It has been known by work of Carter-Keller [1] and Tavgen [2] since the 90s that generalizations of classical, arithmetic matrix groups like $\text{SL}_n(\mathbb{Z})$, so-called split Chevalley groups $G := G(\Phi, R)$, defined using rings $R$ of S-algebraic integers and an irreducible root system $\Phi$ are boundedly generated by root elements (think of elementary matrices). In this context, a subset $T$ of $G$ boundedly generates $G$ iff there is a natural number $N$ such that each element of $G$ can be written as a product with $N$ factors of elements of $T \cup T^{-1} \cup \{1\}$. The smallest such $N$ is denoted by $N(G,T)$. Work by Kedra-Gal [2, Theorem] has further shown that these results can be used to show that a generating collection of conjugacy classes $T$ boundedly generates $G$. Obviously, this raises the question how precisely $N(G,T)$ depends on $T$ and $\Gamma$. One of the early results was the following theorem by Kedra, Libman and Martin:

**Theorem 1.** [3, Corollary 6.2] Let $R$ be a ring of S-algebraic integers of class number 1. Then each collection $T \subset \text{SL}_n(R)$ of finitely many conjugacy classes generating $\text{SL}_n(R)$ boundedly generates $\text{SL}_n(R)$ with $N(\text{SL}_n(R),T) \leq (4n+51) \cdot (4n+4) \cdot |T|$. Further, for each natural number $k$, there is a generating collection of conjugacy classes $T_k$ of $\text{SL}_n(R)$ with $|T_k| = k$ and $N(\text{SL}_n(R),T) \geq k$.

Generally speaking a group $G$ is called strongly bounded, iff for each natural number $k$, the supremum

$$\Delta_k(G) := \sup\{N(G,T) \mid T \subset G \text{ normally generates } G, |T| = k\}$$

is a natural number. In this series of lectures, we will explain how such strong boundedness results can be obtained for split Chevalley groups in a systematic and structural manner by invoking classical results in algebraic K-theory, so-called Sandwich Theorems, together with model-theoretic compactness arguments and explain obstructions to the existence of normally generating subsets of $G(\Phi, R)$. This will naturally divide the talks into three parts:

First, we will present the strong boundedness results for $\text{SL}_n(R)$ for $n \geq 3$. In this part, we will explain in more detail some of the previous results like Theorem 1 or Morris’ [4], a bit of the historical context, that is conjugation-invariant metrics on (hamiltonian) diffeomorphism groups and the archetypical Sandwich Theorem. If time permits, we will also talk about normally generating sets of $\text{SL}_n(R)$ and the only partially understood asymptotics of $\Delta_k(\text{SL}_n(R))$ in terms of $k$ and $n$ and how the bounds $\Delta_k(\text{SL}_n(R))$ compare to similar invariants called conjugacy diameters for finite, simple groups of Lie type.

Second, we will explain how the strong boundedness results generalize to essentially all other cases of $G(\Phi, R)$ using Sandwich Theorems except for $\Phi = \mathbb{C}_2, \mathbb{G}_2$ and $A_1$ [6]. Having seen the methods of the first part this is relatively straightforward and while there are some
differences due to the presence of two root lengths in non-simply-laced $\Phi$, ultimately the strong boundedness results are virtually identical to the ones of $\text{SL}_n(R)$.

Third, we will explain how strong boundedness results and the behavior of $N(G(\Phi, R, T)$ for $\Phi = C_2$ and $G_2$ differ from the higher rank cases: Contrary to the higher rank cases where strong boundedness appears as an almost pure first-order phenomena, in these lower rank cases, one is forced to use additional non first-order arguments and to consider the conjugacy width of certain congruence subgroups. Furthermore, we will construct epimorphisms obstructing the existence of small normally generating sets of $\text{Sp}_4(R)$ and $G_2(R)$ respectively. These epimorphisms will show that the differences between the cases of $\text{Sp}_4(R), G_2(R)$ and the other Chevalley groups are not merely artifacts of our proof strategies but due to actual structural differences between $\text{Sp}_4$ and $G_2$ and the higher rank cases.

These epimorphisms arise due to the presence of bad primes of the ring of $S$-algebraic integers $R$ for the corresponding Chevalley-Demazure group scheme. For example, consider the ring of Kleinian integers $R = \mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$. One can easily see in this case that $2R$ factors as $2R = (\omega) \cdot (\omega - 1)$ for $\omega := \frac{1+\sqrt{-7}}{2}$. But this then implies $R/(\omega) = R/(\omega - 1) = \mathbb{F}_2$ and so an epimorphism $\text{Sp}_4(R) \to \text{Sp}_4(R/(\omega)) \times \text{Sp}_4(R/(\omega - 1)) = \text{Sp}_4(\mathbb{F}_2)^2$ exists. But there is an exceptional isomorphism between $\text{Sp}_4(\mathbb{F}_2)$ and the permutation group $S_6$ and hence there is an epimorphism $\text{Sp}_4(R) \to \text{Sp}_4(\mathbb{F}_2)^2 \to \mathbb{F}_2 \oplus \mathbb{F}_2$. This epimorphism however makes it impossible to find a single conjugacy class that generates $\text{Sp}_4(R)$. If time permits, we will also explain why this type of obstruction is sufficient to classify normally generating subsets of $\text{Sp}_4$ and $G_2$.

Last, we will explain recent results concerning strong boundedness in the case of $\text{SL}_2(R)$ for $R$ a ring with infinitely many units and the more complicated shape that the epimorphisms obstructing the existence of small normal generating sets take in this case.

References


Alexander Trost, Ruhr University Bochum

E-mail address: Alexander.Trost@ruhr-uni-bochum.de