New Perspectives in Hyperplane Arrangements
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Abstracts and references for preparation

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Simplicial arrangements

A simplicial arrangement is a set of linear hyperplanes decomposing the space into simplicial cones. More generally, a Tits arrangement decomposes a certain convex cone into simplicial cones. So far, Tits arrangements appeared (at least) in the following areas of mathematics:
(1) The special case in which the arrangement is crystallographic (this is a strong integrality property, see [Cun11a], [CH15]) can be considered as an invariant of Hopf algebras which is also called a Weyl groupoid. In particular it may be used to classify the so-called Nichols algebras (see for example [Cun18]).
(2) Tits arrangements generalize Coxeter groups and thus preserve some of their properties (see [CMW17]). For example, the complexified complement of a simplicial arrangement is a $K(\pi, 1)$-space (see [Del72]) and thus interesting from a topological point of view.
(3) Like reflection groups, simplicial arrangements produce interesting examples in the context of freeness of the module of derivations (see for example [BC12]). A counter example to the famous conjecture by Terao could be related to a simplicial arrangement.
(4) Simplicial arrangements of small rank play a role in the study of frieze patterns and thus of cluster algebras (see [Cun14]).

In my lecture I will report on old results as well as on recent progress, see for example [Cun11b], [Cun12], [CG15], [CM17], [CG17], [CMW17], [Cun18].

References

[BC12] M. Barakat and M. Cuntz, *Coxeter and crystallographic arrangements are inductively free*, Adv. Math. **229** (2012), no. 1, 691–709.

[CG15] M. Cuntz and D. Geis, *Combinatorial simpliciality of arrangements of hyperplanes*, Beitr. Algebra Geom. **56** (2015), no. 2, 439–458.

[CG17] ______, *Tits arrangements on cubic curves*, ArXiv e-prints (2017), 15 pp.

[CH15] M. Cuntz and I. Heckenberger, *Finite Weyl groupoids*, J. Reine Angew. Math. **702** (2015), 77–108.

[CM17] M. Cuntz and P. Mücksch, *Supersolvable simplicial arrangements*, ArXiv e-prints (2017), 33 pp.

[CMW17] M. Cuntz, B. Mühlherr, and C. J. Weigel, *Simplicial arrangements on convex cones*, Rend. Semin. Mat. Univ. Padova **138** (2017), 147–191, available at `arXiv:1505.08024`.

[Cun11a] M. Cuntz, *Crystallographic arrangements: Weyl groupoids and simplicial arrangements*, Bull. London Math. Soc. **43** (2011), no. 4, 734–744.

[Cun11b] ______, *Minimal fields of definition for simplicial arrangements in the real projective plane*, Innov. Incidence Geom. **12** (2011), 49–60.

[Cun12] ______, *Simplicial arrangements with up to 27 lines*, Discrete Comput. Geom. **48** (2012), no. 3, 682–701.

[Cun14] ______, *Frieze patterns as root posets and affine triangulations*, European J. Combin. **42** (2014), 167–178.

[Cun18] ______, *On subsequences of quiddity cycles and nichols algebras*, J. Algebra **502** (2018), 315 – 327.

[Del72] P. Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math. **17** (1972), 273–302.

Emanuele Delucchi (University of Fribourg)

Introduction to the combinatorics of Abelian arrangements

The combinatorial theory of arrangements of hyperplanes has blossomed since the ‘70s. Central at every turn in this development, from enumerative results about topological dissections to deep connections with topology, geometry, algebra and representation theory, is the notion of matroid.

Recently there has been an interest in studying Abelian arrangements - a class that contains arrangements in the torus and on products of elliptic curves, as well as arrangements of hyperplanes in vector spaces. This leads to the quest for a suitable combinatorial framework which could underpin and boost the general theory in as effective a way as matroids do in the case of hyperplanes.

In this minicourse I will introduce the basics of the combinatorial theory of hyperplane arrangements and discuss recent developments and open problems in the combinatorics of toric and elliptic arrangements, with a view towards a unified treatment.

Preparation:
Sections 2.1 and 2.3 of
P. Orlik, H. Terao: *Arrangements of Hyperplanes,* Springer-Verlag Berlin Heidelberg, 1992
can be a good foretaste or Chapter 2 of
P. Orlik: *Introduction to Arrangements,* CBMS Regional Conference Series in Mathematics, 72, American Mathematical Soc., 1989.

Graham Denham (Western University)

Arrangements, matroids, and toric varieties

Some useful constructions for the study of complex hyperplane arrangements involve toric geometry in an intrinsic way, and this is closely related to the basic role hyperplane arrangements play in tropical geometry. The toric varieties that appear in this way can have interesting and nontrivial underlying combinatorics.

In these lectures, we will introduce and examine basic objects like the Bergman fan of a matroid. We will consider the corresponding geometry of wonderful compactifications. We will see that some aspects of the geometric picture survive beyond hyperplane arrangements in a purely combinatorial setting, such as the remarkable tropical Hodge theory of Adiprasito, Huh and Katz.

Students will not be assumed to have an in-depth background in the theory of hyperplane arrangements or toric varieties.

Michael Falk (Northern Arizona University)

Topology and Combinatorics of Arrangements

Lecture 1: Orlik-Solomon algebras and Tutte polynomials
In this lecture we establish the main properties of the Orlik-Solomon algebra of a complex hyperplane arrangement, using mainly combinatorial arguments along with a couple of elementary results from differential topology. We introduce the fundamental combinatorial invariant of an arrangement, its underlying matroid, and establish the isomorphism of the Orlik-Solomon algebra of the underlying matroid with the cohomology algebra of the complement, along with the identification of betti numbers of the complement with Whitney numbers of the second kind, the no-broken-circuit basis for the Orlik-Solomon algebra, and the formality of the complement.

Lecture 2: Resonance varieties, multinets, master functions
In this lecture we define the resonance varieties of a complex hyperplane arrangement, and prove the characterization of the degree-one resonance variety via neighborly partitions. We will explain the connection with pencils of curves, nets and multinets, critical loci of products of linear forms, and illustrate with several classical examples.

Lecture 3: Local systems, characteristic varieties, higher resonance
In this lecture we explain the connection of resonance varieties with cohomology of local systems. We define the characteristic varieties of a complex hyperplane arrangement, and discuss the tangent cone theorem, Aomoto complex, and propagation of resonance. We close the course with some examples and results concerning resonance varieties in degrees greater than one, including a generalization of neighborly partitions that characterizes a part of the degree p resonance variety of a p-generic arrangement.

References for preparation:

- **Local systems:**
  Section 3.H of
  A. Hatcher, *Algebraic Topology,* Cambridge University Press, 2002.
  and
  Chapter 8: *Local systems on complements of arrangements*
  of a book in progress. Will be sent via e-Mail after registration.

- **Matroids:**
  T. Brylawski and D. G. Kelly, *Matroids and Combinatorial Geometries,* Univ. North Carolina, Chapel Hill, NC (1980).
  Will be sent via e-Mail after registration.

Luis Paris (Université de Bourgogne)

Artin groups and hyperplane arrangements

An *Artin group* is by definition a group with a presentation with relations of the form $sts\cdots = tst\cdots$, where the word in the left hand side and the word in the right hand side have the same length. The standard example of such a group is the braid group. In this mini-course by a *hyperplane arrangement* we mean a finite family of linear hyperplanes in a finite dimensional real vector space. The theory of Artin groups and that of hyperplane arrangements are in some sense born from the same series of papers essentially due to Brieskorn in the early 1970s, and their stories are closely related. The purpose of this mini-course is to explain why.

The first ingredient is the notion of *(real) reflection group.* Let $V = \mathbb{R}^n$ be a finite dimensional real vector space and let W be a finite subgroup of $\mathrm{GL}(V)$ generated by reflections. A classical theorem of Coxeter dated from the 1930s says that such a group admits a presentation with relations of the form $sts\cdots = tst\cdots$ as above together with relations of the form $s^2 = 1$. Notice that by removing the relations $s^2 = 1$ we get an Artin group $A = A_W$. The set of reflecting hyperplanes of the reflections lying in W forms a hyperplane arrangement $\mathcal{A} = \mathcal{A}_W$ called *Coxeter arrangement.* Now, define the *complement of the complexification* of $\mathcal{A}$ to be

$$M(\mathcal{A}) = \mathbb{C}^n \setminus \left(\bigcup_{H\in\mathcal{A}} \mathbb{C}\otimes H \right).$$

A (not trivial) consequence of Coxeter's work is that the reflection group W acts freely on $M(\mathcal{A})$. We denote by $N(\mathcal{A}) = M(\mathcal{A})/W$ the quotient of $M(\mathcal{A})$ by W. By a theorem of Brieskorn the fundamental group of $N(\mathcal{A})$ is the Artin group A_W associated with W. In the case where W is the symmetric group with its standard action on $\mathbb{R}^n$ the Artin group A_W is the braid group and the space $N(\mathcal{A})$ is the space of unordered configurations of n points in $\mathbb{C}$.

This is the beginning of the story and the topic of the mini-course.

Historical references:

- **H. S. M. Coxeter,** *Discrete groups generated by reflections,* Ann. of Math. (2) 35 (1934), no. 3, 588–621.
- **H. S. M. Coxeter,** *The complete enumeration of finite groups of the form $R_i^2 = (R_iR_j)^{k_{i,j}} = 1$,* J. London Math. Soc. 10 (1935), 21–25.
- **E. Brieskorn**, *Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe,* Invent. Math. 12 (1971), 57–61.
- **E. Brieskorn,** *Sur les groupes de tresses [d'après V. I. Arnol'd],* Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401, pp. 21–44. Lecture Notes in Math., Vol. 317, Springer, Berlin, 1973.

Reference to learn:

- **L. Paris,** *$K(\pi, 1)$ conjecture for Artin groups,* Ann. Fac. Sci. Toulouse Math. (6) 23 (2014), no. 2, 361–415.

Masahiko Yoshinaga (Hokkaido University)

Introduction to Catalan arrangements

The Catalan arrangement is a collection of certain parallel translated hyperplanes of the braid arrangements. As the name shows, the number of chambers is related to the so-called Catalan number. Catalan arrangements and its root system generalizations have been intensively studied from algebraic and combinatorial view points. We survey some of these studies with focuses on the notion of free arrangements. In particular, we will discuss recent explicit description of the basis of log vector fields of Catalan arrangements in terms of discrete integrations.

References:

- **C. A. Athanasiadis,** *Generalized Catalan numbers, Weyl groups and arrangements of hyperplanes,* Bull. London Math. Soc. 36 (2004), no. 3, 294–302.
- **P. H. Edelman and V. Reiner,** *Free arrangements and rhombic tilings,* Discrete Comput. Geom. 15 (1996), no. 3, 307–340.
- **A. Postnikov and R. Stanley,** *Deformations of Coxeter hyperplane arrangements,* J. Combin. Theory Ser. A 91 (2000), no. 1-2, 544–597.
- **K. Saito,** *Uniformization of the orbifold of a finite reflection group,* Frobenius manifolds, 265-320, Aspects Math., E36, Vieweg, Wiesbaden, 2004.
- **M. Yoshinaga,** *Worpitzky partitions for root systems and characteristic quasi-polynomials,* Tohoku Math. J. 70 (2018), to appear.
- **M. Yoshinaga,** *Freeness of hyperplane arrangements and related topics,* Annales de la Faculte des Sciences de Toulouse, vol. 23 no. 2 (2014), 483-512. (arXiv:1212.3523)