# Artin groups and hyperplane arrangements III

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Let $\Gamma$ be a Coxeter graph and let $(W, S)$ be its Coxeter system. Let $S^*$ be the free monoid on $S$. Let $w \in W$. A word $\mu = s_1 \cdots s_\ell \in S^*$ is an **expression** of $w$ if the equality $w = s_1 \cdots s_\ell$ holds in $W$. The **length** of $w$, $\lg(w)$, is the minimal length of an expression of $w$. An expression $\mu = s_1 \cdots s_\ell$ of $w$ is **reduced** if $\ell = \lg(w)$.

Let $\mu, \mu' \in S^*$. We say that there is an **elementary M-transformation** joining $\mu$ to $\mu'$ if there exist $\nu_1, \nu_2 \in S^*$ and $s, t \in S$ such that $m_{s,t} \neq \infty$,

$$\mu = \nu_1 \, \Pi(s, t : m_{s,t}) \, \nu_2 \,, \quad \text{and} \quad \mu' = \nu_1 \, \Pi(t, s : m_{s,t}) \, \nu_2 \,.$$

**Theorem** (**Tits [1969]**). Let $w \in W$, and let $\mu, \mu'$ be two reduced expressions of $w$. Then there is a finite sequence of elementary M-transformations joining $\mu$ to $\mu'$.

Let $(A, \Sigma)$ be the Artin system of $\Gamma$. Recall the epimorphism $\theta : A \to W$ which sends $\sigma_s$ to $s$ for all $s \in S$. We define a set-section $\tau : W \to A$ of $\theta$ as follows. Let $w \in W$. Choose a reduced expression $\mu = s_1 \cdots s_\ell$ of $w$. Set

$$\tau(w) = \sigma_{s_1} \cdots \sigma_{s_\ell} \,.$$

The definition of $\tau(w)$ does not depend on the choice of the reduced expression.

**Definition.** Let $X \subset S$.

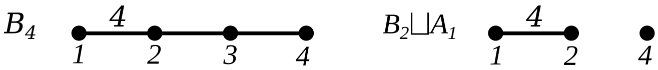
(a) $M_X = (m_{s,t})_{s,t \in X}$.

(b) $\Gamma_X$ is the Coxeter graph of $M_X$.

(c) $W_X$ is the subgroup of $W = W_\Gamma$ generated by $X$. It is called a **Standard parabolic subgroup**.

**Theorem** (**Bourbaki [1968]**). $(W_X, X)$ is the Coxeter system of $\Gamma_X$.

**Example.** Consider the Coxeter graph $\Gamma = B_4$.




Set $S = \{s_1, s_2, s_3, s_4\}$. Let $X = \{s_1, s_2, s_4\}$. Then $\Gamma_X$ is the Coxeter graph $B_2 \sqcup A_1$.

$$W_X = \langle s_1, s_2, s_4 \mid s_1^2 = s_2^2 = s_4^4 = 1\,, \\ (s_1 s_2)^4 = (s_1 s_4)^2 = (s_2 s_4)^2 = 1 \rangle$$

**Definition.** Let $X$, $Y$ be two subsets of $S$. We say that an element $w \in W$ is **$(X, Y)$-minimal** if it is of minimal length in the double-coset $W_X w W_Y$.

**Proposition** (**Bourbaki [1968]**). Let $(W, S)$ be a Coxeter system.

(1) Let $X, Y$ be two subsets of $S$, and let $w \in W$. Then there exists a unique $(X, Y)$-minimal element lying in $W_X w W_Y$.

(2) Let $X \subset S$, and let $w \in W$. Then $w$ is $(\emptyset, X)$-minimal if and only if $\mathrm{lg}(ws) > \mathrm{lg}(w)$ for all $s \in X$.

(3) Let $X \subset S$, and let $w \in W$. Then $w$ is $(\emptyset, X)$-minimal if and only if $\mathrm{lg}(wu) = \mathrm{lg}(w) + \mathrm{lg}(u)$ for all $u \in W_X$.

**Definition.** An (abstract) **simplicial complex** is a pair $\Upsilon = (S, A)$, where $S$ is a set, called **set of vertices**, and $A$ is a set of subsets of $S$, called **set of simplices**, satisfying:

(a) $\emptyset$ is not a simplex, and all the simplices are finite.

(b) All the singletons are simplices.

(c) Any nonempty subset of a simplex is a simplex.

**Definition.** Let $\Upsilon = (S, A)$ be a simplicial complex. Take $B = \{e_s \mid s \in S\}$. $V$ is the real vector space having $B$ as a basis. For $\Delta = \{s_0, s_1, \dots, s_p\}$ in $A$, we set

$$|\Delta| = \{t_0 e_{s_0} + t_1 e_{s_1} + \cdots + t_p e_{s_p} \mid 0 \le t_0, t_1, \dots, t_p \le 1 \text{ and } \sum_{i=0}^{p} t_i = 1\}\,.$$

Note that $|\Delta|$ is a (geometric) simplex of dimension $p$.

The **geometric realization** of $\Upsilon$ is defined to be the following subset of $V$.

$$|\Upsilon| = \bigcup_{\Delta \in A} |\Delta|\,.$$

We endow $|\Upsilon|$ with the **weak topology**.

**Example.** If $(E, \leq)$ is a partially ordered set, then the nonempty finite chains of $E$ form a simplicial complex, called **derived complex** of $(E, \leq)$ and denoted by $E' = (E, \leq)'$.

Now, we take a Coxeter system $(W, S)$.

**Definition.** $\mathcal{S}^f = \{X \subset S \mid W_X \text{ is finite}\}$.

**Lemma.** Let $\preceq$ be the relation on $W \times \mathcal{S}^f$ defined by

$$(u, X) \preceq (v, Y)$$

if

$$X \subset Y\,,\ v^{-1}u \in W_Y\,,\ \text{and } v^{-1}u \text{ is } (\emptyset, X)\text{-minimal.}$$

Then $\preceq$ is a (partial) ordering relation.

**Definition.** The **Salvetti complex** of $\Gamma$, denoted by $\mathrm{Sal}(\Gamma)$, is the geometric realization of the derived complex of $(W \times \mathcal{S}^f, \preceq)$. Note that the action of $W$ on $W \times \mathcal{S}^f$ defined by $w \cdot (u, X) = (wu, X)$ preserves the ordering. Hence, it induces an action of $W$ on $\mathrm{Sal}(\Gamma)$.

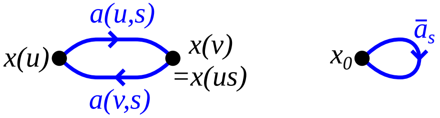
**Theorem** (**Charney–Davis [1995]**). Take a Vinberg system $(W, S)$ and denote by $\Gamma$ the Coxeter graph of $(W, S)$. Then there exists a homotopy equivalence $f : \mathrm{Sal}(\Gamma) \to M(W, S)$ equivariant under the actions of $W$ and that induces a homotopy equivalence $\bar{f} : \mathrm{Sal}(\Gamma)/W \to M(W, S)/W = N(W, S)$.

**Corollary** (**Charney–Davis [1995]**). The homotopy type of $N(W, S)$ (resp. $M(W, S)$) depends only on the Coxeter graph $\Gamma$.

$\mathrm{Sal}(\Gamma)$ and $\mathrm{Sal}(\Gamma)/W = \overline{\mathrm{Sal}}(\Gamma)$ have "cellular decompositions" whose $k$-skeletons for $k = 0, 1, 2$ can be described as follows.

*0-skeleton.* The 0-skeleton of $\mathrm{Sal}(\Gamma)$ is a set $\{x(w) \mid w \in W\}$. The 0-skeleton of $\overline{\mathrm{Sal}}(\Gamma)$ is reduced to a point that we denote by $x_0$.

*1-skeleton.* With $(u,s) \in W \times S$ is associated an edge $a(u,s)$ of $\mathrm{Sal}(\Gamma)$ from $x(u)$ to $x(us)$. So, for $u, v \in W$, if $v = us$ with $s \in S$, there is an edge $a(u,s)$ going from $x(u)$ to $x(v)$, and there is another edge $a(v,s)$ going from $x(v)$ to $x(u)$.



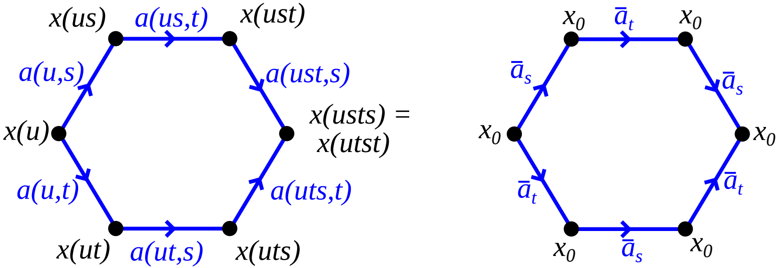

There is no edge joining $x(u)$ and $x(v)$ if $v$ is not of the form $v = us$ with $s \in S$. For each $s \in S$ there is an arrow $\bar{a}_s$ in $\overline{\mathrm{Sal}}(\Gamma)$ from $x_0$ to $x_0$.

*2-skeleton.* Let $s, t \in S$, $s \neq t$. Note that $\{s, t\} \in \mathcal{S}^f$ if and only if $m_{s,t} \neq \infty$. Assume $m = m_{s,t} \neq \infty$. With every $u \in W$ is associated a 2-cell of $\mathrm{Sal}(\Gamma)$, $B(u, \{s, t\})$, whose boundary is

$$a(u, s)\, a(us, t) \cdots a(ut, s)^{-1}\, a(u, t)^{-1}\,.$$

for such a pair $\{s, t\}$ is associated a 2-cell $\bar{B}(\{s, t\})$ of $\overline{\mathrm{Sal}}(\Gamma)$ whose boudary is

$$\bar{a}_s\, \bar{a}_t \cdots \bar{a}_s^{-1}\, \bar{a}_t^{-1} = \Pi(\bar{a}_s, \bar{a}_t : m)\, \Pi(\bar{a}_t, \bar{a}_s : m)^{-1}\,.$$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**Theorem.** We have $\pi_1(\overline{\mathrm{Sal}}(\Gamma), x_0) = A_\Gamma$, $\pi_1(\mathrm{Sal}(\Gamma), x(1)) = CA_\Gamma$. The exact sequence associated with the regular covering $\mathrm{Sal}(\Gamma) \to \overline{\mathrm{Sal}}(\Gamma)$ is

$$1 \longrightarrow CA_\Gamma \longrightarrow A_\Gamma \xrightarrow{\theta} W \longrightarrow 1$$

**Corollary** (**Van der Lek [1983]**). Let $(W, S)$ be a Vinberg system. Let $\Gamma$ be the Coxeter graph of $(W, S)$. Then $\pi_1(N(W, S)) = A_\Gamma$, $\pi_1(M(W, S)) = CA_\Gamma$. The exact sequence associated with the regular covering $M(W, S) \to N(W, S)$ is

$$1 \longrightarrow CA_\Gamma \longrightarrow A_\Gamma \xrightarrow{\theta} W \longrightarrow 1$$

**Definition.** The **Artin monoid** of $\Gamma$ is the monoid $A_\Gamma^+$ defined by

$$A_\Gamma^+ = \langle \Sigma \mid \Pi(\sigma_s, \sigma_t : m_{s,t}) = \Pi(\sigma_t, \sigma_s : m_{s,t}) \\ \text{for all } s,t \in S,\ s \neq t,\ m_{s,t} \neq \infty \rangle^+ \,.$$

**Theorem** (**P. [2002]**). The natural homomorphism $A_\Gamma^+ \to A_\Gamma$ is injective.

Recall the natural epimorphism $\theta : A_\Gamma \to W_\Gamma$ This epimorphism extends to a map $\tilde{\theta} : A_\Gamma \times \mathcal{S}^f \to W_\Gamma \times \mathcal{S}^f$. And $\tilde{\theta}$ extends to the universal cover $\widetilde{\mathrm{Sal}}(\Gamma) \to \mathrm{Sal}(\Gamma)$.

**Definition.** We set $\widetilde{\mathrm{Sal}}^+(\Gamma)$ the subcomplex of $\widetilde{\mathrm{Sal}}(\Gamma)$ generated by $A_\Gamma^+ \times \mathcal{S}^f$.

**Theorem** (**P. [2014]**) $\widetilde{\mathrm{Sal}}^+(\Gamma)$ is contractible.

**Definition.** $\Gamma$ is of **spherical type** if $W_\Gamma$ is finite.

**Corollary** (**Deligne [1972]** – not obvious but true). $\mathrm{Sal}(\Gamma)$ is an Eilenberg MacLane space if $\Gamma$ is of spherical type.

**Theorem** (**Godelle–P. [2012]**). Let $\Gamma$ be a Coxeter graph, let $S$ be its set of vertices, and let $X$ be a subset of $S$. Recall that $\Gamma_X$ denotes the full subgraph of $\Gamma$ generated by $X$. Then there are "natural" maps $\iota_X : \mathrm{Sal}(\Gamma_X) \to \mathrm{Sal}(\Gamma)$ and $\pi_X : \mathrm{Sal}(\Gamma) \to \mathrm{Sal}(\Gamma_X)$ such that $\pi_X \circ \iota_X = \mathrm{Id}$.

*Description of $\iota_X$.* Recall $W_X$ is the subgroup of $W$ generated by $X$. $(W_X, X)$ is the Coxeter system of $\Gamma_X$. Set $\mathcal{S}_X^f = \{Y \subset X \mid W_Y \text{ is finite}\}$. The inclusion $W_X \times \mathcal{S}_X^f \hookrightarrow W \times \mathcal{S}$ induces an embedding $\iota_X : \mathrm{Sal}(\Gamma_X) \hookrightarrow \mathrm{Sal}(\Gamma)$.

*Description of $\pi_X$.* Let $(u, Y) \in W \times \mathcal{S}^f$. We write $u = u_0 u_1$, where $u_0 \in W_X$ and $u_1$ is $(X, \emptyset)$-minimal. Let $Y_0 = X \cap u_1 Y u_1^{-1}$. We set

$$\pi_X(u, Y) = (u_0, Y_0)$$

Note that, since $W_{Y_0} \subset u_1 W_Y u_1^{-1}$, the group $W_{Y_0}$ is finite, thus $Y_0 \in \mathcal{S}_X^f$. Then $\pi_X : \mathrm{Sal}(\Gamma) \to \mathrm{Sal}(\Gamma_X)$ is induced by $\pi_X : W \times \mathcal{S}^f \to W_X \times \mathcal{S}_X^f$.

**Corollary** (**Godelle–P. [2012]**). Let $\Gamma$ be a Coxeter graph, let $S$ be its set of vertices, and let $X$ be a subset of $S$. If $\mathrm{Sal}(\Gamma)$ is an Eilenberg MacLane space, then $\mathrm{Sal}(\Gamma_X)$ is also an Eilenberg MacLane space.

**Corollary** (**Van der Lek [1983]**). Let $\Gamma$ be a Coxeter graph, let $S$ be its set of vertices, and let $X$ be a subset of $S$. Let $\varphi_X : A_{\Gamma_X} \to A_\Gamma$ the natural homomorphism which sends $\sigma_s$ to $\sigma_s$ for all $s \in X$. Then $\varphi_X$ is injective.

THE END of III

Thank you for your attention!