A self-similar process arising from a random walk with random environment in random scenery

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Abstract

In this article we merge two celebrated results by Kesten and Spitzer (1979) and by Kawazu and Kesten (1984). A random walk performs a motion in an iid environment and observes an iid scenery along its path. We assume that the scenery is in the domain of attraction of a stable distribution and prove that the resulting observations satisfy a limit theorem. The resulting limit process is a self-similar stochastic process with non-trivial dependencies.

Keywords: birth-death process, random walk, random scenery, random environment, self similar process
MSC: 60K37, 60F05

1 Introduction

The following model for a random walk in random environment can be found in the physical literature (see Anshelevic and Vologodskii (1981), Alexander et al. (1981), Kawazu and Kesten (1984)). Let $\{\lambda_j; j \in \mathbb{Z}\}$ be a family of positive iid random-variables and $\mathcal{A}$ the σ-algebra generated by those random-variables. Let $\{X(t); t \geq 0\}$ be a continuous-time random walk on $\mathbb{Z}$ having the following asymptotic transition rates for $h \to 0$

$$\mathbb{P}(X(t+h) = j+1 | X(t) = j, \mathcal{A}) = \lambda_j h + o(h) \tag{1}$$
$$\mathbb{P}(X(t+h) = j-1 | X(t) = j, \mathcal{A}) = \lambda_{j-1} h + o(h) \tag{2}$$
$$\mathbb{P}(X(t+h) = j | X(t) = j, \mathcal{A}) = 1 - (\lambda_j + \lambda_{j-1})h + o(h). \tag{3}$$

In other words the process $\{X(t); t \geq 0\}$ is a birth and death process with possibly negative population size, where for a population with j individuals birth occurs at rate λ_j and death at rate λ_{j-1}. We will assume that the process $\{X(t); t \geq 0\}$ starts in zero at time zero. The

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resulting process is symmetric in the sense that the permeability of the edge connecting the vertices j and $j+1$ does not depend on the direction of the motion. This physical background motivates the name of random environment for the sequence $\{\lambda_j; j \in \mathbb{Z}\}$. In the following we denote the distribution of the random environment on the sequence-space by P_λ. The following convergence results are described in Kawazu and Kesten (1984):

KK1: *If $c := \mathbb{E}[\lambda_0^{-1}] < \infty$, then for P_λ-almost all environments the distributions (after conditioning on the environment) of the processes*

$$X_n(t) := \frac{1}{n} X(n^2 t); t \geq 0$$

converge weakly with respect to the Skorohod topology toward the distribution of the process $\{c^{-1/2} B(t); t \geq 0\}$, where $\{B(t); t \geq 0\}$ is standard Brownian motion on $\mathbb{R}$.
(see also Papanicolaou and Varadhan (1981) for some related result)

KK2: *If there exists a slowly varying function L_1 such that*

$$\frac{1}{nL_1(n)} \sum_{j=1}^{n} \frac{1}{\lambda_j} \longrightarrow 1 \quad \textit{in probability},$$

then the distributions of the processes

$$X_n(t) := \frac{1}{n} X(n^2 L_1(n) t)$$

converge weakly with respect to the Skorohod topology toward the distribution of standard Brownian motion.

KK3: *If there exists a slowly varying function L_2 such that the sequence of random variables*

$$R_n := \frac{1}{n^{1/\alpha} L_2(n)} \sum_{j=1}^{n} \frac{1}{\lambda_j}$$

converges in distribution toward a one-sided stable distribution ϑ_α with index $\alpha \in (0,1)$, then the distributions of the processes

$$X_n(t) := \frac{1}{n} X(n^{(1+\alpha)/\alpha} L_2(n) t)$$

converge weakly with respect to the Skorohod topology toward the distribution of a continuous self-similar process $\{X_*(t); t \geq 0\}$ with scaling exponent $\eta = \frac{\alpha}{\alpha+1}$.

Remark: 1) In the next section we will give a representation for the process X_* in terms of a standart Brownian motion and a stable subordinater associated to the measure ϑ_α.
2) We note that the results from Kawazu and Kesten (1984) are generalised in Kawazu (1989). He considered random-walks in random environments defined by the following transition asymptotics

$$\begin{aligned}
&\mathbb{P}(X(t+h) = j+1 | X(t) = j, \mathcal{A}) = (\lambda_j/\eta_j)h + o(h)\\
&\mathbb{P}(X(t+h) = j-1 | X(t) = j, \mathcal{A}) = (\lambda_{j-1}/\eta_j)h + o(h)\\
&\mathbb{P}(X(t+h) = j | X(t) = j, \mathcal{A}) = 1 - ((\lambda_j + \lambda_{j-1})/\eta_j)h + o(h),
\end{aligned}$$

where $\{\eta_j, j \in \mathbb{N}\}$ is an iid family of positive random-variables satisfying suitable assumptions. Similar to the situation studied in Kawazu and Kesten (1984) the resulting random-walks converge toward appropriate continuous processes after scaling.

In Kesten and Spitzer (1979) new classes of continuous self-similar processes are described. Moreover they proved that those processes are weak limits of random walks in random scenery. Those random walks are defined as follows:

Let $\{\xi(x); x \in \mathbb{Z}\}$ and $\{Z_i; i \in \mathbb{N}\}$ be two independent families of iid random variables, where the random variables Z_i are assumed to be $\mathbb{Z}$-valued. One can think of the sequence $\{Z_i; i \in \mathbb{N}\}$ as increments of a classical $\mathbb{Z}$-valued random walk $S_k := \sum_{i=1}^{k} Z_i$. The stationary sequence $\{\xi(S_k); k \in \mathbb{N}\}$ has some non-trivial long range dependencies, if the underlying random walk $\{S_k; k \in \mathbb{N}\}$ is recurrent. This is for example the case, if Z_1 is in the domain of attraction of an α-stable distribution with $\alpha \in (1, 2]$. The random sequence $D(n) := \sum_{k=1}^{n} \xi(S_k)$ is called a random walk in random scenery. In Kesten and Spitzer (1979) the following convergence result was proved for those processes:

KS1: *If $\xi(0)$ is in the domain of attraction of a β-stable distribution with $\beta \in (0, 2]$ and if Z_1 is in the domain of attraction of an α-stable distribution with $\alpha \in (0, 1)$, then the distributions of the processes*

$$D_n(t) := n^{-1/\beta} \sum_{k=1}^{\lfloor nt \rfloor} \xi(S_k)$$

converge weakly with respect to the Skorohod topology toward β-stable Lévy motion.
(see also Spitzer (1976) for a special case)

KS2: *If $\xi(0)$ is in the domain of attraction of a β-stable distribution with $\beta \in (0, 2]$ and if Z_1 is in the domain of attraction of an α-stable distribution with $\alpha \in (1, 2]$, then the distributions of the processes*

$$D_n(t) := n^{-\delta} \sum_{k=1}^{\lfloor nt \rfloor} \xi(S_k)$$

converge weakly with respect to the Skorohod topology toward a continuous self-similar process D_* with scaling exponent $\delta = 1 - \frac{1}{\alpha} + \frac{1}{\alpha\beta}$.

Remark: The statement in KS1 corresponds to the transient case and is not difficult to prove, since in that case the sequence $\{\xi(S_k); k \in \mathbb{N}\}$ has only weak dependencies. This is the reason, why one obtains β-stable Levy noise in the limit. We also mention that the case $\beta = 1$ is still open.

Remark: There exist various generalisations of the results from Kesten and Spitzer (1979). We only mention Shieh (1995) where the limiting process is generalised to higher dimensions, Lang and Nguyen (1983) which deals with multidimensional random walks and some special random scenery, Maejima (1996) where the random scenery belongs to the domain of attraction of operator stable distribution, Arai (2001) where the random scenery belongs to the domain of partial attraction of a semi-stable distribution and Saigo and Takahashi (2005) where the random scenery and the random walk belong to the partial domain of attractions of semi-stable and operator semi-stable distributions.

In this article we investigate, whether it is possible to substitute the classical random walk in the result from Kesten and Spitzer (1979) by the random walk in random environment which was introduced in Kawazu and Kesten (1984). We will restrict our attention to the result KK3, since this is the case where a new type of self-similar process arises at the end. For simplicity and in order to avoid abusive notations we will assume that the slowly varying function L_2 which appears in KK3 is constant and equal to one. The general case with non-constant L_2 can be treated in a similar way.

We now fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is sufficiently large to support a family of iid random variables $\{\lambda_j; j \in \mathbb{Z}\}$, a birth-death process $\{X(t); t \geq 0\}$ with asymptotic transition rates given by equations (1)-(3) and a family of iid random variables $\{\xi(k), k \in \mathbb{Z}\}$.
We assume that the families $\{\xi(k), k \in \mathbb{Z}\}$ and $\{X(t); t \geq 0\}$ are independent and that $t \mapsto X(t)$ is cadlag $\mathbb{P}$-almost surely.
Further, we assume that λ_1^{-1} is in the domain of normal attraction of a one-sided α-stable distribution ϑ_α with $\alpha \in (0,1)$.
Moreover, we assume that $\xi(0)$ is in the domain of normal attraction of a β-stable distribution ϑ_β with $\beta \in (0,2]$. Its characteristic function is given by

$$\psi(\theta) = \exp(-|\theta|^\beta(A_1 + iA_2 \mathrm{sgn}(\theta)),$$

where $0 < A_1 < \infty$ and $|A_1^{-1}A_2| \leq \tan(\pi\beta/2)$. For $\beta > 1$, it follows from those assumptions that $\mathbb{E}[\xi(0)] = 0$.
For $\beta = 1$ we make the further assumption that there exists a $K > 0$ such that

$$\left|\mathbb{E}\left[\xi(0)\mathbb{1}_{[-\rho,\rho]}(\xi(0))\right]\right| \leq K \quad \text{for all } \rho > 0.$$

We can now define the following continuous time version of the random walk in random scenery

$$\Xi(t) := \int_0^t \xi(X(s))ds.$$

In the following we will use the space

$$D[0,\infty) := \{\gamma : [0,\infty) \to \mathbb{R} : \gamma \text{ is cadlag}\}.$$

with the Skorohod topology. We will prove the following theorem in this article:

Theorem 1 *For $\kappa := \frac{1}{\alpha} + \frac{1}{\beta}$ and $k_n := n^{\frac{1+\alpha}{\alpha}}$ the distributions of the processes*

$$\Xi_n(t) := n^{-\kappa} \int_0^{k_n t} \xi(X(s))ds,$$

converge weakly with respect to the Skorohod topology toward the distribution of a self-similar stochastic process $\{\Xi_*(t); t \geq 0\}$ with scaling exponent $\mu = 1 - \frac{\alpha}{\alpha+1} + \frac{\alpha}{(\alpha+1)\beta}$.

Remark: The stochastic process $\{\Xi_*(t); t \geq 0\}$ can be constructed as follows:
Let Z_+ and Z_- be two independent copies of the β-stable Lévy-process which can be associated to the characteristic function

$$\psi(\theta) = \exp\left(-|\theta|^\beta(A_1 + iA_2 \mathrm{sgn}(\theta))\right).$$

Further, let $\{L_*(\tau, x); \tau \geq 0, x \in \mathbb{R}\}$ be the local time of the stochastic process $\{X_*(\tau); \tau \geq 0\}$; i.e.: the random variable $L_*(\tau, x)$ is the derivative with respect to x of the occupation-time

$$\Gamma_*(\tau, (-\infty, x]) := \int_0^\tau \mathbb{1}_{(-\infty,x]}(X_*(\sigma))d\sigma.$$

We will see in the next section that the local time exists for all except a countable number of points $x \in \mathbb{R}$. Moreover for all $\tau \geq 0$ the processes

$$\{L_*(\tau, x-); x \geq 0\} \quad \text{and} \quad \{L_*(\tau, -(x-)); x \geq 0\}$$

are predictable with respect to the natural filtrations of Z_+ resp. Z_-. The following integral representation of the process Ξ_* can be given

$$\Xi_*(\tau) := \int_0^\infty L_*(\tau, x-)dZ_+(x) + \int_0^\infty L_*(\tau, -(x-))dZ_-(x).$$

2 The convergence of the birth death process

The goal of this section is to prove Corollary 2, which is the main ingredient to show that the finite dimensional distributions of Ξ_n converge toward the finite dimensional distributions of Ξ_*. This corollary contains a statement on the weak convergence of certain functionals of the occupation times of the rescaled processes X_n. A result corresponding to Corollary 2 is also proved in Kesten and Spitzer (1979), however, we have to follow a totally different approach, since we do not have so precise information on the potential theory related to the random walk X. Instead we will understand the occupation times of X_n and prove that they converge in an appropriate sense toward the local time of the limit process X_*.

We describe some of the main arguments from the proof in Kawazu and Kesten (1984) for the convergence of the processes

$$X_n(t) := \frac{1}{n} X(n^{\frac{1+\alpha}{\alpha}} t)$$

toward the self-similar process X_* defined in Kawazu and Kesten (1984). We can enlarge our underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in such a way that it contains a standard Brownian motion $\{B(t); t \geq 0\}$ and a cadlag-version of the stable Lévy-subordinator $\{W(x); x \in \mathbb{R}\}$ which can be associated to the one-sided α-stable distribution ϑ_α.
Furthermore we assume that $\{B(t); t \geq 0\}$, $\{W(x); x \in \mathbb{R}\}$, $\{X(t); t \geq 0\}$ and $\{\xi(n); n \in \mathbb{Z}\}$ are independent. Moreover, we assume that $W(0) = 0$ and $B(0) = 0$ hold $\mathbb{P}$-almost surely.
In the future we will denote by $\{L(t, x); t \geq 0, x \in \mathbb{R}\}$ the local time of the Brownian motion $\{B(t); t \geq 0\}$. The process

$$V_*(t) := \int_{\mathbb{R}} L(t, W(x))dx$$

is non-decreasing $\mathbb{P}$-almost surely. Therefore, we can define the following pseudo-inverse

$$W^{-1}(y) := \inf\{x \in \mathbb{R}; W(x) > y\} \quad \text{and} \quad V_*^{-1}(\tau) := \inf\{t \geq 0; V_*(t) > \tau\}.$$

In Kawazu and Kesten (1984) the following representation for the self-similar process X_* is given

$$X_*(\tau) := W^{-1}(B(V_*^{-1}(\tau))).$$

Now we sketch the main arguments from the proof in Kawazu and Kesten (1984). We will need some of those ideas in our proof of the convergence of Ξ_n toward Ξ_*. Their approach is based on the natural scale of the birth death process. One defines

$$S(j) := \begin{cases} \sum_{k=0}^{j-1} \lambda_k^{-1} & \text{for } j > 0 \\ 0 & \text{for } j = 0 \\ -\sum_{k=j}^{-1} \lambda_k^{-1} & \text{for } j < 0. \end{cases}$$

This implies that conditioned on $\mathcal{A} := \{\lambda_j; j \in \mathbb{Z}\}$ the process $S(X(t))$ is on natural scale (see Kawazu and Kesten (1984) p.565). This means that for all $a, b, x \in \mathbb{R}$ with $a < x < b$ one has

$$\mathbb{P}(S(X(t)) \text{ hits } \{a, b\} \text{ first at } a | S(X(0)) = x, \mathcal{A}) = \frac{b - x}{b - a}.$$

It is then possible to represent the process $S(X(t))$ as the time change of standard Brownian motion $\{B(t); t \geq 0\}$ as follows:
One defines $m(dx) := \sum_{i \in \mathbb{Z}} \delta_{S(i)}(dx)$ and

$$V(t) := \int_{\mathbb{R}} L(t, x) m(dx) = \sum_{i \in \mathbb{Z}} L(t, S(i)),$$

where again $\{L(t, x); t \geq 0, x \in \mathbb{R}\}$ is the local time of the standard Brownian motion B. One can see that $\{B(V^{-1}(t)); t \geq 0\}$ and $\{S(X(t)); t \geq 0\}$ are both cadlag and have the same distribution (see Kawazu and Kesten (1984) p.566).
Then one has to scale the above constructions.

$$S_n(x) := n^{-1/\alpha} S(\lfloor nx \rfloor), \quad n \in \mathbb{N}, \; x \in \mathbb{R},$$

where for a positive real number x we denote by $\lfloor x \rfloor$ its integer part. It follows from the assumptions on the environment $\{\lambda_j; j \in \mathbb{Z}\}$ that for $n \to \infty$ the processes $\{S_n(x); x \in \mathbb{R}\}$ converge in distribution toward an α-stable Lévy-process $\{W(x); x \in \mathbb{R}\}$. Moreover, the process W is strictly increasing $\mathbb{P}$-almost surely, since ϑ_α is a one sided stable distribution and $\alpha \in (0, 1)$. By a method given in Skorohod (1956) and Dudley (1968) it is possible to construct a suitable probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with suitable D-valued random variables $\tilde{S}_n$ and $\tilde{W}$ having the properties that $\tilde{S}_n$ converges toward $\tilde{W}$ almost surely with respect to $\tilde{\mathbb{P}}$ and that $\tilde{S}_n$ and $\tilde{W}$ have the same distributions as S_n resp. W (see Kawazu and Kesten (1984) p.567). One then defines

$$\tilde{V}_n(t) := \int_{\mathbb{R}} L(t, x) \tilde{m}_n(dx) \quad \text{and} \quad \tilde{V}_*(t) := \int_{\mathbb{R}} L(t, x) \tilde{m}_*(dx)$$

with

$$\int_{\mathbb{R}} f(x) \tilde{m}_n(dx) := \int_{\mathbb{R}} f(\tilde{S}_n(x)) dx \quad \text{and} \quad \int_{\mathbb{R}} f(x) \tilde{m}_*(dx) := \int_{\mathbb{R}} f(\tilde{W}(x)) dx$$

for all measurable $f \geq 0$. We then define $\tilde{S}_n^{-1}$, $\tilde{W}^{-1}$, $\tilde{V}_n^{-1}$ and $\tilde{V}_*^{-1}$ in the same way as W^{-1} resp. V_*^{-1} above.
In Kawazu and Kesten (1984) (see p.568) they prove that $\{B(\tilde{V}_n^{-1}(t)); t \geq 0\}$ converges $\tilde{\mathbb{P}}$-almost surely toward $\{B(\tilde{V}_*^{-1}(t)); t \geq 0\}$ in the J_1-topology. For convenience we define

$$\tilde{X}_n(t) := \tilde{S}_n^{-1}(B(\tilde{V}_n^{-1}(t))), \quad \tilde{X}_*(t) := \tilde{W}^{-1}(B(\tilde{V}_*^{-1}(t)))$$

We note that the process $\{\tilde{X}_n(t); t \geq 0\}$ is defined on $(\Omega \times \tilde{\Omega}, \mathcal{F} \times \tilde{\mathcal{F}}, , \mathbb{P} \times \tilde{\mathbb{P}})$. It is proved in Kawazu and Kesten (1984) that $\{\tilde{X}_n(t); t \geq 0\}$ converges toward $\{\tilde{X}_*(t); t \geq 0\}$ with respect to the J_1-topology almost surely with respect to $\mathbb{P} \times \tilde{\mathbb{P}}$ (see page 569).
Moreover, for $B_n(t) := n^{-1/2} B(nt)$ one has that (see Kawazu and Kesten page 572)

$$|X_n(t) - S_n^{-1}(B_n(V_n^{-1}(t)))| \leq 1/n$$

and

$$\left\{S_n^{-1}(B_n(V_n^{-1}(t))); t \geq 0\right\} \stackrel{\mathcal{D}}{=} \left\{\tilde{S}_n^{-1}(B(\tilde{V}_n^{-1}(t))); t \geq 0\right\} = \left\{\tilde{X}_n(t); t \geq 0\right\}.$$

If we define $\hat{X}_n(t) := S_n^{-1}(B_n(V_n^{-1}(t)))$ the previous observations imply that both processes $\{X_n(t); t \geq 0\}$ and $\{\hat{X}_n(t); t \geq 0\}$ converge in distribution toward $\{\tilde{X}_*(t); t \geq 0\}$, which has the same distribution as $\{X_*(t); t \geq 0\}$.
In the rest of this section we analyse the distributional behaviour of the occupation times for the process X_n (see Proposition 6). In order to obtain this result we prove an analogue result for the process $\tilde{X}_n$ (see Lemma 5), which can be boiled down to Proposition 4. The advantage of this detour is that we can prove almost sure convergence for the occupation times of the process $\tilde{X}_n$ toward the local time of $\tilde{X}_*$ (see Proposition 3). This result is based on the fact that we have explicit formulas for the occupation times of $\tilde{X}_n$ and the local time of $\tilde{X}_*$ (see Proposition 2 and Corollary 1). The explicit expression of the occupation time of $\tilde{X}_n$ and the local time of $\tilde{X}_*$ unveils that in order to prove Proposition 3 it is sufficient to prove the almost sure convergence of $\tilde{S}_n$ and $\tilde{V}_n^{-1}$ toward $\tilde{W}_*$ resp. $\tilde{V}_*^{-1}$. The convergence of $\tilde{S}_n$ toward $\tilde{W}_*$ holds by construction. The convergence of $\tilde{V}_n$ toward $\tilde{V}_*$ is obtained in Lemma 1 and then used to obtain the convergence of $\tilde{V}_n^{-1}$ toward $\tilde{V}_*^{-1}$ in Lemma 2.

2.1 The local times of X_* and $\tilde{X}_*$

We define the time that the processes $\tilde{X}_*$ and X_* spend in the measurable set A until time τ as

$$\Gamma_*(\tau, A) := \int_0^\tau \mathbb{1}_A(X_*(\sigma)) d\sigma, \text{ resp. } \quad \tilde{\Gamma}_*(\tau, A) := \int_0^\tau \mathbb{1}_A(\tilde{X}_*(\sigma)) d\sigma.$$

We denote by $\{L_*(\tau, x); \tau \geq 0, x \in \mathbb{R}\}$ and $\{\tilde{L}_*(\tau, x); \tau \geq 0, x \in \mathbb{R}\}$ the local times of X_* resp. $\tilde{X}_*$ if they exist. In this subsection we prove that both local times exist almost surely and relate them to the local time $\{L(t, x); t \geq 0, x \in \mathbb{R}\}$ of the underlying Brownian motion $\{B(t); t \geq 0\}$.

Proposition 1 *One has $\mathbb{P}$-almost surely that for $\tau \geq 0$ and all $x \in \mathbb{R}$ that*

$$\Gamma_*(\tau, (-\infty, x)) = \int_{-\infty}^x L(V_*^{-1}(\tau), W(y)) dy.$$

Further, $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely that for all $\tau \geq 0$ and all $x \in \mathbb{R}$ that

$$\tilde{\Gamma}_*(\tau, (-\infty, x)) = \int_{-\infty}^x L(\tilde{V}_*^{-1}(\tau), \tilde{W}(y)) dy.$$

Proof: We have $\mathbb{P}$-almost surely that $x \mapsto W(x)$ is increasing. It follows that the set $\mathcal{N}_1$ of $x \in \mathbb{R}$, where W is not continuous, is countable. We define the set

$$\mathcal{N}_2 := \left\{x \in \mathbb{R} : \ell(\sigma; B(V_*^{-1}(\sigma)) = W(x)) > 0\right\},$$

where ℓ denotes the Lebesgue measure on $\mathbb{R}$. The set $\mathcal{N}_2$ is countable since for $x_1 \neq x_2$ one has that the sets $\{\sigma; B(V_*^{-1}(\sigma)) = W(x_1))\}$ and $\{\sigma; B(V_*^{-1}(\sigma)) = W(x_2))\}$ are disjoined. The statement then follows since there can not be an uncountable number of disjoined subsets of $\mathbb{R}$ with positive Lebesgue measure. Thus the set $\mathcal{N} := \mathcal{N}_1 \cup \mathcal{N}_2$ is countable. Since the function $x \mapsto \Gamma_*(\tau, (-\infty, x))$ is increasing and since

$$x \mapsto \int_{-\infty}^{x} L(V_*^{-1}(\tau), W(y))dy$$

is continuous, it is sufficient to prove the statement of the proposition for $x \in \mathcal{N}^c$.
The fact that W is increasing and continuous in x implies the equivalence of the statement $W(x) > y$ with the statement $\exists z_0 < x : W(z_0) > y$.
The later statement is then equivalent to the statement $W^{-1}(y) := \inf\{z : W(z) > y\} < x$.
This then implies that $\mathbb{I}_{(-\infty,x)}(X_*(\sigma)) = \mathbb{I}_{(-\infty,W(x))}(B(V_*^{-1}(\sigma)))$.
We also note that $t \mapsto V(t)$ is continuous and non-decreasing. This implies $V_* \circ V_*^{-1} = \mathrm{id}_{\mathbb{R}}$.
In the following we want to compute the derivative of the non-decreasing function

$$M : \sigma \mapsto \int_{-\infty}^{x} L(V_*^{-1}(\sigma), W(y))dy.$$

Since W is increasing and continuous in x, we have that $B(V_*^{-1}(\sigma_0)) < W(x)$ implies

$$\sigma \mapsto \int_{x}^{\infty} L(V_*^{-1}(\sigma), W(y))dy \text{ is locally constant, say equal to } c_0, \text{ in a neighbourhood of } \sigma_0.$$

Thus

$$\sigma \mapsto \int_{-\infty}^{x} L(V_*^{-1}(\sigma), W(y))dy = V_*(V_*^{-1}(\sigma)) - c_0 = \sigma - c_0 \text{ in a neighbourhood of } \sigma_0.$$

Moreover, since W is increasing and continuous in x we have that $B(V_*^{-1}(\sigma_0)) > W(x)$ implies

$$\sigma \mapsto \int_{-\infty}^{x} L(V_*^{-1}(\sigma), W(y))dy \text{ is locally constant in a neighbourhood of } \sigma_0.$$

It therefore turns out that

$$M'(\sigma) = \begin{cases} 1 & \text{if } B(V_*^{-1}(\sigma)) < W(x) \\ 0 & \text{if } B(V_*^{-1}(\sigma)) > W(x). \end{cases}$$

Moreover, for all $\sigma_1, \sigma_2 \in \mathbb{R}^+$ with $\sigma_1 \leq \sigma_2$ we have that

$$\int_{-\infty}^{x} L(V_*^{-1}(\sigma_1), W(y))dy \leq \int_{-\infty}^{x} L(V_*^{-1}(\sigma_2), W(y))dy$$

and

$$\int_{x}^{\infty} L(V_*^{-1}(\sigma_1), W(y))dy \leq \int_{x}^{\infty} L(V_*^{-1}(\sigma_2), W(y))dy.$$

This implies that

$$\int_{-\infty}^{x} L(V_*^{-1}(\sigma_2), W(y))dy - \int_{-\infty}^{x} L(V_*^{-1}(\sigma_1), W(y))dy \leq V_*(V_*^{-1}(\sigma_2)) - V_*(V_*^{-1}(\sigma_1)) = \sigma_2 - \sigma_1$$

It follows that

$$\sigma \mapsto \int_{-\infty}^{x} L(V_*^{-1}(\sigma), W(y))dy$$

is Lipshitz-continuous with Lipshitz-constant smaller than one.
Since the set $\{\sigma : B(V_*^{-1}(\sigma)) = W(x)\}$ is a zero set with respect to the Lebesgue measure ℓ for all $x \in \mathcal{N}^c$, it follows that

$$\int_0^\tau \mathbb{I}_{(-\infty,x)}(X_*(\sigma))d\sigma = \int_0^\tau \mathbb{I}_{(-\infty,W(x))}(B(V_*^{-1}(\sigma)))d\sigma = \int_0^\tau M'(\sigma)d\sigma = M(\tau).$$

The second statement is proved in the same way. □

Corollary 1 *One has $\mathbb{P}$-almost surely that the local time $L_*(\tau, x)$ is defined for all $\tau \geq 0$ and for all x, where $x \mapsto W(x)$ is continuous. Further, one has $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely that the local time $\tilde{L}_*(\tau, x)$ is defined for all $\tau \geq 0$ and for all x, where $x \mapsto \tilde{W}(x)$ is continuous. In those points one has*

$$L_*(\tau, x) = L(V_*^{-1}(\tau), W(x)), \quad \textit{resp.} \quad \tilde{L}_*(\tau, x) = L(\tilde{V}_*^{-1}(\tau), \tilde{W}(x)).$$

Proof: Differentiation in Proposition 1 proves the statement of this corollary. □

2.2 The occupation time of $\tilde{X}_n$

For a measurable set $A \subset \mathbb{R}$ we define

$$\hat{\Gamma}_n(t, A) := \int_0^t \mathbb{I}_A(\hat{X}_n(\sigma))d\sigma, \quad \tilde{\Gamma}_n(t, A) := \int_0^t \mathbb{I}_A(\tilde{X}_n(\sigma))d\sigma$$

and

$$\Gamma_n(t, A) := \int_0^t \mathbb{I}_A(X_n(\sigma))d\sigma.$$

This is the time that the processes $\hat{X}_n$, $\tilde{X}_n$ resp X_n spend in the set A until time t. In this section we give an explicit expression for the occupation time of $\tilde{X}_n$ in terms of the local time $\{L(t, x); t \geq 0, x \in \mathbb{R}\}$ of the underlying Brownian motion $\{B(t); t \geq 0\}$.

Proposition 2 *One has $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely for all $\tau \geq 0$ and all $x \in \mathbb{R}$ that*

$$\tilde{\Gamma}_n(\tau, \{x\}) = \begin{cases} \frac{1}{n}L(\tilde{V}_n^{-1}(\tau), \tilde{S}_n(x - \frac{1}{n})) & \text{if } nx \in \mathbb{Z} \\ 0 & \text{if } nx \notin \mathbb{Z}. \end{cases}$$

Proof: First we note that

$$S_n^{-1}(S_n(x)) = x + 1/n \quad \text{for all } x \text{ satisfying } nx \in \mathbb{Z}$$

If we use the fact that $\{B_n(V_n^{-1}(t)); t \geq 0\}\} \overset{\mathcal{D}}{=} \{S_n(X_n(t)); t \geq 0\}$ then we can see that $\{\hat{X}_n(t); t \geq 0\} \overset{\mathcal{D}}{=} \{X_n(t) + 1/n; t \geq 0\}$. Therefore, we see that $\hat{X}_n$ only takes values in the lattice $\frac{1}{n}\mathbb{Z}$ Moreover, we have that $\tilde{S}_n$ and $\tilde{V}_n$ have the same joint distribution as S_n and V_n. Therefore, $\hat{X}_n = S_n^{-1}(B_n(V_n^{-1}(.)))$ has the same distribution as $\tilde{X}_n = \tilde{S}_n^{-1}(B(\tilde{V}_n^{-1}(.)))$. From

this follows that also $\tilde{X}_n$ stays for all time in the countable state space $\{x \in \mathbb{R}; nx \in \mathbb{Z}\}$. This implies that $\tilde{\Gamma}_n(\tau, \{x\}) = 0$ for $nx \notin \mathbb{Z}$. This proves one part of the statement.
For the proof of the other part of the statement we will need the derivative of the function

$$\tilde{M}(\sigma) := \frac{1}{n} L(\tilde{V}_n^{-1}(\sigma), \tilde{S}_n(x - 1/n)).$$

We first collect some useful facts, which help to compute the derivative of $\tilde{M}$.
Since $\tilde{S}_n$ is constant on the intervals $[\frac{k}{n}, \frac{k+1}{n})$ for all $k \in \mathbb{Z}$, we have

$$\tilde{V}_n(t) = \int_{\mathbb{R}} L(t, \tilde{S}_n(x))dx = \frac{1}{n} \sum_{i \in \mathbb{Z}} L(t, \tilde{S}_n(i/n)). \tag{4}$$

Since the $(t, x) \mapsto L(t, x)$ is jointly continuous and non-decreasing $\mathbb{P}$-almost surely (see Boylan (1964) or Getoor and Kesten (1972)), it follows that $t \mapsto \tilde{V}_n(t)$ is continuous and non-decreasing $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely. This then gives rise to

$$\tilde{V}_n \circ \tilde{V}_n^{-1} = \mathrm{id}_{\mathbb{R}+} \qquad \mathbb{P} \times \tilde{\mathbb{P}} - \text{almost surely}. \tag{5}$$

By construction one has for all $b \in \{\tilde{S}_n(x); x \in \mathbb{R}\}$ that $\tilde{S}_n^{-1}(b) = x$ is equivalent to $b = \tilde{S}_n(x - \frac{1}{n})$. Moreover, one has that $B(\tilde{V}_n^{-1}(\sigma)) \in \{\tilde{S}_n(x); x \in \mathbb{R}\}$ for all $\sigma \geq 0$ almost surely with respect to $\mathbb{P} \times \tilde{\mathbb{P}}$. Hence

$$\tilde{X}_n(\sigma) = \tilde{S}_n^{-1}(B(\tilde{V}_n^{-1}(\sigma))) = x \quad \text{is equivalent to} \quad B(\tilde{V}_n^{-1}(\sigma)) = \tilde{S}_n(x - \frac{1}{n}). \tag{6}$$

Moreover, the random variables $\{\lambda_i^{-1}; i \in \mathbb{N}\}$ are positive $\mathbb{P}$-almost surely and therefore

$$\text{the restriction of } x \mapsto \tilde{S}_n(x) \text{ to the set } \frac{1}{n}\mathbb{Z} \text{ is injective almost surely with respect to } \tilde{\mathbb{P}}. \tag{7}$$

Since conditioned on $\mathcal{A} = \sigma\{\lambda_j; j \in \mathbb{N}\}$ the process X is a Markov process, it follows that for $nx \in \mathbb{Z}$ there exist non-negative random variables $a_1 < b_1 < a_2 < b_2 < \ldots$ with the property

$$\Big\{\sigma \geq 0; \tilde{X}_n(\sigma) = x\Big\} = \bigcup_{i \in \mathbb{N}} [a_i, b_i) \quad \mathbb{P} \times \tilde{\mathbb{P}} - \text{a.s.}$$

This implies that for all $\sigma_0 \notin \{a_i; i \in \mathbb{N}\}$ there exists a neighbourhood $\mathcal{U}(\sigma_0)$ containing σ_0 with the property that $\sigma \mapsto \tilde{X}_n(\sigma) = \tilde{S}_n^{-1}(B(\tilde{V}_n^{-1}(\sigma)))$ is constant on $\mathcal{U}(\sigma_0)$. Then (6) and (7) imply that $\sigma \mapsto B(\tilde{V}_n^{-1}(\sigma))$ must be constant on $\mathcal{U}(\sigma_0)$.
Therefore, for $\sigma_0 \notin \{a_i; i \in \mathbb{N}\}$ and $B(\tilde{V}_n^{-1}(\sigma_0)) \neq \tilde{S}_n(x - \frac{1}{n})$ we have $B(\tilde{V}_n^{-1}(\sigma)) \neq \tilde{S}_n(x - \frac{1}{n})$ for all σ in a neighbourhood of σ_0. Hence

$$\sigma \mapsto L(\tilde{V}_n^{-1}(\sigma), \tilde{S}_n(x - 1/n)) \text{ is constant in a neighbourhood of } \sigma_0.$$

The previous argument and the fact that $\tilde{X}_n$ only jumps to nearest neighbours in $\frac{1}{n}\mathbb{Z}$ lead to the fact that $\sigma_0 \notin \{a_i; i \in \mathbb{N}\}$ and $B(\tilde{V}_n^{-1}(\sigma_0)) = \tilde{S}_n(x - \frac{1}{n})$ imply the existence of a suitable $c_0 > 0$ with the property

$$\sigma \mapsto \frac{1}{n} \sum_{z \neq nx-1} L(\tilde{V}_n^{-1}(\sigma), \tilde{S}_n(z/n)) = c_0 \text{ in a neighbourhood of } \sigma_0.$$

Therefore we can use (5) to see that $B(\tilde{V}_n^{-1}(\sigma_0)) = \tilde{S}_n(x - \frac{1}{n})$ implies

$$\sigma \mapsto \frac{1}{n} L(\tilde{V}_n^{-1}(\sigma), \tilde{S}_n(x - 1/n)) = \tilde{V}_n(\tilde{V}_n^{-1}(\sigma)) - c_0 = \sigma - c_0 \text{ in a neighbourhood of } \sigma_0.$$

Consequently the function

$$\tilde{M}(\sigma) := \frac{1}{n} L(\tilde{V}_n^{-1}(\sigma), \tilde{S}_n(x - 1/n))$$

is differentiable for all $\sigma \notin \{a_i; i \in \mathbb{N}\}$ and for $nx \in \mathbb{Z}$ we have

$$\tilde{M}'(\sigma) = \begin{cases} 1 & \text{if } B(\tilde{V}_n^{-1}(\sigma)) = \tilde{S}_n(x - \frac{1}{n}) \\ 0 & \text{if } B(\tilde{V}_n^{-1}(\sigma)) \neq \tilde{S}_n(x - \frac{1}{n}). \end{cases}$$

Moreover, it is possible to prove that the function $\tilde{M}$ is Lipshitz-continuous with Lipshitz-constant one. From those properties, it follows that

$$\int_0^\tau \mathbb{1}_{\{x\}}(\tilde{X}_n(\sigma)) d\sigma = \int_0^\tau \mathbb{1}_{\{\tilde{S}_n(x - \frac{1}{n})\}}(B(\tilde{V}_n^{-1}(\sigma))) d\sigma = \int_0^\tau \tilde{M}'(\sigma) d\sigma = \tilde{M}(\tau).$$

□

2.3 The convergence of the occupation times

In this section we investigate whether the occupation times of $\tilde{X}_n$ converge toward the local time of $\tilde{X}_*$ in an appropriate way as $n \to \infty$. For this we first need some auxiliary results.

Lemma 1 *One has $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely that $\tilde{V}_n(t)$ converges toward $\tilde{V}_*(t)$ for all $t \in \mathbb{R}$.*

Proof: We fix a $T > 0$ and define $w_o := \sup\{x : L(T, x) > 0\}$ and $w_u := \inf\{x : L(T, x) > 0\}$. Those two random-variables are independent of $\tilde{\mathbb{P}}$. We know that $\{\tilde{S}_n(x); x \in \mathbb{R}\}$ converges toward $\{\tilde{W}(x); x \in \mathbb{R}\}$ with respect to the J_1-topology $\tilde{\mathcal{F}}$-almost surely. We note that the local time of Brownian motion $(x, t) \mapsto L(t, x)$ is jointly continuous $\mathbb{P}$-almost surely (see Boylan (1964) or Getoor R.K. and Kesten (1972)).
It follows that $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely $\{L(t, \tilde{S}_n(x)); x \in \mathbb{R}\}$ converges toward $\{L(t, \tilde{W}(x)); x \in \mathbb{R}\}$ with respect to the J_1-topology for all $t \in [0, T]$.
We fix a pair $(\omega, \tilde{\omega}) \in \Omega \times \tilde{\Omega}$ with the property that $\{L(t, \tilde{S}_n(x))(\omega, \tilde{\omega}); x \in \mathbb{R}\}$ converges toward $\{L(t, \tilde{W}(x))(\omega, \tilde{\omega}); x \in \mathbb{R}\}$ with respect to the J_1-topology for all $t \in [0, T]$.
Then there exist suitable $x_u, x_o \in \mathbb{R}$ with $\tilde{W}(x_u) \leq w_u$ and $\tilde{W}(x_o) \geq w_o$, and there exists a sequence of increasing, absolutely continuous, surjective Lipshitz-maps $\lambda_n : [x_u, x_o] \to [x_u, x_o]$ with the properties

$$\sup_{x \in [x_u, x_o]} \left| L(t, \tilde{W}(x)) - L(t, \tilde{S}_n(\lambda_n(x))) \right| \longrightarrow 0 \quad \text{as } n \to \infty$$

and

$$\text{esssup}_{x \in [x_u, x_o]} \left| \lambda_n'(x) - 1 \right| \longrightarrow 0 \quad \text{as } n \to \infty.$$

We should emphasise that the derivative of the function λ_n may not exist everywhere. However, those points where it does not exist form a zero set, since λ_n is an absolutely continuous Lipshitz-function.
Then by change of variables for all $t \in [0,T]$ one has

$$\begin{aligned}
&\int_{x_u}^{x_o} L(t, \tilde{S}_n(x))dx - \int_{x_u}^{x_o} L(t, \tilde{S}_n(\lambda_n(x)))dx \\
= \; &\int_{x_u}^{x_o} L(t, \tilde{S}_n(x)) \left(1 - \frac{1}{\lambda_n'(\lambda_n^{-1}(x))}\right) dx + O\left(\sup_{x\in[x_u,x_o]} |\lambda_n(x) - x|\right).
\end{aligned}$$

It follows from the assumptions on the sequence λ_n that the above difference converges toward zero. Further, we have for all $t \in [0,T]$ that

$$\int_{\mathbb{R}} L(t, \tilde{S}_n(\lambda_n(x)))dx \longrightarrow \int_{\mathbb{R}} L(t, \tilde{W}(x))dx \quad \text{as } n \to \infty.$$

Hence one has $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely that $\tilde{V}_n(t)$ converges toward $\tilde{V}_*(t)$ for all $t \in [0,T]$. Thus we obtain for every $T > 0$ an zero-set N_T in $\Omega \times \tilde{\Omega}$, where this convergence does not hold. The lemma now follows, since the union

$$N_\infty := \bigcup_{T\in\mathbb{N}} U_T$$

is also a zero-set with respect to $\mathbb{P} \times \tilde{\mathbb{P}}$. □

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We call $\tau \in f(\mathbb{R})$ a critical value for f, if there exist at least two distinct points $t_1, t_2 \in \mathbb{R}$ such that $f(t_1) = f(t_2) = \tau$. Further, we call a point $\tau \in f(\mathbb{R})$ a regular value for f if τ is not a critical value. It is straight forward to see, that the preimages of critical values contain an open interval, if the function f is non-decreasing. This implies that the set of critical values of a non-decreasing function is at most countable.

Lemma 2 *One has $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely that $\tilde{V}_n^{-1}(\tau)$ converges toward $\tilde{V}_*^{-1}(\tau)$ for all regular values τ of $\tilde{V}_*$.*

Proof: We note that $\mathbb{P}$-almost surely the local time $L(t,x)$ of the Brownian motion B is continuous and non-decreasing in t for all $x \in \mathbb{R}$ (see Boylan (1964) or Getoor R.K. and Kesten (1972) for the continuity). It follows that $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely the function

$$t \mapsto \tilde{V}_*(t) := \int_{\mathbb{R}} L(t,x) m_*(dx)$$

is continuous and non-decreasing.
Therefore, $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely the function $\tilde{V}_*^{-1}(\tau) := \inf\{t; \tilde{V}(t) > \tau\}$ is strictly increasing and right-continuous.
We use Lemma 1 to fix a pair $(\omega, \tilde{\omega}) \in \Omega \times \tilde{\Omega}$ with the properties that:
(i) $\tau \mapsto \tilde{V}_*^{-1}(\tau)$ is strictly increasing and right-continuous;
(ii) $\tilde{V}_n(t)$ converges toward $\tilde{V}_*(t)$ for all $t \geq 0$.
Since the set where $\tilde{V}_*$ is not continuous is countable, the set where $\tilde{V}_*$ is continuous is a dense set in $[0,\infty)$.

We denote by K the set of critical values of $\tilde{V}_*$. As was pointed out before, K is at most countable. For an arbitrary point $\tau \in [0,\infty) \cap K^c$ and for any $\epsilon > 0$ one can find points $t_{\epsilon,0}, t_{\epsilon,1} \in (\tilde{V}_*^{-1}(\tau) - \epsilon, \tilde{V}_*^{-1}(\tau))$ and $t_{\epsilon,2}, t_{\epsilon,3} \in (\tilde{V}_*^{-1}(\tau), \tilde{V}_*^{-1}(\tau) + \epsilon)$ with the property

$$\tilde{V}_*(t_{\epsilon,0}) < \tilde{V}_*(t_{\epsilon,1}) < \tau < \tilde{V}_*(t_{\epsilon,2}) < \tilde{V}_*(t_{\epsilon,3}).$$

Now we can choose a $\delta > 0$ such that

$$\tilde{V}_*(t_{\epsilon,0}) + \delta < \tilde{V}_*(t_{\epsilon,1}) - \delta < \tilde{V}_*(t_{\epsilon,1}) + \delta < \tau < \tilde{V}_*(t_{\epsilon,2}) - \delta < \tilde{V}_*(t_{\epsilon,2}) + \delta < \tilde{V}_*(t_{\epsilon,3}) - \delta.$$

Since $\tilde{V}_n$ converges toward $\tilde{V}_*$ in all points where $\tilde{V}_*$ is continuous, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\tilde{V}_n(t_{\epsilon,0}) < \tilde{V}_*(t_{\epsilon,0}) + \delta < \tilde{V}_*(t_{\epsilon,1}) - \delta < \tilde{V}_n(t_{\epsilon,1}) < \tilde{V}_*(t_{\epsilon,1}) + \delta < \tau$$

and

$$\tau < \tilde{V}_*(t_{\epsilon,2}) - \delta < \tilde{V}_n(t_{\epsilon,2}) < \tilde{V}_*(t_{\epsilon,2}) + \delta < \tilde{V}_*(t_{\epsilon,3}) - \delta < \tilde{V}_n(t_{\epsilon,3}).$$

By definition of $t_{\epsilon,0}$ we have that $z \leq \tilde{V}_*^{-1}(\tau) - \epsilon$ implies $z \leq t_{\epsilon,0}$. From monotonicity and the first of both inequalities above, it follows that

$$\tilde{V}_n(z) \leq \tilde{V}_n(t_{\epsilon,0}) \leq \tilde{V}_*(t_{\epsilon,0}) + \delta < \tilde{V}_*(t_{\epsilon,1}).$$

We thus have seen that $z \leq \tilde{V}_*^{-1}(\tau) - \epsilon$ implies $\tilde{V}_n(z) < \tilde{V}_*(t_{\epsilon,1})$. If we reverse the implication, we obtain that $\tilde{V}_n(z) \geq \tilde{V}_*(t_{\epsilon,1})$ implies $z > \tilde{V}_*^{-1}(\tau) - \epsilon$. From this implication it follows that

$$\tilde{V}_n^{-1}(\tilde{V}_*(t_{\epsilon,1})) = \inf\{z : \tilde{V}_n(z) > \tilde{V}_*(t_{\epsilon,1})\} > \tilde{V}_*^{-1}(\tau) - \epsilon.$$

For $z = t_{\epsilon,3}$ we have $\tilde{V}_n(z) = \tilde{V}_n(t_{\epsilon,3}) > \tilde{V}_*(t_{\epsilon,2})$. In other words there exists a $z < \tilde{V}_*^{-1}(\tau) + \epsilon$ with $\tilde{V}_n(z) > \tilde{V}_*(t_{\epsilon,2})$. This proves that

$$\tilde{V}_*^{-1}(\tau) + \epsilon > \tilde{V}_n^{-1}(\tilde{V}_*(t_{\epsilon,2})).$$

Altogether, we have proved that for all $n \geq n_0$,

$$\tilde{V}_*^{-1}(\tau) - \epsilon < \tilde{V}_n^{-1}(\tilde{V}_*(t_{\epsilon,1})) < \tilde{V}_n^{-1}(\tilde{V}_*(t_{\epsilon,2})) < \tilde{V}_*^{-1}(\tau) + \epsilon.$$

By monotonicity, for all $n \geq n_0$ and all $\tau' \in [\tilde{V}_*(t_{\epsilon,1}), \tilde{V}_*(t_{\epsilon,2})]$ one has

$$\tilde{V}_*^{-1}(\tau) - \epsilon < \tilde{V}_n^{-1}(\tau') < \tilde{V}_*^{-1}(\tau) + \epsilon.$$

Since $\tau \in [\tilde{V}_*(t_{\epsilon,1}), \tilde{V}_*(t_{\epsilon,2})]$, the proof is complete. $\square$

Lemma 3 *For all $\tau \geq 0$, one has that τ is a regular value of $\tilde{V}_*$ almost surely with respect to $\mathbb{P} \times \tilde{\mathbb{P}}$.*

Proof: By the invariance properties of the Brownian motion we have that for all $\gamma > 0$

$$\{L(t,w); w \in \mathbb{R}, t \geq 0\} \stackrel{\mathcal{D}}{=} \{\gamma^{-1} L(\gamma^2 t, \gamma w); w \in \mathbb{R}, t \geq 0\}.$$

By the invariance of the α-stable Lévy-process that

$$\begin{aligned}\{L(t,\tilde W(x)); x\in\mathbb{R}, t\geq 0\} &\overset{\mathcal{D}}{=} \{\gamma^{-1}L(\gamma^2 t,\gamma\tilde W(x)); x\in\mathbb{R}, t\geq 0\}\\ &\overset{\mathcal{D}}{=} \{\gamma^{-1}L(\gamma^2 t,\tilde W(\gamma^\alpha x)); x\in\mathbb{R}, t\geq 0\}.\end{aligned}$$

Substitution then yields

$$\begin{aligned}\left\{\int_{\mathbb{R}} L(t,\tilde W(x))dx; t\geq 0\right\} &\overset{\mathcal{D}}{=} \left\{\gamma^{-1}\int_{\mathbb{R}} L(\gamma^2 t,\tilde W(\gamma^\alpha x))dx; t\geq 0\right\}\\ &\overset{\mathcal{D}}{=} \left\{\gamma^{-1-\alpha}\int_{\mathbb{R}} L(\gamma^2 t,\tilde W(x))dx; t\geq 0\right\}.\end{aligned}$$

By definition this means that

$$\{\tilde V_*(t); t\geq 0\} \overset{\mathcal{D}}{=} \{\gamma^{-1-\alpha}\tilde V_*(\gamma^2 t); t\geq 0\}.$$

We define ℓ_* to be the image-measure of the Lebesgue-measure ℓ with respect $\tilde V_*$. The previous considerations imply

$$\ell_*(dt) \overset{\mathcal{D}}{=} \gamma^2\ell_*(\gamma^{-1-\alpha}dt).$$

This identity implies that no $\tau>0$ satisfies $\ell_*(\{\tau\})>0$ with a positive probability with respect to $\mathbb{P}\times\tilde{\mathbb{P}}$. To a critical value τ corresponds an interval where $t\mapsto\tilde V_*$ is constant, which implies $\ell_*(\{\tau\})>0$. For a particular point $\tau>0$ this can not happen with positive probability. This finishes the proof of the statement. $\square$

Proposition 3 *For all $\tau\geq 0$ the sequence of functions $x\mapsto L(\tilde V_n^{-1}(\tau),\tilde S_n(x+1/n))$ converges toward the function $x\mapsto L(\tilde V_*^{-1}(\tau),\tilde W(x))$ in the J_1-topology $\mathbb{P}\times\tilde{\mathbb{P}}$-almost surely.*

Proof: It is known that $\tilde S_n$ converges toward $\tilde W$ in the J_1-topology almost surely with respect to $\tilde{\mathbb{P}}$. Moreover, by Lemma 2 and Lemma 3, for all $\tau\geq 0$ the sequence $\tilde V_n^{-1}(\tau)$ converges toward $\tilde V_*^{-1}(\tau)$ almost surely with respect to $\mathbb{P}\times\tilde{\mathbb{P}}$. The proposition follows, since it is well known that $(t,x)\mapsto L(t,x)$ is jointly continuous $\mathbb{P}$-almost surely (see Boylan (1964) or Getoor and Kesten (1972)). $\square$

Lemma 4 *For all $k\in\mathbb{N}$, $\theta_1,...,\theta_k\in\mathbb{R}$ and all $\tau_1,...,\tau_k\geq 0$,the set*

$$\mathcal{C} := \left\{c>0 : \ell\left(x\in\mathbb{R}; \left|\sum_{i=1}^k \theta_i L(\tilde V_*^{-1}(\tau_i),\tilde W(x))\right| = c\right) > 0\right\}$$

is countable $\mathbb{P}\times\tilde{\mathbb{P}}$-almost surely, where ℓ denotes the Lebesgue measure on $\mathbb{R}$.

Proof: It is well known that $x\mapsto\tilde W(x)$ is strictly increasing $\tilde{\mathbb{P}}$-almost surely. For $c>0$ we define the level-sets

$$\mathcal{N}_c := \left\{w\in\mathbb{R}; \left|\sum_{i=1}^k \theta_i L(\tilde V_*^{-1}(\tau_i),w)\right| = c\right\}.$$

Fix a strictly increasing path $f : x\mapsto\tilde W(x)$ and assume that there exist an uncountable number of $c>0$ with the property $\ell(f^{-1}(\mathcal{N}_c))>0$. For $c\neq c'$ the sets $f^{-1}(\mathcal{N}_c)$ and $f^{-1}(\mathcal{N}_{c'})$ are disjoint. We would obtain an uncountable number of disjoint sets with positive Lebesgue measure. This is of course not possible. $\square$

Proposition 4 *For all $k \in \mathbb{N}$, $\theta_1, ..., \theta_k \in \mathbb{R}$ and all $\tau_1, ..., \tau_k \geq 0$ one has $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely that*

$$\frac{1}{n}\mathrm{card}\left\{x \in \mathbb{Z} : n\left|\sum_{i=1}^{k} \theta_i \tilde{\Gamma}_n(\tau_i, \{x/n\})\right| > c\right\} \longrightarrow \ell\left(x \in \mathbb{R} : \left|\sum_{i=1}^{k} \theta_i \tilde{L}_*(\tau_i, x)\right| > c\right) \quad \text{as } n \to \infty$$

for all except a countable number of $c > 0$.

Proof: We can find a $K > 0$ such that $\{y \in \mathbb{R} : L(\tau_i, y) \neq 0 \text{ for all } i = 1, ..., k\}$ is a subset of the interval $(\tilde{W}(-K), \tilde{W}(K))$. By Proposition 2, Proposition 3 and Corollary 1 the sequence

$$\tilde{A}_n(x) := n\left|\sum_{i=1}^{k} \theta_i \tilde{\Gamma}_n(\tau_i, \{x\})\right| = \left|\sum_{i=1}^{k} \theta_i L(\tilde{V}_n^{-1}(\tau_i), \tilde{S}_n(x - 1/n))\right|$$

converges $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely in the J_1-topology toward

$$\tilde{A}_*(x) := \left|\sum_{i=1}^{k} \theta_i \tilde{L}_*(\tau_i, x)\right| = \left|\sum_{i=1}^{k} \theta_i L(\tilde{V}_*^{-1}(\tau_i), \tilde{W}(x))\right|.$$

Then there exists a sequence of continuous increasing maps $\lambda_n : [-K, K] \to [-K, K]$ such that

$$\sup_{x \in [-K,K]} \left|\tilde{A}_*(x) - \tilde{A}_n \circ \lambda_n(x)\right| \longrightarrow 0 \quad \text{as } n \to \infty$$

and such that each λ_n is Lipshitz continuous and satisfies

$$\mathrm{esssup}_{x \in [-K,K]} \left|\lambda_n'(x) - 1\right| \longrightarrow 0.$$

We should emphasise that the derivative of the function λ_n may not exist everywhere. However, those points where the derivative does not exist form a zero-set, since λ_n is an absolutely continuous Lipshitz-function. We note that for suitably large $n \in \mathbb{N}$ one has

$$\begin{aligned}\frac{1}{n}\mathrm{card}\left\{x \in \mathbb{R}; \left|\sum_{i=1}^{k} \theta_i L(\tilde{V}_n^{-1}(\tau_i), \tilde{S}_n(x - 1/n))\right| > c\right\} &= \ell\left(x \in [-K, K]; \tilde{A}_n(x) > c\right) \\ &= \int_{-K}^{K} \mathbb{I}_{(c,\infty)}(\tilde{A}_n(x))dx.\end{aligned}$$

Then it follows that

$$\begin{aligned}&\frac{1}{n}\mathrm{card}\left\{x \in [-K, K]; n\left|\sum_{i=1}^{k} \theta_i \tilde{\Gamma}_n(\tau_i, \{x\})\right| > c\right\} - \int_{-K}^{K} \mathbb{I}_{(c,\infty)}(\tilde{A}_n(\lambda_n(x)))dx \\ = \quad &\int_{-K}^{K} \mathbb{I}_{(c,-\infty)}(\tilde{A}_n(x))dx \left(1 - \frac{1}{\lambda_n'(\lambda_n^{-1}(x))}\right) dx + O\left(\sup_{x \in [-K,K]} |\lambda_n(x) - x|\right).\end{aligned}$$

By the assumptions on the sequence $\{\lambda_n; n \in \mathbb{N}\}$ the previous difference converges toward zero. Furthermore,

$$\int_{-K}^{K} \mathbb{I}_{(c,\infty)}(\tilde{A}_n(\lambda_n(x)))dx \longrightarrow \int_{-K}^{K} \mathbb{I}_{(c,\infty)}(\tilde{A}_*(x))dx \quad \text{as } n \to \infty$$

whenever the set $\{x \in [-K,K]; \tilde{A}_*(s) = c\}$ is a zero-set with respect to the Lebesgue measure ℓ on $\mathbb{R}$. Since this was proved in Lemma 4, the statement of the proposition follows. $\square$

Subsequently we make use of the following notations

$$A_n^+ := \left\{ x \in \mathbb{Z} : \sum_{i=1}^k \theta_i \tilde{\Gamma}_n(\tau_i, \{x/n\}) > 0 \right\} \quad \text{resp.} \quad A_n^- := \left\{ x \in \mathbb{Z} : \sum_{i=1}^k \theta_i \tilde{\Gamma}_n(\tau_i, \{x/n\}) < 0 \right\}$$

and

$$A^+ := \left\{ x \in \mathbb{R} : \sum_{i=1}^k \theta_i \tilde{L}_*(\tau_i, x) > 0 \right\} \quad \text{resp.} \quad A^- := \left\{ x \in \mathbb{R} : \sum_{i=1}^k \theta_i \tilde{L}_*(\tau_i, x) < 0 \right\}.$$

Later we need the following version of Proposition 4:

Proposition 5 *For all $k \in \mathbb{N}$, $\theta_1, ..., \theta_k \in \mathbb{R}$ and all $\tau_1, ..., \tau_k \geq 0$ one has $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely that*

$$\frac{1}{n} \text{card} \left\{ x \in \mathbb{Z} \cap A_n^{\pm} : n \left| \sum_{i=1}^k \theta_i \tilde{\Gamma}_n(\tau_i, \{x/n\}) \right| > c \right\} \longrightarrow \ell \left(x \in \mathbb{R} \cap A^{\pm} : \left| \sum_{i=1}^k \theta_i \tilde{L}_*(\tau_i, x) \right| > c \right)$$

for all except a countable number of $c > 0$.

Proof: The proof uses essentially the same arguments as the proof of Proposition 4. $\square$

Remark: With the same proof as for Proposition 4, we can show that $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely

$$\frac{1}{n} \text{card} \left\{ x \in \mathbb{Z} : n^2 \tilde{\Gamma}_n^2(\tau_i, \{x/n\}) > c \right\} \longrightarrow \ell \left(x \in \mathbb{R} : \tilde{L}_*^2(\tau_i, x) > c \right) \quad \text{as } n \to \infty$$

for all except a countable number of $c > 0$.

2.4 An usefull Lemma on integrated powers of local time

Lemma 5 *For $\tau_1, ..., \tau_k \geq 0$ and $\theta_1, ..., \theta_k \in \mathbb{R}$ the two sequences of random variables*

$$n^{\beta-1} \sum_{x \in \mathbb{Z}} \left| \sum_{i=1}^k \theta_i \tilde{\Gamma}_n(\tau_i, \{x/n\}) \right|^\beta \quad \textit{and} \quad n^{\beta-1} \sum_{x \in \mathbb{Z}} \left(\left| \sum_{i=1}^k \theta_i \tilde{\Gamma}_n(\tau_i, \{x/n\}) \right|^\beta \text{sgn} \left(\sum_{i=1}^k \theta_i \tilde{\Gamma}_n(\tau_i, \{x/n\}) \right) \right)$$

converge $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely toward the random variables

$$\int_{-\infty}^{\infty} \left| \sum_{i=1}^k \theta_i \tilde{L}_*(\tau_i, x) \right|^\beta dx \quad \textit{resp.} \quad \int_{-\infty}^{\infty} \left(\left| \sum_{i=1}^k \theta_i \tilde{L}_*(\tau_i, x) \right|^\beta \text{sgn} \left(\sum_{i=1}^k \theta_i \tilde{L}_*(\tau_i, x) \right) \right) dx.$$

Proof: We use the layer-cake representation of the integrals (see Lieb and Loss (2001)) to write

$$\sum_{x \in \mathbb{Z}} \left| \sum_{i=1}^k \theta_i n \tilde{\Gamma}_n(\tau_i, \{x/n\}) \right|^\beta = \beta \int_0^\infty c^{\beta-1} \text{card} \left\{ x \in \mathbb{Z} : n \left| \sum_{i=1}^k \theta_i \tilde{\Gamma}_n(\tau_i, \{x/n\}) \right| > c \right\} dc$$

and

$$\int_{-\infty}^{\infty}\left|\sum_{i=1}^{k}\theta_i\tilde{L}_*(\tau_i,x)\right|^{\beta}dx=\beta\int_0^{\infty}c^{\beta-1}\ell\left(x\in\mathbb{R}:\left|\sum_{i=1}^{k}\theta_i\tilde{L}_*(\tau_i,x)\right|>c\right)dc.$$

We note that the convergence of $\tilde{V}_n^{-1}(\tau_i)$ toward $\tilde{V}_*^{-1}(\tau_i)$ and the fact that $t\mapsto L(t,y)$ is increasing for every $y\in\mathbb{R}$ imply that there exists an $n_0\in\mathbb{N}$ with

$$L(\tilde{V}_n^{-1}(\tau_i),y)\le L(\tilde{V}_*^{-1}(\tau_i)+1,y)\quad\text{for all }y\in\mathbb{R},\ 1\le i\le k,n\ge n_0.$$

Moreover, for all $i\in\{1,...,k\}$ the functions $y\mapsto L(\tilde{V}_*^{-1}(\tau_i)+1,y)$ are continuous and their supports are contained in $[-K,K]$ for a suitable $K>0$. Hence there exists a $C>0$ such that for $n\ge n_0$ one has

$$n\left|\sum_{i=1}^{k}\theta_i\tilde{\Gamma}_n(\tau_i,\{x/n\})\right|\le\left|\sum_{i=1}^{k}\theta_i L(\tilde{V}_n^{-1}(\tau_i),\tilde{S}_n((x-1)/n))\right|\le\sum_{i=1}^{k}\theta_i\sup_{y\in\mathbb{R}}L(\tilde{V}_*^{-1}(\tau_i)+1,y)\le C.$$

This implies that all the functions

$$c\mapsto\operatorname{card}\left\{x\in\mathbb{Z}:n\left|\sum_{i=1}^{k}\theta_i\tilde{\Gamma}_n(\tau_i,\{x/n\})\right|>c\right\}\quad\text{have support contained in }[0,C].$$

Moreover, for all $c>0$ we have

$$\operatorname{card}\left\{x\in\mathbb{Z}:n\left|\sum_{i=1}^{k}\theta_i\tilde{\Gamma}_n(\tau_i,\{x/n\})\right|>c\right\}\le\operatorname{card}\left\{x\in\mathbb{Z}:-K\le\tilde{S}_n((x-1)/n)\le K\right\}.$$

Since

$$\ell\left(x;\tilde{W}(x)\in\{-K,K\}\right)=0$$

and since $\tilde{S}_n$ converges toward $\tilde{W}$ with respect to the Skorohod metric, we have that

$$\frac{1}{n}\operatorname{card}\left\{x\in\mathbb{Z}:-K\le\tilde{S}_n((x-1)/n)\le K\right\}\longrightarrow\ell\left(x\in\mathbb{R}:-K\le\tilde{W}(x)\le K\right).$$

This implies that there exists a $R>0$ such that for all $n\in\mathbb{N}$ and all $c>0$ we have

$$\frac{1}{n}\operatorname{card}\left\{x\in\mathbb{Z}:n\left|\sum_{i=1}^{k}\theta_i\tilde{\Gamma}_n(\tau_i,\{x/n\})\right|>c\right\}\le R.$$

The first statement of the lemma then follows from dominated convergence and Proposition 4. The second statement is proved in the same way by separating the positive and the negative part of the integrals and using the statements from Proposition 5 instead of Proposition 4. □

Proposition 6 *For $\tau_1,...,\tau_k\ge 0$ and $\theta_1,...,\theta_k\in\mathbb{R}$ the two sequences of random variables*

$$n^{\beta-1}\sum_{x\in\mathbb{Z}}\left|\sum_{i=1}^{k}\theta_i\Gamma_n(\tau_i,\{x/n\})\right|^{\beta}\ \textit{and}\ n^{\beta-1}\sum_{x\in\mathbb{Z}}\left(\left|\sum_{i=1}^{k}\theta_i\Gamma_n(\tau_i,\{x/n\})\right|^{\beta}\operatorname{sgn}\left(\sum_{i=1}^{k}\theta_i\Gamma_n(\tau_i,\{x/n\})\right)\right)$$

converge jointly in distribution toward the random variables

$$\int_{-\infty}^{\infty}\left|\sum_{i=1}^{k}\theta_i L_*(\tau_i,x)\right|^{\beta}dx\quad\textit{resp.}\quad\int_{-\infty}^{\infty}\left(\left|\sum_{i=1}^{k}\theta_i L_*(\tau_i,x)\right|^{\beta}\operatorname{sgn}\left(\sum_{i=1}^{k}\theta_i L_*(\tau_i,x)\right)\right)dx.$$

Proof: We know that

$$\{L_*(t,x); t \geq 0, x \in \mathbb{R}\} \stackrel{\mathcal{D}}{=} \left\{\tilde{L}_*(t,x); t \geq 0, x \in \mathbb{R}\right\}$$

and

$$\left\{S_n^{-1}(B_n(V_n^{-1}(t))); t \geq 0\right\} \stackrel{\mathcal{D}}{=} \left\{\tilde{S}_n^{-1}(B(\tilde{V}_n^{-1}(t))); t \geq 0\right\}.$$

Therefore, by Lemma 5 the sequences of random variables

$$n^{\beta-1}\sum_{x\in\mathbb{Z}}\left|\sum_{i=1}^{k}\theta_i\hat{\Gamma}_n(\tau_i,\{x/n\})\right|^{\beta} \text{ and } n^{\beta-1}\sum_{x\in\mathbb{Z}}\left(\left|\sum_{i=1}^{k}\theta_i\hat{\Gamma}_n(\tau_i,\{x/n\})\right|^{\beta}\operatorname{sgn}\left(\sum_{i=1}^{k}\theta_i\hat{\Gamma}_n(\tau_i,\{x/n\})\right)\right)$$

converge jointly in distribution toward the random variables

$$\int_{-\infty}^{\infty}\left|\sum_{i=1}^{k}\theta_i L_*(\tau_i,x)\right|^{\beta}dx \quad \text{resp.} \quad \int_{-\infty}^{\infty}\left(\left|\sum_{i=1}^{k}\theta_i L_*(\tau_i,x)\right|^{\beta}\operatorname{sgn}\left(\sum_{i=1}^{k}\theta_i L_*(\tau_i,x)\right)\right)dx.$$

Moreover, $S_n^{-1}(S_n(x/n)) = (x+1)/n$ for all $x \in \mathbb{Z}$. This implies that

$$\hat{X}_n(\tau) \stackrel{\mathcal{D}}{=} S_n^{-1}(S_n(X_n(\tau))) = X_n(\tau) + 1/n.$$

Hence we have $\hat{\Gamma}_n(\tau,\{x/n\}) \stackrel{\mathcal{D}}{=} \Gamma_n(\tau,\{(x+1)/n\})$ for all $x \in \mathbb{Z}$. Therefore,

$$n^{\beta-1}\sum_{x\in\mathbb{Z}}\left|\sum_{i=1}^{k}\theta_i\hat{\Gamma}_n(\tau_i,\{x/n\})\right|^{\beta} \stackrel{\mathcal{D}}{=} n^{\beta-1}\sum_{x\in\mathbb{Z}}\left|\sum_{i=1}^{k}\theta_i\Gamma_n(\tau_i,\{x/n\})\right|^{\beta}$$

and

$$\begin{aligned} & n^{\beta-1}\sum_{x\in\mathbb{Z}}\left(\left|\sum_{i=1}^{k}\theta_i\hat{\Gamma}_n(\tau_i,\{x/n\})\right|^{\beta}\operatorname{sgn}\left(\sum_{i=1}^{k}\theta_i\hat{\Gamma}_n(\tau_i,\{x/n\})\right)\right) \\ \stackrel{\mathcal{D}}{=} \;& n^{\beta-1}\sum_{x\in\mathbb{Z}}\left(\left|\sum_{i=1}^{k}\theta_i\Gamma_n(\tau_i,\{x/n\})\right|^{\beta}\operatorname{sgn}\left(\sum_{i=1}^{k}\theta_i\Gamma_n(\tau_i,\{x/n\})\right)\right). \end{aligned}$$

This proves the proposition. □

For the sequel we define the occupation time

$$\Gamma(t,A) := \int_0^t \mathbb{1}_A(X(s))ds$$

of the process X in the measurable set $A \subset \mathbb{R}$. Consequently we have

$$\Xi(t) = \sum_x \Gamma(t,\{x\})\xi(x).$$

We will use this fact and the following corollary in the proofs of the next section.

Corollary 2 *For $\tau_1, ..., \tau_k \geq 0$ and $\theta_1, ..., \theta_k \in \mathbb{R}$ the two sequences of random variables*

$$n^{-1-\frac{\beta}{\alpha}} \sum_{x\in\mathbb{Z}} \left| \sum_{i=1}^{k} \theta_i \Gamma(k_n\tau_i, \{x\}) \right|^{\beta} \text{ and } n^{-1-\frac{\beta}{\alpha}} \sum_{x\in\mathbb{Z}} \left(\left| \sum_{i=1}^{k} \theta_i \Gamma(k_n\tau_i, \{x\}) \right|^{\beta} \operatorname{sgn}\left(\sum_{i=1}^{k} \theta_i \Gamma(k_n\tau_i, \{x\}) \right) \right)$$

converge jointly in distribution toward the random variables

$$\int_{-\infty}^{\infty} \left| \sum_{i=1}^{k} \theta_i L_*(\tau_i, x) \right|^{\beta} dx \quad \text{resp.} \quad \int_{-\infty}^{\infty} \left(\left| \sum_{i=1}^{k} \theta_i L_*(\tau_i, x) \right|^{\beta} \operatorname{sgn}\left(\sum_{i=1}^{k} \theta_i L_*(\tau_i, x) \right) \right) dx.$$

Proof: If we put $k_n := n^{\frac{1+\alpha}{\alpha}}$, for all $n \in \mathbb{N}$ and $x \in \mathbb{Z}$ we have that

$$\Gamma_n(\tau, x/n) = \int_0^{\tau} 1\!\!1_{\{x/n\}}(X_n(t))dt = k_n^{-1} \int_0^{k_n\tau} 1\!\!1_{\{x\}}(X(t))dt = n^{-\frac{\alpha+1}{\alpha}} \Gamma(k_n\tau, \{x\}).$$

The result then follows from Proposition 6. □

3 The finite dimensional distributions

In this section we prove the convergence of the finite dimensional distributions of Ξ_n toward the finite dimensional distributions of Ξ_*. In order to do so we first compute the exact expression of the finite dimensional distributions of Ξ_*. The proofs in this section follow the ideas given in Kesten and Spitzer (1979).

In the introduction we defined

$$\Xi_*(\tau) := \int_0^{\infty} L_*(\tau, x-) dZ_+(x) + \int_0^{\infty} L_*(\tau, -(x-)) dZ_-(x),$$

where $\{Z_+(t); t \geq 0\}$ and $\{Z_-(t); t \geq 0\}$ are independent copies of the β-stable Lévy process, which can be associated to the stable distribution ϑ_β with characteristic function given by

$$\psi(\theta) = \exp\left(-|\theta|^{\beta}(A_1 + iA_2 \operatorname{sgn}(\theta) \right).$$

Lemma 6 *For $t_1, ..., t_k \geq 0$ and $\theta_1, ..., \theta_k \in \mathbb{R}$ we have that*

$$\begin{aligned} \mathbb{E}\left[\exp\left(i \sum_{j=1}^{k} \theta_j \Xi_*(t_j) \right) \right] &= \mathbb{E}\left[\exp\left(-A_1 \int_{-\infty}^{\infty} \left| \sum_{j=1}^{k} \theta_j L_*(t_j, x) \right|^{\beta} dx \right) \right. \\ &\quad \left. \exp\left(-iA_2 \int_{-\infty}^{\infty} \left| \sum_{j=1}^{k} \theta_j L_*(t_j, x) \right|^{\beta} dx \operatorname{sgn}\left(\sum_{j=1}^{k} \theta_j L_*(t_j, x) \right) \right) \right]. \end{aligned}$$

Proof: The proof is similar to that given in Kesten and Spitzer (1979 p.16 ff.). Let ν be the Lévy measure of Z_+. One can truncate the Lévy-measure as follows:

$$\nu_1(B) = \nu(B \cap \{y \in \mathbb{R}; |y| \leq 1\}) \quad \text{and} \quad \nu_2(B) = \nu(B \cap \{y \in \mathbb{R}; |y| > 1\}).$$

Let $M(t)$ and $A(t)$ be independent Lévy-processes with characteristic functions

$$\mathbb{E}\left[e^{i\theta M(t)}\right] = \exp\left(t\int_{|y|\leq 1}\left(e^{i\theta y} - 1 - i\theta y\right)\nu_1(dy)\right)$$

resp.

$$\mathbb{E}\left[e^{i\theta A(t)}\right] = \exp\left(t\int_{|y|\leq 1}\left(e^{i\theta y} - 1\right)\nu_2(dy)\right)$$

such that

$$Z^+(t) = M(t) + A(t) + Dt,$$

where D is a suitable real constant. This decomposition exists and is called the Lévy-Itô representation of Z^+. The advantage of this representation is that $M(t)$ is a martingale and has all moments and $A(t)$ is a process with bounded variation. Since the process $\{L_*(t, x-); x \geq 0\}$ is left-continuous and independent with respect to the filtration $\mathcal{F}_t$ generated by $Z^+(t)$, the process $\{L_*(t, x-); x \geq 0\}$ is $\mathcal{F}_t$-predictable. Moreover, $\{L_*(t, x-); x \geq 0\}$ has bounded support $\mathbb{P}$-almost surely. Therefore, we can find a suitable sequence of partitions $\{x_l^{(n)}; l \in \mathbb{N}\}; n \in \mathbb{N}$ with $x_l^{(n)} < x_{l+1}^{(n)}$ for all $l, n \in \mathbb{N}$ satisfying

$$\lim_{l\to\infty} x_l^{(n)} = \infty \quad \text{and} \quad \lim_{n\to\infty}\max_{l\in\mathbb{N}}\left(x_{l+1}^{(n)} - x_l^{(n)}\right) = 0$$

such that

$$\int_0^\infty L_*(t, x-)dM(x) = \lim_{n\to\infty}\sum_{l=1}^\infty L_*(t, x_l^{(n)}-)\left(M(x_{l+1}^{(n)}) - M(x_l^{(n)})\right) \quad \text{with probability 1}$$

(see Meyer (1976) chap. II sec. 23). Moreover, we can also assume that

$$\int_0^\infty L_*(t, x-)dA(x) = \lim_{n\to\infty}\sum_{l=1}^\infty L_*(t, x_l^{(n)}-)\left(A(x_{l+1}^{(n)}) - A(x_l^{(n)})\right) \quad \text{with probability 1.}$$

From those considerations it follows that there exists a sequence of partitions $(x_l^{(n)})_{l\in\mathbb{N}}$ such that

$$\int_0^\infty L_*(t, x-)dZ_+(x) = \lim_{n\to\infty}\sum_{l=1}^\infty L_*(t, x_l^{(n)}-)\left(Z_+(x_{l+1}^{(n)}) - Z_+(x_l^{(n)})\right) \quad \text{with probability 1.}$$

Since the increments $D_l^{(n)} := Z_+(x_{l+1}^{(n)}) - Z_+(x_l^{(n)}),\ l \in \mathbb{N}$ are independent and have characteristic function

$$\mathbb{E}\left[e^{i\theta D_l^{(n)}}\right] = \exp\left(-(x_{l+1}^{(n)} - x_l^{(n)})|\theta|^\beta(A_1 + iA_2\cdot\operatorname{sgn}(\theta))\right).$$

By dominated convergence we have

$$
\begin{aligned}
& \mathbb{E}\left[\exp\left(i\sum_{j=1}^{k}\theta_j\int_0^{\infty}L_*(t_j,x-)dZ_+(x)\right)\right] \\
= & \lim_{n\to\infty}\mathbb{E}\left[\exp\left(\sum_{l=1}^{\infty}\sum_{j=1}^{k}i\theta_j L_*(t_j,x_l^{(n)}-)\left(Z_+(x_{l+1}^{(n)})-Z_+(x_l^{(n)})\right)\right)\right] \\
= & \lim_{n\to\infty}\mathbb{E}\left[\exp\left(-\sum_{l=1}^{\infty}\left(x_{l+1}^{(n)}-x_l^{(n)}\right)\left|\sum_{j=1}^{k}\theta_j L_*(t_j,x_l^{(n)}-)\right|^{\beta}\right.\right. \\
& \left.\left.\times\left(A_1+iA_2\cdot\operatorname{sgn}\left(\sum_{j=1}^{k}\theta_j L_*(t_j,x_l^{(n)}-)\right)\right)\right)\right]. \\
= & \mathbb{E}\left[\exp\left(-A_1\int_0^{\infty}\left|\sum_{j=1}^{k}\theta_j L_*(t_j,x_l^{(n)})\right|^{\beta}dx\right.\right. \\
& \left.\left.-iA_2\int_0^{\infty}\left|\sum_{j=1}^{k}\theta_j L_*(t_j,x_l^{(n)})\right|^{\beta}\operatorname{sgn}\left(\sum_{j=1}^{k}\theta_j L_*(t_j,x_l^{(n)})\right)dx\right)\right].
\end{aligned}
$$

For Z_- one can proceed with similar arguments. □

Proposition 7 *The finite dimensional distributions of the processes* $\{\Xi_n(t);t\geq 0\}$ *converge toward the finite dimensional distributions of the process* $\{\Xi_*(t);t\geq 0\}$.

Proof: As in the previous sections, we denote $k_n := n^{(1+\alpha)/\alpha}$ and $\kappa := \frac{1}{\alpha}+\frac{1}{\beta}$. We already saw that we can use the occupation time $\{\Gamma(t,\{x\});t\geq 0, x\in\mathbb{R}\}$ of the process $\{X(t);t\geq 0\}$ to represent the process $\{\Xi(t);t\geq 0\}$ as follows

$$\Xi(t)=\sum_{x\in\mathbb{Z}}\Gamma(t,\{x\})\xi(x).$$

It follows that

$$\Xi_n(t)=n^{-\kappa}\Xi(k_n t)=n^{-\kappa}\sum_{x\in\mathbb{Z}}\Gamma(k_n t,\{x\})\xi(x).$$

Let $\varphi(\theta):=\mathbb{E}\left[\exp(i\theta\xi(1))\right]$ be the characteristic function of the scenery random variable $\xi(1)$. It then follows from the above representation that

$$\sum_{j=1}^{k}\theta_j\Xi_n(t_j)=n^{-\kappa}\sum_{x\in\mathbb{Z}}\sum_{j=1}^{k}\theta_j\Gamma(k_n t_j,\{x\})\xi(x)$$

and

$$R_n:=\mathbb{E}\left[\exp\left(i\sum_{j=1}^{k}\theta_j\Xi_n(t_j)\right)\right]=\mathbb{E}\left[\prod_{x\in\mathbb{Z}}\varphi\left(n^{-\kappa}\sum_{j=1}^{k}\theta_j\Gamma(k_n t_j,\{x\})\right)\right].$$

The random-scenery $\{\xi(z); z \in \mathbb{Z}\}$ is in the domain of attraction of a β-stable distribution with characteristic function given by

$$\psi(\theta) = \exp(-|\theta|^{\beta}(A_1 + iA_2 \cdot \mathrm{sgn}(\theta)).$$

This implies that

$$1 - \varphi(\theta) \quad \sim \quad |\theta|^{\beta}(A_1 + iA_2 \cdot \mathrm{sgn}(\theta)) \quad \text{as } \theta \to 0.$$

Thus

$$\log(\varphi(\theta)) \sim \log(\psi(\theta)) \quad \text{as } \theta \to 0.$$

Therefore one has for $|\theta| \leq 1$ that

$$\left| \frac{\log(\varphi(\theta)) - \log(\psi(\theta))}{\log(\psi(\theta))} \right| = o(\theta).$$

If we define

$$\varphi_{x,n} := \varphi\left(n^{-\kappa} \sum_{j=1}^{k} \theta_j \Gamma(k_n t_j, \{x\}) \right)$$

and

$$\psi_{x,n} := \exp\left(-n^{-\kappa\beta} \left| \sum_{j=1}^{k} \theta_j \Gamma(k_n t_j, \{x\}) \right|^{\beta} \left(A_1 + iA_2 \cdot \mathrm{sgn}\left(\sum_{j=1}^{k} \theta_j \Gamma(k_n t_j, \{x\}) \right) \right) \right)$$

for all $x \in \mathbb{Z}$ one has

$$\left| \frac{\log(\varphi_{x,n}) - \log(\psi_{x,n})}{\log(\psi_{x,n})} \right| = o\left(n^{-\kappa} \sum_{j=1}^{k} \theta_j \Gamma(k_n t_j, \{x\}) \right).$$

This implies

$$\begin{aligned} \left| \log\left(\prod_{x\in\mathbb{Z}} \varphi_{x,n} \right) - \log\left(\prod_{x\in\mathbb{Z}} \psi_{x,n} \right) \right| &= \left| \sum_{x\in\mathbb{Z}} \log(\varphi_{x,n}) - \sum_{x\in\mathbb{Z}} \log(\psi_{x,n}) \right| \\ &\leq \sum_{x\in\mathbb{Z}} \log(\psi_{x,n}) \; o\left(n^{-\kappa} \sum_{j=1}^{k} \theta_j \Gamma(k_n t_j, \{x\}) \right). \end{aligned}$$

By Corollary 2 the right side of the previous inequality converges toward zero in probability. The continuity of the logarithm then implies that

$$\left| \prod_{x\in\mathbb{Z}} \varphi_{x,n} - \prod_{x\in\mathbb{Z}} \psi_{x,n} \right| \longrightarrow 0 \ \text{ in probability as } n \to \infty.$$

We use this and dominated convergence to prove that the limit of the sequence $\{R_n; n \in \mathbb{N}\}$ exists and is equal to the limit of the following sequence

$$Q_n := \mathbb{E}\left[\exp\left(-\sum_{x\in\mathbb{Z}} n^{-\kappa\beta}\left|\sum_{j=1}^k \theta_j \Gamma(k_n t_j, \{x\})\right|^\beta \left(A_1 + iA_2 \cdot \operatorname{sgn}\left(\sum_{j=1}^k \theta_j \Gamma(k_n t_j, \{x\})\right)\right)\right)\right].$$

By Corollary 2 and Lemma 6 the sequence $\{Q_n; n \in \mathbb{N}\}$ converges toward

$$\begin{aligned} Q_* &:= \mathbb{E}\left[\exp\left(-\int_{-\infty}^{\infty}\left|\sum_{j=1}^k \theta_j L_*(t_j, x)\right|^\beta \left(A_1 + iA_2 \cdot \operatorname{sgn}\left(\sum_{j=1}^k \theta_j L_*(t_j, x)\right)\right) dx\right)\right] \\ &= \mathbb{E}\left[\exp\left(i\sum_{j=1}^k \theta_j \Xi_*(t_j)\right)\right]. \end{aligned}$$

As we have seen in Lemma 6 that Q_* is the characteristic function for the finite dimensional distributions of $\{\Xi_*(t); t \geq 0\}$. This finishes the proof of the proposition. □

4 The tightness

In this section we prove that the sequence $\{\Xi_n(t); t \geq 0\}$ is tight. The proof of Theorem 1 then follows, since we already obtained the convergence of the finite dimensional distributions in the previous section. The main proof for tightness also follows the ideas given in Kesten and Spitzer (1979). In order to do so we first need some suitable inequalities for the occupation times of X_*. However the proofs of those inequalities differ from those given in Kesten and Spitzer (1979).

Lemma 7 *There exists a function $\epsilon : \mathbb{R}^+ \to \mathbb{R}^+$ with the properties $\epsilon(A) \to 0$ as $A \to \infty$ and*

$$\mathbb{P}\left(\Gamma(s, \{x\}) > 0 \text{ for some } x \text{ with } |x| > As^{\frac{\alpha}{1+\alpha}}\right) \leq \epsilon(A) \quad \text{for all } s \geq 0.$$

Proof: For a positive real number x we denote by $\lceil x \rceil$ the smallest integer which is larger or equal to x. Obviously for all $s \geq 0$ we have

$$\begin{aligned} & \mathbb{P}\left(\Gamma(s, \{x\}) > 0 \text{ for some } x \text{ with } |x| > As^{\frac{\alpha}{1+\alpha}}\right) \\ \leq\ & \mathbb{P}\left(|X(r)| > As^{\frac{\alpha}{1+\alpha}} \text{ for some } r \leq s\right) \\ \leq\ & \mathbb{P}\left(|X(r)| > A\left(\lceil s^{\frac{\alpha}{1+\alpha}}\rceil - 1\right) \text{ for some } r \leq \lceil s^{\frac{\alpha}{1+\alpha}}\rceil^{\frac{1+\alpha}{\alpha}}\right) \\ =\ & \mathbb{P}\left(\left|X\left(\lceil s^{\frac{\alpha}{1+\alpha}}\rceil^{\frac{1+\alpha}{\alpha}} u\right)\right| > A\lceil s^{\frac{\alpha}{1+\alpha}}\rceil - A \text{ for some } u \leq 1\right) \\ \leq\ & \mathbb{P}\left(\sup_{r\leq 1} |X_{n(s)}(r)| > A/2\right) \quad \text{for } s > 1 \end{aligned}$$

with $n(s) := \lceil s^{\frac{\alpha}{1+\alpha}}\rceil \to \infty$ as $s \to \infty$. Since

$$\mathbb{P}\left(\sup_{r\leq 1}|X_n(r)| > A/2\right) \longrightarrow \mathbb{P}\left(\sup_{r\leq 1}|X_*(r)| > A/2\right) \quad \text{as } n \to \infty,$$

we can define

$$\epsilon(A) := \sup_{s\geq 0} \mathbb{P}\left(\sup_{r\leq 1} |X_{n(s)}(r)| > A/2\right) \quad \text{for all } A > 0.$$

This proves the statement of the lemma. $\square$

Lemma 8 *There exists a $C > 0$ such that for all $s \geq 0$ one has*

$$\sum_{x\in\mathbb{Z}} \mathbb{E}\left[\Gamma^2(s,\{x\})\right] \sim C s^{2-\frac{\alpha}{1+\alpha}}.$$

Proof: For a positive real number x we denote by $\lfloor x \rfloor$ its integer part. We know that for $w(s) := \lfloor s^{\frac{\alpha}{\alpha+1}} \rfloor$ one has

$$\frac{(w(s))^{2\frac{\alpha+1}{\alpha}}}{s^2} \sum_{x\in\mathbb{Z}} \Gamma^2_{w(s)}(1,\{x/w(s)\}) = s^{-2} \sum_{x\in\mathbb{Z}} \Gamma^2\big((w(s))^{\frac{\alpha+1}{\alpha}},\{x\}\big) \leq s^{-2} \sum_{x\in\mathbb{Z}} \Gamma^2(s,\{x\})$$

and

$$\begin{aligned} s^{-2} \sum_{x\in\mathbb{Z}} \Gamma^2(s,\{x\}) &\leq s^{-2} \sum_{x\in\mathbb{Z}} \Gamma^2\big((w(s)+1)^{\frac{\alpha+1}{\alpha}},\{x\}\big) \\ &= \frac{(w(s)+1)^{2\frac{\alpha+1}{\alpha}}}{s^2} \sum_{x\in\mathbb{Z}} \Gamma^2_{w(s)+1}(1,\{x/(w(s)+1)\}) \end{aligned}$$

Consequently one has

$$s^{-2} \sum_{x\in\mathbb{Z}} \mathbb{E}\left[\Gamma^2(s,\{x\})\right] \sim \sum_{x\in\mathbb{Z}} \mathbb{E}\left[\Gamma^2_{w(s)}(1,\{x/w(s)\})\right] = \sum_{x\in\mathbb{Z}} \mathbb{E}\left[\tilde{\Gamma}^2_{w(s)}(1,\{x/w(s)\})\right].$$

It follows from the layer cake representation and the remark after the proof of Proposition 5 that

$$w(s) \sum_{x\in\mathbb{Z}} \tilde{\Gamma}^2_{w(s)}(1,\{x/w(s)\}) = \frac{1}{w(s)} \int_0^\infty \operatorname{card}\left\{x\in\mathbb{Z} : w^2(s)\tilde{\Gamma}^2_{w(s)}(1,\{x/w(s)\}) > c\right\} dc$$

converges $\mathbb{P}\times\tilde{\mathbb{P}}$-almost surely toward

$$\int_0^\infty \ell\left(x\in\mathbb{R} : \tilde{L}^2(1,x) > c\right) dc = \int_{\mathbb{R}} \tilde{L}^2_*(1,x)dx.$$

Dominated convergence and Fubini theorems imply that

$$w(s) \sum_{x\in\mathbb{Z}} \mathbb{E}\left[\tilde{\Gamma}^2_{w(s)}(1,\{x/w(s)\})\right] \longrightarrow \int_{\mathbb{R}} \mathbb{E}\left[\tilde{L}^2_*(1,x)\right] dx \quad \text{as } s\to\infty.$$

Therefore

$$w(s)s^{-2} \sum_{x\in\mathbb{Z}} \mathbb{E}\left[\Gamma^2(s,\{x\})\right] \longrightarrow \int_{\mathbb{R}} \mathbb{E}\left[\tilde{L}^2_*(1,x)\right] dx \quad \text{as } s\to\infty.$$

This proves the statement of the lemma. $\square$

Lemma 9 *1) For all $\beta \in (0,2]$ and $\rho > 0$ there exists a $C_1 > 0$ such that as $n \to \infty$ one has*

$$\left|\mathbb{E}\left[\xi(0)\mathbb{1}_{[-\rho,\rho]}(n^{-\frac{1}{\beta}}\xi(0))\right]\right| \sim C_1 n^{\frac{1-\beta}{\beta}}.$$

2) For all $\beta \in (0,2)$ and $\rho > 0$ there exists a $C_2 > 0$ such that as $n \to \infty$ one has

$$\left|\mathbb{E}\left[\xi^2(0)\mathbb{1}_{[-\rho,\rho]}(n^{-\frac{1}{\beta}}\xi(0))\right]\right| \sim C_2 n^{\frac{2-\beta}{\beta}}.$$

Proof: The random variable $\xi(0)$ is in the domain of attraction of a β-stable random variable with characteristic function given by

$$\psi(\theta) = \exp(-|\theta|^\beta(A_1 + iA_2\mathrm{sgn}(\theta)),$$

with $0 < A_1 < \infty$ and $|A_1^{-1}A_2| \leq \tan(\pi\beta/2)$. A consequence of this setting is that for $\beta > 1$ one has $\mathbb{E}[\xi(0)] = 0$. Further, if $\beta \in (0,2]$ then there exist $B_1, B_2 \geq 0$ such that

$$\lim_{\rho\to\infty} \rho^\beta \mathbb{P}(\xi(0) \geq \rho) = B_1 \quad \text{and} \quad \lim_{\rho\to\infty} \rho^\beta \mathbb{P}(\xi(0) \leq -\rho) = B_2.$$

For $\beta = 2$ we have $B_1 = B_2 = 0$ since the decay of the tail-probabilities is exponential in that case. For $\beta \neq 1$ we then have that

$$\begin{aligned}\left|\mathbb{E}\left[\xi(0)\mathbb{1}_{[-\rho,\rho]}(n^{-\frac{1}{\beta}}\xi(0))\right]\right| &= \int_0^{\rho n^{\frac{1}{\beta}}} \mathbb{P}(|\xi(0)| \geq c)dc \\ &\sim (B_1+B_2)\int_0^{\rho n^{\frac{1}{\beta}}} c^{-\beta}dc \\ &= (B_1+B_2)(1-\beta)^{-1}\rho^{1-\beta}n^{\frac{1}{\beta}(1-\beta)}.\end{aligned}$$

This proves the first statement for $\beta \neq 1$. For $\beta = 1$ the statement is just our assumption from the introduction.

Moreover, by similar arguments for $\beta \neq 2$ we have that

$$\begin{aligned}\left|\mathbb{E}\left[\xi^2(0)\mathbb{1}_{[-\rho,\rho]}(n^{-\frac{1}{\beta}}\xi(0))\right]\right| &\sim (B_1+B_2)\int_0^{\rho n^{\frac{1}{\beta}}} c^{1-\beta}dc \\ &= (B_1+B_2)(2-\beta)^{-1}\rho^{2-\beta}n^{\frac{1}{\beta}(2-\beta)}.\end{aligned}$$

This finishes the proof of the second statement. $\square$

Proposition 8 *The distributions of the sequence $\{\Xi_n; n \in \mathbb{N}\}$ are tight with respect to the Skorohod topology.*

Proof: We follow the method given in Kesten and Spitzer (1979). Let $\epsilon > 0$ be given. By Lemma 7 there exists an $A > 0$ such that $\epsilon\left(AT^{-\frac{\alpha}{1+\alpha}}\right) \leq \epsilon/4$. This implies that

$$\begin{aligned}&\mathbb{P}\left(\Xi_n(t) \neq n^{-\kappa}\sum_{|x|\leq An}\Gamma(k_nt,\{x\})\xi(x) \text{ for some } t \leq T\right) \\ \leq\ &\mathbb{P}\left(\Gamma(k_nT,\{x\}) > 0 \text{ for some } x \text{ with } |x| > Ak_n^{\frac{\alpha}{1+\alpha}}\right) \\ \leq\ &\epsilon\left(AT^{-\frac{\alpha}{1+\alpha}}\right) \\ \leq\ &\epsilon/4.\end{aligned}$$

There exists a $\rho_0 > 0$ with the property that for all $\rho > \rho_0$ and all $n \in \mathbb{N}$ one has

$$3An(1 - \mathbb{P}\left(-\rho n^{\frac{1}{\beta}} \leq \xi(0) \leq \rho n^{\frac{1}{\beta}}\right) \leq \epsilon/4.$$

This is valid since for suitable $B_1, B_2 \geq 0$ we have

$$\lim_{\rho\to\infty} \rho^\beta \mathbb{P}(\xi(0) \geq \rho) = B_1 \quad \text{and} \quad \lim_{\rho\to\infty} \rho^\beta \mathbb{P}(\xi(0) \leq -\rho) = B_2.$$

We define for all $x \in \mathbb{Z}$ the random variables

$$\bar{\xi}_n(x) := \xi(x)\mathbb{1}_{[-\rho,\rho]}(n^{-\frac{1}{\beta}}\xi(x))$$

and

$$E_n := n^{-\kappa}\frac{1}{T}\mathbb{E}\left[\sum_{x\in\mathbb{Z}}\Gamma(k_nt,\{x\})\bar{\xi}_n(x)\right] = n^{-\kappa}\frac{1}{T}\mathbb{E}\left[\sum_{x\in\mathbb{Z}}\Gamma(k_nt,\{x\})\mathbb{E}\left[\bar{\xi}_n(x)\right]\right]$$

and

$$\bar{\Xi}_n(t) := n^{-\kappa}\sum_{x\in\mathbb{Z}}\Gamma(k_nt,\{x\})\left(\bar{\xi}_n(x) - \mathbb{E}\left[\bar{\xi}_n(x)\right]\right).$$

Claim 1) The family of random variables $\{E_n(t); n \in \mathbb{N}\}$ is bounded. This is true, since by Lemma 9 we have

$$\left|\sum_{x\in\mathbb{Z}}\Gamma(k_nt,\{x\})\mathbb{E}\left[\bar{\xi}_n(x)\right]\right| = \left|\mathbb{E}\left[\bar{\xi}_n(0)\right]\right|\sum_{x\in\mathbb{Z}}\Gamma(k_nt,\{x\}) = k_nt\left|\mathbb{E}\left[\bar{\xi}_n(0)\right]\right| \leq Ctn^{\frac{\alpha+1}{\alpha}}n^{\frac{1}{\beta}(1-\beta)}$$

and $\frac{\alpha+1}{\alpha} + \frac{1}{\beta}(1-\beta) - \kappa = 0$.
Claim 2) For all $\eta > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ one has

$$\mathbb{P}\left(\sup_{t\leq T}|\Xi_n(t) - \bar{\Xi}_n(t) - E_nt| > \frac{\eta}{2}\right) \leq \frac{\epsilon}{2}.$$

To see this, we first note that

$$\Xi_n(t) - \bar{\Xi}_n(t) - E_nt \quad = \quad n^{-\kappa}\sum_{x\in\mathbb{Z}}\Gamma(k_nt,\{x\})\left(\xi(x) - \bar{\xi}_n(x)\right),$$

since

$$\begin{aligned}
&\Xi_n(t) - \bar{\Xi}_n(t) - E_nt - n^{-\kappa}\sum_{x\in\mathbb{Z}}\Gamma(k_nt,\{x\})\left(\xi(x) - \bar{\xi}_n(x)\right)\\
=\;& n^{-\kappa}\left(\sum_{x\in\mathbb{Z}}\Gamma(k_nt,\{x\})\mathbb{E}\left[\bar{\xi}(x)\right] - \frac{t}{T}\mathbb{E}\left[\sum_{x\in\mathbb{Z}}\Gamma(k_nt,\{x\})\mathbb{E}\left[\bar{\xi}(x)\right]\right]\right)\\
=\;& n^{-\kappa}\mathbb{E}\left[\bar{\xi}(0)\right]\left(\sum_{x\in\mathbb{Z}}\Gamma(k_nt,\{x\}) - \frac{t}{T}\mathbb{E}\left[\sum_{x\in\mathbb{Z}}\Gamma(k_nt,\{x\})\right]\right)\\
=\;& n^{-\kappa}\mathbb{E}\left[\bar{\xi}(0)\right]\left(k_nt - \frac{t}{T}k_nT\right)\\
=\;& 0.
\end{aligned}$$

Lemma 9 implies that

$$\begin{aligned}
& \mathbb{P}\left(n^{-\kappa}\sum_{x\in\mathbb{Z}}\Gamma(k_nt,\{x\})\left(\xi(x)-\bar{\xi}_n(x)\right)\neq 0 \text{ for some } t\leq T\right) \\
\leq\ & \mathbb{P}\left(\Gamma(k_nT,\{x\})>0 \text{ for some } x \text{ with } |x|>Ak_n^{\frac{\alpha}{1+\alpha}}\right) \\
& +\mathbb{P}\left(\xi(x)\neq\bar{\xi}_n(x) \text{ for some } |x|\leq Ak_n^{\frac{\alpha}{1+\alpha}}\right) \\
\leq\ & \epsilon\left(AT^{-\frac{\alpha}{1+\alpha}}\right)+3Ak_n^{\frac{\alpha}{1+\alpha}}\mathbb{P}\left(\xi(0)\neq\bar{\xi}_n(0)\right) \\
\leq\ & \frac{\epsilon}{4}+3An\left(1-\mathbb{P}\left(-\rho n^{\frac{1}{\beta}}\leq\xi(0)\leq\rho n^{\frac{1}{\beta}}\right)\right) \\
\leq\ & \frac{\epsilon}{2}.
\end{aligned}$$

Claim 3) There exists a $K_0>0$ such that for all $n\in\mathbb{N}$ one has

$$\mathbb{E}\left[\left|\bar{\Xi}_n(t_2)-\bar{\Xi}_n(t_1)\right|^2\right]\leq C_0(t_2-t_1)^{2-\frac{1+\alpha}{\alpha}}.$$

We define the σ-field $\mathcal{X}=\{X(t);t\geq 0\}$. Then it follows from the independence of $\{X(t);t\geq 0\}$ and $\{\xi(x);x\in\mathbb{Z}\}$ that

$$\begin{aligned}
& \mathbb{E}\left[\left(\sum_{x\in\mathbb{Z}}(\Gamma(k_nt_2,\{x\})-\Gamma(k_nt_1,\{x\}))\bar{\xi}_n(x)\right)^2\right] \\
=\ & \mathbb{E}\left[\mathbb{E}\left[\left(\sum_{x\in\mathbb{Z}}(\Gamma(k_nt_2,\{x\})-\Gamma(k_nt_1,\{x\}))\bar{\xi}_n(x)\right)^2\middle|\,\mathcal{X}\right]\right] \\
=\ & \mathbb{E}\left[\sum_{x\in\mathbb{Z}}(\Gamma(k_nt_2,\{x\})-\Gamma(k_nt_1,\{x\}))^2\mathbb{E}\left[\bar{\xi}_n^2(x)\middle|\,\mathcal{X}\right]\right] \\
=\ & \sum_{x\in\mathbb{Z}}\mathbb{E}\left[(\Gamma(k_nt_2,\{x\})-\Gamma(k_nt_1,\{x\}))^2\right]\mathbb{E}\left[\bar{\xi}_n^2(x)\right]
\end{aligned}$$

This implies

$$\begin{aligned}
\mathbb{E}\left[\left|\bar{\Xi}_n(t_2)-\bar{\Xi}_n(t_1)\right|^2\right] &\leq n^{-2\kappa}\sum_{x\in\mathbb{Z}}\mathbb{E}\left[(\Gamma(k_nt_2,\{x\})-\Gamma(k_nt_1,\{x\}))^2\right]\mathbb{E}\left[\bar{\xi}_n^2(x)\right] \\
&= n^{-2\kappa}\mathbb{E}\left[\sum_{x\in\mathbb{Z}}(\Gamma(k_nt_2,\{x\})-\Gamma(k_nt_1,\{x\}))^2\right]\mathbb{E}\left[\bar{\xi}_n^2(0)\right].
\end{aligned}$$

Conditioned on $\mathcal{A}:=\{\lambda_i;i\in\mathbb{Z}\}$ the process X has the strong Markov-property. Using this one can prove that for $t_1\leq t_2$ the conditional distribution of $\sum_x(\Gamma(t_2,\{x\})-\Gamma(t_1,\{x\}))^2$ with respect to $\mathcal{A}$ equals the conditional distribution of $\sum_x \Gamma^2(t_2-t_1,\{x\})$ with respect to $\mathcal{A}$. Hence

$$\begin{aligned}
\mathbb{E}\left[\sum_{x\in\mathbb{Z}}(\Gamma(t_2,\{x\})-\Gamma(t_1,\{x\}))^2\right] &= \mathbb{E}\left[\mathbb{E}\left[\sum_{x\in\mathbb{Z}}(\Gamma(t_2,\{x\})-\Gamma(t_1,\{x\}))^2\middle|\mathcal{A}\right]\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[\sum_{x\in\mathbb{Z}}\Gamma^2(t_2-t_1,\{x\})\middle|\mathcal{A}\right]\right]\\
&= \mathbb{E}\left[\sum_{x\in\mathbb{Z}}\Gamma^2(t_2-t_1,\{x\})\right].
\end{aligned}$$

By Lemma 8 it follows that

$$\begin{aligned}
\mathbb{E}\left[\sum_{x\in\mathbb{Z}}(\Gamma(k_nt_2,\{x\})-\Gamma(k_nt_1,\{x\}))^2\right] &\leq Ck_n^{2-\frac{\alpha}{1+\alpha}}(t_2-t_1)^{2-\frac{\alpha}{1+\alpha}}\\
&= Cn^{2\frac{1+\alpha}{\alpha}-1}(t_2-t_1)^{2-\frac{\alpha}{1+\alpha}}.
\end{aligned}$$

Moreover, we know that

$$\mathbb{E}\left[\bar{\xi}_n^2(0)\right]\leq \tilde{C}n^{(2-\beta)\frac{1}{\beta}}.$$

Altogether, we obtain

$$\mathbb{E}\left[\left|\bar{\Xi}_n(t_2)-\bar{\Xi}_n(t_1)\right|^2\right]\leq C_0n^{(2-\beta)\frac{1}{\beta}}n^{-2\kappa}n^{2\frac{1+\alpha}{\alpha}-1}(t_2-t_1)^{2-\frac{\alpha}{1+\alpha}}.$$

Since $(2-\beta)\frac{1}{\beta}-2\kappa+2\frac{1+\alpha}{\alpha}-1=0$, the claim 3 follows.
Since $2-\frac{\alpha}{1+\alpha}>1$ the tightness in the Skorohod topology of the family $\{\Xi_n; n\in\mathbb{N}\}$ now follows from the claims 1, 2, 3 and a theorem from Billingsley (1968) (see p.95). □

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