Combinatorics and Topology of Hyperplane Arrangements
Ruhr-Universität Bochum
September 10-14, 2018
Michael J. Falk

Two (and a half) corrections from Lecture III

- As observed in the lecture, one has a disjoint union: $U' = U \sqcup U''$, where U, U', and U'' are the complements of the arrangement $\mathcal{A}$ and its deletion $\mathcal{A}'$ and contraction $\mathcal{A}''$, respectively. The long exact sequence of the pair (U', U), in cohomology (with complex coefficients), looks like

$$\cdots \to H^p(U') \to H^p(U) \to H^{p+1}(U', U) \to \cdots.$$

  (In the lecture I wrote $H^p(U', U)$.) Using excision and the tubular neighborhood theorem, and the fact that the normal bundle of U'' in U' is trivial, one has $H^p(U', U) \cong H^p(U'' \times (\mathbb{C}, \mathbb{C}^\times)$. The relative cohomology $H^k(\mathbb{C}, \mathbb{C}^\times)$ is isomorphic to $\mathbb{C}$ if $k = 2$ and vanishes otherwise. Then the Künneth formula gives $H^{p+1}(U', U) \cong H^{p-1}(U'')$. The rest of the argument given in lecture is correct.

- In the lecture I wrote two relations among Poincaré polynomials:

$$P(\mathcal{A}, t) = P(\mathcal{A}', t) + tP(\mathcal{A}'', t),$$

  and

$$P(\mathcal{A}_1 \oplus \mathcal{A}_2, t) = P(\mathcal{A}_1, t)P(\mathcal{A}_2, t),$$

  and said this implies $P(\mathcal{A}, t)$ is an evaluation of the Tutte polynomial of the underlying matroid of $\mathcal{A}$, obtained by evaluating $P(\mathcal{A}, t)$ for the loop and coloop. This is incorrect. Let $Q(\mathcal{A}, t) = t^{-r(\mathcal{A})}P(\mathcal{A}, t)$, where $r(\mathcal{A})$ denotes the rank of (the underlying matroid of) $\mathcal{A}$. Then from the equality above one concludes $Q(\mathcal{A}, t) = Q(\mathcal{A}', t) + Q(\mathcal{A}'', t)$ if $r(\mathcal{A}') = r(\mathcal{A})$, i.e., if the deleted hyperplane is not an "isthmus" of the underlying matroid (or a "separator" of $\mathcal{A}$), and $Q(\mathcal{A}_1 \oplus \mathcal{A}_2, t) = Q(\mathcal{A}_1, t)Q(\mathcal{A}_2, t)$. (The latter equation covers the case when the deleted element is an isthmus; the identity $P(\mathcal{A}, t) = P(\mathcal{A}', t) + tP(\mathcal{A}'', t)$ still holds in this case.) These two identities for Q imply that $Q(\mathcal{A}, t)$ is a Tutte-Grothendieck invariant of the underlying matroid, and therefore is an evaluation of the Tutte polynomial $T_M(x, y)$:

$$Q(\mathcal{A}, t) = T_M(Q(\mathsf{loop}), Q(\mathsf{coloop})),$$

  and one calculates $Q(\mathsf{loop}) = 1$ and $Q(\mathsf{coloop}) = t^{-1}(1 + t) = t^{-1} + 1$. Thus

$$P(\mathcal{A}, t) = t^{r(\mathcal{A})}T_M(1, t^{-1} + 1).$$

- As Luis pointed out, the identity $\mathrm{nbc}(\mathcal{A}) = \mathrm{nbc}(\mathcal{A}') \cup (S \cup \{e_n\} \mid S \in \mathrm{nbc}(\mathcal{A}'')\}$ is nonsense - the second set on the right is not a subset of the set on the left. For this identity I was actually thinking of the underlying matroids, with ground sets $\{1, \ldots, n\}$ for $\mathcal{A}$, and $\{1, \ldots, n-1\}$ for both $\mathcal{A}'$ and $\mathcal{A}''$, so that all three sets are collections of subsets of $\{1, \ldots, n\}$. With this interpretation the statement is correct. Note that this requires to consider $\mathcal{A}''$ as an arrangement with multiple copies of the same hyperplane (formulated as the "Ziegler multiplicity" in Masahiko's lecture). As pointed out, the OS algebra of the multiarrangement is the same as for the underlying "simple" arrangement, and $\mathrm{nbc}(\mathcal{A}'')$ reflects that.

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