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Lectures on Garside Theory

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The philosophy of Garside's theory as developed in the past decades is that, under some assumptions, a group can be realised as a group of fractions of a monoid in which there exist divisibility relations that provide relevant information about the Garside group.

Garside structures first arose out of observations of properties of Artin's braid group that were made in Garside's Oxford thesis [7] and his article [8]. It was then realised that Garside's approach extend to all Artin (or, better, Artin–Tits) groups of spherical type independently by Brieskorn–Saito and Deligne in two adjacent articles in the Inventiones [2] and [6]. At the end of the 1990's, Dehornoy and Paris [5] defined the notion of Gaussian groups and Garside groups which leads, in "a natural, but slowly emerging program" as stated in [4], to Garside theory.

Garside groups are desirable since they admit efficient solutions to the Word and Conjugacy Problems. They enjoy important group-theoretical [4], homological, and homotopical properties [3]. Garside theory is also important within geometric group theory. Actually, there is a way to construct Garside groups from intervals in a given group (see [1]). The resulting Garside structures are called interval Garside structures.

The first lectures are devoted to introduce Garside monoids and groups, as well as Garside's normal form that served as a paradigmatic example in the theory of automatic groups (developed by Cannon, Thurston, and others) and the notion of a Garside family. We also develop a number of interesting applications of the theory such as the construction of aspherical spaces related to the Garside group and Krammer's algebraic proof of the linearity of braid groups [9]. Although Garside theory emerged from the theories of Coxeter and Artin groups, it is important to note here that Garside theory appears in different (and sometimes surprising) locations such as the Deligne–Lusztig varieties in reductive groups theory, and in the set-theoretic solutions of Yang–Baxter equations.

The middle lectures introduce the theories of Coxeter and Artin groups, as well as complex reflection and braid groups. For special families of Artin groups and for complex braid groups, we establish how Garside theory enables us to understand them by using the so-called standard and dual approaches. In one of the lectures, we would also like to highlight the vast void in the theory of Artin groups when Garside structures are not applicable.

In the last lectures, we shift attention to the geometry and topology of complements of hyperplane arrangements. Based on recent inventions of McCammond–Sulway [10], Paolini–Salvetti [11] were able to employ the generalised non-crossing partition posets within the dual Garside approach in order to solve the long-standing famous $K(\pi, 1)$ conjecture in the case of affine Artin groups. Together with Deligne's result [6], this provides a general hope for future developments to prove the $K(\pi, 1)$ conjecture in full generality by setting it in the framework of Garside theory.

References

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