Generating Functions Quadratic at Infinity (Part I)

Ex. B smooth manifold, $\pi : T^*B \to B$ cotangent bundle.

Define a 1-form λ_{can} on T^*B by

$$\lambda_{can}(X) := \alpha(d\pi(X)) \quad , \quad X \in T_{(q,\alpha)}T^*B .$$

Let $q_1, \dots, q_n$ be local coordinates on B and $q_1, \dots, q_n, p_1, \dots, p_n$ the associated local coordinates on T^*B. Then

$$\lambda_{can} = \sum_{i=1}^{n} p_i \, dq_i \quad \rightsquigarrow \quad \omega_{can} := -d\lambda_{can} = \sum_{i=1}^{n} dq_i \wedge dp_i .$$

Hence (T^*B, ω_{can}) is an exact symplectic manifold.

1. Symp(M,ω) and Ham(M,ω)

Let (M,ω) be a symplectic manifold. Denote with

$$\mathrm{Symp}(M,\omega) := \{ \varphi : M \to M \mid \varphi \text{ diffeo. and } \varphi^*\omega = \omega \}$$

the group of symplectomorphisms on (M,ω).

A smooth (time-dependent) Hamiltonian $H_t : M \to \mathbb{R}$, $t \in [0,1]$, induces a family $\{\varphi_t\}_{t\in[0,1]} \in \mathrm{Symp}(M,\omega)$ by

$$\begin{cases} \frac{d}{dt}\varphi_t = X_{H_t} \circ \varphi_t , & \leftarrow \text{"flow" of } H_t \\ \varphi_0 = id, \end{cases}$$

where X_{H_t} is the (time-dep.) Hamiltonian vector field defined by $\qquad \iota_{X_{H_t}} \omega = \omega(X_{H_t}, \cdot) = -dH_t.$

Note that X_{H_t} is uniquely defined since ω is non-degenerate.

$\varphi \in \mathrm{Diff}(M)$ is called Hamiltonian (symplectomorphism) if there exists a (time-dep.) Hamiltonian $H_t : M \to \mathbb{R}$ s.t. $\varphi = \varphi_1$, i.e. φ is the time-1-Hamiltonian flow of H_t. We denote with

$$\mathrm{Ham}(M, \omega) := \{\varphi : M \to M \mid \varphi \text{ Hamiltonian}\}$$

the group of Hamiltonian symplectomorphisms.

Rmk The vector space of Hamiltonian vector fields is infinite-dimensional. In particular, $\mathrm{Ham}(M,\omega) \subseteq \mathrm{Symp}(M,\omega)$ is an infinite-dimensional Lie group.

A symplectomorphism φ of an exact symplectic manifold $(M, \omega = -d\lambda)$ is called exact (wrt. λ) if $\varphi^*\lambda - \lambda$ is exact.

We will now see that every Hamiltonian symplectomorphism of an exact symplectic manifold is exact.

Lemma 1 Let $\{\varphi_t\}_{t \in [0,1]}$ be a symplectic isotopy of an exact symplectic manifold $(M, \omega = -d\lambda)$. Then φ_t is the flow of a (time-dep.) Hamiltonian iff

$$\varphi_t^* \lambda - \lambda = dS_t$$

for a smooth family of functions $S_t : M \to \mathbb{R}$. In this case

$$S_t = \int_0^t (\lambda(X_s) + H_s) \circ \varphi_s \, ds,$$

where X_t is the vector field generating φ_t and $H_t : M \to \mathbb{R}$ the corresponding Hamiltonian.

2. Lagrangian Submanifolds

(M, ω) $2n$-dim symplectic manifold.

Def A Lagrangian submanifold of M is an n-dim submanifold $L \subset M$ such that $i_L^* \omega = 0$, where $i_L : L \hookrightarrow M$ denotes the inclusion. A Lagrangian submanifold L of an exact symplectic manifold $(M, \omega = -d\lambda)$ is said to be exact (wrt. λ) if the (closed) 1-form $i_L^* \lambda$ is exact.

Rmk 1) $L \subset M$ submanifold with $i_L^* \omega = 0 \overset{\text{linear algebra}}{\Rightarrow} \dim L \leq n$

2) $L \subset (M, \omega = -d\lambda)$ exact Lagrangian, $\varphi \in \mathrm{Symp}(M, \omega)$ exact

$\Rightarrow \varphi(L) \subset M$ exact Lagrangian since

$$(\varphi \circ i_L)^* \lambda = i_L^* \underbrace{(\varphi^* \lambda - \lambda)}_{\text{exact}} + \underbrace{i_L^* \lambda}_{\text{exact}}.$$

Ex. (M_1, ω_1), (M_2, ω_2) symplectic manifolds. Then the twisted product $\overline{M}_1 \times M_2 := (M_1 \times M_2, (-\omega_1) \oplus \omega_2)$ is a symplectic mfd.
(M, ω) sympl. mfd. Then the diagonal

$$\Delta := \{(q, q) \mid q \in M\} \subseteq \overline{M} \times M$$

is a Lagrangian submanifold.

Given a diffeomorphism $\varphi : M \to M$, then its graph

$$gr(\varphi) := \{ (q, \varphi(q)) \mid q \in M \} \subseteq \bar{M} \times M$$

is Lagrangian iff $\varphi \in Symp(M, \omega)$. Indeed, identify

$T_{(q,\varphi(q))} \bar{M} \times M \cong T_q M \times T_{\varphi(q)} \bar{M}$, then $T_{(q,\varphi(q))}(gr(\varphi)) \cong gr(d\varphi(q))$

and therefore

$$(-\omega) \oplus \omega \big((\xi, d\varphi(\xi)), (\eta, d\varphi(\eta)) \big) = -\omega(\xi, \eta) + \varphi^* \omega(\xi, \eta)$$

$\Rightarrow$ $gr(\varphi)$ Lagrangian iff $\varphi^* \omega = \omega$.

Example $\alpha : B \to T^* B$ 1-form (viewed as a section).

Then its graph $\Gamma_\alpha := \{ (q, \alpha_q) \mid q \in B \} \subseteq T^* B$ is

1) Lagrangian iff $d\alpha = 0$,

2) exact Lagrangian iff $\alpha = df$, $f \in C^\infty(B)$.

proof $i_\alpha : B \to T^* B$, $q \mapsto (q, \alpha_q)$ is a diffeomorphism onto its image $R(i_\alpha) = \Gamma_\alpha$. Compute for $X \in T_q B$:

$$i_\alpha^* \lambda_{can}(X) = \lambda_{can}(di_\alpha(X)) = \alpha(d\pi(di_\alpha(X))) = \alpha(X). \qquad \blacksquare$$

In particular, the 0-section $B_0 := \{ (q, 0) \mid q \in B \} \subseteq T^* B$ is exact Lagrangian.

Rmk If $\varphi \in Ham(T^* B, \omega_{can})$, then $\varphi(B_0)$ is exact Lagrangian by Lemma 1. If, in addition, $\varphi(B_0)$ is a section (e.g. if φ is C^1-close to id), then $\exists f \in C^\infty(B)$ such that $\varphi(B_0) = \Gamma_{df}$. In this case, f is called a generating function for $\varphi(B_0)$.

Note that (transverse) intersections of $\varphi(B_0)$ with B_0 correspond to (non-degenerate) critical points of f.

$\rightsquigarrow |\varphi(B_0) \cap B_0| = |\mathrm{crit}(f)| \geq \mathrm{crit}(B) := \min\{|\mathrm{crit}(f)| \mid f \in C^\infty(B)\}$.

"Lagrangian" Arnold conjecture (for cotangent bundles)

Let $\varphi \in \mathrm{Ham}(T^*B, \omega_{can})$, B compact. Then

$$|\varphi(B_0) \cap B_0| \geq \mathrm{crit}(B).$$

If, in addition, $\varphi(B_0) \pitchfork B_0$, then

$$|\varphi(B_0) \cap B_0| \geq \sum_{i=0}^{n} \dim H_i(B; \mathbb{Z}_2).$$

i-th Betti number

Rmk Above: true if φ is C^1-small. ✓

Can be proven in general using g.f.q.i (see below).

3. Coisotropic Submanifolds, Characteristic Foliations and Symplectic Reduction

Let (M,ω) be a symplectic manifold. Given a compact orientable hypersurface $S \subseteq M$ can define a line bundle $\mathcal{L}$ over S by

$$\mathcal{L}_p := (T_pS)^\omega = \{ \xi \in T_pM \mid \omega_p(\eta, \xi) = 0 \ \forall \eta \in T_pS \}.$$

Then $\mathcal{L}$ determines a 1-dimensional foliation of S, wich is called the characteristic foliation.

Ex. $H: M \to \mathbb{R}$ Hamiltonian, $s \in \mathbb{R}$ regular value and $S := H^{-1}(s)$, then $\omega(X_H, Y) = dH(Y) = 0$ for all Y tangent to S.

Hence $\mathcal{L}_p = \mathrm{span}\{X_H(p)\}$ and the leaves of the foliation are the integral curves of X_H.

More generally, let $N \subseteq M$ be a coisotropic submanifold, i.e. we have $(TN)^\omega \subseteq TN$.

Lemma 2 The distribution $(TN)^\omega \subseteq TN$ is integrable, i.e. for all $X, Y \in \Gamma(TN)$ with values in $(TN)^\omega$ the Lie bracket $[X,Y]$ also takes values in $(TN)^\omega$.

proof Let $q \in N$, $v \in T_qN$ and $Z \in \Gamma(TN)$ with $Z(q) = v$. Then

$$0 = d\omega(X, Y, Z)$$

$$= \mathcal{L}_X(\omega(Z,Y)) + \mathcal{L}_Y(\omega(X,Z)) + \mathcal{L}_Z(\omega(Y,X))$$
$$+ \omega([X,Y],Z) + \omega([Y,Z],X) + \omega([Z,X],Y)$$
$$= \omega([X,Y],Z).$$

Where we have used $\omega(X,V) = \omega(Y,V) = 0 \ \forall\, V \in \Gamma(TN)$ to conclude the last equality. ∎

By the Frobenius theorem, this implies that there is a foliation on N that integrates the distribution $(TN)^\omega$. This foliation is called the characteristic foliation of N. We define an equivalence relation $\sim$ on N by $p_0 \sim p_1$ iff p_0 and p_1 lie on the same leaf, i.e. iff there exists a smooth path $\gamma : [0,1] \to N$ s.t. $\gamma(0) = p_0$, $\gamma(1) = p_1$ and $\dot\gamma(t) \in (T_{\gamma(t)}N)^\omega \ \forall t$. Assume now that the resulting quotient $\bar{M}_N := N/\sim$ is a smooth manifold and denote by $\pi : N \to \bar{M}_N$. the projection.

Lemma 3 There is a symplectic form $\bar\omega$ on $\bar{M}_N$ such that $\pi^*\bar\omega = i^*\omega$, where $i : N \hookrightarrow M$ is the inclusion.

The symplectic manifold $(\bar{M}_N, \bar\omega)$ is called the symplectic reduction of (M,ω) at the coisotropic submanifold N.

Lemma 4 If L is a Lagrangian submanifold of M wich is transverse to N, then $\bar{L} := \pi(L \cap N)$ is an immersed Lagrangian submanifold of $(\bar{M}_N, \bar\omega)$.

4. Generating Functions for Lagrangian Submanifolds of the Cotangent Bundle

We have seen that any exact Lagrangian submanifold L of T^*B that is a section is the graph of a differential of a smooth function $f: B \to \mathbb{R}$. In this case f is called a generating function for L and (non-degenerate) critical points of f correspond to (transverse) intersections of L with the 0-section B. $\rightsquigarrow$ Link between symplectic geometry of L and the Morse theory of f. However, being a section is very restrictive.

<u>Goal</u>: generalize this idea (i.e. the notion of a generating function), to associate a generating function to every Lagrangian submanifold of T^*B that is Hamiltonian isotopic to B_0.

Let $p: E \to B$ be a fiber bundle, and consider its fiber conormal bundle

$$N_E := \{ (e,\eta) \in T^*E \mid \eta|_{\ker dp(e)} = 0 \},$$

i.e. the space of 1-forms that vanish in the vertical direction. Assume that $F: E \to \mathbb{R}$ is a smooth function such that $\Gamma_{dF} \pitchfork N_E$ (i.e. the graph of the differential of F is transverse to the fiber normal bundle). The set of fiber critical points of F is

$$\Sigma_F = \{ e \in E \mid e \text{ is a critical point of } F|_{p^{-1}(p(e))} \}$$
$$= \{ e \in E \mid dF|_{T_e\, p^{-1}(p(e))} = 0 \}.$$

Note that, if we identify $i : \Sigma_F \to T^*E$, $e \mapsto (e, dF(e))$, then $i(\Sigma_F) = \Gamma_{dF} \cap N_E$. Thus, since $\Gamma_{dF} \pitchfork N_E$, $i(\Sigma_F) \subseteq T^*E$ is a smooth manifold of dimension

$$\dim i(\Sigma_F) = \underbrace{\dim \Gamma_{dF}}_{= \dim E} + \underbrace{\dim N_E}_{= \dim E + \dim B} - \underbrace{\dim T^*E}_{= 2 \dim E} = \dim B.$$

In particular, $\overset{\pi_E : T^*E \to E}{\pi_E}(i(\Sigma_F)) = \Sigma_F \subseteq E$ is a smooth submanifold with $\dim \Sigma_F = \dim B$. Given a point $e \in \Sigma_F$ we can associate to it an element $v^*(e) \in T^*_{p(e)} B$ by defining

$$v^*(e)(X) := dF(e)[\hat{X}]$$

for $X \in T_{p(e)} B$, where $\hat{X}$ is any lift of X, i.e. $\hat{X} \in T_e E$ with $p_*(\hat{X}) = X$. This is well-defined: If Y is a lift of X, then $\hat{X} - Y \in \ker p_*$ is vertical and therefore $dF(e)[\hat{X} - Y] = 0$ since $e \in \Sigma_F$.

The map

$$i_F : \Sigma_F \to T^*B, \quad e \mapsto (p(e), v^*(e))$$

is an exact Lagrangian immersion, i.e. an immersion and $L_F = i_F(\Sigma_F)$ is an exact Lagrangian submanifold with $i_F^* \lambda_{can} = d(F|_{\Sigma_F})$.

Indeed, take $X \in T_e \Sigma_F$, then

$$i_F^* \lambda_{can}(X) = \lambda_{can}(\underbrace{di_F(X)}_{\in T_{(e,v^*(e))}T^*B})$$

$$= v^*(e)\,(d\pi(di_F(X)))$$

$$\overset{\pi \circ i_F = p}{=} v^*(e)\,(dp(X))$$

$$= dF(e)[X].$$

In this case, F is called a generating function for L_F.

Rmk If $E = B$ and $p = id : B \to B$ is the identity, then $i_F(\Sigma_F) = \Gamma_{dF}$. Thus this construction indeed generalizes the fact that every smooth function $f : B \to \mathbb{R}$ generates the (exact) Lagrangian submanifold Γ_{df} of T^*B.

Ex. $E = B \times \mathbb{R}^N$, $p : B \times \mathbb{R}^N \to B$, $(q,v) \mapsto q$, $F : B \times \mathbb{R}^N \to B$.

Then

$$i_F(\Sigma_F) = \left\{ (q,p) \in T^*B \;\middle|\; \underbrace{\exists\, \xi \in \mathbb{R}^N \text{ s.t. } \frac{\partial F}{\partial \xi}(q,\xi) = 0}_{\substack{\text{condition that there exists}\\ \text{a fiber critical point in the}\\ \text{fiber over } q \in B}},\ \frac{\partial F}{\partial q}(q,\xi) = p \right\}.$$

Rmk The above construction can be equivalently described with the concepts introduced in section 3: N_E is a coisotropic submanifold of T^*E, and the symplectic reduction at N_E can be naturally identified with T^*B. Since $\Gamma_{dF} \pitchfork N_E$, Lemma 4 implies that $\overline{\Gamma}_{dF} := \pi(\Gamma_{dF} \cap N_E) \subseteq T^*B$ (after natural identification) is an immersed Lagrangian submanifold. $\overline{\Gamma}_{dF}$ coincides with L_F

Lemma 5 (Non-degenerate) critical points of F correspond to (transverse) intersections of L_F with the 0-section B_0.

Note that Lemma 7 does not necessarily implies that L_F intersects B_0, since E is not necessarily compact.

in this case, the lower bound in the Lagrangian Arnold conjecture for $\varphi \in \mathrm{Ham}(M,\omega)$ with $\varphi(B_0) = L_F$ is trivial

$\rightsquigarrow$ need generating functions with some condition at infinity that makes them behave as functions that are defined on a compact manifold.

preliminary

Def A generating function $F: E \to \mathbb{R}$ is said to be quadratic at infinity if $p: E \to B$ is a vector bundle (of finite rank) and F coincides with a non-degenerate quadratic form $Q: E \to \mathbb{R}$ outside a compact subset.

Theorem Let B be a compact manifold. If L is a Lagrangian submanifold of T^*B that is Hamiltonian isotopic to the 0-section (i.e. $L = \varphi(B_0)$ for some $\varphi \in \mathrm{Ham}(M,\omega)$) then it has a generating function quadratic at infinity.

More generall, if $L \subseteq T^*B$ has a g.f.q.i. and φ_t is a Hamiltonian flow of T^*B, then there exists a continuous family of g.f.q.i. $F_t : E \to \mathbb{R}$ s.t. $L_{F_t} = \varphi_t(L)$.

Assuming this theorem, (a weaker version of) the Lagrangian Arnold conjecture (in cotangent bundles) follows from the following theorem.

Theorem Let B be a compact mfd, $p: E \to B$ a vector bundle and $F: E \to \mathbb{R}$ a smooth function quadratic at infinity. Then $\operatorname{crit}(F)$ is bounded below by the cup-length of B. Moreover, if we assume that, in addition, F is a Morse function, then

$$\operatorname{crit}(F) \geq \sum_{i=1}^{n} \dim H_i(B, \mathbb{Z}_2).$$