On inductively free and additionally free arrangements

Sven Wiesner

Oberseminar on Arrangements and Symmetries
Sommersemester 2021

Let $\mathcal{A}(W)$ be the hyperplane arrangement consisting of all the reflecting hyperplanes of a finite complex reflection group W acting on a vectorspace V. The class of inductively free arrangements $\mathcal{IF}$ got introduced by Orlik and Terao in [OT]. In [CH] it was shown to be a combinatorial property. Abe showed in [A] that the Addition-Deletion-Theorem is combinatorial too and defined the class of additionally free arrangements $\mathcal{AF}$. Those properties are the same if the dimension of V is lower than 4.

Hoge and Röhrle gave two examples of additionally free arrangements that are not inductively free. To construct the first example $\mathcal{B}$ they took an inductively free subarrangement $\mathcal{A}$ of $\mathcal{A}(E_7)$ containing 32 hyperplanes and exponents $\{1,5,5,5,5,5,6\}$. Removing one further hyperplane provides an arrangement $\mathcal{B}$ that is additionally free with exponents $\{1,5,5,5,5,5,5\}$, but fails to be inductively free. They also show that the rank 6 restriction of $\mathcal{B}$ with matching exponents $\{1,5,5,5,5,5\}$ is additionally free without being inductively free. To construct an other example $\mathcal{D}$ they restrict $\mathcal{B}$ to the intersection of $\ker(x_1)$ and $\ker(x_6)$. The arrangement $\mathcal{D}$ is free with exponents $\{1,5,5,5,5\}$ and up to isomorphism only one restriction of $\mathcal{D}$ got matching exponents to use the Addition-Deletion-Theorem on. This restriction is free, but fails to be additionally free (in particular inductively free), because none of its restrictions got matching exponents. Removing a hyperplane $H \in \mathcal{D}$ such that the restriction on H is free and got matching exponents results in an inductively free arrangement. In particular the arrangement $\mathcal{D}$ satisfies $\mathcal{D} \in \mathcal{AF} \backslash \mathcal{IF}$. Those examples of additionally free arrangements that fail to be inductively free are the only ones known for now.

In this talk I will present results obtained in my Master Thesis concerning the coincidence of inductive and additional freeness of subarrangements of $\mathcal{A}(W)$ where W is one of the groups $G_{29}, H_4, G_{31}, G_{32}$ or $G(r,r,4)$.

References

[A] T. Abe; *Deletion theorem and combinatorics of hyperplane arrangements*, Math. Ann. 373 (2019), no. 1-2, 581–595.

[CH] M. Cuntz and T. Hoge; *Free but not recursively free arrangements*, Proc. Amer. Math. Soc. 143 (2015), 35-40.

[HR] T. Hoge and G. Röhrle; *Some Remarks On Free Arrangements*, Tôhoku Math. J. **73**, no. 2, (2021).

[KU] J. P. S. Kung; *Critical Problems*
in: J. E. Bonin, J. G. Oxley und B. Servatius; *Theory of Matroids*
Amer. Math. Soc., Providence, RI, 1996.

[LT] G. I. Lehrer and D. E. Taylor; *Unitary Reflection Groups*, Springer-Verlag, Berlin 1992.

[OT] P. Orlik and H. Terao; *Arrangements of Hyperplanes*, Springer-Verlag, Berlin 1992.

[OX] J. Oxley; *Matroid Theory*, Oxford University Press, USA 2011.

[T] S. Tohaneanu; *Free arrangements with low exponents*,
https://arxiv.org/abs/1707.07091.