Artin groups and hyperplane arrangements II

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Definition. For a nonempty open convex cone I in a real vector space V, and a hyperplane arrangement $\mathcal{A}$ in I, we set

$$M(\mathcal{A}) = (I \times I) \setminus \left(\bigcup_{H \in \mathcal{A}} H \times H \right) .$$

If (W, S) is a Vinberg system and $\mathcal{A}$ is the Coxeter arrangement of (W, S), then we set $M(W, S) = M(\mathcal{A})$. Note that W acts freely and properly discontinuously on $M(W, S)$. We set

$$N(W, S) = M(W, S)/W .$$

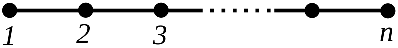
Theorem (**Van der Lek [1983]**). Let (W, S) be a Vinberg system, and let Γ be the Coxeter graph of the pair (W, S). Then $\pi_1(N(W, S)) = A_\Gamma$, $\pi_1(M(W, S)) = CA_\Gamma$, and the short exact sequence associated with the regular covering $M(W, S) \to N(W, S)$ is

$$1 \longrightarrow CA_\Gamma \longrightarrow A_\Gamma \xrightarrow{\theta} W_\Gamma \longrightarrow 1 \ .$$

Definition. A space X is an **Eilenberg MacLane space** for a discrete group G if the fundamental group of X is G and the universal cover of X is contractible. We also say that X is **aspherical** or that it is a $K(G,1)$ **space**. Eilenberg MacLane spaces play a prominent role in cohomology of groups.

Conjecture ($K(\pi,1)$ **conjecture**). Let (W,S) be a Vinberg system, and let Γ be the Coxeter graph of the pair (W,S). Then $N(W,S)$ is an Eilenberg MacLane space for A_Γ.

Example. Consider the symmetric group $\mathfrak{S}_{n+1}$ acting on the vector space $V = \mathbb{R}^{n+1}$ by permutation of the coordinates. For $i, j \in \{1, \dots, n+1\}$, $i \neq j$, we set $H_{i,j} = \{x \in V \mid x_i = x_j\}$. For $i \in \{1, \dots, n\}$, $s_i = (i, i+1)$ is a reflection with respect to $H_{i,i+1}$. Recall that The pair $(\mathfrak{S}_{n+1}, S)$ is a Vinberg system, and its associated Coxeter graph is A_n.



In this case we have $I = \bar{I} = V$. The set $\mathcal{R}$ of reflections coincides with the set of transpositions, and $\mathcal{A} = \{H_{i,j} \mid 1 \leq i < j \leq n+1\}$.

We identify $V \times V$ with $\mathbb{C}^{n+1} = \mathbb{C} \otimes V$. Then

$$M(\mathfrak{S}_{n+1}, S) = \mathbb{C}^{n+1} \setminus \left(\bigcup_{i<j} \mathbb{C} \otimes H_{i,j} \right)$$

is the **space of ordered configurations** of $n+1$ points in $\mathbb{C}$.

$$N(\mathfrak{S}_{n+1}, S) = M(\mathfrak{S}_{n+1}, S)/\mathfrak{S}_{n+1}$$

is the **space of (non-ordered) configurations** of $n+1$ points in $\mathbb{C}$.

Theorem (**Artin [1947]**) $\pi_1(N(\mathfrak{S}_{n+1}, S)) = \mathcal{B}_{n+1}$, the braid group on $n+1$ strands.

Definition. Let $f, g \in \mathbb{C}[x]$ be two non-constant polynomials. Set

$$\begin{aligned} f &= a_0 x^m + a_1 x^{m-1} + \cdots + a_m\,, \quad a_0 \neq 0 \\ g &= b_0 x^n + b_1 x^{n-1} + \cdots + b_n\,, \quad b_0 \neq 0\,. \end{aligned}$$

The **Sylvester matrix** of f and g is

$$\mathrm{Sylv}(f,g) = \begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \ddots & \vdots & b_1 & b_0 & \ddots & \vdots \\ \vdots & a_1 & \ddots & 0 & \vdots & b_1 & \ddots & 0 \\ a_m & \vdots & \ddots & a_0 & b_n & \vdots & \ddots & b_0 \\ 0 & a_m & & a_1 & 0 & b_n & & b_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_m & 0 & \cdots & 0 & b_n \end{pmatrix}$$

Definition. The **resultant** of f and g is

$$\mathrm{Res}(f, g) = \det(\mathrm{Sylv}(f, g))\,.$$

Theorem. f and g have a common root if and only if $\mathrm{Res}(f, g) = 0$.

Corollary. Let $f \in \mathbb{C}[x]$ be a polynomial of degree $d \geq 2$. Then f has a multiple root if and only if $\mathrm{Res}(f, f') = 0$.

Definition. $\mathrm{Res}(f, f')$ is the **discriminant** of f. It is denoted by $\mathrm{Disc}(f)$.

Example. If $f = ax^2 + bx + c$, then $\mathrm{Disc}(f) = b^2 - 4ac$.

Definition. $\mathbb{C}_n[x]$ is the set of monic polynomials of degree n. Note that $\mathbb{C}_n[x] \simeq \mathbb{C}^n$. $\mathrm{Disc} : \mathbb{C}_n[x] \to \mathbb{C}$ is an algebraic function. Thus

$$\mathcal{D} = \{f \in \mathbb{C}_n[x]; f \text{ has a multiple root}\} = \{f \in \mathbb{C}_n[x]; \mathrm{Disc}(f) = 0\}$$

is an algebraic hypersurface. $\mathcal{D}$ is the **n-th discriminant**.

Proposition. $N(\mathfrak{S}_n, S) = \mathbb{C}_n[x] \setminus \mathcal{D}$.

Proof. Let $\Phi : M(\mathfrak{S}_n) \to \mathbb{C}_n[x] \setminus \mathcal{D}$ be

$$\Phi(z_1, \dots, z_n) = (x - z_1) \cdots (x - z_n)\,.$$

Then Φ is surjective and we have $\Phi(u) = \Phi(v)$ if and only if there exists $\chi \in \mathfrak{S}_n$ such that $v = \chi\, u$. Thus $\mathbb{C}_n[x] \setminus \mathcal{D} \simeq M(\mathfrak{S}_n)/\mathfrak{S}_n = N(\mathfrak{S}_n)$. $\square$

Theorem (**Fox–Neuwirth [1962]**). $N(\mathfrak{S}_n)$ is an Eilenberg MacLane space for $\mathcal{B}_n$.

Proof. We use (and do not prove) the following three statements.

(1) Let $X \to Y$ be a covering map. Then X is an Eilenberg MacLane space if and only if Y is an Eilenberg MacLane space.

(2) Let $X \to B$ be a locally trivial fibration map with connected fiber F. If B and F are both Eilenberg MacLane spaces, then X is an Eilenberg MacLane space, too.

(3) Any graph is an Eilenberg MacLane space.

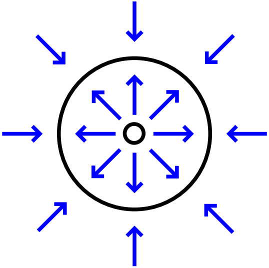
By (1), in order to prove that $N(\mathfrak{S}_n)$ is an Eilenberg MacLane space, it suffices to prove that $M(\mathfrak{S}_n)$ is an Eilenberg MacLane space. We show that $M(\mathfrak{S}_n)$ is an Eilenberg MacLane space by induction on n.

Suppose $n = 2$. Then

$$M(\mathfrak{S}_2) = \mathbb{C}^2 \setminus \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 = z_2\} \simeq \mathbb{C} \times \mathbb{C}^*\,.$$

$\mathbb{C}$ is an Eilenberg MacLane space because it is contractible.

The circle is a deformation retract of $\mathbb{C}^*$,



thus $\mathbb{C}^*$ has the same homotopy type as the circle, therefore $\mathbb{C}^*$ is an Eilenberg MacLane space by (3). By (2) we conclude that $M(\mathfrak{S}_2)$ is an Eilenberg MacLane space.

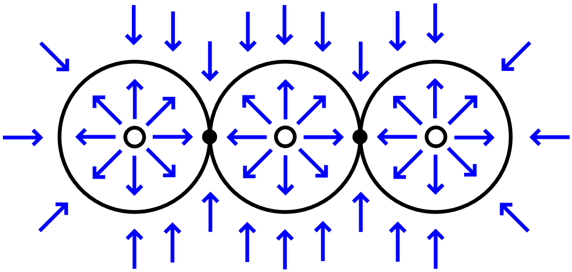
Suppose that $M(\mathfrak{S}_n)$ is an Eilenberg MacLane space.

$$\begin{array}{ccc} M(\mathfrak{S}_{n+1}) & \to & M(\mathfrak{S}_n) \\ (z_1, \dots, z_n, z_{n+1}) & \mapsto & (z_1, \dots, z_n) \end{array}$$

is a locally trivial fibration. The fiber above $(1, \dots, n)$ is

$$\{(1, \dots, n, z_{n+1}) \mid z_{n+1} \not\in \{1, \dots, n\}\} \simeq \mathbb{C} \setminus \{1, \dots, n\}\,.$$

There is a graph which is a deformation retract of $\mathbb{C} \setminus \{1, \ldots, n\}$.



Thus $\mathbb{C} \setminus \{1, \ldots, n\}$ is an Eilenberg MacLane space by (3). We conclude by (2) that $M(\mathfrak{S}_{n+1}, S)$ is an Eilenberg MacLane space. $\square$

Γ be a Coxeter graph, and (W, S) be its Coxeter system. Take an abstract set $\{e_s \mid s \in S\}$. Denote by V the real vector space having $\{e_s \mid s \in S\}$ as a basis. Define $B : V \times V \to \mathbb{R}$ by

$$B(e_s, e_t) = \begin{cases} -\cos(\frac{\pi}{m_{s,t}}) & \text{if } m_{s,t} \neq \infty \\ -1 & \text{if } m_{s,t} = \infty \end{cases}$$

For $s \in S$ define $\rho_s \in \mathrm{GL}(V)$ by

$$\rho_s(x) = x - 2\,B(x, e_s)e_s\,, \quad x \in V\,.$$

Then ρ_s is a linear reflection for all $s \in S$. $S \to \mathrm{GL}(V)$, $s \mapsto \rho_s$, induces a linear representation $\rho : W \to \mathrm{GL}(V)$.

V^* be the dual space of V. Recall that any linear map $f \in \mathrm{GL}(V)$ determines a linear map $f^t \in \mathrm{GL}(V^*)$ defined by

$$\langle f^t(\alpha), x\rangle = \langle \alpha, f(x)\rangle$$

The **dual representation** $\rho^* : W \to \mathrm{GL}(V^*)$ of ρ is defined by

$$\rho^*(w) = (\rho(w)^t)^{-1}$$

For $s \in S$, we set $H_s = \{\alpha \in V^* \mid \langle \alpha, e_s\rangle = 0\}$. Let

$$\bar{C}_0 = \{\alpha \in V^* \mid \langle \alpha, e_s\rangle \geq 0 \text{ for all } s \in S\}\,.$$

Theorem (**Tits, Bourbaki [1968]**).

(1) $\rho : W \to \mathrm{GL}(V)$ and $\rho^* : W^* \to \mathrm{GL}(V^*)$ are faithful.

(2) $\bar{C}_0$ is a simplicial cone whose walls are H_s, $s \in S$. $\rho^*(s)$ is a linear reflection whose fixed hyperplane is H_s, for all $s \in S$. We have $\rho^*(w)C_0 \cap C_0 = \emptyset$ for all $w \in W \setminus \{1\}$.

In particular, $(\rho^*(W), \rho^*(S))$ is a Vinberg system whose associated Coxeter graph is Γ.

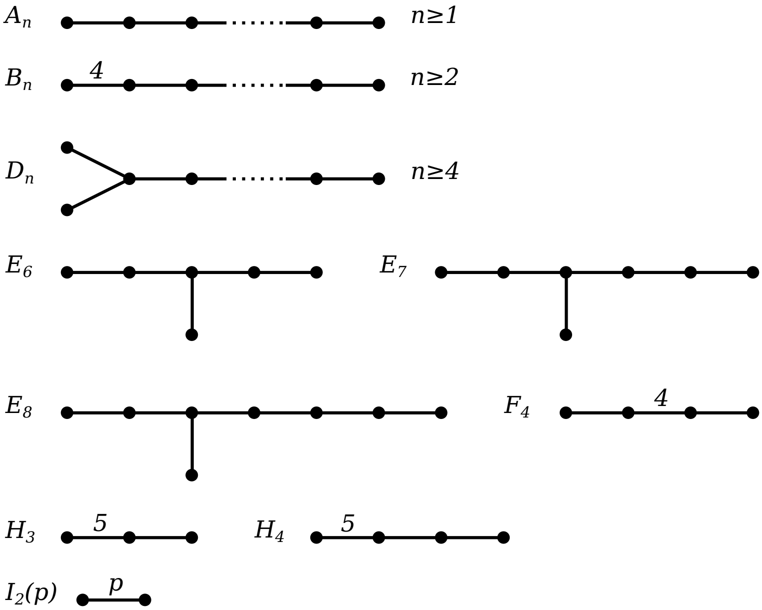
Γ is of **spherical type** if W_Γ is finite.

Observation. If $\Gamma_1, \dots, \Gamma_\ell$ are the connected components of Γ, then $W_\Gamma = W_{\Gamma_1} \times \cdots \times W_{\Gamma_\ell}$. In particular, Γ is of spherical type if and only if all its connected components are of spherical type.

Theorem (**Coxeter [1934, 1935]**).

(1) Γ is of spherical type if and only if the bilinear form $B : V \times V \to \mathbb{R}$ is positive definite.

(2) The spherical type connected Coxeter graphs are precisely those listed in the following figure.


A_n
$n \geq 1$
B_n
4
$n \geq 2$
D_n
$n \geq 4$
E_6
E_7
E_8
F_4
4
H_3
5
H_4
5
$I_2(p)$
p

Theorem (**Deligne [1972]**). Let (W, S) be a Vinberg system. If W is finite, then $N(W)$ is an Eilenberg MacLane space.

THE END of II

Thank you for your attention!