SIMPLICIAL ARRANGEMENTS

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A simplicial arrangement is a set of linear hyperplanes decomposing the space into simplicial cones. More generally, a Tits arrangement decomposes a certain convex cone into simplicial cones. So far, Tits arrangements appeared (at least) in the following areas of mathematics:
(1) The special case in which the arrangement is crystallographic (this is a strong integrality property, see [Cun11a], [CH15]) can be considered as an invariant of Hopf algebras which is also called a Weyl groupoid. In particular it may be used to classify the so-called Nichols algebras (see for example [Cun18]).
(2) Tits arrangements generalize Coxeter groups and thus preserve some of their properties (see [CMW17]). For example, the complexified complement of a simplicial arrangement is a $K(\pi, 1)$-space (see [Del72]) and thus interesting from a topological point of view.
(3) Like reflection groups, simplicial arrangements produce interesting examples in the context of freeness of the module of derivations (see for example [BC12]). A counter example to the famous conjecture by Terao could be related to a simplicial arrangement.
(4) Simplicial arrangements of small rank play a role in the study of frieze patterns and thus of cluster algebras (see [Cun14]).

In this lecture I report on old results as well as on recent progress, see for example [Cun11b], [Cun12], [CG15], [CM17], [CG17], [CMW17], [Cun18].

1. Simplicial arrangements

1.1. Arrangements and combinatorics.

Definition 1.1. Let K be a field, $r \in \mathbb{N}$, and $V := K^r$. An **arrangement of hyperplanes** (or **r-arrangement**) $(\mathcal{A}, V)$ (or $\mathcal{A}$ for short) is a finite set of hyperplanes $\mathcal{A}$ in V.

Definition 1.2. Let $r \in \mathbb{N}$, $V := \mathbb{R}^r$, and $\mathcal{A}$ an arrangement in V. Let $\mathcal{K}(\mathcal{A})$ be the set of connected components (**chambers**) of $V \setminus \bigcup_{H \in \mathcal{A}} H$. If every chamber K is an **open simplicial cone**, i.e. there exist $\alpha_1^\vee, \ldots, \alpha_r^\vee \in V$ such that

$$K = \left\{ \sum_{i=1}^{r} a_i \alpha_i^\vee \mid a_i > 0 \quad \text{for all} \quad i = 1, \ldots, r \right\} =: \langle \alpha_1^\vee, \ldots, \alpha_r^\vee \rangle_{>0},$$

then $\mathcal{A}$ is called a **simplicial arrangement**.

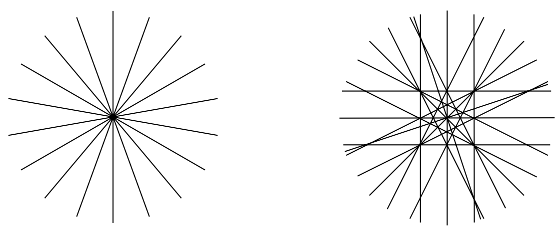


Figure 1. A simplicial arrangement in $\mathbb{R}^2$, a representation of a simplicial arrangement in $\mathbb{R}^3$ in the projective plane.

Example 1.3. (1) Figure 1 displays examples for $r = 2$ and $r = 3$.

(2) Let W be a real reflection group, R the set of roots of W. For $\alpha \in V^*$ we write $\alpha^\perp = \ker(\alpha)$. Then $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$ is a simplicial arrangement.

Definition 1.4. Let $\mathcal{A}$ be an arrangement. For $X \leq V$, we define the **localization**

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\}$$

of $\mathcal{A}$ at X, and the **restriction of** $\mathcal{A}$ to X, $(\mathcal{A}^X, X)$, where

$$\mathcal{A}^X := \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X\}.$$

Remark 1.5. If $\mathcal{A}$ is simplicial, then all localizations and restrictions to elements of its intersection lattice are simplicial.

Proposition 1.6. *Let $\mathcal{A}$ be a central essential arrangement of hyperplanes in $\mathbb{R}^r$, $r \geq 2$. Then $\mathcal{A}$ is simplicial if and only if*

$$r|\mathcal{K}(\mathcal{A})| = 2 \sum_{H \in \mathcal{A}} |\mathcal{K}(\mathcal{A}^H)|. \tag{1}$$

Remark 1.7. By Zaslavsky's theorem, $|\mathcal{K}(\mathcal{A})| = (-1)^r \chi_{\mathcal{A}}(-1)$ which depends only on the intersection lattice of $\mathcal{A}$. Thus simpliciality is a purely combinatorial property.

1.2. **History.** Achievements so far (possibly incomplete):

- Definition of simplicial arrangements (Melchior 1941).
- Catalogue of simplicial arrangements in the real projective plane (Grünbaum 1971, 2009, 2013).
- Simplicial arrangements are $K(\pi, 1)$ (Deligne 1972).
- Finite Weyl groupoids (C., Heckenberger 2009-2010).
- Simplicial arrangements with up to 27 lines (C. 2012).
- Some affine simplicial arrangements (C. 2014).
- Tits arrangements and Weyl groupoids (C., Mühlherr 2017).
- Supersolvable simplicial arrangements (C., Mücksch 2017).
- Tits arrangements on a cubic curve (C., Geis 2017).
- Free simplicial arrangements (Geis 2018).

2. Reflection groupoids

2.1. **Reflections and Cartan matrices.**

Definition 2.1. Let K be a field, $r \in \mathbb{N}$, $V := K^r$, and H a hyperplane in V. A **reflection** on V at H is a $\sigma \in \mathrm{GL}(V)$, $\sigma \neq \mathrm{id}$ of finite order which fixes H. Notice that the eigenvalues of σ are 1 and ζ for some root of unity $\zeta \in K$.

Lemma 2.2. *Let $\mathcal{A}$ be a simplicial arrangement and K a chamber, i.e. there is a basis $B = \{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$ of V such that $K = \langle B \rangle_{>0}$. Let K' be another chamber with*

$$\overline{K} \cap \overline{K'} = \langle \alpha_2^\vee, \ldots, \alpha_r^\vee \rangle_{\geq 0}.$$

Then there is a unique $\beta^\vee \in V$ with

$$K' = \langle B' \rangle_{>0}, \quad B' = \{\beta^\vee, \alpha_2^\vee, \ldots, \alpha_r^\vee\}, \quad \textit{and} \quad |B^* \cap -B'^*| = 1.$$

Proof. Choose $\beta^\vee \in V$ such that $K' = \langle \beta^\vee, \alpha_2^\vee, \ldots, \alpha_r^\vee \rangle_{>0}$. Let $\mu_1, \ldots, \mu_r \in \mathbb{R}$ be such that $\beta^\vee = \sum_{i=1}^r \mu_i \alpha_i^\vee$ (notice $\mu_1 \neq 0$). Let $B'^* = \{\beta_1, \ldots, \beta_r\}$ be the dual basis of $\{\beta^\vee, \alpha_2^\vee, \ldots, \alpha_r^\vee\}$, and $B^* = \{\alpha_1, \ldots, \alpha_r\}$ be dual to B. Then $\beta_1 = \frac{1}{\mu_1}\alpha_1$ and $\beta_j = -\frac{\mu_j}{\mu_1}\alpha_1 + \alpha_j$ for $j > 1$. To obtain $|B^* \cap -B'^*| = 1$ we need $-\alpha_1 = \beta_1 \in B'^*$ and hence $\mu_1 = -1$, $\beta_1 = -\alpha_1$ and $\beta_j = \mu_j \alpha_1 + \alpha_j$ for $j > 1$. Thus a $\beta^\vee$ as desired exists and is unique. $\square$

Corollary 2.3. *Using the notation of the proof of Lemma 2.2, the map*

$$\sigma : V^* \to V^*, \quad \alpha_i \mapsto \beta_i$$

is a reflection. With respect to B^*, it becomes the matrix

$$\begin{pmatrix} -1 & \mu_2 & \dots & \mu_r \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}.$$

Example 2.4. Let $R = \{(1,0),(0,1),(1,2)\} \in (\mathbb{R}^2)^*$, $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$. Then $K = \langle B\rangle_{>0}$ is a chamber if $B = \{\alpha_1^\vee = (1,0), \alpha_2^\vee = (0,1)\}$, $K' = \langle B'\rangle_{>0}$ with $B' = \{\tilde{\beta}^\vee = (-2,1), \alpha_2^\vee = (0,1)\}$ is an adjacent chamber. To obtain $\mu_1 = -1$, we need to choose $\beta^\vee = (-1, \frac{1}{2})$, hence $\mu_2 = \frac{1}{2}$. The unique reflection σ is

$$\begin{pmatrix} -1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

with respect to B^*.

Definition 2.5. Let $\mathcal{A}$ be a simplicial arrangement, $K = \langle B\rangle_{>0}$, $B = \{\alpha_1^\vee, \dots, \alpha_r^\vee\}$ a chamber, and $B^* = \{\alpha_1, \dots, \alpha_r\}$ be dual to B. Then by Corollary 2.3, there are reflections $\sigma_1, \dots, \sigma_r$, represented by

$$\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ \mu_{i,1} & \cdots & -1 & \cdots & \mu_{i,r} \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix},$$

for certain $\mu_{i,j} \in \mathbb{R}$, $i \neq j$ with respect to B^* and uniquely determined by K, B and its adjacent chambers.

The matrix $C^{K,B} = (c_{i,j})_{1\le i,j\le r}$ with

$$c_{i,j} := \begin{cases} -\mu_{i,j} & \text{if } i \neq j \\ 2 & \text{if } i = j \end{cases}$$

is called the **Cartan matrix** of (K, B) in $\mathcal{A}$. Note that

$$\sigma_i(\alpha_j) = \alpha_j - c_{i,j}\alpha_i$$

for all $1 \le i, j \le r$. We sometimes write $\sigma_i^{K,B}$ to emphasize that σ_i depends on K and B.

Example 2.6. (1) Let $\mathcal{A}$ be as in Example 2.4. Then the Cartan matrix of (K, B) is

$$C^{K,B} = \begin{pmatrix} 2 & -\frac{1}{2} \\ -2 & 2 \end{pmatrix}.$$

(2) If W is a Weyl group with root system R, then all Cartan matrices of (K, B) when B^* is a set of simple roots for the chamber K are equal and coincide with the classical Cartan matrix of W.

Definition 2.7. Let $\mathcal{A}$ be a simplicial arrangement in $V = \mathbb{R}^r$. We construct a category $\mathcal{C}(\mathcal{A})$ with

- objects: $\mathrm{Obj}(\mathcal{C}(\mathcal{A})) = \{B^* = (\alpha_1, \dots, \alpha_r) \mid \langle B\rangle_{>0} \in \mathcal{K}(\mathcal{A})\}$ (where the simple systems are ordered).
- morphisms: for each $B^* = (\alpha_1, \dots, \alpha_r) \in \mathrm{Obj}(\mathcal{C}(\mathcal{A}))$ and $i = 1, \dots, r$ there is a morphism $\sigma_i^{K,B} \in \mathrm{Mor}(B, (\sigma_i^{K,B}(\alpha_1), \dots, \sigma_i^{K,B}(\alpha_r)))$. All other morphisms are compositions of the generators $\sigma_i^{K,B}$.

A **reflection groupoid** $\mathcal{W}(\mathcal{A})$ of $\mathcal{A}$ is a connected component of $\mathcal{C}(\mathcal{A})$. A **Weyl groupoid** is a reflection groupoid for which all Cartan matrices are integral.

2.2. **Crystallographic arrangements.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$, $|\mathcal{A}| = n$ be simplicial. For each H_i, $i = 1, \dots, n$ we choose an element $\beta_i \in V^*$ such that $H_i = \beta_i^\perp$ and let $R := \{\pm\beta_1, \dots, \pm\beta_n\} \subseteq V^*$. For each chamber $K \in \mathcal{K}(\mathcal{A})$ set

$$\begin{aligned} B^K &= \{\alpha \in R \mid \forall x \in K \,:\, \alpha(x) \geq 0,\ \langle \alpha^\perp \cap \overline{K} \rangle = \alpha^\perp\} \\ &= \{\text{ “normal vectors in } R \text{ of the walls of } K \\ &\qquad \text{pointing to the inside” }\}. \end{aligned}$$

If $\alpha_1^\vee, \dots, \alpha_r^\vee$ is the dual basis to $B^K = \{\alpha_1, \dots, \alpha_r\}$, then $K = \langle \alpha_1^\vee, \dots, \alpha_r^\vee \rangle_{>0}$ since $\mathcal{A}$ is simplicial.

Definition 2.8. Let $\mathcal{A}$ be a simplicial arrangement in V and $R \subseteq V^*$ a finite set such that $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$ and $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$ for all $\alpha \in R$. We call $(\mathcal{A}, R)$ a **crystallographic arrangement** if for all $K \in \mathcal{K}(\mathcal{A})$:

$$R \subseteq \sum_{\alpha \in B^K} \mathbb{Z}\alpha.$$

Two crystallographic arrangements $(\mathcal{A}, R)$, $(\mathcal{A}', R')$ in V are called **equivalent** if there exists $\psi \in \mathrm{Aut}(V^*)$ with $\psi(R) = R'$. We then write $(\mathcal{A}, R) \cong (\mathcal{A}', R')$.

Example 2.9. (1) Let R be the set of roots of the root system of a crystallographic Coxeter group. Then $(\{\alpha^\perp \mid \alpha \in R\}, R)$ is a crystallographic arrangement.
(2) If $R_+ := \{(1,0), (3,1), (2,1), (5,3), (3,2), (1,1), (0,1)\}$, then $\{\alpha^\perp \mid \alpha \in R_+\}$ is a crystallographic arrangement.

Remark 2.10. Weyl groupoids are reflection groupoids of crystallographic arrangements.

Definition 2.11. Fix an object B in a reflection groupoid $\mathcal{C}(\mathcal{A})$. Then

$$R^B = \{\gamma_B(\varphi(\alpha)) \mid \alpha \in B',\ \varphi \in \mathrm{Mor}(B', B)\} \subseteq \mathbb{R}^r$$

where $\gamma_B : V \to \mathbb{R}^r$ is the coordinate map with respect to B, is the set of **real roots** (at B). The collection $(R^B)_B$ is denoted by $\mathcal{R}^{\mathrm{re}}(\mathcal{W})$. A real root $\alpha \in R^B$, is called **positive** (resp. **negative**) if $\alpha \in \mathbb{R}^r_{\geq 0}$ (resp. $\alpha \in \mathbb{R}^r_{\leq 0}$).

Let $\{\alpha_1, \dots, \alpha_r\}$ be the standard basis of $\mathbb{R}^r$. We call the α_i **simple roots**.

Definition 2.12. Let $\mathcal{W}(\mathcal{A})$ be a reflection groupoid and A the set of objects. The real roots of $\mathcal{W}(\mathcal{A})$ are called a **root system** if they satisfy:

(R1) $R^a = R^a_+ \cup -R^a_+$, where $R^a_+ = R^a \cap \mathbb{R}^r_{\geq 0}$, for all $a \in A$.
(R2) $R^a \cap \mathbb{R}\alpha_i = \{\alpha_i, -\alpha_i\}$ for all $i = 1, \dots, r$, $a \in A$.
(R3) $\sigma_i^a(R^a) = R^{\sigma_i(a)}$ for all $i = 1, \dots, r$, $a \in A$.
(R4) If $i, j \in \{1, \dots, r\}$ and $a \in A$ such that $i \neq j$ and $m^a_{i,j}$ is finite, then $(\sigma_i\sigma_j)^{m^a_{i,j}}(a) = a$.

Here, $m^a_{i,j} = |R^a \cap (\mathbb{R}_{\geq 0}\alpha_i + \mathbb{R}_{\geq 0}\alpha_j)|$.

A root system is called **finite** if for all $a \in A$ the set R^a is finite.

2.3. **Frieze patterns and quiddity cycles.**

Example 2.13.

$$\begin{array}{cccccccccccc}
0 & 1 & 1 & 3 & 2 & 1 & 0 & & & & & \\
 & 0 & 1 & 4 & 3 & 2 & 1 & 0 & & & & \\
 & & 0 & 1 & 1 & 1 & 1 & 1 & 0 & & & \\
 & & & 0 & 1 & 2 & 3 & 4 & 1 & 0 & & \\
 & & & & 0 & 1 & 2 & 3 & 1 & 1 & 0 & \\
 & & & & & 0 & 1 & 2 & 1 & 2 & 1 & 0
\end{array}$$

Definition 2.14. Let R be a subset of a commutative ring.

(1) A **frieze pattern** over R is an array $\mathcal{F}$ of the form

$$\begin{array}{cccccccccccc}
 & & \ddots & & & & & \ddots & & & & \\
0 & 1 & c_{i-1,i+1} & c_{i-1,i+2} & \cdots & \cdots & c_{i-1,n+i} & 1 & 0 & & & \\
 & 0 & 1 & c_{i,i+2} & c_{i,i+3} & \cdots & \cdots & c_{i,n+i+1} & 1 & 0 & & \\
 & & 0 & 1 & c_{i+1,i+3} & c_{i+1,i+4} & \cdots & \cdots & c_{i+1,n+i+2} & 1 & 0 & \\
 & & & & \ddots & & & & \ddots & & &
\end{array}$$

where $c_{i,j}$ are numbers in R, and such that every (complete) adjacent 2×2 submatrix has determinant 1. We call n the **height** of the frieze pattern $\mathcal{F}$. We say that the frieze pattern $\mathcal{F}$ is **periodic** with period $m>0$ if $c_{i,j}=c_{i+m,j+m}$ for all i,j.
(2) A frieze pattern is called **tame** if every adjacent 3×3-submatrix has determinant 0.

The following matrices are the key to the structure of tame frieze patterns.

Definition 2.15. For c in a commutative ring, let

$$\eta(c) := \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix}.$$

Remark 2.16. Notice that up to a transposition, $\eta(c)$ may be viewed as a **reflection**:

$$\eta(c)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & c \\ 0 & 1 \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\eta(c) = \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}.$$

We will see later why this implies that frieze patterns may be seen as reflection groupoids.

Proposition 2.17. *Let R be a commutative ring.*

(1) Let $\mathcal{F}=(c_{i,j})$ be a tame frieze pattern over R of height n, $c_k := c_{k,k+2}$ for $k\in\mathbb{Z}$. Then $\mathcal{F}$ is periodic with period $m=n+3$ and

$$\prod_{k=1}^{m}\eta(c_k) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \textit{and} \quad c_{i,j+2} = (M_{i,j})_{1,1}, \quad \textit{where} \quad M_{i,j} := \prod_{k=i}^{j}\eta(c_k).$$

(2) Let $(c_1,\dots,c_m)\in R^m$ satisfy $\prod_{k=1}^{m}\eta(c_k) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and let $M_{i,j} := \prod_{k=i}^{j}\eta(c_k)$ be as above. Then the $(a_{i,j+2})_{i,j} = ((M_{i,j})_{1,1})_{i,j}$ (where $i-1\le j\le m+i-3$) defines a periodic tame frieze pattern over R with period m and height $m-3$.

Definition 2.18. Let R be a subset of a commutative ring and $\lambda\in\{\pm 1\}$. A **λ-quiddity cycle** over R is a sequence $(c_1,\dots,c_m)\in R^m$ satisfying

$$\prod_{k=1}^{m}\eta(c_k) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda\mathrm{id}. \tag{2}$$

A (-1)-quiddity cycle is called a **quiddity cycle** for short.

Example 2.19. Consider the commutative ring $\mathbb{C}$ and $R=\mathbb{C}$.

(1) $(0,0)$ is the only λ-quiddity cycle of length 2.
(2) $(1,1,1)$ and $(-1,-1,-1)$ are the only λ-quiddity cycles of length 3.
(3) $(t,2/t,t,2/t)$, t a unit and $(a,0,-a,0)$, a arbitrary, are the only λ-quiddity cycles of length 4 (check this as an exercise).

Remark 2.20. Let $\underline{c}=(c_1,\dots,c_m)$ be a λ-quiddity cycle. Then for any $\sigma\in D_n$, the cycle $\underline{c}^\sigma$ is a λ-quiddity cycle as well.

The following useful lemma is inspired by old results on continued fractions.

Lemma 2.21. *Let $(c_1,\dots,c_m)\in\mathbb{C}^m$ such that $\prod_{j=1}^{m}\eta(c_j)$ is a scalar multiple of the identity matrix. Then there are two different indices $j,k\in\{1,\dots,m\}$ with $|c_j|<2$ and $|c_k|<2$.*

Using $\eta(a)\eta(b) = \eta(a+1)\eta(1)\eta(b+1)$ for all a, b we conclude:

Corollary 2.22. *Frieze patterns with entries in $\mathbb{N}$ correspond to triangulations of polygons by non-intersecting diagonals.*

3. Finite Weyl groupoids

3.1. Rank two.

Remark 3.1. The set of quiddity cycles with entries in $\mathbb{N}_{>0}$ corresponds to the sequences of Cartan entries of finite Weyl groupoids of rank two.

Definition 3.2. A reflection groupoid and its root system are called **irreducible** if all Cartan matrices are indecomposable.

These are the roots of irreducible finite Weyl groupoids of rank two.

Definition 3.3. Define *$\mathcal{F}$-sequences* as finite sequences of length ≥ 3 with entries in $\mathbb{N}_0^2$ given by the following recursion.

(1) $((0,1),(1,1),(1,0))$ is an $\mathcal{F}$-sequence.
(2) If $(v_1,\dots,v_n)$ is an $\mathcal{F}$-sequence, then
$$(v_1,\dots,v_i,v_i+v_{i+1},v_{i+1},\dots,v_n)$$
are $\mathcal{F}$-sequences for $i=1,\dots,n-1$.
(3) Every $\mathcal{F}$-sequence is obtained recursively by (1) and (2).

Proposition 3.4. *Crystallographic arrangements of rank two correspond to $\mathcal{F}$-sequences: each $\mathcal{F}$-sequence is a set of roots of a Weyl groupoid.*

Corollary 3.5. *Assume that $\mathcal{R}^{re}(\mathcal{W}(\mathcal{A}))$ is a finite root system of a Weyl groupoid. Then for all objects a and $\alpha \in R^a_+$, either α is simple or it is the sum of two positive roots in R^a_+.*

Corollary 3.6. *Let R^a be the set of roots at an object a of a finite Weyl groupoid of rank two and $(c^a_{i,j})$ the Cartan matrix at a. Then*
$$c_{i,j} = -\max\{k \in \mathbb{N}_{\geq 0} \mid k\alpha_i + \alpha_j \in R^a\}$$
for $i \neq j$. Moreover, $k\alpha_i + \alpha_j \in R^a$ for $k = 0,\dots,-c_{i,j}$.

3.2. Rank three.

Let A be the set of objects of an irreducible reflection groupoid and $a \in A$. The last observation on root systems of rank two implies the following central result:

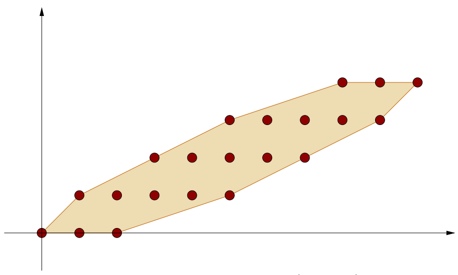
Theorem 3.7. *Let $\alpha \in R^a_+$. Then either α is simple, or it is the sum of two positive roots.*

Theorem 3.8. *Let $\alpha_1,\dots,\alpha_r$ be the simple roots at a. Then $\sum_{i=1}^r \alpha_i \in R^a$.*

View the roots as elements of $\mathbb{Z}^3$ with the lexicographic ordering $\leq$.

Lemma 3.9. *Let $\alpha, \beta \in R^a_+$ be minimal in $\langle\alpha,\beta\rangle_{\mathbb{Q}} \cap R^a_+$ with respect to $\leq$. Then $\langle\alpha,\beta\rangle_{\mathbb{Q}} \cap R^a \subseteq \pm(\mathbb{N}_0\alpha + \mathbb{N}_0\beta)$.*

Theorem 3.10 (Weak Convexity)**.** *Let $a \in A$ and $\alpha,\beta,\gamma \in R^a_+$. If $\det(\alpha,\beta,\gamma)^2 = 1$ and α,β are the two smallest elements of $\langle\alpha,\beta\rangle \cap R^a_+$, then either α,β,γ are simple, or one of $\gamma-\alpha$, $\gamma-\beta$ is in R^a.*



Roots on the plane $(1,*,*)$

Example: $\alpha = (0,1,0)$, $\beta = (0,0,1)$, $\gamma = (1,7,3)$

Theorem 3.11 (Required roots)**.** *Let $a \in A$ and $\alpha, \beta, \gamma \in R^a_+$. Assume that* $\det(\alpha, \beta, \gamma)^2 = 1$ *and that $\gamma - \alpha, \gamma - \beta \notin R^a$. Then $\gamma + \alpha \in R^a$ or $\gamma + \beta \in R^a$.*

Theorem 3.12 (Bound for the Cartan entries)**.** *All entries of the Cartan matrices are greater or equal to -7.*

Remark 3.13. In fact, all entries of the Cartan matrices are greater or equal to -6.

The preceding theorems suggest the following algorithm.

Function **Enumerate**(R)

(1) If R defines a crystallographic arrangement, output R and continue.
(2) $Y := \{\alpha + \beta \mid \alpha, \beta \in R,\ \alpha \neq \beta\} \backslash R$.
(3) For all $\alpha \in Y$ with $\alpha > \max R$:
  (a) Compute all "subgroupoids of rank two" in $R \cup \{\alpha\}$.
  (b) If all Cartan entries are ≥ -7, all rank two root sets are $\mathcal{F}$-sequences, and the "Weak Convexity" is satisfied, then call **Enumerate**$(R \cup \{\alpha\})$.

Remark 3.14. We use Theorem "Required roots" as a further obstruction.

The algorithm terminates and yields the result:

Theorem 3.15 (Cuntz, Heckenberger (2012))**.** *Up to equivalences, there are* 55 *irreducible crystallographic arrangements of rank three.*

3.3. **Higher rank.** With the knowledge about rank three, we enumerate crystallographic arrangements in ranks four to eight with a similar algorithm.

An analysis of Dynkin diagrams leads to a complete classification in rank > 8.

Theorem 3.16 (Cuntz, Heckenberger (2015))**.** *There are exactly three families of irreducible crystallographic arrangements:*

(1) The family of rank two parametrized by triangulations of a convex n-gons by non-intersecting diagonals.
(2) For each rank $r > 2$, arrangements of type A_r, B_r, C_r and D_r, and a further series of $r - 1$ arrangements.
(3) Further 74 *"sporadic" arrangements of rank r, $3 \leq r \leq 8$.*

Remark 3.17. The proof of the classification relies on enumerations by the computer. In rank three, approximately 60.000.000 cases need to be considered. A "short" proof is work in progress. At least an enumeration of all potential parabolics seems to be possible.

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