STRONG BOUNDEDNESS OF S-ARITHMETIC, SPLIT CHEVALLEY GROUPS — SANDWICH THEOREMS, COMPACTNESS AND BAD PRIMES

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It has been known by work of Carter-Keller [1] and Tavgen [5] since the 90s that generalizations of classical, arithmetic matrix groups like $\mathrm{SL}_n(\mathbb{Z})$, so-called split Chevalley groups $G := G(\Phi, R)$, defined using rings R of S-algebraic integers and an irreducible root system Φ are boundedly generated by root elements (think of elementary matrices). In this context, a subset T of G *boundedly generates* G iff there is a natural number N such that each element of G can be written as a product with N factors of elements of $T \cup T^{-1} \cup \{1\}$. The smallest such N is denoted by $N(G,T)$. Work by Kedra-Gal [2, Theorem] has further shown that these results can be used to show that a generating collection of conjugacy classes T boundedly generates G. Obviously, this raises the question how precisely $N(G,T)$ depends on T and Γ. One of the early results was the following theorem by Kedra, Libman and Martin:

Theorem 1. [3, Corollary 6.2] *Let R be a ring of S-algebraic integers of class number 1. Then each collection $T \subset \mathrm{SL}_n(R)$ of finitely many conjugacy classes generating $\mathrm{SL}_n(R)$ boundedly generates $\mathrm{SL}_n(R)$ with $N(\mathrm{SL}_n(R),T) \leq (4n+51)\cdot(4n+4)\cdot|T|$. Further, for each natural number k, there is a generating collection of conjugacy classes T_k of $\mathrm{SL}_n(R)$ with $|T_k| = k$ and $N(\mathrm{SL}_n(R),T) \geq k$.*

Generally speaking a group G is called *strongly bounded*, iff for each natural number k, the supremum

$$\Delta_k(G) := \sup\{N(G,T) \mid T \subset G \text{ normally generates } G, |T| = k\}$$

is a natural number. In this series of lectures, we will explain how such strong boundedness results can be obtained for split Chevalley groups in a systematic and structural manner by invoking classical results in algebraic K-theory, so-called Sandwich Theorems, together with model-theoretic compactness arguments and explain obstructions to the existence of normally generating subsets of $G(\Phi, R)$. This will naturally divide the talks into three parts:

First, we will present the strong boundedness results for $\mathrm{SL}_n(R)$ for $n \geq 3$. In this part, we will explain in more detail some of the previous results like Theorem 1 or Morris' [4], a bit of the historical context, that is conjugation-invariant metrics on (hamiltonian) diffeomorphism groups and the archetypical Sandwich Theorem. If time permits, we will also talk about normally generating sets of $\mathrm{SL}_n(R)$ and the only partially understood asymptotics of $\Delta_k(\mathrm{SL}_n(R))$ in terms of k and n and how the bounds $\Delta_k(\mathrm{SL}_n(R))$ compare to similar invariants called *conjugacy diameters* for finite, simple groups of Lie type.

Second, we will explain how the strong boundedness results generalize to essentially all other cases of $G(\Phi, R)$ using Sandwich Theorems except for $\Phi = C_2, G_2$ and A_1 [6]. Having seen the methods of the first part this is relatively straightforward and while there are some differences due to the presence of two root lengths in non-simply-laced Φ, ultimately the strong boundedness results are virtually identical to the ones of $\mathrm{SL}_n(R)$.

Third, we will explain how strong boundedness results and the behavior of $N(G(\Phi, R), T)$ for $\Phi = C_2$ and G_2 differ from the higher rank cases: Contrary to the higher rank cases where strong boundedness appears as an almost pure first-order phenomena, in these lower rank cases, one is forced to use additional non first-order arguments and to consider the conjugacy width of certain congruence subgroups. Furthermore, we will construct epimorphisms obstructing the existence of small normally generating sets of $\mathrm{Sp}_4(R)$ and $G_2(R)$ respectively. These epimorphisms will show that the differences between the cases of $\mathrm{Sp}_4(R), G_2(R)$ and the other Chevalley groups are not merely artifacts of our proof strategies but due to actual structural differences between Sp_4 and G_2 and the higher rank cases.

These epimorphisms arise due to the presence of bad primes of the ring of S-algebraic integers R for the corresponding Chevalley-Demazure group scheme. For example, consider the ring of Kleinian integers $R = \mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$. One can easily see in this case that $2R$ factors as $2R = (\omega) \cdot (\omega - 1)$ for $\omega := \frac{1+\sqrt{-7}}{2}$. But this then implies $R/(\omega) = R/(\omega - 1) = \mathbb{F}_2$ and so an epimorphism $\mathrm{Sp}_4(R) \to \mathrm{Sp}_4(R/(\omega)) \times \mathrm{Sp}_4(R/(\omega - 1)) = \mathrm{Sp}_4(\mathbb{F}_2)^2$ exists. But there is an exceptional isomorphism between $\mathrm{Sp}_4(\mathbb{F}_2)$ and the permutation group S_6 and hence there is an epimorphism $\mathrm{Sp}_4(R) \to \mathrm{Sp}_4(\mathbb{F}_2)^2 \to \mathbb{F}_2 \oplus \mathbb{F}_2$. This epimorphism however makes it impossible to find a single conjugacy class that generates $\mathrm{Sp}_4(R)$. If time permits, we will also explain why this type of obstruction is sufficient to classify normally generating subsets of Sp_4 and G_2.

Last, we will explain recent results concerning strong boundedness in the case of $\mathrm{SL}_2(R)$ for R a ring with infinitely many units and the more complicated shape that the epimorphisms obstructing the existence of small normal generating sets take in this case.

References

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