf{d}(L_p^{-1})^*\operatorname{vec}(G^{-1}A^{(0)}G^{-1}) &= -\frac{1}{m}\sum_{j=1}^{m}(L_p^{-1})^*\,\mathbf{d}\operatorname{vec}(G^{-1}\lambda_j^D I_X(\lambda_j)\lambda_j^D G^{-1})\\ &= -\frac{1}{m}\sum_{j=1}^{m}((L_p^{-1})^*\operatorname{vec}(G^{-1}(\mathbf{d}\,\lambda_j^D)I_X(\lambda_j)\lambda_j^D G^{-1}\\ &\quad + G^{-1}\lambda_j^D I_X(\lambda_j)(\lambda_j)(\mathbf{d}\,\lambda_j^D)G^{-1})\\ &= -\frac{1}{m}\sum_{j=1}^{m}\log(\lambda_j)(L_p^{-1})^*((\lambda_j^D I_X(\lambda_j)\lambda_j^D G^{-1})'\otimes G^{-1})\\ &\quad + ((G^{-1})'\otimes G^{-1}\lambda_j^D I_X(\lambda_j)\lambda_j^D))E'_{p,p^2}\operatorname{vec}(\mathbf{d}\,\widetilde{D}),\end{aligned}$$

where

$$\begin{aligned}\operatorname{vec}(G^{-1}(\mathbf{d}\,\lambda^D)I_X(\lambda)\lambda^D G^{-1}) &= \log(\lambda)\operatorname{vec}(G^{-1}\lambda^D(\mathbf{d}\,D)I_X(\lambda)\lambda^D G^{-1})\\ &= \log(\lambda)((I_X(\lambda)\lambda^D G^{-1})'\otimes G^{-1}\lambda^D)\operatorname{vec}(\mathbf{d}\,D)\end{aligned}$$

$$= \log(\lambda)((\lambda^D I_X(\lambda)\lambda^D G^{-1})' \otimes G^{-1})E'_{p,p^2} \operatorname{vec}(\mathbf{d}\,\widetilde{D}).$$

The relation (1.20) for $\mathbf{D}_{\widetilde{D}}(\mathbf{D}_{\widetilde{D}}\,\ell)$ follows from

$$\begin{aligned}
\mathbf{d}\,E_{p,p^2} \operatorname{vec}(G^{-1}A^{(1)}) &= \frac{1}{m}\sum_{j=1}^{m} \log(\lambda_j) E_{p,p^2}\,\mathbf{d}\operatorname{vec}(G^{-1}\lambda_j^D I_X(\lambda_j)\lambda_j^D) \\
&= \frac{1}{m}\sum_{j=1}^{m} \log(\lambda_j) E_{p,p^2} \operatorname{vec}(G^{-1}(\mathbf{d}\,\lambda_j^D) I_X(\lambda_j)\lambda_j^D + G^{-1}\lambda_j^D I_X(\lambda_j)(\mathbf{d}\,\lambda_j^D)) \\
&= E_{p,p^2}\Big(((A^{(2)})' \otimes G^{-1}) + (I_p \otimes (A^{(2)}G^{-1})')\Big)E'_{p,p^2} \operatorname{vec}(\mathbf{d}\,\widetilde{D}), \\
\mathbf{d}\,E_{p,p^2} \operatorname{vec}(A^{(1)}G^{-1}) &= E_{p,p^2}\Big(((A^{(2)}G^{-1})' \otimes I_p) + ((G^{-1})' \otimes A^{(2)}G^{-1})\Big)E'_{p,p^2} \operatorname{vec}(\mathbf{d}\,\widetilde{D}),
\end{aligned}$$

since for example

$$\begin{aligned}
\operatorname{vec}((\mathbf{d}\,\lambda^D) I_X(\lambda)\lambda^D G^{-1}) &= ((\lambda^D I_X(\lambda)\lambda^D G^{-1})' \otimes I_p) \operatorname{vec}(\mathbf{d}\,D) \\
&= \log(\lambda)((\lambda^D I_X(\lambda)\lambda^D G^{-1})' \otimes I_p)E'_{p,p^2} \operatorname{vec}(\mathbf{d}\,\widetilde{D}).
\end{aligned}$$

□

Lemma 1.2. *The inverse of the information matrix* (1.6) *can be written as in* (1.11).

Proof: By using Magnus and Neudecker (2007), p. 12, the block structure of the information matrix leads to the inverse

$$I(\theta)^{-1} = \begin{pmatrix} M_G^{-1} + M_G^{-1}M_{G,D}S_{G,D}^{-1}M_{G,D}^{*}M_G^{-1} & -M_G^{-1}M_{G,D}S_{G,D}^{-1} \\ -S_{G,D}^{-1}M_{G,D}^{*}M_G^{-1} & S_{G,D}^{-1} \end{pmatrix}, \tag{1.26}$$

where the so-called Schur complement is defined as $S_{G,D} = M_D - M_{G,D}^{*}M_G^{-1}M_{G,D}$ with

$$M_G^{-1} = L_p(G' \otimes G)L_p^{*}$$

and M_G, $M_{G,D}$ and M_D are given in (1.7), (1.8) and (1.9), respectively. We can write

$$\begin{aligned}
-M_{G,D}^{*}M_G^{-1} &= T_1((L_p^{-1})^{*}((I_p \otimes G^{-1}) + ((G^{-1})' \otimes I_p))E'_{p,p^2})^{*}L_p(G' \otimes G)L_p^{*} \\
&= T_1 E_{p,p^2}((I_p \otimes G^{-1}) + ((G^{-1})' \otimes I_p))(G' \otimes G)L_p^{*} \\
&= T_1 E_{p,p^2}((G' \otimes I_p) + (I_p \otimes G))L_p^{*} \\
&= T_1 E_{p,p^2}(G \oplus G')L_p^{*}.
\end{aligned} \tag{1.27}$$

In the view of (1.27), (1.8) and (1.9), the Schur complement can be simplified to

$$\begin{aligned}
S_{G,D} &= M_D - M_{G,D}^{*}M_G^{-1}M_{G,D} \\
&= M_D - T_1^2 E_{p,p^2}(G \oplus G')L_p^{*}(L_p^{-1})^{*}((I_p \otimes G^{-1}) + ((G^{-1})' \otimes I_p))E'_{p,p^2} \\
&= M_D - T_1^2 E_{p,p^2}(G \oplus G')((I_p \otimes G^{-1}) + ((G^{-1})' \otimes I_p))E'_{p,p^2} \\
&= (T_2 - T_1^2)E_{p,p^2}(G \oplus G')(G^{-1} \oplus (G^{-1})')E'_{p,p^2} \\
&= (T_2 - T_1^2)(1/T_2)M_D.
\end{aligned}$$

We write its inverse as

$$(S_{G,D})^{-1} = \frac{1}{T_2 - T_1^2} T_2 M_D^{-1}. \tag{1.28}$$

Whether M_D^{-1} can be expressed in closed form remains an open question. Combining (1.27) and (1.28) yields

$$-M_G^{-1} M_{G,D} S_{G,D}^{-1} = \frac{T_1}{T_2 - T_1^2} L_p(G \oplus G') E_{p,p^2} T_2 M_D^{-1} \tag{1.29}$$

and

$$\begin{aligned} & M_G^{-1} + M_G^{-1} M_{G,D} S_{G,D}^{-1} M_{G,D}^{*} M_G^{-1} \\ &= L_p(G' \otimes G) L_p^{*} + \frac{T_1^2}{T_2 - T_1^2} T_2 L_p(G \oplus G') E_{p,p^2}' M_D^{-1} E_{p,p^2} (G \oplus G') L_p^{*}. \end{aligned} \tag{1.30}$$

The relations (1.28), (1.29) and (1.30) give (1.11). □

Lemma 1.3. *The inverse of the information matrix* (1.11) *can be simplified to* (1.12).

Proof: To obtain the representation (1.12), we refrain from taking the derivatives with respect to $\widetilde{D} = (d_1, \dots, d_p)'$ and instead take the derivatives componentwise with respect to d_k for $k \in \{1, \dots, p\}$. This leads to

$$M_D = \mathrm{E}\left(\frac{\partial^2 \ell}{\partial d_k \partial d_l}\right)_{k,l=1,\dots,p} = T_2 2(G \odot G^{-1} + I_p), \tag{1.31}$$

$$M_{G,D} = \mathrm{E}\left(\frac{\partial}{\partial d_k} \mathbf{D}_{\widetilde{G}}\, \ell\right)_{k=1,\dots,p} = -T_1 (L_p^{-1})^{*} ((G^{-1})' \otimes G^{-1}) R \tag{1.32}$$

with R as in (1.13).

To verify (1.31) and (1.32), note that the negative log-likelihood in (1.1) can be written as

$$\ell(\theta) = \log|G| - 2\left(\sum_{i=1}^{p} d_i\right) \frac{1}{m} \sum_{j=1}^{m} \log \lambda_j + \frac{1}{|G|} S_X(\theta),$$

with $S_X(\theta) = \sum_{ij=1}^{p} A_{ij}^{(0)} \widehat{g}_{ij}$, where $A_{kl}^{(r)}$ denotes the klth component of (1.14) and the inverse of $G^{-1} = (\widehat{g}_{ij})_{i,j=1,\dots,p}$ can be written as

$$\widehat{g}_{ij} = \frac{1}{|G|} \sum_{\sigma \in S_p \wedge \sigma_i = j} \operatorname{sign}(\sigma) \prod_{r \in \mathcal{S}_i} g_{r\sigma_r},$$

where S_k denotes the symmetric group and $\mathcal{S}_p := \{1, \dots, m\} \setminus \{p\}$. The formula is a consequence of using Cramer's rule for the inverse of G.

The negative score function is written as

$$\begin{aligned} \frac{\partial \ell}{\partial d_l} &= -2T_1 + \frac{\partial}{\partial d_l} \frac{1}{|G|} \sum_{i,j=1}^{p} A_{ij}^{(0)} \sum_{\sigma \in S_l \wedge \sigma_i = j} \operatorname{sign}(\sigma) \prod_{r \in \mathcal{S}_i} g_{r\sigma_r} \\ &= -2T_1 + \frac{1}{|G|} \sum_{i,j=1}^{p} A_{ij}^{(1)} (\mathbb{1}_{\{l=i\}} + \mathbb{1}_{\{l=j\}}) \sum_{\sigma \in S_l \wedge \sigma_i = j} \operatorname{sign}(\sigma) \prod_{r \in \mathcal{S}_i} g_{r\sigma_r}. \end{aligned}$$

Taking the second derivative with respect to d_k, we obtain

$$\begin{aligned}\frac{\partial^2 \ell}{\partial d_k \partial d_l} &= \frac{\partial}{\partial d_k}\left(-2T_1 + \frac{1}{|G|}\sum_{i,j=1}^{p} A_{ij}^{(1)}(\mathbb{1}_{\{l=i\}} + \mathbb{1}_{\{l=j\}}) \sum_{\sigma\in S_p \wedge \sigma_i = j} \operatorname{sign}(\sigma) \prod_{r\in\mathcal{S}_i} g_{r\sigma_r}\right)\\ &= \frac{1}{|G|}\sum_{i,j=1}^{p} A_{ij}^{(2)}(\mathbb{1}_{\{l=i\}} + \mathbb{1}_{\{l=j\}})(\mathbb{1}_{\{k=i\}} + \mathbb{1}_{\{k=j\}}) \sum_{\sigma\in S_p \wedge \sigma_i = j} \operatorname{sign}(\sigma) \prod_{r\in\mathcal{S}_i} g_{r\sigma_r}.\end{aligned}$$

Taking the expected value and using (1.15) gives

$$\begin{aligned}\mathrm{E}\left(\frac{\partial^2 \ell}{\partial d_k \partial d_l}\right)_{k,l=1,\dots,p} &= \sum_{i,j=1}^{p} T_2 g_{ij}(\mathbb{1}_{\{l=i\}} + \mathbb{1}_{\{l=j\}})(\mathbb{1}_{\{k=i\}} + \mathbb{1}_{\{k=j\}}) \frac{1}{|G|} \sum_{\sigma\in S_p \wedge \sigma_i = j} \operatorname{sign}(\sigma) \prod_{r\in\mathcal{S}_i} g_{r\sigma_r}\\ &= T_2 \sum_{i,j=1}^{p} g_{ij}\widehat{g}_{ij}(\mathbb{1}_{\{l=i\}} + \mathbb{1}_{\{l=j\}})(\mathbb{1}_{\{k=i\}} + \mathbb{1}_{\{k=j\}})\\ &= T_2 \begin{cases} 2g_{lk}\widehat{g}_{lk}, & \text{if } l \neq k,\\ 2g_{ll}\widehat{g}_{ll} + 2\Re(\sum_{j=1}^{p} g_{lj}\widehat{g}_{lj}), & \text{if } l = k.\end{cases}\\ &= T_2 2(G \odot G^{-1} + I_p).\end{aligned}$$

We get a representation for the matrix $M_{G,D}$ by taking the derivative of (1.16) with respect to d_k for $k = 1, \dots, p$ as

$$\begin{aligned}\frac{\partial}{\partial d_k}\mathbf{D}_{\widetilde{G}}\,\ell &= -(L_p^{-1})^* \frac{\partial}{\partial d_k}\operatorname{vec}(G^{-1}A^{(0)}G^{-1})\\ &= -(L_p^{-1})^*((G^{-1})' \otimes G^{-1})\frac{\partial}{\partial d_k}\operatorname{vec}(A^{(0)})\\ &= -(L_p^{-1})^*((G^{-1})' \otimes G^{-1})\operatorname{vec}(X_k)\end{aligned} \tag{1.33}$$

with $X_k = (A_{rs}^{(1)}(\mathbb{1}_{\{k=r\}} + \mathbb{1}_{\{k=s\}}))_{r,s=1,\dots,p}$, since

$$\begin{aligned}\frac{\partial}{\partial d_k}\operatorname{vec}(A^{(0)}) &= \frac{\partial}{\partial d_k}\operatorname{vec}(\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{d_r+d_s} I_{X,rs}(\lambda_j))_{r,s=1,\dots,p}\\ &= \operatorname{vec}(\frac{1}{m}\sum_{j=1}^{m}\log(\lambda_j)(\mathbb{1}_{\{k=r\}} + \mathbb{1}_{\{k=s\}})\lambda_j^{d_r+d_s} I_{X,rs}(\lambda_j))_{r,s=1,\dots,p}\\ &= \operatorname{vec}((A_{rs}^{(1)}(\mathbb{1}_{\{k=r\}} + \mathbb{1}_{\{k=s\}}))_{r,s=1,\dots,p}).\end{aligned}$$

Taking the expected value of (1.33) leads to

$$\begin{aligned}\mathrm{E}\left(\frac{\partial}{\partial d_k}\mathbf{D}_{\widetilde{G}}\,\ell\right) &= -T_1(L_p^{-1})^*((G^{-1})' \otimes G^{-1})\operatorname{vec}((g_{rs}(\mathbb{1}_{\{k=r\}} + \mathbb{1}_{\{k=s\}}))_{r,s=1,\dots,p})\\ &= -T_1(L_p^{-1})^*((G^{-1})' \otimes G^{-1})\operatorname{vec}(G) \odot \operatorname{vec}(Y_k)\\ &= -T_1(L_p^{-1})^*((G^{-1})' \otimes G^{-1})R\end{aligned}$$

with $Y_k = (\mathbb{1}_{\{k=r\}} + \mathbb{1}_{\{k=s\}})_{r,s=1,\dots,p}$ and R as in (1.13).

Using formula (1.26) for the inverse of the information matrix, (1.31) and (1.32) gives

$$S_{G,D}^{-1} = \frac{T_1}{T_2 - T_1^2} M_D^{-1} = \frac{1}{T_2 - T_1^2} Z_G^{-1},$$

$$-M_G^{-1} M_{G,D} S_{G,D}^{-1} = L_p (G' \otimes G) L_p^* T_1 (L_p^{-1})^* ((G^{-1})' \otimes G^{-1}) R M_D^{-1} = \frac{T_1}{T_2 - T_1^2} L_p R Z_G^{-1},$$

$$M_G^{-1} M_{G,D} S_{G,D}^{-1} M_{G,D}^* M_G^{-1} = \frac{T_1^2}{T_2 - T_1^2} L_p R Z_G^{-1} R^* L_p^*$$

with $M_G^{-1} = L_p (G' \otimes G) L_p^*$, which leads to the desired representation of (1.12) of $I(\theta)^{-1}$. □

1.2 Proof of Remark [2.1]

Remark [2.1] states that Theorem [2.1] can also be written in terms of the precision matrix $P = G^{-1}$.

The parameter vector of interest is given now by

$$\theta_P = \begin{pmatrix} \operatorname{vec}(\widetilde{P}) \\ \widetilde{D} \end{pmatrix}, \tag{1.34}$$

where $\widetilde{P}$ is defined from P in the same way that $\widetilde{G}$ and G relate; see (1.3). We prove that the information matrix

$$I(\theta) = \begin{pmatrix} M_P & M_{P,D} \\ M_{P,D}^* & M_D \end{pmatrix} \tag{1.35}$$

with

$$M_{P,D} = \mathrm{E}(\mathbf{D}_{\widetilde{P}}(\mathbf{D}_{\widetilde{P}}\, \ell)), \;\; M_P = M_{P,P}, \;\; M_D = M_{D,D},$$

coincides with (1.6) written in terms of (1.7), (1.8) and (1.9) by replacing G with P and changing the sign of the off-diagonal block matrices.

The resulting blocks of (1.35) can be expressed as

$$M_P = (L_p^{-1})^* (G' \otimes G) L_p^{-1}, \tag{1.36}$$

$$M_{P,D} = T_1 (L_p^{-1})^* (G \oplus G') E'_{p,p^2}, \tag{1.37}$$

For the negative score function $\mathbf{D}_{\theta_P}\, \ell(D,G)$, we calculate $\mathbf{D}_{\widetilde{P}}\, \ell(D,G)$ as

$$\mathbf{D}_{\widetilde{P}}\, \ell = ((L_p^{-1})^* \operatorname{vec}(G))^* - ((L_p^{-1})^* \operatorname{vec}(A^{(0)}))^*.$$

To get the information matrix, we need to calculate the second derivative matrices, which leads to

$$\mathbf{D}_{\widetilde{P}}(\mathbf{D}_{\widetilde{P}}\, \ell) = (L_p^{-1})^* (G' \otimes G) L_p^{-1}, \tag{1.38}$$

$$\mathbf{D}_{\widetilde{D}}(\mathbf{D}_{\widetilde{P}}\, \ell) = (L_p^{-1})^* ((A^{(1)} \otimes I_p) + (I_p \otimes A^{(1)})) E'_{p,p^2}. \tag{1.39}$$

Taking the expected values of the second derivatives (1.38) and (1.39) gives the information matrix (1.35) with blocks (1.36), (1.37). The detailed arguments are omitted since they are similar to those in the proof of Theorem [2.1].

1.3 Proof of Remark [2.2]

We first prove the upper bound

$$\big[2(G \odot G^{-1} + I_p)\big]^{-1}_{kk} \leq \frac{1}{4}, \ \ k = 1, \ldots, p. \tag{1.40}$$

For Hermitian matrices A and B, we write $A \succeq B$ if $A - B$ is positive semidefinite and $A \succ B$ if $A - B$ is positive definite. We use $\lambda_j(A)$ for the jth eigenvalue of A.

Note that $2(G \odot G^{-1} + I_p) \succ 0$ since $G \succ 0$; see Theorem 7.5.3. in Horn and Johnson (2013). For this reason,

$$\big[2(G \odot G^{-1} + I_p)\big]^{-1} \preceq \frac{1}{4} I_p \Leftrightarrow 2(G \odot G^{-1} + I_p) \succeq 4 * I_p \Leftrightarrow 2(G \odot G^{-1} - I_p) \succeq 0, \tag{1.41}$$

where Corollary 7.7.4. (a) in Horn and Johnson (2013) was used for the first equivalence. The last relation in (1.41) is proven in Theorem 7.7.17. (c) in Horn and Johnson (2013). Corollary 7.7.4. (c) in Horn and Johnson (2013) implies

$$\lambda_j(\big[2(G \odot G^{-1} + I_p)\big]^{-1}) \leq \frac{1}{4}\lambda_j(I_p) = \frac{1}{4}$$

and so

$$\big[2(G \odot G^{-1} + I_p)\big]^{-1}_{kk} \leq \max_{1 \leq j \leq p} \lambda_j(\big[2(G \odot G^{-1} + I_p)\big]^{-1}) \leq \frac{1}{4}.$$

We conjecture that the diagonal entries in (1.40) also have a lower bound as

$$\frac{1}{4p} \leq \big[2(G \odot G^{-1} + I_p)\big]^{-1}_{kk}, \ \ k = 1, \ldots, p.$$

This is suggested by at least the following two observations. First, recall that $\big[2(G \odot G^{-1} + I_p)\big]^{-1}_{kk}$ is the (normalized) asymptotic variance of the estimator $\widehat{d}_k$. We expect this variance to be smallest when the dependence between the kth component series and the rest $(p-1)$ series is strongest, so that estimation of d_k benefits most from these other series. This extreme dependence can be thought as having p copies of the kth series or the kth series of length $N \cdot p$. The usual local Whittle estimator of d_k for the series of such length would have its asymptotic variance as $(1/4)(1/(mp)) = (1/(4p))(1/m)$, where m is the index of the Fourier frequency when the sample size is N. Thus, up to the normalization by $1/m$, the asymptotic variance in this extreme case is suggested as $1/(4p)$. Second, in smaller dimensions p, we have calculated $\big[2(G \odot G^{-1} + I_p)\big]^{-1}_{kk}$ assuming $g_{kl} = (g_{kk}g_{ll})^{1/2}\delta$ with δ close to 1, reflecting extreme dependence among the component series. As $\delta \uparrow 0$, we confirmed that $\big[2(G \odot G^{-1} + I_p)\big]^{-1}_{kk}$ indeed converges to $1/(4p)$.

1.4 Details for Remark [2.3]

As discussed in Remark [2.3] the matrix G in $f(\lambda) \sim \lambda^{-D} G \lambda^{-D}$, as $\lambda \to 0^+$, can also be parametrized in terms of polar coordinates. To distinguish between the two parametrization, we use Z instead of G. Then, the parametrization reads

$$Z = (\omega_{kl} e^{\mathrm{sign}(k-l) i \phi_{kl}})_{k,l=1,\ldots,p}$$

with the so-called phase parameters $\phi_{kl} \in (-\pi/2, \pi/2)$ and $\omega_{kl} \in \mathbb{R}$.

As in the proof of Theorem [2.1], we introduce a suitable representation of the parameter vector. Set $\Phi = (\operatorname{sign}(k-l)i\phi_{kl})_{k,l=1,\dots,p}$, $\Omega = (\omega_{kl})_{k,l=1,\dots,p}$ and

$$\widetilde{\Omega} := \operatorname{vech}(\Omega) = D_p^+ \operatorname{vec}(\Omega), \quad \widetilde{\Phi} := W_{p,p^2} \operatorname{vec}(\Phi),$$

where

$$W_{p,p^2} = (-i)(\mathbf{0}_p, V_1, \mathbf{0}_{p\times 2}, \dots, \mathbf{0}_{p\times(p-1)}, V_{p-1}, \mathbf{0}_{p\times p})$$

with $V_q = (v_{(q-1)p+q}, \dots, v_{(q-1)p+p-q})$, where v_i denotes the ith standard basis vector of $\mathbb{R}^p$ and $\mathbf{0}_{p\times q}$ is a $p \times q$-matrix with all entries equal to zero. $W_{p,p^2}^+ = (W_{p,p^2}^* W_{p,p^2})^{-1} W_{p,p^2}^* = -W_{p,p^2}^*$ is the pseudoinverse of W_{p,p^2}. The parameter vector can then be written as $\theta = ((\widetilde{\Omega})', (\widetilde{\Phi})', (\widetilde{D})')$.

The information matrix is now given by a 3×3 block matrix

$$I(\theta) = \begin{pmatrix} M_\Omega & M_{\Omega,\phi} & M_{\Omega,D} \\ M_{\Omega,\phi}^* & M_\phi & M_{\phi,D} \\ M_{\Omega,D}^* & M_{\phi,D}^* & M_D \end{pmatrix}, \tag{1.42}$$

where

$$M_{\Omega,\phi} = \mathrm{E}(\mathbf{D}_{\widetilde{\Phi}}(\mathbf{D}_{\widetilde{\Omega}})), \quad M_{\Omega,D} = \mathrm{E}(\mathbf{D}_{\widetilde{D}}(\mathbf{D}_{\widetilde{\Omega}})), \quad M_{\phi,D} = \mathrm{E}(\mathbf{D}_{\widetilde{D}}(\mathbf{D}_{\widetilde{\Phi}})),$$

and $M_\Omega = M_{\Omega,\Omega}, M_\phi = M_{\phi,\phi}, M_D = M_{D,D}$. As proved in Lemma 1.4 below,

$$M_\Omega = D_p^*(\operatorname{diag}(\operatorname{vec}(\Gamma)))^*((Z^{-1})' \otimes Z^{-1}) \operatorname{diag}(\operatorname{vec}(\Gamma)) D_p, \tag{1.43}$$
$$M_{\Omega,\phi} = D_p^*(\operatorname{diag}(\operatorname{vec}(\Gamma)))^*((Z^{-1})' \otimes Z^{-1}) \operatorname{diag}(\operatorname{vec}(Z)) W_{p,p^2}, \tag{1.44}$$
$$M_{\Omega,D} = -T_1 D_p^*(\operatorname{diag}(\operatorname{vec}(\Gamma)))^*(Z^{-1} \oplus (Z^{-1})') E_{p,p^2}', \tag{1.45}$$
$$M_\phi = W_{p,p^2}^+ \operatorname{diag}(\operatorname{vec}(\overline{Z}))((Z^{-1})' \otimes Z^{-1}) \operatorname{diag}(\operatorname{vec}(Z)) W_{p,p^2}, \tag{1.46}$$
$$M_{\phi,D} = T_1 W_{p,p^2}^+(\operatorname{diag}(\operatorname{vec}(Z)))^*(Z^{-1} \oplus (Z^{-1})') E_{p,p^2}', \tag{1.47}$$
$$M_D = T_2 E_{p,p^2}(Z \oplus Z')(Z^{-1} \oplus (Z^{-1})') E_{p,p^2}', \tag{1.48}$$

where

$$\Gamma = (e^{\operatorname{sign}(k-l)i\phi_{kl}})_{k,l=1,\dots,p}. \tag{1.49}$$

As noted in Remark [2.3], it is quite challenging to derive an explicit formula for the inverse of the information matrix. The formula to derive the inverse of a 2×2 block matrix can be used to get a formula for the inverse of a 3×3 block matrix. Therefore, rewrite $I(\theta)$ in terms of a 2×2 block matrix

$$I(\theta) = \begin{pmatrix} N_{\Omega,\phi} & O_{\Omega,D} \\ O_{\Omega,D}^* & M_D \end{pmatrix}$$

with $O_{\Omega,\phi,D} = (M_{\Omega,D} \;\; M_{\phi,D})'$ and

$$N_{\Omega,\phi} = \begin{pmatrix} M_\Omega & M_{\Omega,\phi} \\ M_{\Omega,\phi}^* & M_\phi \end{pmatrix}.$$

Note that

$$M_\Omega^{-1} = D_p^+(\operatorname{diag}(\operatorname{vec}(\Gamma)))^{-1}(Z' \otimes Z)(\operatorname{diag}(\operatorname{vec}(\Gamma))^*)^{-1}(D_p^+)^*.$$

Then, the Schur complement can be computed as

$$\widetilde{S} = M_\phi - M_{\Omega,\phi}^* M_\Omega^{-1} M_{\Omega,\phi} = 0.$$

Since $\widetilde{S}$ is not invertible, we use

$$N_{\Omega,\phi}^{-1} = \begin{pmatrix} S_1^{-1} & -S_1^{-1} M_{\Omega,\phi} M_\phi^{-1} \\ -M_\phi^{-1} M_{\Omega,\phi}^* S_1^{-1} & M_\phi^{-1} + M_\phi^{-1} M_{\Omega,\phi}^* S_1^{-1} M_{\Omega,\phi} M_\phi^{-1} \end{pmatrix}$$

with $S_1 = M_\Omega - M_{\Omega,\phi} M_\phi^{-1} M_{\Omega,\phi}^*$. Then, the inverse can be computed as

$$I^{-1}(\theta) = \begin{pmatrix} N_{\Omega,\phi}^{-1} + N_{\Omega,\phi}^{-1} O_{\Omega,D} S_2^{-1} O_{\Omega,D}^* N_{\Omega,\phi}^{-1} & -N_{\Omega,\phi}^{-1} O_{\Omega,D} S_2^{-1} \\ -S_2^{-1} O_{\Omega,D}^* N_{\Omega,\phi}^{-1} & S_2^{-1} \end{pmatrix}$$

with

$$S_2 = M_D - O_{\Omega,D}^* N_{\Omega,\phi}^{-1} O_{\Omega,D}.$$

However, it is an open question if one can find further simplifications to achieve a more explicit representation of $I^{-1}(\theta)$ in terms of the parametrization $Z = (\omega_{kl} e^{\operatorname{sign}(k-l) i \phi_{kl}})_{k,l=1,\dots,p}$.

Lemma 1.4. *The information matrix* (1.42) *is given by* (1.43)–(1.48).

Proof: In order to obtain the negative score function $\mathbf{D}_\theta\, \ell(\theta)$, we calculate the first derivatives in the proof below as

$$\mathbf{D}_{\widetilde{\Omega}} = (D_p^* (\operatorname{diag}(\operatorname{vec}(\Gamma)))^* \operatorname{vec}(Z^{-1}))^* - (D_p^* (\operatorname{diag}(\operatorname{vec}(\Gamma)))^* \operatorname{vec}(Z^{-1} A^{(0)} Z^{-1}))^*, \tag{1.50}$$

$$\mathbf{D}_{\widetilde{\Phi}} = (W_{p,p^2}^+ (\operatorname{diag}(\operatorname{vec}(Z)))^* \operatorname{vec}(Z^{-1}))^* + (W_{p,p^2}^+ (\operatorname{diag}(\operatorname{vec}(Z)))^* \operatorname{vec}(Z^{-1} A^{(0)} Z^{-1}))^*. \tag{1.51}$$

Note that the first derivative with respect to $\widetilde{D}$ can be obtained by replacing G with Z in (1.17). As shown in the proof below, the second derivative matrices can be written as

$$\begin{aligned}
\mathbf{D}_{\widetilde{\Omega}}(\mathbf{D}_{\widetilde{\Omega}}) &= D_p^* (\operatorname{diag}(\operatorname{vec}(\Gamma)))^* \Big(- ((Z^{-1})' \otimes Z^{-1}) \\
&\quad + ((Z^{-1} A^{(0)} Z^{-1})' \otimes Z^{-1}) + ((Z^{-1})' \otimes Z^{-1} A^{(0)} Z^{-1}) \Big) \operatorname{diag}(\operatorname{vec}(\Gamma)) D_p, && (1.52) \\
\mathbf{D}_{\widetilde{\Phi}}(\mathbf{D}_{\widetilde{\Omega}}) &= D_p^* (\operatorname{diag}(\operatorname{vec}(\Gamma)))^* \Big(- ((Z^{-1})' \otimes Z^{-1}) \\
&\quad - ((Z^{-1} A^{(0)} Z^{-1})' \otimes Z^{-1}) - ((Z^{-1})' \otimes Z^{-1} A^{(0)} Z^{-1}) \Big) \operatorname{diag}(\operatorname{vec}(Z)) W_{p,p^2}, && (1.53) \\
\mathbf{D}_{\widetilde{D}}(\mathbf{D}_{\widetilde{\Omega}}) &= -D_p^* (\operatorname{diag}(\operatorname{vec}(\Gamma)))^* ((A^{(1)} Z^{-1})' \otimes Z^{-1}) + ((Z^{-1})' \otimes Z^{-1} A^{(1)})) E_{p,p^2}', && (1.54) \\
\mathbf{D}_{\widetilde{\Phi}}(\mathbf{D}_{\widetilde{\Phi}}) &= W_{p,p^2}^+ \operatorname{diag}(\operatorname{vec}(\overline{Z})) \Big(\operatorname{diag}(\operatorname{vec}(Z^{-1})) \overline{W_{p,p^2}} - ((Z^{-1})' \otimes Z^{-1}) \operatorname{diag}(\operatorname{vec}(Z)) W_{p,p^2} \Big) \\
&\quad - W_{p,p^2}^+ \operatorname{diag}(\operatorname{vec}(Z^{-1} A^{(0)} Z^{-1})) \operatorname{diag}(\operatorname{vec}(\overline{Z})) \overline{W_{p,p^2}} \\
&\quad + W_{p,p^2}^+ \operatorname{diag}(\operatorname{vec}(\overline{Z})) \Big(((Z^{-1} A^{(0)} Z^{-1})' \otimes Z^{-1}) \\
&\quad + ((Z^{-1})' \otimes Z^{-1} A^{(0)} Z^{-1}) \Big) \operatorname{diag}(\operatorname{vec}(Z)) W_{p,p^2}, && (1.55) \\
\mathbf{D}_{\widetilde{D}}(\mathbf{D}_{\widetilde{\Phi}}) &= W_{p,p^2}^+ (\operatorname{diag}(\operatorname{vec}(Z)))^* (((A^{(1)} Z^{-1})' \otimes Z^{-1}) + ((Z^{-1})' \otimes Z^{-1} A^{(1)})) E_{p,p^2}'. && (1.56)
\end{aligned}$$

Taking the expected value of the second derivatives (1.52)–(1.56) and using (1.15) gives the information matrix (1.42) with the blocks (1.43)–(1.48).

For further calculations, note that

$$\operatorname{vec}(\mathbf{d}\, Z) = \operatorname{vec}(\Gamma \odot \mathbf{d}\, \Omega) = \operatorname{vec}(\Gamma) \odot \operatorname{vec}(\mathbf{d}\, \Omega) = \operatorname{diag}(\operatorname{vec}(\Gamma)) D_p \operatorname{vec}(\mathbf{d}\, \widetilde{\Omega}),$$

$$\mathrm{vec}(\mathbf{d}\,Z) = \mathrm{vec}(Z \odot \mathbf{d}\,\Phi) = \mathrm{vec}(Z) \odot \mathrm{vec}(\mathbf{d}\,\Phi) = \mathrm{diag}(\mathrm{vec}(Z)) W_{p,p^2}\,\mathrm{vec}(\mathbf{d}\,\widetilde{\Phi}),$$

where we used (1.49) and the notation $\Phi = (\mathrm{sign}(k-l) i\phi_{kl})_{k,l=1,\dots,p}$. We next prove (1.50), (1.51) and (1.52)–(1.56).

The differential (1.50) can be derived as

$$\begin{aligned}
\mathbf{d}\,\frac{1}{m}\sum_{j=1}^{m}\log|\lambda_j^{-D} Z \lambda_j^{-D}| &= (\mathrm{vec}(Z^{-1}))^*\,\mathrm{vec}(\mathbf{d}\,Z)\\
&= (\mathrm{vec}(Z^{-1}))^*\,\mathrm{diag}(\mathrm{vec}(\Gamma)) D_p\,\mathrm{vec}(\mathbf{d}\,\widetilde{\Omega})\\
&= (D_p^*(\mathrm{diag}(\mathrm{vec}(\Gamma)))^*\,\mathrm{vec}(Z^{-1}))^*\,\mathrm{vec}(\mathbf{d}\,\widetilde{\Omega}),\\
\mathbf{d}\,\frac{1}{m}\sum_{j=1}^{m}\mathrm{tr}(I_X(\lambda_j)\lambda_j^{D} Z^{-1}\lambda_j^{D}) &= -(\mathrm{vec}(Z^{-1}A^{(0)}Z^{-1}))^*\,\mathrm{vec}(\mathbf{d}\,Z)\\
&= -(D_p^*(\mathrm{diag}(\mathrm{vec}(\Gamma)))^*\,\mathrm{vec}(Z^{-1}A^{(0)}Z^{-1}))^*\,\mathrm{vec}(\mathbf{d}\,\widetilde{\Omega})
\end{aligned}$$

and (1.51) as

$$\begin{aligned}
\frac{1}{m}\sum_{j=1}^{m}\log|\lambda_j^{-D} Z \lambda_j^{-D}| &= \mathrm{tr}(Z^{-1}\,\mathbf{d}\,Z)\\
&= (\mathrm{vec}(Z^{-1}))^*\,\mathrm{vec}(\mathbf{d}\,Z)\\
&= (\mathrm{vec}(Z^{-1}))^*\,\mathrm{diag}(\mathrm{vec}(Z)) W_{p,p^2}\,\mathrm{vec}(\mathbf{d}\,\widetilde{\Phi})\\
&= (W_{p,p^2}^{+}(\mathrm{diag}(\mathrm{vec}(Z)))^*\,\mathrm{vec}(Z^{-1}))^*\,\mathrm{vec}(\mathbf{d}\,\widetilde{\Phi}),\\
\mathbf{d}\,\frac{1}{m}\sum_{j=1}^{m}\mathrm{tr}(I_X(\lambda_j)\lambda_j^{D} Z^{-1}\lambda_j^{D}) &= \mathbf{d}\,\mathrm{tr}(A^{(0)}Z^{-1})\\
&= -\,\mathrm{tr}(Z^{-1}A^{(0)}Z^{-1}(\mathbf{d}\,Z))\\
&= (W_{p,p^2}^{+}(\mathrm{diag}(\mathrm{vec}(Z)))^*\,\mathrm{vec}(Z^{-1}A^{(0)}Z^{-1}))^*\,\mathrm{vec}(\mathbf{d}\,\widetilde{\Phi}).
\end{aligned}$$

The second derivative matrix $\mathbf{D}_{\widetilde{\Omega}}(\mathbf{D}_{\widetilde{\Omega}})$ in (1.52) follows from

$$\begin{aligned}
&\mathbf{d}\,D_p^*(\mathrm{diag}(\mathrm{vec}(\Gamma)))^*\,\mathrm{vec}(Z^{-1})\\
&= -D_p^*(\mathrm{diag}(\mathrm{vec}(\Gamma)))^*\,\mathrm{vec}(Z^{-1}(\mathbf{d}\,Z)Z^{-1})\\
&= -D_p^*(\mathrm{diag}(\mathrm{vec}(\Gamma)))^*((Z^{-1})'\otimes Z^{-1})\,\mathrm{diag}(\mathrm{vec}(\Gamma)) D_p\,\mathrm{vec}(\mathbf{d}\,\widetilde{\Omega}),\\
-\,&\mathbf{d}\,D_p^*(\mathrm{diag}(\mathrm{vec}(\Gamma)))^*\,\mathrm{vec}(Z^{-1}A^{(0)}Z^{-1})\\
&= -D_p^*(\mathrm{diag}(\mathrm{vec}(\Gamma)))^*\Big(((Z^{-1}A^{(0)}Z^{-1})'\otimes Z^{-1})\\
&\qquad + ((Z^{-1})'\otimes Z^{-1}A^{(0)}Z^{-1})\Big)\,\mathrm{diag}(\mathrm{vec}(\Gamma)) D_p\,\mathrm{vec}(\mathbf{d}\,\widetilde{\Omega}).
\end{aligned}$$

The second derivative matrix $\mathbf{D}_{\widetilde{\Phi}}(\mathbf{D}_{\widetilde{\Omega}})$ in (1.53) is a consequence of

$$\begin{aligned}
&\mathbf{d}\,D_p^*(\mathrm{diag}(\mathrm{vec}(\Gamma)))^*\,\mathrm{vec}(Z^{-1})\\
&= -D_p^*(\mathrm{diag}(\mathrm{vec}(\Gamma)))^*((Z^{-1})'\otimes Z^{-1}),\mathrm{vec}(\mathbf{d}\,Z)\\
&= -D_p^*(\mathrm{diag}(\mathrm{vec}(\Gamma)))^*((Z^{-1})'\otimes Z^{-1})\,\mathrm{diag}(\mathrm{vec}(Z)) W_{p,p^2}\,\mathrm{vec}(\mathbf{d}\,\widetilde{\Phi})
\end{aligned}$$

$$
\begin{aligned}
&- \mathbf{d}\, D_p^*(\mathrm{diag}(\mathrm{vec}(\Gamma)))^* \,\mathrm{vec}(Z^{-1}A^{(0)}Z^{-1}) \\
&\quad = D_p^*(\mathrm{diag}(\mathrm{vec}(\Gamma)))^* \Big(((Z^{-1}A^{(0)}Z^{-1})' \otimes Z^{-1}) \\
&\qquad + ((Z^{-1})' \otimes Z^{-1}A^{(0)}Z^{-1}) \Big) \,\mathrm{diag}(\mathrm{vec}(Z)) W_{p,p^2} \,\mathrm{vec}(\mathbf{d}\,\widetilde{\Phi}).
\end{aligned}
$$

The relation (1.54) for $\mathbf{D}_{\widetilde{D}}(\mathbf{D}_{\widetilde{\Omega}})$ can be proven as (1.19), since we are taking the derivative with respect to $\widetilde{D}$, the calculations are independent of the underlying parametrization.

The second derivative $\mathbf{D}_{\widetilde{\Phi}}(\mathbf{D}_{\widetilde{\Phi}})$ in (1.55) with respect to the phase parameter vector follows from

$$
\begin{aligned}
&\mathbf{d}\, W_{p,p^2}^+ (\mathrm{diag}(\mathrm{vec}(Z)))^* \,\mathrm{vec}(Z^{-1}) \\
&= W_{p,p^2}^+ \Big((\mathbf{d}\,\mathrm{vec}(\overline{Z})) \odot \mathrm{vec}(Z^{-1}) + \mathrm{vec}(\overline{Z}) \odot \mathbf{d}\,\mathrm{vec}(Z^{-1}) \Big) \\
&= W_{p,p^2}^+ \Big(\mathrm{vec}(Z^{-1}) \odot \mathrm{diag}(\mathrm{vec}(\overline{Z})) \overline{W_{p,p^2}} \,\mathrm{vec}(\mathbf{d}\,\widetilde{\Phi}) - \mathrm{vec}(\overline{Z}) \odot \mathrm{vec}(Z^{-1}(\mathbf{d}\,Z)Z^{-1}) \Big) \\
&= W_{p,p^2}^+ \Big(\mathrm{diag}(\mathrm{vec}(Z^{-1}))\, \mathrm{diag}(\mathrm{vec}(\overline{Z})) \overline{W_{p,p^2}} \\
&\quad - \mathrm{diag}(\mathrm{vec}(\overline{Z}))((Z^{-1})' \otimes Z^{-1})\, \mathrm{diag}(\mathrm{vec}(Z)) W_{p,p^2} \Big) \,\mathrm{vec}(\mathbf{d}\,\widetilde{\Phi}),
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{d}\, W_{p,p^2}^+ (\mathrm{diag}(\mathrm{vec}(Z)))^* \,\mathrm{vec}(Z^{-1}A^{(0)}Z^{-1}) \\
&= -W_{p,p^2}^+ \,\mathrm{diag}(\mathrm{vec}(Z^{-1}A^{(0)}Z^{-1}))\, \mathrm{diag}(\mathrm{vec}(\overline{Z})) \overline{W_{p,p^2}} \,\mathrm{vec}(\mathbf{d}\,\widetilde{\Phi}) \\
&\quad + W_{p,p^2}^+ \,\mathrm{diag}(\mathrm{vec}(\overline{Z})) \Big(((Z^{-1}A^{(0)}Z^{-1})' \otimes Z^{-1}) \\
&\quad + ((Z^{-1})' \otimes Z^{-1}A^{(0)}Z^{-1}) \Big) \,\mathrm{diag}(\mathrm{vec}(Z)) W_{p,p^2} \,\mathrm{vec}(\mathbf{d}\,\widetilde{\Phi})
\end{aligned}
$$

The proof for the last second derivative $\mathbf{D}_{\widetilde{D}}(\mathbf{D}_{\widetilde{\Phi}})$ in (1.56) is omitted but follows the calculations to derive (1.19).

□

2 Details for Section [III]

We recall here the setting of Section [III] and give the proof of Corollary [3.1], which is a consequence of Theorem [2.1].

We consider the hypothesis testing problem

$$
H_0 : r_{1,kl} = r_{2,kl} = 0 \quad \text{for all} \quad k \neq l. \tag{2.1}
$$

The parameters entering the null hypothesis can be obtained from $\widetilde{G}$ by eliminating the diagonal elements of G. Therefore, we introduce the matrix

$$
\mathcal{E}_{p(p-1),p^2} = (\mathbf{0}_{p(p-1)}, V_1, \mathbf{0}_{p(p-1)}, \ldots, \mathbf{0}_{p(p-1)}, V_{p-1}, \mathbf{0}_{p(p-1)}) \tag{2.2}
$$

with $V_q = (v_{1+(q-1)p}, \ldots, v_{p+(q-1)p})$, where v_i denotes the ith standard basis vector of $\mathbb{R}^{p(p-1)}$ and $\mathbf{0}_{p(p-1)}$ is a $p(p-1)$-vector with all entries equal to zero. Then, the vector of parameters of interest can be written as

$$
\vartheta = \mathcal{E}_{p(p-1),p^2} \,\mathrm{vec}(\widetilde{G}),
$$

and similarly $\widehat{\vartheta}$ for the local Whittle estimators.

2.1 Proof of Corollary [3.1]

The asymptotic normality result can be obtained under the null hypothesis (2.1) by replacing the matrix G with $\mathcal{G} = \text{diag}(g_{11}, \ldots, g_{pp})$ in the upper left block of (1.12). This block is given by

$$M_G^{-1} + \frac{T_1^2}{T_2 - T_1^2} L_p R Z_G^{-1} R^* L_p^* \tag{2.3}$$

with $M_G^{-1} = L_p(G' \otimes G)L_p^*$ and $Z_G = 2(G \odot G^{-1} + I_p)$.

Considering the two summands in (2.3) separately with G replaced by $\mathcal{G}$ leads to

$$L_p R Z_{\mathcal{G}}^{-1} R^* L_p^* = \frac{1}{4} L_p R R^* L_p^* = (Q_{kl})_{k,l=1,\ldots,p} \tag{2.4}$$

with

$$Q_{kl} = \begin{cases} g_{rr}^2, & \text{if } k, l = 1 + (p+1)(r-1), \text{ for } r = 1, \ldots p, \\ 0, & \text{otherwise,} \end{cases}$$

since $Z_{\mathcal{G}}^{-1} = (2(\mathcal{G} \odot \mathcal{G}^{-1} + I_p))^{-1} = (1/4)I_p$ and

$$\begin{aligned} M_{\mathcal{G}}^{-1} &= L_p(\mathcal{G}' \otimes \mathcal{G})L_p^* \\ &= \frac{1}{4}(J_{p^2} I_{p^2} + J_{p^2}^* K_p)(\mathcal{G} \otimes \mathcal{G})(J_{p^2} I_{p^2} + J_{p^2}^* K_p)^* \\ &= \frac{1}{4}\Big(J_{p^2}(\mathcal{G} \otimes \mathcal{G})J_{p^2}^* + J_{p^2}(\mathcal{G} \otimes \mathcal{G})K_p^* J_{p^2} + J_{p^2}^* K_p(\mathcal{G} \otimes \mathcal{G})J_{p^2}^* + J_{p^2}^* K_p(\mathcal{G} \otimes \mathcal{G})K_p^* J_{p^2}\Big) \\ &= \frac{1}{4}\Big(2\Re(J_{p^2}(\mathcal{G} \otimes \mathcal{G})J_{p^2}^*) + 2\Re(J_{p^2}(\mathcal{G} \otimes \mathcal{G})K_p J_{p^2}^*)\Big) \\ &= \frac{1}{2}(\mathcal{G} \otimes \mathcal{G}), \end{aligned} \tag{2.5}$$

where we used Theorem 9 in Magnus and Neudecker (2007), p. 55 for the equality before last and the last equality follows since the real part of the second summand zero.

Then, combining (2.4) and (2.5), Theorem [2.1] implies

$$\sqrt{m}\widehat{\vartheta} \xrightarrow{\text{d}} \mathcal{N}(0, C_0)$$

with

$$C_0 = \frac{1}{2}\mathcal{E}_{p(p-1),p^2}(\mathcal{G} \otimes \mathcal{G})\mathcal{E}_{p(p-1),p^2}'.$$

3 Additional comments for Section [V]

Our numerical study assesses the performance of the fractal non-connectivity test of Section [3]. We provide here additional details regarding the discussion in the article.

First, we recall the underlying time series model for the simulation study. For the empirical size calculations, we simulate a fractally non-connected series with the spectral density $f_X(\lambda) = (f_{X,kl}(\lambda))_{k,l=1,\ldots,p}$ given by

$$f_{X,kk}(\lambda) = \frac{\sigma_{kk}}{2\pi}|1 - e^{-i\lambda}|^{-2d_k}, \quad f_{X,kl}(\lambda) = \frac{\sigma_{kl}}{2\pi}(1 - e^{-i\lambda})^{-\delta_k}(1 - e^{i\lambda})^{-\delta_l}, \tag{3.1}$$

for $k < l$, where $0 < \delta_k < d_k < 1/2$, $k = 1, \ldots, p$, and $\sigma_{kl} \neq 0$, $\sigma_{kk} > 0$. For the empirical power calculations, we use a fractally connected model and take the same spectral density as in (3.1) but

	$X_{1,n}$	$X_{2,n}$	$X_{3,n}$	$X_{4,n}$	$X_{5,n}$
$X_{1,n}$		0.73	0.227	0.082	0.048
$X_{2,n}$			0.082	0.09	0.065
$X_{3,n}$				0.065	0.08
$X_{4,n}$					0.066
$X_{5,n}$					

Table 1: Empirical sizes for the pairwise fractal non-connectivity test applied to each pair of the five dimensional time series with $n = 1, \dots, 1000$.

with $\delta_k = d_k$. The autocovariance functions for these series can be computed explicitly and the Gaussian series can be generated exactly following Helgason et al. (2011). For the simulation study, we take $p = 5$,

$$d = (d_1, \dots, d_5) = (0.1, 0.2, 0.25, 0.3, 0.4), \quad \delta_k = (\delta_1, \dots, \delta_5) = 0.1 \cdot d,$$

$$\Sigma = (\sigma_{kl})_{k,l=1\dots,p} = \begin{pmatrix} 1 & 0.1 & 0.5 & 0.2 & 0.1 \\ & 1 & 0.2 & 0.4 & 0.1 \\ & & 1 & 0.1 & 0.2 \\ & & & 1 & 0.05 \\ & & & & 1 \end{pmatrix} \tag{3.2}$$

and the sample size $N = 1000$. The entries below the main diagonal of the symmetric Σ are omitted.

Table [1] in the article presents the empirical sizes and powers of the fractal non-connectivity test as functions of the tuning parameters m in the local Whittle estimation. As can be seen from the table, the test is oversized even for smaller numbers of frequencies m.

To shed light on this observation, we fix the number of frequencies to $m = N^{0.45}$ and consider pairwise testing. In other words, we apply the fractal connectivity test to each possible pair $(X_{k,n}, X_{l,n})$, $k, l = 1, \dots, 5$, $n = 1, \dots, 1000$, of the five dimensional time series. The empirical sizes are reported in Table 1. Comparing these sizes to the respective off-diagonal elements of Σ in (3.2), one may observe that the individual pairwise tests perform well for smaller off diagonal elements in Σ. In contrast, for the pair associated with $\sigma_{13} = 0.5$, the test performs poorly.

Our global fractal connectivity test also gets affected by the (relative) magnitudes of the entries of Σ. To argue this point, we again consider the pairs $(X_{k,n}, X_{l,n})$, $k, l = 1, \dots, 5$, $n = 1, \dots, 1000$, and calculate the empirical probabilities for rejecting a pairwise hypothesis falsely given that another pairwise hypothesis has already been rejected falsely; see Figure 1.

Focussing on the second row in Figure 1, one may observe higher values compared to the rest of the empirical probabilities. The second row presents the empirical probabilities that pairwise test rejects falsely given that the hypothesis of the pair $(X_{1,n}, X_{3,n})$ being fractally non-connected has already been rejected. The pair $(X_{1,n}, X_{3,n})$ is associated with $\sigma_{13} = 0.5$ in (3.2).

Regarding our global fractal non-connectivity test, the discussion above shows that the dependence structure imposed by Σ might increase the size of the test: single larger entries of Σ will tend to lead to a rejection in the pairwide testing and as a result, also to a rejection of the global hypothesis (2.1).

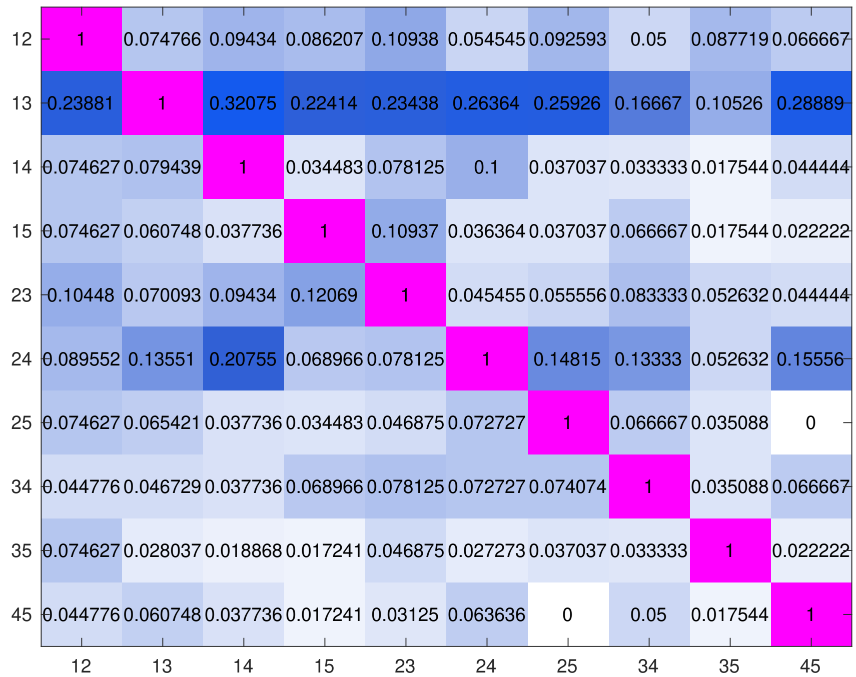

Figure 1: The empirical probabilities for rejecting a pairwise hypothesis (horizontal) falsely given another pairwise hypothesis (vertical) has already been rejected falsely. The labeling kl on the horizontal and vertical axes denote the respective pairs $(X_{k,n}, X_{l,n})$, $k, l = 1, \dots, 5$.

References

C. Baek, S. Kechagias, and V. Pipiras. Asymptotics of bivariate local Whittle estimators with applications to fractal connectivity. Preprint, 2019. To appear in *Journal of Statistical Planning and Inference*. Available at http://pipiras.web.unc.edu.

H. Helgason, V. Pipiras, and P. Abry. Fast and exact synthesis of stationary multivariate gaussian time series using circulant embedding. *Signal Processing*, 91(5):1123–1133, 2011.

R. A. Horn and C. R. Johnson, editors. *Matrix Analysis*. Cambridge University Press, New York, NY, USA, 2013.

J. Magnus and H. Neudecker. *Matrix Differential Calculus with Applications in Statistics and Econometrics*. Probabilistics and Statistics. Wiley, 2007.

P. M. Robinson. Multiple local Whittle estimation in stationary systems. *The Annals of Statistics*, 36(5):2508–2530, 10 2008.

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