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Towards the Linearity of Complex Braid Groups

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Complex reflection groups are finite groups generated by complex reflections, where a complex reflection is a linear transformation of finite order that fixes a hyperplane pointwise. These groups include finite real relection groups, also known as finite Coxeter groups. It is well known that every complex reflection group is a direct product of irreducible ones. Irreducible complex reflection groups have been classified by Shephard and Todd [14] in 1954. The classification includes the infinite 3-parameter series $G(de, e, n)$ that can be easily described in terms of monomial matrices, and 34 exceptional groups denoted by G_4, G_5, $\ldots$, G_{37}.

Broué, Malle, and Rouquier [3] managed to attach a complex braid group to each complex reflection group, and to establish presentations for almost all these groups including all the infinite series. Their constructions generalise the notion of Artin–Tits groups attached to real reflection groups. Many discoveries unveil the fact that nice properties of these objects in the case of real reflection groups could be extended to the general case of complex reflection groups.

Denote by $B(de, e, n)$ the complex braid groups related to the infinite series $G(de, e, n)$. Our discussion concerns these families of complex braid groups and is focused on the investigation of two important research directions:

I- Construct natural Garside structures.

II- Construct faithful Krammer representations.

It is widely believed that the right approach to study Artin–Tits groups is via Garside structures. Actually, in his 1969 thesis [8], Garside solved the word and conjugacy problems in the usual braid group. In 1972, his results have been generalised independently by Brieskorn–Saito and Deligne to all Artin–Tits groups related to finite Coxeter groups. Furthermore, at the end of the 1990's, Dehornoy and Paris defined the notion of Gaussian groups and Garside groups which leads to Garside theory (see [7]). In Winter Semester 2020/21, the author has developed Garside theory, and presented the richness of the subject (this is I). It is recommended to revise the general definitions within this theory.

Unfortunately, the presentations introduced by Broué, Malle, and Rouquier [3] for the complex braid groups $B(de, e, n)$ do not give rise to Garside structures. Therefore, it is interesting to search for (possibly various) Garside structures for these groups. In his PhD thesis [12], the author has obtained interval Garside structures for $B(e, e, n)$ that derive from natural and explicit intervals in the associated complex reflection group (see also [13]). This requires the elaboration of a combinatorial technique in order to determine geodesic normal forms in $G(e, e, n)$ over an appropriate generating set obtained earlier by Corran–Picantin. We will now discuss II and reveal how these Garside structures will be employed in our discussion.

Both Bigelow [1] and Krammer [9, 10] proved that the classical braid group

B_n is linear, that is there exists a faithful linear representation of finite dimension. This result has been extended to all Artin-Tits groups associated to finite Coxeter groups independently by Cohen–Wales and Digne by generalising Krammer's representation as well as Krammer's faithfulness proof. Note that for the case of the classical braid group, the representations of B_n occur in earlier work of Lawrence. The faithfulness criterion used by Krammer can be stated for a Garside group. It provides necessary conditions to prove that a linear representation of a Garside group is faithful.

Consider now a complex braid group $B(de, e, n)$. For $d > 1$, $e \geq 1$ and $n \geq 2$, it is known that the group $B(de, e, n)$ injects in the finite-type Artin–Tits group $B(de, 1, n)$; see [3]. Since $B(de, 1, n)$ is linear, we have $B(de, e, n)$ is linear for $d > 1$, $e \geq 1$ and $n \geq 2$. Note that $B(1, 1, n)$, $B(2, 2, n)$ and $B(e, e, 2)$ are finite-type Artin–Tits groups. All of them are then linear. The only remaining cases in the general series are when $d = 1$, $e > 2$ and $n > 2$. This arises the following question: Is $B(e, e, n)$ linear for all $e > 2$ and $n > 2$? In our attempt to provide an answer to this question, we shift attention to a particular aspect of Krammer's representations that we explain in the next paragraph.

Zinno [15] observed that Krammer's representation of the classical braid group B_n factors through the BMW (Birman-Murakami-Wenzl) algebra introduced in [2, 11]. In [5], Cohen-Gijsbers-Wales defined a BMW algebra for Artin–Tits groups of type ADE and showed that the faithful representation constructed by Cohen–Wales in [6] factors through their BMW algebra. In [4], Chen defined a BMW algebra for the dihedral groups, based on which he defined a BMW algebra for any Coxeter group.

Attempting to make a similar approach in order to explicitly construct faithful irreducible representations for the complex braid groups $B(e, e, n)$, we define a BMW algebra for type (e, e, n) that we denote by $\mathrm{BMW}(e, e, n)$. This definition is inspired from the Garside monoids established in the author's PhD, and is a generalisation of the definitions of the BMW algebras for the dihedral groups and for type ADE of Coxeter groups. Moreover, we describe $\mathrm{BMW}(e, e, n)$ as a deformation of a certain algebra that we call the Brauer algebra of type (e, e, n) that we denote by $\mathrm{Br}(e, e, n)$. For $e = 1$, we recover the usual algebra of Brauer diagrams.

We are able to construct explicit linear (finite dimensional and absolutely irreducible) representations for some cases of the complex braid groups $B(e, e, n)$. Actually, they are irreducible representations of the corresponding BMW algebras. Our method uses the computation of a Gröbner basis from the list of (non-commutative) polynomials that describe the relations of $\mathrm{BMW}(e, e, n)$. We put forward many arguments and prove a number of properties that allow us to believe that these explicit representations are good candidates to be called Krammer representations for the associated complex braid groups. Finally, we conjecture that these representations are faithful and propose a number of conjectures related to the structure of the BMW algebra.

We establish the necessary background to accurately describe all these results and motivate our study in a series of two lectures. In the second lecture, we propose a research program that seeks to construct faithful Krammer representations for the complex braid groups.

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