Algebraic Groups and Buildings

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1 Introduction

In this lecture series we provide an approach to Borel-Tits theory from the perspective of buildings. We deal with abstract groups acting highly transitive on spherical buildings. The group of rational points of an isotropic semi-simple algebraic group over a field admits such an action. The groups considered in these lectures can be characterized as groups endowed with a spherical RGD-system (where RGD stands for *root group datum*). The axioms for an RGD-system were formulated by Tits in 1992 (see [16]). There had been, however, early predecessors of RGD-systems that played an important role in Tits' work on algebraic groups (see Section 2 of [12]).

The main objectives of the lecture series are the following.

1. Tits' simplicity proof for groups with an irreducible RGD-system,
2. An outline of the proof of the Borel-Tits theorem on unipotent subgroups based on the center theorem for spherical buildings,
3. Existence of an RGD-system in isotropic, semi-simple algebraic groups over fields via descent in buildings.

We remark that a proof of the Borel-Tits theorem using buildings has been indicated in [12]. The proof of this theorem given later in [2] is based on different arguments.

We start with a brief introduction to buildings. Using Weyl-transitive actions on buildings we provide a first version of Tits' simplicity criterion. We then study spherical buildings and introduce Moufang structures on them. Moufang structures are just an alternative way of dealing with RGD-systems. Their definition is slightly more general and more suitable for our purposes. Using Moufang structures, we are able to present Tits' simplicity criterion as given in [13]. We then consider convex sets of chambers in spherical buildings in order to discuss the center theorem. Using the center theorem we will be able to prove a building-theoretic version of the Borel-Tits theorem on unipotent subgroups. The next goal is to develop a theory of descent in spherical buildings. In a first step we study fixed-point sets of automorphisms in buildings. A group Γ acting on a building $\mathcal{B}$ will be called a descent group if its fixed point set is a building. We associate to any descent group a (combinatorial) Tits index. If $\mathcal{B}$ admits a Moufang structure, then there is also a canonical Moufang structure on the fixed-point building and hence an RGD-system in its group of automorphisms.

The content of these notes is based on work (or at least ideas) of Jacques Tits. The main reference for the building theoretic version of Galois descent for semi-simple algebraic groups is Chapter 3 in [5]. We remark that the combinatorial version of Borel-Tits theory plays a central role in recent joint work with Richard Weiss ([8], [9].

2 Examples of buildings

2.1 Notation for graphs

By a graph we always mean a simplicial graph. Thus a graph Γ is a pair (V, E) consisting of a set V and a set $E \subseteq \{X \subseteq V \mid |X| = 2\}$. The elements of V (resp. E) are called the vertices (resp. edges) of Γ. For the rest of this subsection $\Gamma = (V, E)$ is a graph.

For a vertex $v \in V$ we denote the set of its neighbors by Γ_v, hence $\Gamma_v := \{w \in V \mid \{v, w\} \in E\}$.

A path (of length k) is a sequence $\pi := (v_0, v_1, \ldots, v_k)$ of vertices such that $v_{i-1} \neq v_{i+1}$ and $v_{i-1}, v_{i+1} \in \Gamma_{v_i}$ for all $1 \leq i < k$. Its length k will is denoted by $\ell(\pi)$. The path π is called a circuit if $k \geq 3$ and $v_0 = v_k$.

Let $v, w \in V$ be vertices. A path from v to w is a path $(v = v_0, v_1, \ldots, v_k = w)$ and $\mathrm{Path}(v, w)$ denotes the set of all paths from v to w. The distance $\mathrm{dist}(v, w)$ between v and w is defined as follows:

$$\mathrm{dist}(v, w) := \begin{cases} \infty, & \text{if } \mathrm{Path}(v, w) = \emptyset, \\ \min\{\ell(\pi) \mid \pi \in \mathrm{Path}(v, w)\}, & \text{if } \mathrm{Path}(v, w) \neq \emptyset. \end{cases}$$

A path $\pi = (v_0, v_1, \ldots, v_k)$ is called minimal if $\mathrm{dist}(v_0, v_k) = k$.

The graph Γ is connected if $\mathrm{dist}(v, w) \neq \infty$ for all $v, w \in V$. The diameter of Γ is defined by $\mathrm{diam}(\Gamma) := \sup\{\mathrm{dist}(v, w) \mid v, w \in V\}$ and its girth is $\mathrm{girth}(\Gamma) := \min\{\ell(\pi) \mid \pi \text{ a circuit of } \Gamma\}$. The graph Γ is called bipartite if there is a partition $V = X \cup Y$ such that $|e \cap X| = 1$ for each edge. Recall that Γ is bipartite if and only if there are no circuits of odd length.

A set $C \subseteq V$ is called a clique of Γ if any two vertices in C form an edge of Γ.

2.2 The building $A_n(K)$

For $1 \leq n \in \mathbf{N}$ the set $\{1, \ldots, n\}$ is denoted with $[n]$. Let $1 \leq n \in \mathbf{N}$, K a field and X a vector space over K of dimension $n + 1$.

We define the set $V := \mathcal{V}(X) := \{U \leq X \mid \{0\} \neq U \neq X\}$, join two elements $U \neq W \in V$ by an edge if $U \subset W$ or $W \subset U$ and we denote the corresponding graph by $\Gamma(X)$.

A flag of X is a clique of $\Gamma(X)$ and for each flag we set $\mathrm{typ}(F) := \{\dim_K U \mid U \in F\} \subseteq 2^{[n]}$. A chamber of X is a maximal flag, that is, a flag of type $[n]$ and $\mathcal{C}$ denotes the set of all chambers. For a chamber c and $1 \le i \le n$ we denote the subspace of dimension i in c by c^i.

Let $B = (b_1, \dots, b_{n+1})$ be an ordered base of X. We put $c(B) := \{\langle b_1, \dots, b_i \rangle \mid 1 \le i \le n\} \in \mathcal{C}$. For a permutation $\pi \in \mathrm{Sym}(n+1)$ we put $B_\pi = (b_{\pi(1)}, \dots, b_{\pi(n+1)})$. The apartment associated to B is the set $\Sigma(B) := \{c(B_\pi) \mid \pi \in \mathrm{Sym}(n+1)\}$.

Lemma 2.1: *Let $c, d \in \mathcal{C}$.*

a) There exist an ordered basis B of X and a permutation $\pi \in \mathrm{Sym}(n+1)$ such that $c = c(B)$ and $d = c(B_\pi)$;

b) If an ordered basis B' of X and $\pi' \in \mathrm{Sym}(n+1)$ are such that $c = c(B')$ and $d = c(B'_{\pi'})$, then $\pi = \pi'$.

Proof: See Example 7.4 in [18]. □

The previous lemma provides a mapping $\delta : \mathcal{C} \times \mathcal{C} \to \mathrm{Sym}(n+1)$ and we have for $c, d \in \mathcal{C}$:

1. $\delta(d, c) = \delta(c, d)^{-1}$;
2. for $1 \le k \le n$ we have $\delta(c, d) = (k\ k+1)$ if and only if $c^i = d^i$ for all $i \ne k$ and $c^k \ne d^k$.

We put $A_n(K) := (\mathcal{C}, \delta)$.

2.3 Buildings of type $I_2(m)$

Let $2 \le m \in \mathbf{N}$. A generalized m-gon is a bipartite graph Γ such that $2\,\mathrm{diam}(\Gamma) = 2m = \mathrm{girth}(\Gamma)$. A generalized m-gon is called thick if each of its vertices has at least three neighbors.

Remarks: 1. The generalized 2-gons are precisely the complete bipartite graphs such that there are at least two vertices of each colour.

2. The thick generalized 3-gons are precisely the incidence graphs of projective planes. In particular, if $\dim_K X = 3$, then the graph $\Gamma(X)$ defined in the previous subsection is a generalized 3-gon.

3. Examples of generalized 4-gons arise from non-degenerate hermitian or quadratic forms of Witt-index 2. There are also examples related to exceptional algebraic groups.

4. Using free constructions one can show that thick generalized m-gons exist for all $2 \le m \in \mathbf{N}$. By a famous theorem of Feit and Higman one knows that thick *finite* generalized polygons only exist for $m = 2, 3, 4, 6$ or 8.

Some basic observations: Let $2 \leq m \in \mathbf{N}$ and let $\Gamma = (V, E)$ be a generalized m-gon. An apartment of Γ is a circuit of length $2m$.

1. Let $\pi = (v_0, \ldots, v_k)$ be a path in Γ. If $k \leq m$, then π is minimal; if $k < m$, then π is the only minimal path from v_0 to v_k and if $k = m + 1$ then π is contained in a unique apartment.

2. Any two edges are contained in an apartment.

3. Let $v, w \in V$ be at distance m. Then the following holds: For each neighbor $x \in \Gamma_v$ there exists a unique neighbor of v at distance $m - 2$ and it is denoted by $\mathrm{proj}_w^v\, x$. The mappings $\mathrm{proj}_w^v \colon \Gamma_v \to \Gamma_w, v \mapsto \mathrm{proj}_w^v\, x$ and $\mathrm{proj}_v^w \colon \Gamma_w \to \Gamma_v, x \mapsto \mathrm{proj}_v^w\, x$ are inverse bijections.

Opposition in generalized m-gons: Let $2 \leq m$ and let $\Gamma = (V, E)$ be a generalized m-gon. Two vertices of Γ are called opposite if their distance is equal to m, and the set of vertices opposite $v \in V$ is denoted by v^{op}. Two edges $e = \{v, w\}, f \in E$ are called opposite if $v^{\mathrm{op}} \cap f \neq \emptyset \neq w^{\mathrm{op}} \cap f$.

Note that two edges lie in a unique circuit of length $2m$, hence they determine a unique apartment. Given an edge e and an apartment Σ, then there exists at least one edge in Σ which is opposite to e.

3 Coxeter systems

Let W be a group and let $S = S^{-1} \subseteq W$ such that $W = \langle S \rangle$. Let $F(S) \coloneqq \bigcup_{k \in \mathbf{N}} S^k$ be the free monoid on S, $\ell \colon F(S) \to \mathbf{N}$ its length function and $\pi \colon F(S) \to W$ the product map.

We call $f \in F(S)$ a representation of $w \in W$ if $\pi(f) = w$ and denote the set of all representations of w by $\mathrm{Rep}(w)$. We put $\ell(w) \coloneqq \min\{\ell(f) \mid f \in \mathrm{Rep}(w)\}$. The word $f \in \mathrm{Rep}(w)$ is called a reduced representation of w if $\ell(f) = \ell(w)$. A word $f \in F(S)$ is called reduced if it is a reduced representation of $\pi(f)$.

Definition: Let W and S be as above. The pair (W, S) is called a Coxeter system if the following conditions are satisfied for all $w \in W$ and $s, t \in S$:

(CS1) $s^2 = 1 \neq s$;

(CS2) $\ell(ws) \neq \ell(w)$;

(CS3) if $\ell(sw) = \ell(w) + 1 = \ell(wt)$ then $\ell(swt) = \ell(w) + 2$ or $sw = wt$.

Examples: 1. Let $1 \leq n \in \mathbf{N}$, let $s_k \coloneqq (k\ k+1) \in \mathrm{Sym}(n+1)$ for $k \in [n]$ and $S \coloneqq \{s_k \mid k \in [n]\}$. Then $(\mathrm{Sym}(n+1), S)$ is the Coxeter system of type A_n.

2. Let $2 \leq m \in \mathbf{N}$ and let $D_{2m} = \langle s, t \rangle$ be the dihedral group of order $2m$. Then $(D_{2m}, \{s, t\})$ is the Coxeter system of type $I_2(m)$.

3. Any finite reflection group W admits a generating set S such that (W, S) is a Coxeter system.

Conventions and Definitions: Let (W, S) be a Coxeter system.

1. The rank of (W, S) is the cardinality of S. In these notes all Coxeter systems are assumed to have finite rank.

2. The Coxeter matrix of (W, S) is the matrix $M := (m_{st})_{s,t \in S}$ where m_{st} denotes the order of st. The diagram of (W, S) is the edge labeled graph with vertex set S in which two vertices s and t are joined by an edge labeled by m_{st} if $m_{st} \geq 3$. There is the convention that one omits the label if $m_{st} = 3$ and edges with label 4 are represented by a double bond.

3. The Coxeter system is called irreducible if its diagram is connected.

4. An automorphism of (W, S) is an automorphism of W stabilizing S; it induces an automorphism of the diagram.

Basic facts on Coxeter systems: Let (W, S) be a Coxeter system.

1. $W \cong \langle S \mid ((st) = 1)_{s,t \in S} \rangle$;

2. For $J \subseteq S$ the pair $(\langle J \rangle, J)$ is a Coxeter system. If $(s_1, \ldots, s_k)$ is a reduced representation (in (W, S)) of an element $w \in \langle J \rangle$, then $s_i \in J$ for all $1 \leq i \leq k$. In particular, the restriction to $\langle J \rangle$ of the length function of (W, S) is the length function of $(\langle J \rangle, J)$.

3. Let $w \in W$ and $J \subseteq S$. Then there exists a unique element $v \in w\langle J \rangle$ such that $\ell(x) = \ell(v) + \ell(v^{-1}x)$ for all $x \in w\langle J \rangle$.

4 Buildings

Definition: Let (W, S) be a Coxeter system. A building of type (W, S) is a pair $\mathcal{B} = (\mathcal{C}, \delta)$ consisting of a set $\mathcal{C}$ and a mapping $\delta \colon \mathcal{C} \times \mathcal{C} \to W, (c, d) \mapsto \delta(c, d)$, such that the following conditions are satisfied for all $c, d \in \mathcal{C}$ where $w := \delta(c, d)$.

(Bu1) $w = 1$ if and only if $c = d$;

(Bu2) if $d' \in \mathcal{C}$ is such that $\delta(d, d') = s \in S$, then $\delta(c, d') \in \{w, ws\}$ and $\delta(c, d') = ws$ if $\ell(ws) = \ell(w) + 1$;

(Bu3) for each $s \in S$ there is a chamber $d' \in \mathcal{C}$ such that $\delta(d, d') = s$ and $\delta(c, d') = ws$.

Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of type (W, S). The elements of $\mathcal{C}$ are called the chambers of $\mathcal{B}$ and δ is called its Weyl-distance. The rank of $\mathcal{B}$ is the cardinality of S and $\mathcal{B}$ is irreducible if (W, S) is irreducible.

Examples:

1. Let $1 \leq n \in \mathbf{N}$ and K be a field. Then $A_n(K)$ is a building of type A_n.
2. Let $2 \leq m \in \mathbf{N}$, let $\Gamma = (V, E)$ be a generalized m-gon and let $(D_{2m}, \{s, t\})$ be the Coxeter system of type $I_2(m)$. Let $V = X \cup Y$ be the bipartite partition. There exists a unique Weyl-distance on $\mathcal{C} := E$ such that the following holds for any two edges $e, f \in E$: $\delta(e, f) = s$ if and only if $\emptyset \neq e \cap f \subseteq X$ and $\delta(e, f) = t$ if and only if $\emptyset \neq e \cap f \subseteq Y$.
3. A building of rank 1 (that is a building of type $(\langle s \rangle, \{s\})$) is just a set $\mathcal{C}$ of cardinality at least 2 where $\delta(c, d) = s$ if $c \neq d$.
4. Let (W, S) be a Coxeter system. Setting $\mathcal{C} := W$ and $\delta(c, d) := c^{-1}d \in W$ for any two $c, d \in W$ one obtains the Coxeter building of type (W, S).

The chamber graph and galleries: Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of type (W, S). Two chambers $c, d \in \mathcal{B}$ are called adjacent (resp. s-adjacent) if $\delta(c, d) \in S$ (resp. $\delta(c, d) = s \in S$) and the chamber graph is $\mathrm{Cham}(\mathcal{B}) = (\mathcal{C}, E)$ where $E = \{\{c, d\} \mid \delta(c, d) \in S\}$. A gallery is a path $\gamma = (c_0, \ldots, c_k)$ in $\mathrm{Cham}(\mathcal{B})$ and its type is the word $\mathrm{typ}(\gamma) = (s_1, \ldots, s_k)$ where $s_i := \delta(c_{i-1}, c_i)$. A minimal gallery is a minimal path in $\mathrm{Cham}(\mathcal{B})$.

Proposition 4.1: *A gallery γ is minimal if and only if its type is reduced.*

Proof: This follows from Lemma 5.16 b) and Exercise 5.20 in [1]. □

Let $c, d \in \mathcal{C}$. A (minimal) gallery from c to d is a (minimal) path from c to d in $\mathrm{Cham}(\mathcal{B})$. We put $\ell(c, d) = \mathrm{dist}(c, d)$ (in $\mathrm{Cham}(\mathcal{B})$).

Proposition 4.2: *Let $c, d \in \mathcal{C}$ and $w := \delta(c, d) \in W$. Then $\gamma \mapsto \mathrm{typ}(\gamma)$ is a bijection from the set of minimal galleries onto the set of reduced representations of w. In particular, $\ell(c, d) = \ell(\delta(c, d))$.*

Proof: See Lemma 5.16 b) in [1]. □

Residues and projections: Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of type (W, S).

For a chamber $c \in \mathcal{C}$ and a subset J of S, the set $R_J(c) := \{d \in \mathcal{C} \mid \delta(c, d) \in \langle J \rangle\}$ is called the J-residue of c.

Proposition 4.3: *Let $c \in \mathcal{C}$, $J \subseteq S$ and $R = R_J(c)$. Then the following hold.*

a) For all $x, y \in R$ we have $\delta(x, y) \in \langle J \rangle$ and (R, δ_R) is a building of type $(\langle J \rangle, J)$ where δ_R is the restriction of δ on R.

b) For $K \subseteq S$ we have $R \cap R_K(c) = R_{(J \cap K)}(c)$.

Proof: See Lemma 5.29 and Corollary 5.30 in [1] for a) and Exercise 5.32 for b). □

A residue of $\mathcal{B}$ is a subset R of $\mathcal{C}$ such that $R = R_J(c)$. It follows from the previous proposition that the subset J of S is uniquely determined by the set R. It is called the type of R and denoted by $\mathrm{typ}(R)$. The rank of R is the cardinality of $\mathrm{typ}(R)$.

A panel is a residue of rank 1 and an s-panel is a residue of type $\{s\}$. The building $\mathcal{B}$ is called thick if each panel contains at least three chambers.

Proposition 4.4: *Let R be a residue and $c \in \mathcal{C}$. Then there exists a unique chamber $d \in R$ such that $\ell(c, x) = \ell(c, d) + \ell(d, x)$ for all $x \in R$.*

If $J = typ(R)$ and $w = \delta(c, d)$, then w is the shortest element in $w\langle J\rangle$.

PROOF: See Proposition 5.34 in [1]. □

Let c and R be as in the previous proposition. Then the chamber d is called the projection of c onto R and denoted by $\mathrm{proj}_R\, c$.

Isometries and Automorphisms: Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of type (W, S).

Let θ be an automorphism of (W, S). A θ-isometry of $\mathcal{B}$ is a permutation α of $\mathcal{C}$ such that $\delta(\alpha(c), \alpha(d)) = \theta(\delta(c, d))$ for all $c, d \in \mathcal{C}$. An automorphism of $\mathcal{B}$ is a θ-isometry of $\mathcal{B}$ for some automorphism θ of (W, S) and the group of automorphisms of $\mathcal{B}$ is denoted by $\mathrm{Aut}(\mathcal{B})$. An isometry of $\mathcal{B}$ is a id_W-isometry and the group of isometries of $\mathcal{B}$ is denoted by $\mathrm{Spe}(\mathcal{B})$. It is the kernel of a natural homomorphism $\mathrm{typ}\colon \mathrm{Aut}(\mathcal{B}) \to \mathrm{Aut}(W, S)$. Note that $\ell(c, d) = \ell(\alpha(c), \alpha(d))$ for all $c, d \in \mathcal{C}$ and all automorphisms α.

The following technical lemma will be used in the proof of the simplicity criterion.

Lemma 4.5: *Let $G \leq \mathrm{Spe}(\mathcal{B})$ be transitive on $\mathcal{C}$ and R be a J-residue. Suppose that M is a normal subgroup of G which stabilizes R and is transitive on R. Then $[J, S \setminus J] = 1$.*

5 Weyl-transitive actions

Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a thick building of type (W, S).

Let G be a group acting on $\mathcal{B}$ by isometries. The action of G is called Weyl-transitive if G is transitive on $X_w = \{(c, d) \in \mathcal{C}^2 \mid \delta(c, d) = w\}$ for each $w \in W$.

1. A Weyl-transitive action of a group G on a building of rank 1 is just a 2-transitive action on the set of chambers. Hence, the stabilizer of a chamber in G is a maximal subgroup of G in the rank 1 situation.

2. A Weyl-transitive action is transitive on the set of chambers and hence also on the set of J-residues of $\mathcal{B}$ for each $J \subseteq S$.

3. Suppose the action of G on $\mathcal{B}$ is Weyl-transitive. Let R be a residue and P the stabilizer of R in G. Then the action of P on R is Weyl-transitive on the building R.

Proposition 5.1: *Suppose that the group G acts Weyl-transitively by isometries on the building $\mathcal{B}$. Let $c \in \mathcal{C}$ and B denote the stabilizer of c in G. Let $H \leq G$ be a subgroup containing B. Then H is the stabilizer of a $R_J(c)$ for some $J \subseteq S$.*

Proof: Let Ω be the H-orbit of c and let $J := \{s \in S \mid \delta(c,d) = s \text{ for some } d \in \Omega\}$.

Let $s \in J$, let P denote the s-panel containing c and let U be the stabilizer of P in G. Since B is maximal in U and $H \cap U$ contains B properly, it follows that U is contained in H. Thus H is transitive on P. This shows that H is transitive on $R_J(c)$.

Let $d \in \Omega$, $w := \delta(c,d)$ and $s \in S$ such that $\ell(ws) = \ell(w) - 1$. Then one shows that $s \in J$ and hence $d \in R_J(c)$ (by induction on $\ell(c,d)$). □

Proposition 5.2: *Let $\mathcal{B}$ be irreducible and let G be a group acting Weyl-transitively on $\mathcal{B}$. Let M be a normal subgroup of G which is not contained in the kernel of the action. Then the following hold:*

a) The group M is transitive on $\mathcal{C}$.

b) Let $c \in \mathcal{C}$ and B the stabilizer of c in G. If G is perfect and B is solvable, then $M = G$.

6 Spherical buildings

Definition: A Coxeter system (W,S) is called spherical if W is a finite group.

Proposition 6.1: *Let (W,S) be a spherical Coxeter system. Then there exists a unique element $\rho \in W$ such that $\ell(w) \le \ell(\rho)$ for all $w \in W$. The element ρ has the following properties.*

a) $\rho^2 = 1$;

b) $\ell(w) + \ell(w^{-1}\rho) = \ell(\rho)$ for all $w \in W$;

c) $\rho S \rho = S$.

Remarks:

1. Let (W,S) be spherical Coxeter system. By the previous proposition the map $s \mapsto \rho s \rho$ is an automorphism of the diagram of (W,S) whose square is the identity.

2. The longest element in the Coxeter system of type A_n is the involution sending k onto $n+2-k$ for all $k \in [n+1]$; the longest element in $I_2(m)$ is the alternating product of length m.

Lemma 6.2: *Let (W,S) be a Coxeter system, $w \in W$ and $J = \{s \in S \mid \ell(ws) = \ell(w) - 1\}$. Then $\langle J \rangle$ is finite and if ρ_J denotes the longest element in $(\langle J \rangle, J)$, then $\ell(w\rho_J) = \ell(w) - \ell(\rho_J)$ and $w\rho_J$ is the shortest element in $w\langle J \rangle$.*

In particular, if $\ell(ws) = \ell(w) - 1$ for all $s \in S$, then (W,S) is spherical and w is the longest element in (W,S).

Opposite chambers and apartments: Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of spherical type (W,S) and let ρ denote the longest element in (W,S).

1. Two chambers c, d are called opposite if $\delta(c, d) = \rho$ and c^{op} denotes the set of all chambers opposite to c.

2. Two chambers c, d in the building $A_n(K)$ are opposite precisely if $X = c^k \oplus d^{n+1-k}$ for all $k \in [n]$.

3. For any two chambers $c, d \in \mathcal{C}$, $\Sigma(c, d)$ denotes the set of all chambers which are on a minimal gallery from c to d. An apartment of $\mathcal{B}$ is a subset Σ of $\mathcal{C}$ such that $\Sigma = \Sigma(c, d)$ for two opposite chambers c, d.

Proposition 6.3: *Let $\Sigma \subseteq \mathcal{C}$ be an apartment. Then the following hold.*

a) If $c \in \Sigma$ and P is a panel containing c, then $|P \cap \Sigma| = 2$;

b) for each chamber $d \in \mathcal{C}$ we have $|d^{\mathrm{op}} \cap \Sigma| \neq \emptyset$.

Opposite residues and projections: Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of spherical type (W, S) and let ρ denote the longest element in (W, S).

Two residues R and T of $\mathcal{B}$ are called opposite if there exist chambers $c \in R, d \in T$ such that c is opposite to d and if $\mathrm{typ}(T) = \rho\,\mathrm{typ}(R)\rho$. Thus, two opposite residues have the same rank.

For two opposite residues R and T we denote the restriction of proj_T to R by proj_T^R.

Proposition 6.4: *Let R and T be opposite residues in $\mathcal{B}$. Then the following hold.*

a) The mappings proj_T^R and proj_R^T are adjacency-preserving bijections inverse to each other.

b) If $c \in R$ and $d \in T$, then c is opposite d in $\mathcal{B}$ if and only if c is opposite $\mathrm{proj}_R\, d$ in the building R.

7 Moufang structures

Throughout this section $\mathcal{B} = (\mathcal{C}, \delta)$ is a building of spherical type (W, S) and ρ denotes the longest element in (W, S).

Definition: An automorphism u of $\mathcal{B}$ is called unipotent, if it satisfies the following conditions:

(U1) $u \in \mathrm{Spe}(\mathcal{B})$ and u fixes a chamber of $\mathcal{B}$;

(U2) if u fixes two adjacent chambers, then u fixes all chambers in the panel containing them.

Lemma 7.1: *If a unipotent automorphism of $\mathcal{B}$ fixes two opposite chambers, then it is the identity.*

Definition: A Moufang structure on $\mathcal{B}$ is a family $\mathcal{U} = (U_c)_{c \in \mathcal{C}}$ of unipotent subgroups of $\mathrm{Aut}(\mathcal{B})$ such that the following conditions are satisfied for each $c \in \mathcal{C}$:

(MS1) The group U_c stabilizes c and acts transitively on c^{op};

(MS2) if $u \in U_c$ stabilizes a chamber d, then $u \in U_d$;

(MS3) for $g \in U_c$ and $d \in \mathcal{C}$ we have $gU_dg^{-1} = U_{g(d)}$.

Note that the action of U_c on c^{op} is regular by the previous lemma.

For a Moufang structure $\mathcal{U} = (U_c)_{c \in \mathcal{C}}$ the subgroup of $\mathrm{Spe}(\mathcal{B})$ generated by the U_c is denoted by $G(\mathcal{U})$.

Tits' simplicity theorem

Proposition 7.2: *Suppose that $\mathcal{B}$ is thick, that $\mathcal{U} = (U_c)_{c \in \mathcal{C}}$ is a Moufang structure on $\mathcal{B}$ and let $G := \langle U_c \mid c \in \mathcal{C} \rangle$. Then the following hold:*

a) G is Weyl-transitive on Δ.

b) If (W, S) is irreducible, the U_cs are solvable and G is perfect, then G is simple.

Remarks on Moufang structures: Let $\mathcal{B} = (\mathcal{C}, \delta)$ be thick building of type (W, S).

1. If $[s, S] \neq 1$ for all $s \in S$, then $\mathcal{B}$ admits at most one Moufang structure. This follows from Theorem 4.1.1 in [15].
2. If (W, S) is irreducible and of rank at least 3, then $\mathcal{B}$ admits a Moufang structure (which is unique by the previous remar.k). This follows from Theorem 4.1.2 in [15].
3. All Moufang structures for irreducible spherical buildings of rank at least 2 are known by the classification of Moufang polygons due to Tits and Weiss in [17]. It turns out that in all examples the U_cs are nilpotent and that G is perfect except for three small cases.
4. Moufang structures on rank 1 buildings are precisely the Moufang sets (also known as rank 1 groups). The classification of all proper Moufang sets is an open problem. In all known examples the U_cs are nilpotent and G is perfect except for some small cases. A basic reference for Moufang sets is [3].

8 The center theorem

Convex subsets of buildings and subbuildings: Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of type (W, S). A subset $\mathcal{X}$ of $\mathcal{C}$ is called convex if $\Sigma(c, d) = \{x \in \mathcal{C} \mid \ell(c, x) + \ell(x, d) = \ell(c, d)\} \subseteq \mathcal{X}$ for all $c, d \in \mathcal{X}$.

Let $\mathcal{X}$ of $\mathcal{C}$ and let $\delta_{\mathcal{X}}$ denote the restriction of δ on $\mathcal{X}$. Then $\mathcal{X}$ is called a subbuilding if $(\mathcal{X}, \delta_{\mathcal{X}})$ is a building of type (W, S).

Remarks: 1. Let $H \le \mathrm{Spe}(\mathcal{B})$. Then the set of its fixed points in $\mathcal{C}$ is convex.

2. Let R be a J-residue of $\mathcal{B}$. Then (R, δ_R) is a building of type $(\langle J \rangle, J)$, but it is not a subbuilding if $J \ne S$.

Lemma 8.1: *Let $\mathcal{X} \subseteq \mathcal{C}$ be a convex subset of chambers. Then the following hold.*

a) If $R \cap \mathcal{X} \ne \emptyset$ for some residue R, then $\mathrm{proj}_R\, x \in \mathcal{X}$ for all $x \in \mathcal{X}$;

b) if $|P \cap \mathcal{X}| \ne 1$ for each panel P of $\mathcal{B}$, then $\mathcal{X}$ is a subbuilding;

c) if (W, S) is spherical and if $\mathcal{X}$ contains a pair of opposite chambers, then $\mathcal{X}$ is a subbuilding of $\mathcal{B}$.

Centers of chamber sets: Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of type (W, S) and $\mathcal{X} \subseteq \mathcal{C}$. A center of $\mathcal{X}$ in $\mathcal{B}$ is a residue $R \ne \mathcal{C}$ such that the stabilizer of $\mathcal{X}$ in $\mathrm{Aut}(\mathcal{B})$ stabilizes R.

Proposition 8.2 (Center theorem for spherical buildings): *Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of spherical type (W, S) and $\mathcal{X} \subseteq \mathcal{C}$ be a convex set of chambers that is not a subbuilding of $\mathcal{B}$. Then there exists a center of $\mathcal{X}$.*

Remarks (about the proof): 1. The proof can be reduced to the case that (W, S) is irreducible. The cases $|S| \le 2$ are trivial. The cases $A_n, B_n = C_n, D_n$ are dealt in [6] and the cases F_4 and E_6 are dealt in [4].

2. The hardest case is E_8 and it has been dealt by Ramos-Cuevas in [10] (which includes also the case E_7).

3. A proof that is almost uniform for all cases can be found in [7]. It uses the fact that all irreducible buildings of rank at least 3 are Moufang. The key argument is based on deep results of Timmesfeld about groups generated by abstract root groups [11].

Application of the center theorem to unipotent groups

Throughout this subsection $\mathcal{B} = (\mathcal{C}, \delta)$ is a thick building of spherical type (W, S). A group $U \le \mathrm{Aut}(\mathcal{B})$ is called unipotent if all $u \in U$ are unipotent and if U fixes at least one chamber.

Proposition 8.3: *Let $U \le \mathrm{Aut}(\mathcal{B})$ be a non-trivial unipotent group and let C denote its centralizer in $\mathrm{Aut}(\mathcal{B})$. Then there exists a residue $R \ne \mathcal{C}$ such that C stabilizes R and such that U stabilizes all chambers in R.*

Proof: Let $\mathcal{X}$ be the set of chambers fixed by U and let D be the stabilizer of $\mathcal{X}$ in $\mathrm{Aut}(\mathcal{B})$. Since U is non-trivial and unipotent, $\mathcal{X}$ is not a subbuilding. By the center theorem, there exist non-trivial residues stabilized by D. Let R be a minimal residue stabilized by D. We have $U \le D$ and $C \le D$ and therefore U and C both stabilize

R. Let $x \in \mathcal{X}$, then U stabilizes x and hence also the chamber $\operatorname{proj}_R x$. Let U_1 be the image of U in $\operatorname{Aut}(R)$. Then U_1 is a unipotent subgroup of $\operatorname{Aut}(R)$ and the set of fixed chambers of U_1 is $\mathcal{X} \cap R$. Let D_1 be the image of D in $\operatorname{Aut}(R)$. Then D_1 stabilizes $R \cap \mathcal{X}$.

Assume, by contradiction, that $R \cap \mathcal{X} \neq R$. Then the center theorem applied to the building R yields the existence of a residue R_1 properly contained in R which is stabilized by D_1. As D_1 is the image of D in $\operatorname{Aut}(R)$, D stabilizes R_1. Hence R_1 is a residue properly contained in R and stabilized by D. This contradicts our choice of R.

We conclude that $R \subseteq \mathcal{X}$ which means that U fixes all chambers in R. □

Let $\mathcal{U} = (U_c)_{c \in \mathcal{C}}$ be a Moufang structure on $\mathcal{B}$. Then $\operatorname{Aut}_{\mathcal{U}}(\mathcal{B})$ denotes the group of all automorphisms of $\mathcal{B}$ which normalize the Moufang structure $\mathcal{U}$.
By the first remark on Moufang structures above, $\operatorname{Aut}_{\mathcal{U}}(\mathcal{B}) = \operatorname{Aut}(\mathcal{B})$ if $[s, S] \neq 1$ for all $s \in S$.

Corollary 8.4: *Let $\mathcal{U} = (U_c)_{c \in \mathcal{C}}$ be a Moufang structure on $\mathcal{B}$, let $U \leq U_c$ for some $c \in \mathcal{C}$ and let $\Gamma \leq \operatorname{Aut}_{\mathcal{U}}(\mathcal{B})$ be such that Γ centralizes U. Then there is a residue R stabilized by Γ such that $U \leq \cap_{d \in R} U_d$.*

9 Parallel residues

Throughout this section $\mathcal{B} = (\mathcal{C}, \delta)$ is a building of type (W, S).

Lemma 9.1: *Let $R, T \subseteq \mathcal{C}$ be residues of $\mathcal{B}$. Then $\operatorname{proj}_T R := \{\operatorname{proj}_T x \mid x \in R\}$ is a residue contained in T.*

Proof: See Lemma 5.36 in [1]. □

Definition: Let R, T be residues of $\mathcal{B}$. Then R and T are said to be parallel if $\operatorname{proj}_R T = R$ and $\operatorname{proj}_T R = T$.

Remarks:
1. Two opposite residues in a spherical building are parallel.
2. Parallelism is not an equivalence relation on the set of residues. It is an equivalence relation if $\mathcal{B}$ is a Coxeter building.

Proposition 9.2: *Let R, T be residues of $\mathcal{B}$ which are parallel and let $c \in R$. Let $J = \operatorname{typ}(R) \subseteq S$ and $w := \delta(c, \operatorname{proj}_T c)$. Then the following hold.*

a) $\operatorname{proj}^T_R \colon T \to R$ and $\operatorname{proj}^R_T \colon R \to T$ are adjacence-preserving bijections which are inverse to each other;

b) for each $x \in R$ we have $\delta(x, \operatorname{proj}_T x) = w$;

c) typ$(T) = wJw^{-1}$.

Definition: Let R and T be parallel residues in $\mathcal{B}$. Then we put $\delta(R, T) := \delta(c, \operatorname{proj}_R c)$ for some chamber $c \in R$.

10 Fixed point sets in buildings

Throughout this section $\mathcal{B} = (\mathcal{C}, \delta)$ is a building of type (W, S), $\Gamma \leq \mathrm{Aut}(\mathcal{B})$ and $\Theta \coloneqq \mathrm{typ}(\Gamma) \leq \mathrm{Aut}(W, S)$.

Definition: 1. A Γ-residue is a residue of $\mathcal{B}$ stabilized by Γ;

2. a Γ-chamber is a minimal Γ-residue;

3. the set of all Γ-chambers is denoted by $\tilde{\mathcal{C}}$.

Some observations: 1. Suppose that R is a Γ-residue and that T is a residue containing R. Then T is a Γ-residue if an only if $\mathrm{typ}(T)$ is stabilized by Θ.

2. Suppose that R and T are both Γ-residues. Then $\mathrm{proj}_R T$ is a Γ-residue.

3. Any two Γ-chambers are parallel.

Lemma 10.1: *Suppose that (W, S) is spherical and let ρ_J denote the longest element in $(\langle J \rangle, J)$ for each $J \subseteq S$. Let R is a Γ-chamber and set $A \coloneqq \mathrm{typ}(R)$. Suppose that $S \setminus A$ is a Θ-orbit. Then the following hold.*

a) If T is a Γ-chamber with $T \neq R$, then T and R are opposite residues in $\mathcal{B}$ and $S \setminus \mathrm{typ}(T)$ is a Θ-orbit in S;

b) if there exist at least three Γ-chambers, then $\rho_A \rho = \rho \rho_A$ and all Γ-chambers have type A.

11 Tits indices

Definition: A Tits index is a triple $\mathbf{T} = ((W, S), \Theta, A)$ where (W, S) is a Coxeter system, $\Theta \leq \mathrm{Aut}(W, S)$ and $A \subseteq S$ satisfy the following conditions:

(Ti1) A is spherical and Θ stabilizes A;

(Ti2) for each $s \in S \setminus A$ the set $A \cup \Theta(s)$ is spherical and ρ_A commutes with $\rho_{A \cup \Theta(s)}$.

Let $\mathbf{T} = ((W, S), \Theta, A)$ be a Tits index. For each $s \in S \setminus A$ we set $\tilde{s} \coloneqq \rho_A \rho_{\Theta(s) \cup A}$. Note that $\tilde{s} = \tilde{t}$ if s and t are in the same Θ-orbit.

We put $\tilde{S} \coloneqq \{\tilde{s} \mid s \in S \setminus A\}$ and $\tilde{W} \coloneqq \langle \tilde{S} \rangle \leq W$.

Lemma 11.1: *The pair $(\tilde{W}, \tilde{S})$ is a Coxeter system.*

Proof: This is Theorem 20.32 a) in [5]. □

Remarks: 1. Let $\mathbf{T} = ((W, S), \Theta, A)$ be a Tits index. The Coxeter system (W, S) is called the absolute Coxeter system of $\mathbf{T}$, $(\tilde{W}, \tilde{S})$ is called the relative Coxeter system of $\mathbf{T}$ and A is called the anisotropic kernel of $\mathbf{T}$.

2. The Tits index $\mathbf{T} = ((W, S), \Theta, A)$ is called of inner type if Θ is trivial and it is called quasi-split if $A = \emptyset$.

3. Tits indices can be represented by decorated diagrams as in the Tables II of [14]). The Tits-indices here are defined in a purely combinatorial way and only a few of them arise as indices of semi simple algebraic groups. There is one index of absolute type F_4 which not an index of a semi simple group, but arises as index of a pseudo-reductive group.

12 Descent groups

Throughout this section $\mathcal{B} = (\mathcal{C}, \delta)$ is a spherical building of type (W, S) and $\Gamma \leq \mathrm{Aut}(\mathcal{B})$. Furthermore, we let $\tilde{\mathcal{C}}$ denote the set of all Γ-chambers and $\Theta := \mathrm{typ}(\Gamma) \leq \mathrm{Aut}(W, S)$.

Theorem 12.1: *Suppose that each Γ-residue which is not a Γ-chamber contains at least three Γ-chambers. Let $\tilde{c} \in \tilde{\mathcal{C}}$ and $A := \mathrm{typ}(\tilde{c}) \subseteq S$. Then the following hold.*

a) $\mathbf{T} := ((W, S), \Theta, A)$ *is a Tits-index.*

b) If $(\tilde{W}, \tilde{S})$ is the relative Coxeter system of $\mathbf{T}$, then $\delta(\tilde{c}, \tilde{d}) \in \tilde{W}$ for all $\tilde{c}, \tilde{d} \in \tilde{\mathcal{C}}$.

c) Let $\tilde{\delta} \colon \tilde{\mathcal{C}} \times \tilde{\mathcal{C}} \to \tilde{W}, (\tilde{c}, \tilde{d}) \mapsto \delta(\tilde{c}, \tilde{d})$, then $\tilde{\mathcal{B}} = (\tilde{\mathcal{C}}, \tilde{\delta})$ is a thick building of type $(\tilde{W}, \tilde{S})$.

From now on we a assume that $\mathcal{U} = (U_c)_{c\in}$ is a Moufang structure on $\mathcal{B}$. For each residue R of $\mathcal{B}$ we set $U_R := \bigcap_{x \in R} U_x$.

Lemma 12.2: *For each residue R of $\mathcal{B}$ the group U_R is sharply transitive on the set R^{op}.*

Assume now that $\Gamma \leq \mathrm{Aut}_{\mathcal{U}}(\mathcal{B})$. Then Γ normalizes U_R for each Γ-residue R. For $\tilde{c} \in \tilde{\mathcal{C}}$ we denote the centralizer of Γ in $U_{\tilde{c}}$ by $\tilde{U}_{\tilde{c}}$ and we set $\tilde{\mathcal{U}} := (\tilde{U}_{\tilde{c}})_{\tilde{c} \in \tilde{\mathcal{C}}}$.

Theorem 12.3: *Let $\Gamma \leq \mathrm{Aut}_{\mathcal{U}}(\mathcal{B})$ be a descent group. Then $\tilde{\mathcal{U}} := (\tilde{U}_{\tilde{c}})_{\tilde{c} \in \tilde{\mathcal{C}}}$ is a Moufang structure on $\tilde{\mathcal{B}}$.*

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