September 28, 2009 LEAST SQUARES ESTIMATION FOR A PERIODIC MEAN REVERSION PROCESS

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ABSTRACT. In this paper we propose a least squares estimator for the drift parameters of a generalized Ornstein-Uhlenbeck process

 $dX_t = (L(t) - \alpha X_t) dt + \sigma dB_t, \quad t \ge 0.$

The estimator is defined as limit of ordinary least squares estimators for a time discretized version of this stochastic differential equation. We give an explicit formula for the time-continuous least squares estimator. In the case of a periodic mean reversion function L(t), we prove consistency and asymptotic normality of our estimator.

1. INTRODUCTION

The Ornstein-Uhlenbeck process is defined as solution of the stochastic differential equation

$$dX_t = \alpha(\mu - X_t)dt + \sigma dB_t, \quad t \ge 0,$$

where α and σ are positive constants, $\mu \in \mathbb{R}$ and $(B_t)_{t\geq 0}$ Brownian motion. Originally introduced by Ornstein and Uhlenbeck (1932) as a model for particle motion in a fluid, this process is now widely used in many areas of application. The main characteristic of the Ornstein-Uhlenbeck process is the tendency to return towards the long-term equilibrium μ . This property, known as *mean-reversion*, is found in many real life processes, e.g. in commodity and energy price processes, see e.g. Geman (2005).

In many real-life applications, however, the assumption of a constant mean level is not adequate due to seasonality patterns or a long-term trend of the process. Thus we propose the following generalized Ornstein-Uhlenbeck process, defined as solution to the stochastic differential equation

(1)
$$dX_t = (L(t) - \alpha X_t)dt + \sigma dB_t, \quad t \ge 0,$$

where L(t) is a time-dependent mean reversion level and where α, σ are positive constants. Note that model (1) differs from the original Ornstein-Uhlenbeck process in the position of α within the drift term. However, model (1) can easily be transformed to a process with drift term $\alpha(\tilde{L}(t) - X_t)dt$ where $\tilde{L}(t) = L(t)/\alpha$. The advantage of (1) compared with the process provided with the drift $\alpha(L(t) - X_t)dt$ is the simplification of the study of the estimators.

In this paper we make a parametric model for the mean reversion function L(t). We assume that

(2)
$$L(t) = \sum_{i=1}^{p} \mu_i \varphi_i(t)$$

where the basis functions $\varphi_1(t), \ldots, \varphi_p(t)$ are known and μ_1, \ldots, μ_p are unknown parameters. In addition, the mean reversion rate α is assumed to be unknown. In contrast, the

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diffusion parameter σ is assumed to be known which is a common assumption in the field of drift parameter estimation for a time-continuous diffusion. This is due to the fact that the measures corresponding to different diffusion parameters are singular so that σ can be computed from a single continuous-time observation.

Parameter estimation for continuously observed diffusion processes is a well-established area of research, for which a variety of techniques have been proposed. Kutoyants [8] describes several estimation techniques for ergodic time-homogenous diffusion processes. The trajectory fitting estimator, originally introduced by Kutoyants [7], has been extended to certain non-ergodic time-homogenous diffusions by Dietz [1] and by Hu et al. [4]. The trajectory fitting estimator is based on the least squares fitting of the parameters applied to the integral form of the stochastic differential equation (1),

(3)
$$X_t = X_0 + \int_0^t (L(s) - \alpha X_s) ds + \sigma B_t,$$

by solving the minimization problem

$$\theta_{TF} := \operatorname{argmin}_{\theta \in \Theta} \int_0^T \left(X_t - X_0 - \int_0^t (L(s) - \alpha X_s) ds \right)^2 dt$$

where $\theta = (\mu_1, \ldots, \mu_p, \alpha)^t$. In contrast with this, we propose a least squares method directly applied to the stochastic differential equation (1). A formal definition will be based on a least squares estimation in a discretized version of (1), followed by a limit as the discretization step converges to 0. We show that this estimator can be explicitly computed and we establish the asymptotic distribution in the case of periodic mean reversion function.

2. TIME-CONTINUOUS LEAST SQUARES ESTIMATOR

In this section we introduce a least squares estimator for the parameters of a generalized Ornstein-Uhlenbeck process, based on a time-continuous observation scheme. We first discretize the Ornstein-Uhlenbeck process, consider the least squares estimator for the discretized process and then take the limit as the discretization step converges to zero. The resulting estimator can be viewed as a time-continuous version of the least squares method.

The stochastic differential equation (1) can be discretized to the difference equation

(4)
$$X_{(i+1)\Delta t} - X_{i\Delta t} = \left(\sum_{j=1}^{p} \mu_j \varphi_j(i\Delta t) - \alpha X_{i\Delta t}\right) \Delta t + \sigma (B_{(i+1)\Delta t} - B_{i\Delta t}), \quad i = 1, \dots, N,$$

where $N = \lfloor T/\Delta t \rfloor - 1$ and where Δt denotes the constant time increment. Here $\lfloor x \rfloor$ denotes the integer part of x. The structure of (4) is similar to that of the classical linear model, given by $Y_i = \sum_{j=1}^p \beta_j b_{ij} + \varepsilon_i$, $i = 1, \ldots, n$, where $Y = (Y_1, \ldots, Y_n)^t$ is the vector of observations, $B = (b_{ij})$ the design matrix, $\beta = (\beta_1, \ldots, \beta_p)^t$ the vector of unknown parameters and where ε_i are the error terms. The least squares estimator for β is based on the minimization of the functional $(\beta_1, \ldots, \beta_p)^t \mapsto \sum_{i=1}^n (Y_i - \sum_{j=1}^p \beta_j b_{ij})^2$, and is explicitly given by the formula $\hat{\beta} = (B^t B)^{-1} B^t Y$. The main distinction between (4) and the standard linear model lies in the fact that the right of (4) depends on $X_{i\Delta t}$ which is an observed element of the data. Nevertheless, this idea leads to an adequate estimator, as will be shown in Section 3. Analogy with the classical linear model leads to the task of minimization of the functional

$$q(\mu_1,\ldots,\mu_p,\alpha)^t := \sum_{i=0}^N \left(X_{(i+1)\Delta t} - X_{i\Delta t} - \left(\sum_{j=1}^p \mu_j \varphi_j(i\Delta t) - \alpha X_{i\Delta t}\right) \Delta t \right)^2.$$

Lemma 2.1. The solution vector $\hat{\theta}_{\Delta t} = \hat{\theta} = (\hat{\mu}_1, \dots, \hat{\mu}_p, \hat{\alpha})$ of the minimization problem $q(\theta) \to \min$ is given by

$$\hat{\theta}_{\Delta t} = Q_{T,\Delta t}^{-1} P_{T,\Delta t},$$

where $Q_{T,\Delta t} \in \mathbb{R}^{(p+1)\times(p+1)}$ and $P_{T,\Delta t} \in \mathbb{R}^{p+1}$ are defined as

$$Q_{T,\Delta t} = \Delta t \begin{pmatrix} G_{T,\Delta t} & a_{T,\Delta t} \\ a_{T,\Delta t}^t & b_{T,\Delta t} \end{pmatrix}, \quad P_{T,\Delta t} = \begin{pmatrix} \sum_{i=0}^N \varphi_1(i\,\Delta t)(X_{(i+1)\Delta t} - X_{i\Delta t}) \\ \vdots \\ \sum_{i=0}^N \varphi_p(i\,\Delta t)(X_{(i+1)\Delta t} - X_{i\Delta t}) \\ -\sum_{i=0}^N X_{i\,\Delta t}(X_{(i+1)\Delta t} - X_{i\Delta t}) \end{pmatrix},$$

and where $G_{T,\Delta t} = \left(\sum_{i=0}^{N} \varphi_j(i\,\Delta t)\varphi_k(i\,\Delta t)\right)_{1\leq j,k\leq p} \in \mathbb{R}^{p\times p}, \ b_{T,\Delta t} = \sum_{i=0}^{N} X_{i\Delta t}^2 \text{ and } a_{T,\Delta t} = -\left(\sum_{i=0}^{N} \varphi_1(i\,\Delta t)X_{i\,\Delta t}, \dots, \sum_{i=0}^{N} \varphi_p(i\,\Delta t)X_{i\,\Delta t}\right)^t.$

Proof. By general theory of least squares estimation in linear models, the solution to the minimization problem $q(\theta) \rightarrow \min$ is given by

(5)
$$\hat{\theta} = (A^t A)^{-1} A^t D$$

where $\theta = (\mu_1, \ldots, \mu_p, \alpha)^t$ and where

$$A = \Delta t \begin{pmatrix} \varphi_1(0) & \dots & \varphi_p(0) & -X_0 \\ \varphi_1(\Delta t) & \dots & \varphi_p(\Delta t) & -X_{\Delta t} \\ \vdots & \ddots & \vdots & \vdots \\ \varphi_1(N\Delta t) & \dots & \varphi_p(N\Delta t) & -X_{N\Delta t} \end{pmatrix}, \quad D = \begin{pmatrix} X_{\Delta t} - X_0 \\ X_{2\Delta t} - X_{\Delta t} \\ \vdots \\ X_{(N+1)\Delta t} - X_{N\Delta t} \end{pmatrix}.$$

Hence, the products in equation (5) can be calculated to be

$$A^t D = \Delta t P_{T,\Delta t}$$

and

$A^{t}A = (\Delta t)^{2} \begin{pmatrix} G_{T,\Delta t} & a_{T,\Delta t} \\ a_{T,\Delta t}^{t} & b_{T,\Delta t} \end{pmatrix} = \Delta t Q_{T,\Delta t}.$

Thus we get $\hat{\theta} = (A^t A)^{-1} A^t D = Q_{T,\Delta t}^{-1} P_{T,\Delta t}$.

Now a continuous-time estimator can be derived from the least squares estimator by considering $\Delta t \to 0$. Regarding the entries of $P_{T,\Delta t}$ as $\Delta t \to 0$ we obtain

$$\sum_{i=0}^{N} \varphi_j(i\Delta t) \cdot (X_{(i+1)\Delta t} - X_{i\Delta t}) \to \int_0^T \varphi_j(t) dX_t$$

and

$$\sum_{i=0}^{N} X_{i\Delta t} \cdot (X_{(i+1)\Delta t} - X_{i\Delta t}) \to \int_{0}^{T} X_{t} dX_{t}$$

Note that both integrals are Itô type integrals and that the convergence holds in L_2 . For the entries of $Q_{T,\Delta t}$ we have (as $\Delta t \to 0$):

$$\sum_{i=0}^{N} \varphi_l(i\Delta t)\varphi_m(i\Delta t)\Delta t \to \int_0^T \varphi_l(t)\varphi_m(t)dt,$$
$$\sum_{i=0}^{N} \varphi_j(i\Delta t)X_{i\Delta t}\Delta t \to \int_0^T \varphi_j(t)X_tdt$$

and

$$\sum_{i=0}^{N} X_{i\Delta t}^2 \Delta t \to \int_0^T X_t^2 dt.$$

We have thus proved the following proposition.

Proposition 2.2. As $\Delta t \to 0$, the least squares estimator $\hat{\theta}_{\Delta t}$ converges to $Q_T^{-1}P_T$, where $Q_T \in \mathbb{R}^{(p+1)\times(p+1)}$ and $P_T \in \mathbb{R}^{p+1}$ are defined as

(6)
$$Q_T = \begin{pmatrix} G_T & a_T \\ a_T^t & b_T \end{pmatrix},$$

(7)
$$P_T = \left(\int_0^T \varphi_1(t) dX_t, \dots, \int_0^T \varphi_p(t) dX_t, -\int_0^T X_t dX_t\right)^t$$

and where $G_T = (\int_0^T \varphi_j(t)\varphi_k(t)dt)_{1 \le j,k \le p} \in \mathbb{R}^{p \times p}$, $a_T = -(\int_0^T \varphi_1(t)X_t dt, \dots, \int_0^T \varphi_p(t)X_t dt)^t$, and $b_T = \int_0^T X_t^2 dt$.

Definition 2.3. We define the continuous-time least squares estimator $\hat{\theta}_{LS}$ for θ by

(8)
$$\hat{\theta}_{LS} := Q_T^{-1} P_T,$$

where Q_T and P_T are defined as in Proposition 2.2.

Remark. Note that our continuous least squares estimator is equal to the classical maximum likelihood estimator in the case of the generalized mean reversion model introduced in (1). The reason for that identity is the fact that the likelihood ratio \mathcal{L} of a general diffusion process

$$dX_t = S(\theta, t, X_t)dt + dB_t, \quad 0 \le t \le T,$$

has the following form, see Kutoyants [6], for example:

$$\mathcal{L}(\theta, X_t) = \exp\left(\int_0^T S(\theta, t, X_t) dX_t - \frac{1}{2} \int_0^T S(\theta, t, X_t)^2 dt\right).$$

The maximum likelihood estimator is defined as the maximum of the functional

$$\theta \mapsto \mathcal{L}(\theta, X_t)$$

and the partial derivatives of the logarithm of this likelihood ratio are

(9)
$$\frac{\partial}{\partial \theta_i} \ln(\mathcal{L}(\theta, X_t)) = \int_0^T \frac{\partial}{\partial \theta_i} S(\theta, t, X_t) dX_t - \int_0^T S(\theta, t, X_t) \frac{\partial}{\partial \theta_i} S(\theta, t, X_t) dt.$$

The drift function of our generalized mean reversion model is given by

$$S(\theta, t, X_t) = \sum_{i=1}^{p} \mu_i \varphi_i(t) - \alpha X_t$$

such that

$$\frac{\partial}{\partial \theta_i} S(\theta, t, X_t) = \begin{cases} \varphi_i(t), & \text{for } \theta_i = \mu_i, i = 1 \dots, p; \\ -X_t, & \text{for } \theta_i = \alpha. \end{cases}$$

Setting the partial derivatives of the likelihood ratio in (9) equal to zero yields the maximum likelihood estimator which can be seen to be equal to the least squares estimator presented in (8). In summary, we have the same result as in the ordinary linear model, namely the fact that the maximum likelihood and the least squares methodology provide the same estimator.

3. LEAST SQUARES ESTIMATION FOR A PERIODIC MEAN REVERSION FUNCTION

In many applications, the data display regular seasonal effects. These can be modeled by assuming that the mean-reversion function L(t) is periodic, i.e. that

(10)
$$L(t+\nu) = L(t)$$

where ν is the period observed in the data. The resulting stochastic process exhibits a cyclical evolution due to the periodicity of this mean reversion mechanism. Combining the assumption of periodicity with the parametric model (2) leads to the requirement

(11)
$$\varphi_j(t+\nu) = \varphi_j(t).$$

By applying Gram-Schmidt orthogonalization, we may assume without loss of generality that $\varphi_1(t), \ldots, \varphi_p(t)$ form an orthonormal system in $L_2([0, \nu], \frac{1}{\nu}d\lambda)$, i.e. that

(12)
$$\int_0^\nu \varphi_j(t)\varphi_k(t)dt = \begin{cases} \nu, & j=k\\ 0, & j\neq k \end{cases}$$

In the rest of this paper we will assume that we observe an integral multiple of the period length, i.e. that

 $T = N \nu$,

for some integer N. Moreover, we will assume without loss of generality that $\nu = 1$.

Under the above assumptions, the matrix Q_T , defined in (6), simplifies to

(13)
$$Q_T = \begin{pmatrix} T I_{p \times p} & a_T \\ a_T^t & b_T \end{pmatrix}$$

The inverse of a matrix Q of this special form can be explicitly computed by the following lemma.

Lemma 3.1. The inverse of the matrix Q_T , given in (13), is given by

(14)
$$Q_T^{-1} = \frac{1}{T} \begin{pmatrix} I_p + \gamma_T \Lambda_T \Lambda_T^t & -\gamma_T \Lambda_T \\ -\gamma_T \Lambda_T^t & \gamma_T \end{pmatrix}$$

where

(15)
$$\Lambda_{T,i} = \frac{1}{T} \int_0^T \varphi_i(t) X_t dt, \quad i = 1, \dots, p$$

(16)
$$\gamma_T = \left(\frac{1}{T}\int_0^T X_t^2 dt - \sum_{i=1}^p \Lambda_{T,i}^2\right)$$

and $\Lambda_T = (\Lambda_{T,1}, \ldots, \Lambda_{T,p})^t$.

Proof. We make use of the following formula for the inverse of a partitioned matrix which can be deduced from the Frobenius matrix inversion formula, cf. Gantmacher [2], p. 73. Alternatively the formula can also be verified directly. We have for $a \in \mathbb{R}^p$, $b \in \mathbb{R}$

$$\begin{pmatrix} I_p & a \\ a^t & b \end{pmatrix}^{-1} = \begin{pmatrix} I_p + \frac{1}{b - \|a\|^2} a a^t & -\frac{1}{b - \|a\|^2} a \\ -\frac{1}{b - \|a\|^2} a^t & \frac{1}{b - \|a\|^2} \end{pmatrix},$$

where $\|\cdot\|$ denotes the usual Euclidean norm on \mathbb{R}^p . With the notation introduced above, we can write Q as follows,

$$Q = T \begin{pmatrix} I_p & \Lambda_T \\ \Lambda_T^t & \frac{1}{T} \int_0^T X_t^2 dt \end{pmatrix}$$

and thus apply the above formula for the calculation of Q_T^{-1} .

We can now formulate our main results about the asymptotic behavior of the least squares estimator in the generalized Ornstein-Uhlenbeck process with periodic mean reversion function.

Theorem 1. Let $\{X_t, 0 \leq t \leq T\}$ be observations of a generalized mean reversion process with a periodic mean reversion function as introduced in (2), satisfying (11) and (12). Then the least squares estimator given in (8) is consistent, i.e.

$$\theta_{LS} \rightarrow \theta$$
, almost surely,

as $T \to \infty$.

For the description of the asymptotic distribution of $\hat{\theta}_{LS}$, we have to introduce some matrices. We define the $(p+1) \times (p+1)$ matrices C and Σ by

(17)
$$C = \begin{pmatrix} I_p + \gamma \Lambda \Lambda^t & -\gamma \Lambda \\ -\gamma \Lambda^t & \gamma \end{pmatrix}$$

(18)
$$\Sigma_0 = \begin{pmatrix} I_p & \Lambda \\ \Lambda^t & \omega \end{pmatrix}$$

where the entries are defined by

(19)
$$\Lambda_i = \int_0^1 \varphi_i(t) \tilde{h}(t) dt, \quad i = 1, \dots, p$$

(20)
$$\gamma = \left(\int_0^1 (\tilde{h}(t))^2 dt + E(\tilde{Z}_1)^2 - \sum_{i=1}^p \Lambda_i^2 \right)^{-1}$$

(21)
$$\omega = \int_0^1 (\tilde{h}(t))^2 dt + \frac{1}{2\alpha}$$

and $\Lambda = (\Lambda_1, \ldots, \Lambda_p)^t$. Here, the function $\tilde{h} : [0, \infty) \to \mathbb{R}$ and the process \tilde{Z}_t are defined by

(22)
$$\tilde{h}(t) = e^{-\alpha t} \sum_{j=1}^{p} \mu_j \int_{-\infty}^{t} e^{\alpha s} \varphi_j(s) ds$$

(23)
$$\tilde{Z}_t = \sigma e^{-\alpha t} \int_{-\infty}^t e^{\alpha s} d\tilde{B}_s,$$

where $(\tilde{B}_s)_{s\in\mathbb{R}}$ denotes a two-sided Brownian motion, i.e.

$$\tilde{B}_s := B_s \mathbb{1}_{\mathbb{R}_+}(s) + \bar{B}_{-s} \mathbb{1}_{\mathbb{R}_-}(s)$$

with $(B_s)_{s>0}$ and $(\overline{B}_s)_{s>0}$ two independent standard Brownian motions.

Theorem 2. Let $\{X_t, 0 \leq t \leq T\}$ be observations of a generalized mean reversion process with a periodic mean reversion function as introduced in (2), satisfying (11) and (12). The least squares estimator $\hat{\theta}_{LS}$, defined in (8), is asymptotically normal. More precisely,

$$\sqrt{T}(\hat{\theta}_{LS} - \theta) \to N(0, C \Sigma_0 C^t),$$

where C and Σ_0 are defined as in (17) and (18).

The proofs of these theorems require a number of auxiliary results, which will be given in the next two sections.

4. Proof of Theorem 1

The proofs of Theorem 1 and Theorem 2 make use of a representation of the least squares estimator that will be established in the following proposition.

Proposition 4.1. The least squares estimator $\hat{\theta}_{LS}$, defined in (8), can be written as

(24)
$$\hat{\theta}_{LS} = \theta + \sigma Q_T^{-1} R_T,$$

where

(25)
$$R_T := \begin{pmatrix} \int_0^T \varphi_1(t) dB_t \\ \vdots \\ \int_0^T \varphi_p(t) dB_t \\ -\int_0^T X_t dB_t \end{pmatrix},$$

and where Q_T is defined in (6).

Proof. By definition, we have

$$\hat{\theta}_{LS} = Q_T^{-1} P_T,$$

where Q_T and P_T are defined as in (6) and (7). We rewrite this by making use of (1). In fact, the stochastic differential equation

$$dX_t = \left(\sum_{j=1}^p \mu_j \varphi_j(t) - \alpha X_t\right) dt + \sigma dB_t$$

leads to

$$\int_0^T \varphi_i(t) dX_t = \sum_{j=1}^p \mu_j \int_0^T \varphi_i(t) \varphi_j(t) dt - \alpha \int_0^T \varphi_i(t) X_t dt + \sigma \int_0^T \varphi_i(t) dB_t, \quad i = 1, \dots, p,$$

$$\int_0^T X_t dX_t = \sum_{j=1}^p \mu_j \int_0^T X_t \varphi_j(t) dt - \alpha \int_0^T X_t^2 dt + \sigma \int_0^T X_t dB_t.$$

Hence, it follows that

$$P_T = \begin{pmatrix} \int_0^T \varphi_1(t) dX_t \\ \vdots \\ \int_0^T \varphi_p(t) dX_t \\ -\int_0^T X_t dX_t \end{pmatrix} = Q_T \theta + \sigma R_T$$

so that $\hat{\theta}_{LS} = \theta + \sigma Q_T^{-1} R_T$.

In what follows, we will show that $Q_T^{-1} R_T$ converges to zero almost surely, as $T \to \infty$. In order to do so, we write

$$Q_T^{-1} R_T = \left(T Q_T^{-1}\right) \left(\frac{1}{T} R_T\right).$$

We will show that $T Q_T^{-1}$ converges almost surely to a finite limit and that $\frac{1}{T}R_T$ converges almost surely to zero. Both of these results require a number of auxiliary results which will be proved first.

Lemma 4.2. The solution of the stochastic differential equation (1) has the explicit representation

(26)
$$X_t = e^{-\alpha t} X_0 + h(t) + Z_t,$$

where

$$h(t) = e^{-\alpha t} \int_0^t e^{\alpha s} L(s) ds = e^{-\alpha t} \sum_{i=1}^p \mu_i \int_0^t e^{\alpha s} \varphi_i(s) ds$$

and

$$Z_t = \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dB_s.$$

Proof. The Itô lemma states for $Y_t = g(t, X_t)$ that

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2$$

which reduces for $g(t, x) = e^{\alpha t} x$ to

$$dY_t = \alpha e^{\alpha t} X_t dt + e^{\alpha t} dX_t.$$

Plugging (1) in this equation gives

$$dY_t = e^{\alpha t} (L(t)dt + \sigma dB_t).$$

Integrating and multiplying by $e^{-\alpha t}$ finishes the proof of the lemma.

The process $(X_t)_{t\geq 0}$ is not stationary, since we have chosen an arbitrary initial value. Thus we are unable to apply the ergodic theorem. In order to solve this problem, we will next introduce a stationary solution to the stochastic differential equation (1). We define the process

(27)
$$\tilde{X}_t = \tilde{h}(t) + \tilde{Z}_t$$

where $\tilde{h}(t)$ and \tilde{Z}_t are defined in (22) and (23). As usual we denote by C[0,1] the space of real valued continuous functions on [0,1].

Lemma 4.3. The sequence of C[0,1]-valued random variables

(28)
$$W_k(s) := X_{k-1+s}, \ 0 \le s \le 1,$$

is stationary and ergodic.

Proof. We denote by \tilde{h}_0 the restriction of the function \tilde{h} to [0,1]. Since the function \tilde{h} is periodic, we have the decomposition

$$W_{k}(t) = \tilde{h}(k-1+t) + \sigma e^{-\alpha(k-1+t)} \int_{-\infty}^{k-1+t} e^{\alpha s} d\tilde{B}_{s}$$

= $\tilde{h}_{0}(t) + \sigma e^{-\alpha(k-1+t)} \int_{k-1}^{k-1+t} e^{\alpha s} d\tilde{B}_{s} + \sigma \sum_{l=-\infty}^{k-1} e^{-\alpha(k-1+t)} \int_{l-1}^{l} e^{\alpha s} d\tilde{B}_{s}.$

Making use of the time shifted Brownian motion

$$\tilde{B}_s^{(l)} := \tilde{B}_{s+l}$$

yields

$$W_{k}(t) = \tilde{h}_{0}(t) + \sigma e^{-\alpha t} \int_{0}^{t} e^{\alpha s} d\tilde{B}_{s}^{(k)} + \sigma \sum_{l=-\infty}^{k-1} e^{-\alpha(k-l+t)} \int_{0}^{1} e^{\alpha s} d\tilde{B}_{s}^{(l)}$$

$$= \tilde{h}_{0}(t) + \sigma e^{-\alpha t} \int_{0}^{t} e^{\alpha s} d\tilde{B}_{s}^{(k)} + \sigma \sum_{j=-\infty}^{0} e^{-\alpha(1+t-j)} \int_{0}^{1} e^{\alpha s} d\tilde{B}_{s}^{(j+k-1)}.$$

Consequently, this can be written as

$$W_k(\cdot) = \tilde{h}_0(\cdot) + F_0(Y_k) + \sum_{l=-\infty}^{0} e^{\alpha(j-1)} F(Y_{j+k-1})$$

where we used the a.s. defined functionals

$$F_0: C[0,1] \to C[0,1]; \ \omega \mapsto \left(t \mapsto \sigma e^{-\alpha t} \int_0^t e^{\alpha s} d\omega(s)\right),$$
$$F: C[0,1] \to C[0,1]; \ \omega \mapsto \left(t \mapsto \sigma e^{-\alpha t} \int_0^1 e^{\alpha s} d\omega(s)\right).$$

and the C[0, 1]-valued random variables

$$Y_l = (s \mapsto \tilde{B}_s^{(l)} - \tilde{B}_0^{(l)}, \ 0 \le s < 1).$$

The series $(Y_l)_{l \in \mathbb{Z}}$ consists of independent and identically distributed random variables. This implies that $(W_k)_{k \in \mathbb{N}}$ is stationary and ergodic since each element of this sequence can be represented as a measurable function $G : (C[0,1])^{\mathbb{N}} \to C[0,1]$ of elements of the iid sequence $(Y_l)_{l \in \mathbb{Z}}$, i.e.

$$W_k = G(Y_k, Y_{k-1}, \ldots).$$

Lemma 4.4. As $t \to \infty$ one has

(29)
$$\left| \tilde{X}_t - X_t \right| \to 0, \quad a.s.$$

Proof. We have

$$\begin{aligned} \left| \tilde{X}_t - X_t \right| &\leq e^{-\alpha t} |X_0| + \left| \tilde{h}(t) - h(t) \right| + \left| \tilde{Z}_t - Z_t \right| \\ &\leq e^{-\alpha t} |X_0| + e^{-\alpha t} \sum_{i=1}^p \mu_i \int_{-\infty}^0 e^{\alpha s} \varphi_i(s) ds + e^{-\alpha t} \int_{-\infty}^0 e^{\alpha s} d\tilde{B}_s. \end{aligned}$$

Obviously, the three terms on the right side converge toward zero as $t \to \infty$.

Proposition 4.5. As $T \to \infty$, we have

$$T Q_T^{-1} \to C, \quad a.s.$$

where C is the matrix defined in (17).

Proof. We first consider the entries of the vector Λ_T , i.e. $\frac{1}{T} \int_0^T X_t \varphi_j(t) dt$. From Lemma 4.4 we may conclude that

$$\frac{1}{T} \int_0^T \tilde{X}_t \varphi_j(t) dt - \frac{1}{T} \int_0^T X_t \varphi_j(t) dt \to 0,$$

almost surely. Moreover, we get by the ergodic theorem

$$\frac{1}{T}\int_0^T \tilde{X}_t \varphi_j(t) dt = \frac{1}{T}\sum_{k=1}^T \int_{k-1}^k \tilde{X}_t \varphi_j(t) dt \longrightarrow E\left(\int_0^1 \tilde{X}_t \varphi_j(t) dt\right) = \int_0^1 \tilde{h}(t)\varphi_j(t) dt.$$

Thus we have established convergence of $\Lambda_{T,j}$, $1 \leq j \leq p$. In order to determine the limit of γ_T , it suffices to consider the term $\frac{1}{T} \int_0^T X_t^2 dt$. Again, it follows from Lemma 4.4 that

$$\frac{1}{T}\int_0^T \tilde{X}_t^2 dt - \frac{1}{T}\int_0^T X_t^2 dt \to 0.$$

Moreover, again by the ergodic theorem, we get

$$\begin{aligned} \frac{1}{T} \int_0^T \tilde{X}_t^2 dt &= \frac{1}{T} \sum_{k=1}^T \int_{k-1}^k \tilde{X}_t^2 dt \\ &\to E\left(\int_0^1 \tilde{X}_t^2 dt\right) \\ &= E\left(\int_0^1 (\tilde{h}(t) + \tilde{Z}_t)^2 dt\right) \\ &= E\left(\int_0^1 (\tilde{h}(t))^2 dt + \int_0^1 \tilde{h}(t) \tilde{Z}_t dt + \int_0^1 \tilde{Z}_t^2 dt\right) = \int_0^1 (\tilde{h}(t))^2 dt + E(\tilde{Z}_1)^2. \end{aligned}$$

By Bessel's inequality, we have

$$\sum_{i=1}^p \Lambda_i^2 \le \int_0^1 (\tilde{h}(t))^2 dt$$

and thus $(\int_0^1 (\tilde{h}(t))^2 dt + E(\tilde{Z}_1)^2 - \sum_{i=1}^p \Lambda_i^2) \ge E(\tilde{Z}_1)^2 > 0$. This proves the assertion of the proposition.

Lemma 4.6. The sequence $\frac{1}{\sqrt{T}}R_T$ is bounded in L^2 .

Proof. Note that

$$\frac{1}{\sqrt{T}}\int_0^T \varphi_i(t) dB_t$$

is L_2 -bounded because

(30)
$$\operatorname{Var}\left[\frac{1}{\sqrt{T}}\int_{0}^{T}\varphi_{i}(t)dB_{t}\right] = \operatorname{Var}\left[\frac{1}{\sqrt{T}}\int_{0}^{T}\varphi_{i}(t)dB_{t}\right] = \frac{1}{T}\int_{0}^{T}\varphi_{i}^{2}(t)dt = 1.$$

For the last entry of $\frac{1}{\sqrt{T}}R_T$ we have to prove the boundedness of

$$\begin{aligned} \operatorname{Var} \left[\frac{1}{\sqrt{T}} \int_0^T X_t dB_t \right] \\ &= \frac{1}{T} \operatorname{E} \left[\int_0^T X_t^2 dt \right] \\ &= \frac{1}{T} \operatorname{E} \left[\int_0^T e^{-\alpha t} X_0 h(t) + e^{-\alpha t} X_0 Z_t + e^{-2\alpha t} X_0^2 + h(t) Z_t + h(t)^2 + Z_t^2 dt \right]. \end{aligned}$$

Since Z_t is a zero-mean random variable the expectation of the second and fourth term is zero. Moreover, the variance

$$\mathbf{E}[Z_t^2] = \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha t})$$

is bounded and justifies

$$\sup_{T\geq 0} \frac{1}{T} \mathbf{E} \left[\int_0^T Z_t^2 dt \right] < \infty.$$

Moreover, the function

$$h(t) = e^{-\alpha t} \sum_{i=1}^{p} \mu_i \int_0^t e^{\alpha s} \varphi_i(t) dt$$

is bounded due to the periodicity of $\varphi_i(t)$, $i = 1, \ldots, p$. The boundedness of h(t) gives

$$\sup_{T \ge 0} \frac{1}{T} \mathbb{E} \left[\int_0^T e^{-\alpha t} X_0 h(t) dt \right] < \infty$$

and

$$\sup_{T \ge 0} \frac{1}{T} \mathbb{E}\left[\int_0^T h(t)^2 dt\right] < \infty$$

This finishes the proof of the L^2 -boundedness of $\frac{1}{\sqrt{T}}R_T$.

Proposition 4.7. As $T \to \infty$, we have

(31)
$$\lim_{T \to \infty} \frac{1}{T} R_T = 0, \ almost \ surely.$$

Proof. Observe that R_T is a martingale; thus we get by using Doob's maximal inequality for submartingales that for any $\epsilon > 0$

$$P(\sup_{2^{k} \le T \le 2^{k+1}} \frac{1}{T} |R_{T}| \ge \epsilon) \le P(\sup_{2^{k} \le T \le 2^{k+1}} |R_{T}| \ge \epsilon 2^{k})$$
$$\le \frac{4}{\epsilon^{2} 2^{2k}} E |R_{2^{k+1}}|^{2} = O(2^{-k}).$$

11

Applying the Borel-Cantelli theorem, we obtain $\limsup_{T\to\infty} \frac{1}{T} |R_T| \leq \epsilon$, almost surely, and thus we have shown that $R_T/T \to 0$.

Proof of Theorem 1. This follows directly from Proposition 4.5 and Proposition 4.7. \Box

5. Proof of Theorem 2

In the proof of Theorem 2 we use again the representation (24), i.e. $\hat{\theta}_{LS} - \theta = \sigma Q_T^{-1} R_T$, which we rewrite as

$$\sqrt{T}\frac{\hat{\theta}_{LS}-\theta}{\sigma} = \sqrt{T}Q_T^{-1}R_T = (TQ_T^{-1})\frac{1}{\sqrt{T}}R_T.$$

By Proposition 4.5, TQ_T^{-1} converges almost surely to the matrix C. Then, by Slutsky's theorem, Theorem 2 is an immediate corollary of the following proposition.

Proposition 5.1. Under the assumptions of Theorem 2, we have, as $T \to \infty$,

$$\frac{1}{\sqrt{T}}R_T \xrightarrow{\mathcal{D}} N(0, \Sigma_0),$$

where Σ_0 is the matrix defined in (18).

The remaining part of this section is devoted to the proof of this proposition. Recall that

$$\frac{1}{\sqrt{T}}R_T = \begin{pmatrix} \frac{1}{\sqrt{T}} \int_0^T \varphi_1(t) dB_t \\ \vdots \\ \frac{1}{\sqrt{T}} \int_0^T \varphi_p(t) dB_t \\ -\frac{1}{\sqrt{T}} \int_0^T X_t dB_t \end{pmatrix}.$$

Since the basis functions $\varphi_1, \ldots, \varphi_p$ are orthonormal, the first p entries of the vector $\frac{1}{\sqrt{T}}R_T$ are independent, normally distributed random variables with mean zero and variance 1. Thus it remains to investigate the asymptotic distribution of the last entry

$$\frac{1}{\sqrt{T}} \int_0^T X_t dB_t,$$

and its joint distribution with the first p components.

By Lemma 4.2, the process $(X_t)_{t\geq 0}$ can be expressed as

$$X_t = e^{-\alpha t} X_0 + h(t) + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dB_s$$

and thus we have

(32)
$$\frac{1}{\sqrt{T}} \int_0^T X_t dB_t = \frac{X_0}{\sqrt{T}} \int_0^T e^{-\alpha t} dB_t + \frac{1}{\sqrt{T}} \int_0^T h(t) dB_t + \sigma \frac{1}{\sqrt{T}} \int_0^T \int_0^t e^{\alpha(s-t)} dB_s dB_t.$$

The first term on the right hand side converges to 0 in probability, as

$$\operatorname{Var}\left(\frac{1}{\sqrt{T}}\int_{0}^{T}e^{-\alpha t}dB_{t}\right) = \frac{1}{T}\int_{0}^{T}e^{-2\alpha t}dt \longrightarrow 0.$$

The second term is normally distributed with mean zero and variance

$$\frac{1}{T} \int_0^T (h(t))^2 dt \longrightarrow \int_0^1 (\tilde{h}(t))^2 dt.$$

The asymptotic distribution of the third term, as well as its joint distribution with any stochastic integral $\int_0^T \varphi(t) dB_t$, will be evaluated next.

Proposition 5.2. Let $\varphi : [0, \infty) \to \mathbb{R}$ be an L_2 -function, for which

$$\sigma_{\varphi}^2 := \lim_{T \to \infty} \frac{1}{T} \int_0^T (\varphi(t))^2 dt$$

exists. Then, as $T \to \infty$,

$$\frac{1}{\sqrt{T}} \left(\int_0^T \int_0^t e^{\alpha(s-t)} dB_s dB_t, \int_0^T \varphi(t) dB_t \right) \xrightarrow{\mathcal{D}} N(0, \left(\begin{array}{cc} \frac{1}{2\alpha} & 0\\ 0 & \sigma_{\varphi}^2 \end{array} \right)),$$

where N(0, A) denotes a bivariate normal distribution with mean vector 0 and covariance matrix A.

Proof. Application of the time change formula for stochastic integrals twice, cf. Øksendal [10] (Theorem 8.5.7, p. 148), for $g(\tau) := T\tau$, $g'(\tau) = T$, results in

$$\frac{1}{\sqrt{T}} \int_0^T \int_0^t e^{\alpha(s-t)} dB_s dB_t = \int_0^1 \int_0^t e^{\alpha(s-Tt)} dB_s dB_t^{(T)}$$
$$= \sqrt{T} \int_0^1 \int_0^t e^{\alpha T(s-t)} dB_s^{(T)} dB_t^{(T)}$$

where $B_t^{(T)} = \frac{1}{\sqrt{T}} B_{Tt}$. Therefore, it is sufficient to study the asymptotic distribution of

$$\sqrt{T} \int_0^1 \int_0^t e^{\alpha T(s-t)} dW_s dW_t$$

where $(W_t)_{t\geq 0}$ denotes a Brownian motion with the same distribution as $(B_t^{(T)})_{t\geq 0}$. The symmetrization theorem for double Wiener integrals, cf. Kuo [5] (Theorem 9.2.8, p. 154), provides the identity

(33)
$$\sqrt{T} \int_0^1 \int_0^t e^{\alpha T(s-t)} dW_s dW_t = \frac{\sqrt{T}}{2} \int_0^1 \int_0^1 e^{-\alpha T|s-t|} dW_s dW_t$$

By Lemma 5.3 we obtain

(34)
$$\sqrt{T} \int_0^1 \int_0^1 e^{-\alpha T|s-t|} dW_s dW_t \stackrel{\mathcal{D}}{=} \sum_{j=1}^\infty \lambda_{T,j} (\xi_{T,j}^2 - 1)$$

where $(\lambda_{T,j})_{j\in\mathbb{N}}$ is the set of eigenvalues of the integral operator with kernel $f_T(s,t) = \sqrt{T} e^{-\alpha T |s-t|}$ and where $\xi_{T,j} = \int_0^1 e_{T,j}(t) dW_t$. Here we denote by $e_{T,j}(t)$ the eigenfunction associated to the eigenvalue λ_i . By Lemma 5.4 the eigenvalues have the properties

$$\lim_{T \to \infty} \sum_{j=1}^{\infty} \lambda_{T,j}^2 = \frac{1}{\alpha}$$
$$\lim_{T \to \infty} \max_{j \ge 1} |\lambda_{T,j}| = 0.$$

Define $\xi_T := \frac{1}{\sqrt{T}} \int_0^T \varphi(t) dB_t$. Note that $\xi_T, \xi_{T,j}, j \ge 1$ are jointly normally distributed and that $(\xi_{T,j})_{j\ge 1}$ are iid standard normally distributed random variables. Projecting ξ_T onto

the space spanned by the random variables $(\xi_{T,j})_{j\geq 1}$, we can write

$$\xi_T = \xi_{T,0} + \sum_{j=1}^{\infty} \alpha_{T,j} \xi_{T,j},$$

where $\xi_{T,0}$ is independent of $(\xi_{T,j})_{j\geq 1}$. Define $\sigma_T^2 := \frac{1}{T} \int_0^T \varphi^2(t) dt$ and $\sigma_{T,0}^2 := \operatorname{Var}(\xi_{T,0})$ and note that

$$\sigma_T^2 = \sigma_{T,0}^2 + \sum_{j=1}^{\infty} \alpha_{T,j}^2 \longrightarrow \sigma_{\varphi}^2.$$

We will now apply the Cramér-Wold device to prove convergence of the joint distribution of ξ_T and $\sum_{j=1}^{\infty} \lambda_{T,j}(\xi_{T,j}^2 - 1)$. Let $\mu_1, \mu_2 \in \mathbb{R}$; we will show that

$$\mu_1 \xi_T + \mu_2 \sum_{j=1}^{\infty} \lambda_{T,j} (\xi_{T,j}^2 - 1) \xrightarrow{\mathcal{D}} N(0, \mu_1^2 \sigma_{\varphi}^2 + 2\mu_2^2 \frac{1}{\alpha}).$$

In order to do so, we compute the characteristic function of the left hand side and note that

$$\mu_1 \xi_T + \mu_2 \sum_{j=1}^{\infty} \lambda_{T,j} (\xi_{T,j}^2 - 1) = \mu_1 \xi_{T,0} + \sum_{j=1}^{\infty} (\mu_1 \alpha_{T,j} \xi_{T,j} + \mu_2 \lambda_{T,j} (\xi_{T,j}^2 - 1)).$$

If Z is standard normally distributed, the characteristic function of $a Z + b (Z^2 - 1)$ is given by

$$\psi(t) = (1 - 2itb)^{-1/2} \exp\left(-ibt - \frac{a^2t^2}{2(1 - 2ibt)}\right)$$

Thus the characteristic function of $\mu_1\xi_{T,0} + \sum_{j=1}^{\infty} (\mu_1\alpha_{T,j}\xi_{T,j} + \mu_2\lambda_{T,j}(\xi_{T,j}^2 - 1))$ equals

$$\psi_T(t) = e^{-\frac{1}{2}\mu_1^2 \sigma_{T,0}^2 t^2} \prod_{j=1}^{\infty} \left\{ (1 - 2i\mu_2 \lambda_{T,j} t)^{-1/2} \exp\left(-i\mu_2 \lambda_{T,j} t - \frac{(\mu_1 \alpha_{T,j})^2 t^2}{2(1 - 2i\mu_2 \lambda_{T,j} t)}\right) \right\}.$$

Taking logarithms and using Taylor expansion, we obtain

$$\log \psi_{T}(t) = -\frac{1}{2}\mu_{1}^{2}\sigma_{T,0}^{2}t^{2} - \sum_{j=1}^{\infty} \left(\frac{1}{2}\log(1-2i\mu_{2}\lambda_{T,j}t) + i\mu_{2}\lambda_{T,j}t + \frac{\mu_{1}^{2}\alpha_{T,j}^{2}t^{2}}{2(1-2i\mu_{2}\lambda_{T,j}t)}\right)$$

$$= -\frac{1}{2}\mu_{1}^{2}\sigma_{T,0}^{2}t^{2} - \sum_{j=1}^{\infty}(\mu_{2}^{2}\lambda_{T,j}^{2} + \frac{1}{2}\mu_{1}^{2}\alpha_{T,j}^{2})t^{2} + o(1)$$

$$= -\frac{1}{2}\left(\mu_{1}^{2}\sigma_{T,0}^{2} + \sum_{j=1}^{\infty}\mu_{1}^{2}\alpha_{T,j}^{2} + 2\sum_{j=1}^{\infty}\mu_{2}^{2}\lambda_{T,j}^{2}\right)t^{2} + o(1)$$

$$\to -\frac{1}{2}\left(\mu_{1}^{2}\sigma_{\varphi}^{2} + \mu_{2}^{2}\frac{2}{\alpha}\right)t^{2}.$$

Note that the right hand side is the logarithm of the characteristic function of a normal distribution with mean 0 and variance $\mu_1^2 \sigma_{\varphi}^2 + \mu_2^2 \frac{2}{\alpha}$.

Lemma 5.3. Let $f : [0,1]^2 \to \mathbb{R}$ be a symmetric continuous kernel and let $(\lambda_i)_{i\geq 1}$ and $(e_i(t))_{i\geq 1}$ denote the set of eigenvalues and corresponding eigenfunctions of the integral operator $G_f : L^2[0,1] \to L^2[0,1]$ with kernel f, i.e. $G_fg(x) = \int_0^1 g(y)f(x,y)dy$. Then

$$\int_0^1 \int_0^1 f(s,t) dW_s dW_t = \sum_{i=1}^\infty \lambda_i (\xi_i^2 - 1),$$

where

$$\xi_i = \int_0^1 e_i(t) dW_t$$

The random variables $(\xi_i)_{i\in\mathbb{N}}$ are independent and standard normally distributed random variables.

Proof. Since the kernel f is continuous and symmetric the operator G_f is self-adjoint and compact. By Mercer's Theorem it holds that the kernel can be represented as

(35)
$$f(s,t) = \sum_{i=1}^{\infty} \lambda_i e_i(s) e_i(t)$$

where λ_i and e_i , $i \in \mathbb{N}$, are the eigenvalues and eigenfunctions of the integral operator G_T , i.e.

$$\int_0^1 f(s,t)e_i(s)ds = \lambda_i e_i(t), \qquad i \in \mathbb{N}$$

Moreover, it holds that the functions e_i , $i \in \mathbb{N}$, form an orthonormal basis of $L^2[0, 1]$. Define the random variables

$$\xi_i := \int_0^1 e_i(t) \, dW_t, \qquad i \in \mathbb{N},$$

and note that $(\xi_i)_{i\geq 1}$ is an iid sequence of standard normally distributed random variables. It follows by (35) that

$$\int_{0}^{1} \int_{0}^{1} f(s,t) dW_{s} dW_{t} = \sum_{i=1}^{\infty} \lambda_{i} \int_{0}^{1} \int_{0}^{1} e_{i}(s) e_{i}(t) dW_{s} dW_{t} = \sum_{i=1}^{\infty} \lambda_{i} (\xi_{i}^{2} - 1).$$

The last equality follows by Itô's Theorem which states that

$$\int_{0}^{1} \int_{0}^{1} e_{i}(s)e_{i}(t)dW_{s}dW_{t} = H_{2}\left(\int_{0}^{1} e_{i}(t)\,dW_{t}\right)$$

where H_2 is the second Hermite polynomial, i.e. $H_2(x) = x^2 - 1$.

We now consider the kernel $f_T: [0,1] \to \mathbb{R}$, defined by

(36)
$$f_T(s,t) = \sqrt{T}e^{-\alpha T|s-t|}, \quad s,t \in [0,1].$$

Lemma 5.4. Let $(\lambda_{T,i})_{i\geq 1}$, denote the set of eigenvalues of the integral operator with kernel (36). Then we have

(37)
$$\lim_{T \to \infty} \sum_{i=1}^{\infty} \lambda_{T,i}^2 = \frac{1}{\alpha}$$

(38)
$$\lim_{T \to \infty} \max_{i \ge 1} |\lambda_{T,i}| = 0.$$

Proof. Note that the operator G_{f_T} is self-adjoint and bounded so that its eigenvalues are real-valued, and

$$\max_{i \ge 1} \lambda_{T,i}^2 = \sup_{g \in L^2[0,1]: \|g\| = 1} \|G_{f_T}g\|^2$$

where $\|\cdot\|$ denotes the standard L_2 -norm on $L^2[0,1]$. By an equality in Lax [9] (Theorem 2, p. 176) we get

$$\sup_{g \in L^2[0,1]: \|g\|=1} \|G_{f_T}g\| \le \sup_{t \in [0,1]} \int_0^1 |f_T(s,t)| ds$$

Simple integration yields

$$\int_0^1 |f_T(s,t)| ds = \int_0^1 \sqrt{T} e^{-\alpha T |s-t|} ds = \frac{1}{\alpha T} (2 - e^{-\alpha T t} - e^{-\alpha T (1-t)}),$$

and thus it follows that

(39)
$$\max_{i\in\mathbb{N}}\lambda_{T,i} \le \frac{2}{\alpha T} \to 0 \quad \text{as } T \to \infty.$$

The assertion of Mercer's Theorem given in (35) and the orthonormality of the eigenvalues provide the identity

$$\sum_{i=1}^{\infty} \lambda_i^2 = \int_0^1 \int_0^1 f_T(s,t)^2 ds dt = \frac{1}{2\alpha} \left(2 + \frac{1}{T\alpha} (e^{-2\alpha T} - 1) \right)$$

where the last equality is obtained by simple integration.

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16