Artin groups and hyperplane arrangements I

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Definition. S a finite set. **Coxeter matrix** over S is $M = (m_{s,t})_{s,t \in S}$ such that:

(a) $m_{s,s} = 1$ for all $s \in S$;

(b) $m_{s,t} = m_{t,s} \in \{2, 3, 4, \dots\} \cup \{\infty\}$ for all $s, t \in S$, $s \neq t$.

Definition. **Coxeter graph**, $\Gamma = \Gamma(M)$.

Labelled graph defined as follows.

(a) S is the set of vertices of Γ.

(b) Two vertices $s, t \in S$ are connected by an edge if $m_{s,t} \geq 3$.

(c) This edge is labelled by $m_{s,t}$ if $m_{s,t} \geq 4$.

Example. A Coxeter matrix:

$$M = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix}$$

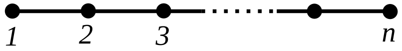
Its Coxeter graph:

Definition. **Coxeter system** of Γ is $(W, S) = (W_\Gamma, S)$, where

$$W_\Gamma = \left\langle S \;\middle|\; \begin{array}{c} s^2 = 1 \text{ for all } s \in S \\ (st)^{m_{s,t}} = 1 \text{ for all } s, t \in S,\ s \neq t,\ m_{s,t} \neq \infty \end{array} \right\rangle$$

The group W_Γ is called the **Coxeter group** of Γ.

Example. The Coxeter graph A_n:



The Coxeter group:

$$\left\langle s_1, \dots, s_n \;\middle|\; \begin{array}{c} s_i^2 = 1 \text{ for } 1 \le i \le n \\ (s_i s_{i+1})^3 = 1 \text{ for } 1 \le i \le n-1 \\ (s_i s_j)^2 = 1 \text{ for } |i-j| \ge 2 \end{array} \right\rangle$$

$W = \mathfrak{S}_{n+1}$, symmetric group, where s_i is $(i, i+1)$.

Definition. If a, b are two letters and m is an integer ≥ 2, we set

$$\Pi(a, b : m) = \begin{cases} (ab)^{\frac{m}{2}} & \text{if } m \text{ is even} \\ (ab)^{\frac{m-1}{2}} a & \text{if } m \text{ is odd} \end{cases}$$

So, $\Pi(a, b : 2) = ab$, $\Pi(a, b : 3) = aba$, $\Pi(a, b : 4) = abab$, and so on.

Lemma 1. *Let Γ be a Coxeter graph. Then W_Γ has the following presentation.*

$$W_\Gamma = \left\langle S \;\middle|\; \begin{array}{c} s^2 = 1 \text{ for all } s \in S \\ \Pi(s, t : m_{s,t}) = \Pi(t, s : m_{s,t}) \text{ for all } s, t \in S, \\ s \neq t,\ m_{s,t} \neq \infty \end{array} \right\rangle$$

Proof. It suffices to prove that the relation $(st)^m = 1$ is equivalent to the relation $\Pi(s,t:m) = \Pi(t,s:m)$ modulo the relations $s^2 = 1$ for all $s \in S$. We prove that for $m = 2$ and $m = 3$. Assume $m = 2$.

$$(st)^2 = stst = 1 \;\Leftrightarrow\; st = t^{-1}s^{-1} = ts \;\Leftrightarrow\; \Pi(s,t:2) = \Pi(t,s:2)\,.$$

Assume $m = 3$.

$$\begin{gathered}(st)^3 = ststst = 1 \;\Leftrightarrow\; sts = t^{-1}s^{-1}t^{-1} = tst \\ \Leftrightarrow\; \Pi(s,t:3) = \Pi(t,s:3)\,.\end{gathered}$$

$\square$

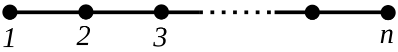
Definition. Let $\Sigma = \{\sigma_s \mid s \in S\}$. The **Artin system** of Γ is $(A, \Sigma) = (A_\Gamma, \Sigma)$, where

$$A_\Gamma = \langle \Sigma \mid \Pi(\sigma_s, \sigma_t : m_{s,t}) = \Pi(\sigma_t, \sigma_s : m_{s,t}) \\ \text{for all } s, t \in S,\ s \neq t,\ m_{s,t} \neq \infty \rangle$$

The group A_Γ is called the **Artin group** of Γ.

Thanks to Lemma 1, the map $\Sigma \to S$, $\sigma_s \mapsto s$, induces an epimorphism $\theta : A_\Gamma \to W_\Gamma$. The kernel of θ is the **colored Artin group** of Γ and is denoted by CA_Γ.

Example. The Coxeter graph A_n:



The Artin group:

$$\left\langle \sigma_1, \dots, \sigma_n \;\middle|\; \begin{array}{c} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \le i \le n-1 \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \ge 2 \end{array} \right\rangle .$$

This is the **braid group** $\mathcal{B}_{n+1}$ on $n+1$ strands. The colored Artin group is the **pure braid group** $\mathcal{PB}_{n+1}$.

Open questions.

(1) Are the Artin groups torsion free?

(2) How is the center of an Artin group?

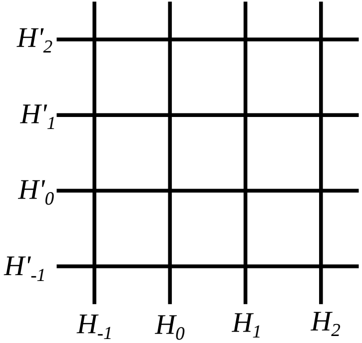
(3) Have Artin groups solvable word problem?

Definition. I is a nonempty open convex cone in a finite dimensional real vector space V. A **hyperplane arrangement** in I is a family $\mathcal{A}$ of linear hyperplanes of V satisfying

(a) $H \cap I \neq \emptyset$ for all $H \in \mathcal{A}$;

(b) $\mathcal{A}$ is **locally finite** in I, that is, for all $x \in I$, there is an open neighborhood U_x of x in I such that the set $\{H \in \mathcal{A} \mid H \cap U_x \neq \emptyset\}$ is finite.

Example. Set $V = \mathbb{R}^3$ and $I = \{(x, y, z) \in V \mid z > 0\}$. $H_k = \{x = kz\}$, and $H'_k = \{y = kz\}$. $\mathcal{A} = \{H_k,\ H'_k \mid k \in \mathbb{Z}\}$ is a hyperplane arrangement in I.



Definition. V be a finite dimensional real vector space. A **reflection** on V is a linear transformation on V of order 2 which fixes a hyperplane.

Definition. Let $\bar{C}_0$ be a closed convex polyhedral cone in V with nonempty interior. Let C_0 be the interior of $\bar{C}_0$. A **wall** of $\bar{C}_0$ is the support of a (codimensional 1) face of $\bar{C}_0$, that is, a hyperplane of V generated by that face.

Definition. Let $H_1, \dots, H_n$ be the walls of $\bar{C}_0$. For each $i \in \{1, \dots, n\}$ we take a reflection s_i which fixes H_i, and we denote by W the subgroup of $\mathrm{GL}(V)$ generated by $S = \{s_1, \dots, s_n\}$. The pair (W, S) is called a **Vinberg system** if $wC_0 \cap C_0 = \emptyset$ for all $w \in W \setminus \{1\}$. In that case, the group W is called a **reflection group** in Vinberg sense, S is called the **canonical generating system** for W, and C_0 is called the **fundamental chamber** of (W, S).

Theorem (**Vinberg [1971]**). Let (W, S) be a Vinberg system. We set

$$\bar{I} = \bigcup_{w \in W} w\, \bar{C}_0\,.$$

Then the following statements hold.

(1) (W, S) is a Coxeter system.

(2) $\bar{I}$ is a convex cone with nonempty interior.

(3) The interior I of $\bar{I}$ is stable under the action of W, and W acts properly discontinuously on I.

(4) Let $x \in I$ be such that $W_x = \{w \in W \mid w(x) = x\}$ is different from $\{1\}$. Then there exists a reflection r in W such that $r(x) = x$.

Remark.

(1) There is a difference in the theorem between the pair (W, S), viewed as a Vinberg system, and the pair (W, S), viewed a Coxeter system. In the first case, W is some specific subgroup of a linear group, while, in the second case, W is just an abstract group.

(2) Any Coxeter system appears as a Vinberg system (this is due to **Tits**), but this representation is not unique in general.

Definition. The above cone I is called the **Tits cone** of the Vinberg system (W, S). Denote by $\mathcal{R}$ the set of reflections lying in W. For $r \in \mathcal{R}$ we denote by H_r the fixed hyperplane of r, and we set $\mathcal{A} = \{H_r \mid r \in \mathcal{R}\}$. By the above, $\mathcal{A}$ is a hyperplane arrangement in the Tits cone I. It is called the **Coxeter arrangement** of (W, S).

Example. Consider the symmetric group $\mathfrak{S}_{n+1}$ acting on the space $V = \mathbb{R}^{n+1}$ by permutations of the coordinates. Let

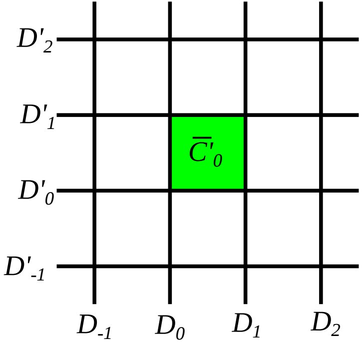
$$\bar{C}_0 = \{x \in V \mid x_1 \le x_2 \le \cdots \le x_{n+1}\}\,.$$

For $i, j \in \{1, \dots, n+1\}$, $i \neq j$, we set $H_{i,j} = \{x \in V \mid x_i = x_j\}$. Then $\bar{C}_0$ is a convex polyhedral cone whose walls are $H_{1,2}, H_{2,3}, \dots, H_{n,n+1}$. For $i \in \{1, \dots, n\}$, $s_i = (i, i+1)$ is a reflection whose fixed hyperplane is $H_{i,i+1}$. Then $(\mathfrak{S}_{n+1}, \{s_1, \dots, s_n\})$ is a Vinberg system. In this case we have

$$\bar{I} = \bigcup_{w \in \mathfrak{S}_{n+1}} w\bar{C}_0 = V\,.$$

So, $I = V$. The set $\mathcal{R}$ of reflections coincides with the set of transpositions, thus $\mathcal{A} = \{H_{i,j} \mid 1 \le i < j \le n+1\}$.

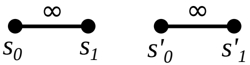
Example. Consider the affine Euclidean plane $\mathbb{E}^2$. D_k is the affine line of equation $x = k$, and D'_k the affine line of equation $y = k$.



s_k is the orthogonal affine reflection with respect to D_k, and s'_k is the orthogonal affine reflection with respect to D'_k. W the subgroup of the orthogonal affine group of $\mathbb{E}^2$ generated by $\{s_k, s'_k \mid k \in \mathbb{Z}\}$. We have

$$W = \langle s_0, s_1, s'_0, s'_1 \mid s_0^2 = s_1^2 = {s'_0}^2 = {s'_1}^2 = 1\,,$$
$$(s_0 s'_0)^2 = (s_0 s'_1)^2 = (s_1 s'_0)^2 = (s_1 s'_1)^2 = 1\rangle\,.$$

This is the Coxeter group of



We embed $\mathbb{E}^2$ in $\mathbb{R}^3$ via the map $(x, y) \mapsto (x, y, 1)$, and we denote by $\mathrm{Aff}(\mathbb{E}^2)$ the affine group of $\mathbb{E}^2$. Recall that, for $f \in \mathrm{Aff}(\mathbb{E}^2)$, there are a unique linear transformation $f_0 \in \mathrm{GL}(\mathbb{R}^2)$ and a unique vector $u \in \mathbb{R}^2$ such that $f = T_u \circ f_0$, where T_u is the translation relative to u. Recall also that there is an embedding $\mathrm{Aff}(\mathbb{E}^2) \hookrightarrow \mathrm{GL}(\mathbb{R}^3)$ defined by

$$f \mapsto \begin{pmatrix} f_0 & u \\ 0 & 1 \end{pmatrix} .$$

In this way, the group W can be regarded as a subgroup of $\mathrm{GL}(\mathbb{R}^3)$. We denote by H_k the linear plane of $\mathbb{R}^3$ generated by D_k, and we denote by H_k' the linear plane generated by D_k'. Then s_k is a linear reflection whose fixed hyperplane is H_k, and s_k' is a linear reflection whose fixed hyperplane is H_k'.

Let

$$\bar{C}'_0 = \{(x, y) \in \mathbb{E}^2 \mid 0 \leq x, y \leq 1\}.$$

Let $\bar{C}_0$ be the cone over $\bar{C}'_0$. This is a closed convex polyhedral cone whose walls are H_0, H_1, H'_0, H'_1. Observe that $wC_0 \cap C_0 = \emptyset$ for all $w \in W \setminus \{1\}$, thus (W, S) is a Vinberg system, where $S = \{s_0, s_1, s'_0, s'_1\}$. We have

$$\bar{I} = \bigcup_{w \in W} w\bar{C}_0 = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\} \cup \{(0, 0, 0)\},$$

thus

$$I = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}.$$

On the other hand,

$$\mathcal{A} = \{H_k, H'_k \mid k \in \mathbb{Z}\}.$$

THE END of I

Thank you for your attention!