SIMPLICIAL ARRANGEMENTS

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A simplicial arrangement is a set of linear hyperplanes decomposing the space into simplicial cones. More generally, a Tits arrangement decomposes a certain convex cone into simplicial cones. So far, Tits arrangements appeared (at least) in the following areas of mathematics:

1. The special case in which the arrangement is crystallographic (this is a strong integrality property, see [Cun11a, CH15]) can be considered as an invariant of Hopf algebras which is also called a Weyl groupoid. In particular it may be used to classify the so-called Nichols algebras (see for example [Cun18]).

2. Tits arrangements generalize Coxeter groups and thus preserve some of their properties (see [CMW17]). For example, the complexified complement of a simplicial arrangement is a $K(\pi, 1)$-space (see [Del72]) and thus interesting from a topological point of view.

3. Like reflection groups, simplicial arrangements produce interesting examples in the context of freeness of the module of derivations (see for example [BC12]). A counter example to the famous conjecture by Terao could be related to a simplicial arrangement.

4. Simplicial arrangements of small rank play a role in the study of frieze patterns and thus of cluster algebras (see [Cun14]).

In this lecture I report on old results as well as on recent progress, see for example [Cun11b, Cun12, CG15, CMI7, CG17, CMW17, Cun18].

1. Simplicial arrangements

1.1. Arrangements and combinatorics.

**Definition 1.1.** Let $K$ be a field, $r \in \mathbb{N}$, and $V := K^r$. An arrangement of hyperplanes (or $r$-arrangement) $(\mathcal{A}, V)$ (or $\mathcal{A}$ for short) is a finite set of hyperplanes $\mathcal{A}$ in $V$.

**Definition 1.2.** Let $r \in \mathbb{N}$, $V := \mathbb{R}^r$, and $\mathcal{A}$ an arrangement in $V$. Let $\mathcal{K}(\mathcal{A})$ be the set of connected components (chambers) of $V \setminus \bigcup_{H \in \mathcal{A}} H$. If every chamber $K$ is an open simplicial cone, i.e. there exist $\alpha_1^{\vee}, \ldots, \alpha_r^{\vee} \in V$ such that

$$K = \left\{ \sum_{i=1}^{r} a_i \alpha_i^{\vee} \mid a_i > 0 \text{ for all } i = 1, \ldots, r \right\} =: (\alpha_1^{\vee}, \ldots, \alpha_r^{\vee})_{>0},$$

then $\mathcal{A}$ is called a simplicial arrangement.

**Example 1.3.** (1) Figure 1 displays examples for $r = 2$ and $r = 3$.  

![Figure 1. A simplicial arrangement in $\mathbb{R}^2$, a representation of a simplicial arrangement in $\mathbb{R}^3$ in the projective plane.](image)
(2) Let $W$ be a real reflection group, $R$ the set of roots of $W$. For $\alpha \in V^*$ we write $\alpha^\perp = \ker(\alpha)$. Then $A = \{\alpha^\perp \mid \alpha \in R\}$ is a simplicial arrangement.

**Definition 1.4.** Let $A$ be an arrangement. For $X \subseteq V$, we define the localization
$$A_X := \{H \in A \mid X \subseteq H\}$$
of $A$ at $X$, and the restriction of $A$ to $X$, $(A^X, X)$, where
$$A^X := \{X \cap H \mid H \in A \setminus A_X\}.$$**Remark 1.5.** If $A$ is simplicial, then all localizations and restrictions to elements of its intersection lattice are simplicial.

**Proposition 1.6.** Let $A$ be a central essential arrangement of hyperplanes in $\mathbb{R}^r$, $r \geq 2$. Then $A$ is simplicial if and only if

$$(1) \quad r|\mathcal{K}(A)| = 2 \sum_{H \in A} |\mathcal{K}(A^H)|.$$**Remark 1.7.** By Zaslavsky’s theorem, $|\mathcal{K}(A)| = (-1)^r \chi_A(-1)$ which depends only on the intersection lattice of $A$. Thus simplicity is a purely combinatorial property.

### 1.2. History.
Achievements so far (possibly incomplete):

- Definition of simplicial arrangements (Melchior 1941).
- Simplicial arrangements are $K(\pi, 1)$ (Deligne 1972).
- Finite Weyl groupoids (C., Heckenberger 2009-2010).
- Simplicial arrangements with up to 27 lines (C. 2012).
- Some affine simplicial arrangements (C. 2014).
- Tits arrangements and Weyl groupoids (C., Mühlherr 2017).
- Supersolvable simplicial arrangements (C., Mücksch 2017).
- Some affine simplicial arrangements (C., Geis 2017).
- Finite Weyl groupoids (C., Heckenberger 2009-2010).
- Free simplicial arrangements (Geis 2018).

### 2. Reflection groupoids

**Definition 2.1.** Let $K$ be a field, $r \in \mathbb{N}$, $V := K^r$, and $H$ a hyperplane in $V$. A reflection on $V$ at $H$ is a $\sigma \in \text{GL}(V)$, $\sigma \neq \text{id}$ of finite order which fixes $H$. Notice that the eigenvalues of $\sigma$ are 1 and $\zeta$ for some root of unity $\zeta \in K$.

**Lemma 2.2.** Let $A$ be a simplicial arrangement and $K$ a chamber, i.e. there is a basis $B = \{\alpha_1^\perp, \ldots, \alpha_r^\perp\}$ of $V$ such that $K = \langle B \rangle_{>0}$. Let $K'$ be another chamber with
$$K \cap K' = \langle \alpha_2^\perp, \ldots, \alpha_r^\perp \rangle_{>0}.$$Then there is a unique $\beta^\perp \in V$ with
$$K' = \langle B' \rangle_{>0}, \quad B' = \{\beta^\perp, \alpha_2^\perp, \ldots, \alpha_r^\perp\}, \quad \text{and} \quad |B^* \cap -B^*| = 1.$$**Proof.** Choose $\beta^\perp \in V$ such that $K' = \langle \beta^\perp, \alpha_2^\perp, \ldots, \alpha_r^\perp \rangle_{>0}$. Let $\mu_1, \ldots, \mu_r \in \mathbb{R}$ be such that $\beta^\perp = \sum_{i=1}^r \mu_i \alpha_i^\perp$ (notice $\mu_1 \neq 0$). Let $B^* = \{\beta_1, \ldots, \beta_r\}$ be the dual basis of $\{\beta^\perp, \alpha_2^\perp, \ldots, \alpha_r^\perp\}$, and $B^* = \{\alpha_1, \ldots, \alpha_r\}$ be dual to $B$. Then $\beta_1 = \frac{1}{\mu_1} \alpha_1$ and $\beta_j = -\frac{\mu_j}{\mu_1} \alpha_1 + \alpha_j$ for $j > 1$. To obtain $|B^* \cap -B^*| = 1$ we need $-\alpha_1 = \beta_1 \in B^*$ and hence $\mu_1 = -1$, $\beta_1 = -\alpha_1$ and $\beta_j = \mu_j \alpha_1 + \alpha_j$ for $j > 1$. Thus a $\beta^\perp$ as desired exists and is unique. □
Corollary 2.3. Using the notation of the proof of Lemma 2.2, the map 
\[ \sigma : V^* \to V^*, \quad \alpha_i \mapsto \beta_i \]
is a reflection. With respect to \( B^* \), it becomes the matrix
\[
\begin{pmatrix}
-1 & \mu_2 & \ldots & \mu_r \\
0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{pmatrix}.
\]

Example 2.4. Let \( R = \{(1,0),(0,1),(1,2)\} \in (\mathbb{R}^2)^*, A = \{\alpha^\perp | \alpha \in R\} \). Then \( K = \langle B \rangle_{>0} \) is a chamber if \( B = \{\alpha_1^\perp = (1,0), \alpha_2^\perp = (0,1)\} \), \( K' = \langle B' \rangle_{>0} \) with \( B' = \{\beta^\perp = (-2,1), \alpha_2^\perp = (0,1)\} \) is an adjacent chamber. To obtain \( \mu_i = -1 \), we need to choose \( \beta^\perp = (-1,\frac{1}{2}) \), hence \( \mu_2 = \frac{1}{2} \).

\[ \text{The unique reflection } \sigma \text{ is} \]
\[
\begin{pmatrix}
-1 & \frac{1}{2} \\
0 & 1
\end{pmatrix}
\]
with respect to \( B^* \).

Definition 2.5. Let \( \mathcal{A} \) be a simplicial arrangement, \( K = \langle B \rangle_{>0}, B = \{\alpha_1^\perp, \ldots, \alpha_r^\perp\} \) a chamber, and \( B^* = \{\alpha_1, \ldots, \alpha_r\} \) be dual to \( B \). Then by Corollary 2.3 there are reflections \( \sigma_1, \ldots, \sigma_r \), represented by
\[
\begin{pmatrix}
1 & & & & \cdot & & & & & \\
& & & & 0 & & & & & \\
& & \mu_i,1 & \cdot & \cdot & \cdot & \cdot & & & \\
& & & & & & & & & \\
0 & & & & & & & & & 1
\end{pmatrix},
\]
for certain \( \mu_{i,j} \in \mathbb{R}, i \neq j \) with respect to \( B^* \) and uniquely determined by \( K, B \) and its adjacent chambers.

The matrix \( C^{K,B} = (c_{i,j})_{1 \leq i,j \leq r} \) with
\[
c_{i,j} := \begin{cases} 
-\mu_{i,j} & \text{if } i \neq j \\
2 & \text{if } i = j
\end{cases}
\]
is called the Cartan matrix of \( (K, B) \) in \( \mathcal{A} \). Note that
\[
\sigma_i(\alpha_j) = \alpha_j - c_{i,j} \alpha_i
\]
for all \( 1 \leq i, j \leq r \). We sometimes write \( \sigma_i^{K,B} \) to emphasize that \( \sigma_i \) depends on \( K \) and \( B \).

Example 2.6. (1) Let \( \mathcal{A} \) be as in Example 2.4. Then the Cartan matrix of \( (K, B) \) is
\[
C^{K,B} = \begin{pmatrix} 2 & -\frac{1}{2} \\ -2 & \frac{1}{2} \end{pmatrix}.
\]

(2) If \( W \) is a Weyl group with root system \( R \), then all Cartan matrices of \( (K, B) \) when \( B^* \) is a set of simple roots for the chamber \( K \) are equal and coincide with the classical Cartan matrix of \( W \).

Definition 2.7. Let \( \mathcal{A} \) be a simplicial arrangement in \( V = \mathbb{R}^r \). We construct a category \( \mathcal{C}(\mathcal{A}) \) with
- objects: \( \text{Obj}(\mathcal{C}(\mathcal{A})) = \{B^* = \{\alpha_1, \ldots, \alpha_r\} | \langle B \rangle_{>0} \in K(\mathcal{A})\} \) (where the simple systems are ordered).
- morphisms: for each \( B^* = \{\alpha_1, \ldots, \alpha_r\} \in \text{Obj}(\mathcal{C}(\mathcal{A})) \) and \( i = 1, \ldots, r \) there is a morphism \( \sigma_i^{K,B} \in \text{Mor}(B, (\sigma_i^{K,B}(\alpha_1), \ldots, \sigma_i^{K,B}(\alpha_r))) \). All other morphisms are compositions of the generators \( \sigma_i^{K,B} \).

A reflection groupoid \( \mathcal{W}(\mathcal{A}) \) of \( \mathcal{A} \) is a connected component of \( \mathcal{C}(\mathcal{A}) \). A Weyl groupoid is a reflection groupoid for which all Cartan matrices are integral.
2.2. Crystallographic arrangements. Let \( A = \{H_1, \ldots, H_n\} \), \(|A| = n\) be simplicial. For each \( H_i, i = 1, \ldots, n\) we choose an element \( \beta_i \in V^* \) such that \( H_i = \beta_i^\perp \) and let \( R := \{\pm \beta_1, \ldots, \pm \beta_n\} \subseteq V^* \). For each chamber \( K \in \mathcal{K}(A) \) set
\[
B^K = \{ \alpha \in R \mid \forall x \in K : \alpha(x) \geq 0, \ (\alpha^\perp \cap K) = \alpha^\perp \} = \{ \text{“normal vectors in } R \text{ of the walls of } K \text{ pointing to the inside”} \}.
\]

If \( \alpha_1^\vee, \ldots, \alpha_r^\vee \) is the dual basis to \( B^K = \{\alpha_1, \ldots, \alpha_r\} \), then \( K = \langle \alpha_1^\vee, \ldots, \alpha_r^\vee \rangle_{>0} \) since \( A \) is simplicial.

**Definition 2.8.** Let \( A \) be a simplicial arrangement in \( V \) and \( R \subseteq V^* \) a finite set such that \( A = \{\alpha^\perp \mid \alpha \in R\} \) and \( R \alpha \cap R = \{\pm \alpha\} \) for all \( \alpha \in R \). We call \((A, R)\) a **crystallographic arrangement** if for all \( K \in \mathcal{K}(A) \):
\[
R \subseteq \bigoplus_{\alpha \in B^K} \mathbb{Z} \alpha.
\]

Two crystallographic arrangements \((A, R)\), \((A', R')\) in \( V \) are called **equivalent** if there exists \( \psi \in \text{Aut}(V^*) \) with \( \psi(R) = R' \). We then write \((A, R) \cong (A', R')\).

**Example 2.9.** (1) Let \( R \) be the set of roots of the root system of a crystallographic Coxeter group. Then \( \{\alpha^\perp \mid \alpha \in R\}, R \) is a crystallographic arrangement.

(2) If \( R_+ := \{(1, 0), (3, 1), (2, 1), (5, 3), (3, 2), (1, 1), (0, 1)\} \), then \( \{\alpha^\perp \mid \alpha \in R_+\} \) is a crystallographic arrangement.

**Remark 2.10.** Weyl groupoids are reflection groupoids of crystallographic arrangements.

**Definition 2.11.** Fix an object \( B \) in a reflection groupoid \( \mathcal{C}(A) \). Then
\[
R^B = \{ \gamma_B(\varphi(\alpha)) \mid \alpha \in B', \ \varphi \in \text{Mor}(B', B) \} \subseteq \mathbb{R}^r
\]
where \( \gamma_B : V \to \mathbb{R}^r \) is the coordinate map with respect to \( B \), is the set of **real roots** (at \( B \)). The collection \((R^B)_B\) is denoted by \( \mathcal{R}^\mathbb{R}(\mathcal{W}) \). A real root \( \alpha \in R^B \), is called **positive** (resp. **negative**) if \( \alpha \in \mathbb{R}_{>0}^r \) (resp. \( \alpha \in \mathbb{R}_{<0}^r \)).

Let \( \{\alpha_1, \ldots, \alpha_r\} \) be the standard basis of \( \mathbb{R}^r \). We call the \( \alpha_i \) **simple roots**.

**Definition 2.12.** Let \( \mathcal{W}(A) \) be a reflection groupoid and \( A \) the set of objects. The real roots of \( \mathcal{W}(A) \) are called a **root system** if they satisfy:

(R1) \( R^a = R^a_+ \cup -R^a_+ \), where \( R^a_+ = R^a \cap \mathbb{R}_{>0}^r \), for all \( a \in A \).

(R2) \( R^a \cap R \alpha_i = \{\alpha_i, -\alpha_i\} \) for all \( i = 1, \ldots, r, \ a \in A \).

(R3) \( \sigma_i^a(R^a) = R^{a_i(a)} \) for all \( i = 1, \ldots, r, \ a \in A \).

(R4) If \( i, j \in \{1, \ldots, r\} \) and \( a \in A \) such that \( i \neq j \) and \( m_{i,j}^a \) is finite, then \( (\sigma_i \sigma_j)^{m_{i,j}^a}(a) = a \).

Here, \( m_{i,j}^a = \left| R^a \cap (\mathbb{R}_{>0} \alpha_i + \mathbb{R}_{>0} \alpha_j) \right| \).

A root system is called **finite** if for all \( a \in A \) the set \( R^a \) is finite.

2.3. Frieze patterns and quiddity cycles.

| 0 1 2 3 1 0 |
| 0 1 4 3 2 1 0 |
| 0 1 1 1 1 0 |
| 0 1 2 3 4 1 0 |
| 0 1 2 3 1 1 0 |
| 0 1 2 1 2 1 0 |

**Example 2.13.**

**Definition 2.14.** Let \( R \) be a subset of a commutative ring.
A **frieze pattern** over \( R \) is an array \( F \) of the form

\[
\begin{array}{ccccccc}
\ddots & 0 & 1 & c_{i-1,i} & c_{i-1,i+1} & \cdots & \cdots & c_{i-1,n+i} & 1 & 0 \\
0 & 1 & c_{i,i+2} & c_{i,i+3} & \cdots & \cdots & c_{i,n+i+1} & 1 & 0 \\
0 & 1 & c_{i+1,i+3} & c_{i+1,i+4} & \cdots & \cdots & c_{i+1,n+i+2} & 1 & 0 \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{array}
\]

where \( c_{i,j} \) are numbers in \( R \), and such that every (complete) adjacent \( 2 \times 2 \) submatrix has determinant 1. We call \( n \) the **height** of the frieze pattern \( F \). We say that the frieze pattern \( F \) is **periodic** with period \( m > 0 \) if \( c_{i,j} = c_{i+m,j+m} \) for all \( i, j \).

(2) A frieze pattern is called **tame** if every adjacent \( 3 \times 3 \)-submatrix has determinant 0.

The following matrices are the key to the structure of tame frieze patterns.

**Definition 2.15.** For \( c \) in a commutative ring, let

\[
\eta(c) := \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix}.
\]

**Remark 2.16.** Notice that up to a transposition, \( \eta(c) \) may be viewed as a **reflection**:

\[
\eta(c) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & c \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \eta(c) = \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}.
\]

We will see later why this implies that frieze patterns may be seen as reflection groupoids.

**Proposition 2.17.** Let \( R \) be a commutative ring.

1. Let \( F = (c_{i,j}) \) be a tame frieze pattern over \( R \) of height \( n \), \( c_k := c_{k,k+2} \) for \( k \in \mathbb{Z} \). Then \( F \) is periodic with period \( m = n + 3 \) and

\[
\prod_{k=1}^{m} \eta(c_k) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad c_{i,j+2} = (M_{i,j})_{1,1}, \quad \text{where} \quad M_{i,j} := \prod_{k=i}^{j} \eta(c_k).
\]

2. Let \( (c_1, \ldots, c_m) \in R^m \) satisfy \( \prod_{k=1}^{m} \eta(c_k) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \), and let \( M_{i,j} := \prod_{k=i}^{j} \eta(c_k) \) be as above. Then the \( (a_{i,j+2})_{i,j} = ((M_{i,j})_{1,1})_{i,j} \) (where \( i - 1 \leq j \leq m + i - 3 \)) defines a periodic tame frieze pattern over \( R \) with period \( m \) and height \( m - 3 \).

**Definition 2.18.** Let \( R \) be a subset of a commutative ring and \( \lambda \in \{ \pm 1 \} \). A **\( \lambda \)-quiddity cycle** over \( R \) is a sequence \((c_1, \ldots, c_m) \in R^m \) satisfying

\[
\prod_{k=1}^{m} \eta(c_k) = \lambda \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} = \lambda \text{id}.
\]

A \((-1)\)-quiddity cycle is called a **quiddity cycle** for short.

**Example 2.19.** Consider the commutative ring \( \mathbb{C} \) and \( R = \mathbb{C} \).

1. \((0, 0)\) is the only \( \lambda \)-quiddity cycle of length 2.
2. \((-1, -1, -1)\) and \((1, 1, 1)\) are the only \( \lambda \)-quiddity cycles of length 3.
3. \((t, 2/t, t, 2/t), \ t \) a unit and \((a, 0, -a, 0), \ a \) arbitrary, are the only \( \lambda \)-quiddity cycles of length 4 (check this as an exercise).

**Remark 2.20.** Let \( \mathbb{C} = (c_1, \ldots, c_m) \) be a \( \lambda \)-quiddity cycle. Then for any \( \sigma \in D_n \), the cycle \( \sigma^\mathbb{C} \) is a \( \lambda \)-quiddity cycle as well.

The following useful lemma is inspired by old results on continued fractions.

**Lemma 2.21.** Let \( (c_1, \ldots, c_m) \in \mathbb{C}^m \) such that \( \prod_{j=1}^{m} \eta(c_j) \) is a scalar multiple of the identity matrix. Then there are two different indices \( j, k \in \{1, \ldots, m\} \) with \( |c_j| < 2 \) and \( |c_k| < 2 \).
Using $\eta(a)\eta(b) = \eta(a + 1)\eta(b + 1)$ for all $a, b$ we conclude:

**Corollary 2.22.** Frieze patterns with entries in $\mathbb{N}$ correspond to triangulations of polygons by non-intersecting diagonals.

### 3. Finite Weyl Groupoids

#### 3.1. Rank two.

**Remark 3.1.** The set of quiddity cycles with entries in $\mathbb{N}_{>0}$ corresponds to the sequences of Cartan entries of finite Weyl groupoids of rank two.

**Definition 3.2.** A reflection groupoid and its root system are called **irreducible** if all Cartan matrices are indecomposable.

These are the roots of irreducible finite Weyl groupoids of rank two.

**Definition 3.3.** Define $\mathcal{F}$-sequences as finite sequences of length $\geq 3$ with entries in $\mathbb{N}_{>0}$ given by the following recursion.

1. $((0,1),(1,1),(1,0))$ is an $\mathcal{F}$-sequence.
2. If $(v_1,\ldots,v_n)$ is an $\mathcal{F}$-sequence, then
   
   $$(v_1,\ldots,v_i,v_i + v_{i+1},v_{i+1},\ldots,v_n)$$

   are $\mathcal{F}$-sequences for $i = 1,\ldots,n - 1$.
3. Every $\mathcal{F}$-sequence is obtained recursively by (1) and (2).

**Proposition 3.4.** Crystallographic arrangements of rank two correspond to $\mathcal{F}$-sequences: each $\mathcal{F}$-sequence is a set of roots of a Weyl groupoid.

**Corollary 3.5.** Assume that $R^{\alpha}(\mathcal{W}(A))$ is a finite root system of a Weyl groupoid. Then for all objects $a$ and $\alpha \in R^a_+$, either $\alpha$ is simple or it is the sum of two positive roots in $R^a_+$.

**Corollary 3.6.** Let $R^a_+$ be the set of roots at an object $a$ of a finite Weyl groupoid of rank two and $(c_{i,j}^a)$ the Cartan matrix at $a$. Then

$$c_{i,j} = -\max\{k \in \mathbb{N}_{\geq 0} \mid k\alpha_i + \alpha_j \in R^a\}$$

for $i \neq j$. Moreover, $k\alpha_i + \alpha_j \in R^a$ for $k = 0,\ldots,-c_{i,j}$.

#### 3.2. Rank three.

Let $A$ be the set of objects of an irreducible reflection groupoid and $a \in A$. The last observation on root systems of rank two implies the following central result:

**Theorem 3.7.** Let $\alpha \in R^a_+$. Then either $\alpha$ is simple, or it is the sum of two positive roots.

**Theorem 3.8.** Let $\alpha_1,\ldots,\alpha_r$ be the simple roots at $a$. Then $\sum_{i=1}^r \alpha_i \in R^a$.

View the roots as elements of $\mathbb{Z}^3$ with the lexicographic ordering $\leq$.

**Lemma 3.9.** Let $\alpha, \beta \in R^a_+$ be minimal in $\langle \alpha, \beta \rangle_\mathcal{W} \cap R^a_+$ with respect to $\leq$. Then $\langle \alpha, \beta \rangle_\mathcal{W} \cap R^a \subseteq \pm(\mathbb{N}_0 \alpha + \mathbb{N}_0 \beta)$.

**Theorem 3.10 (Weak Convexity).** Let $a \in A$ and $\alpha, \beta, \gamma \in R^a_+$. If $\det(\alpha, \beta, \gamma)^2 = 1$ and $\alpha, \beta$ are the two smallest elements of $\langle \alpha, \beta \rangle \cap R^a_+$, then either $\alpha, \beta, \gamma$ are simple, or one of $\gamma - \alpha, \gamma - \beta$ is in $R^a_+$.
Theorem 3.11 (Required roots). Let $a \in A$ and $\alpha, \beta, \gamma \in R^a_\alpha$. Assume that $\det(\alpha, \beta, \gamma)^2 = 1$ and that $\gamma - \alpha, \gamma - \beta \notin R^a$. Then $\gamma + \alpha \in R^a$ or $\gamma + \beta \in R^a$.

Theorem 3.12 (Bound for the Cartan entries). All entries of the Cartan matrices are greater or equal to $-7$.

Remark 3.13. In fact, all entries of the Cartan matrices are greater or equal to $-6$.

The preceding theorems suggest the following algorithm.

Function Enumerate($R$)

(1) If $R$ defines a crystallographic arrangement, output $R$ and continue.
(2) $Y := \{\alpha + \beta \mid \alpha, \beta \in R, \alpha \neq \beta\}\setminus R$. 
(3) For all $\alpha \in Y$ with $\alpha > \max R$:
   (a) Compute all “subgroupoids of rank two” in $R \cup \{\alpha\}$.
   (b) If all Cartan entries are $\geq -7$, all rank two root sets are $\mathcal{F}$-sequences, and the “Weak Convexity” is satisfied, then call Enumerate($R \cup \{\alpha\}$).

Remark 3.14. We use Theorem “Required roots” as a further obstruction.

The algorithm terminates and yields the result:

Theorem 3.15 (Cuntz, Heckenberger (2012)). Up to equivalences, there are 55 irreducible crystallographic arrangements of rank three.

3.3. Higher rank. With the knowledge about rank three, we enumerate crystallographic arrangements in ranks four to eight with a similar algorithm.

An analysis of Dynkin diagrams leads to a complete classification in rank $> 8$.

Theorem 3.16 (Cuntz, Heckenberger (2015)). There are exactly three families of irreducible crystallographic arrangements:

(1) The family of rank two parametrized by triangulations of a convex $n$-gons by non-intersecting diagonals.
(2) For each rank $r > 2$, arrangements of type $A_r, B_r, C_r$ and $D_r$, and a further series of $r - 1$ arrangements.
(3) Further 74 “sporadic” arrangements of rank $r$, $3 \leq r \leq 8$.

Remark 3.17. The proof of the classification relies on enumerations by the computer. In rank three, approximately 60,000,000 cases need to be considered. A “short” proof is work in progress. At least an enumeration of all potential parabolics seems to be possible.

References