Transversality

(Reading course on Floer Homology).

Let (M,ω) be a closed symplectic manifold such that $\pi_2(M)=0$. Fix J an almost complex structure on M compatible with ω, i.e. $\langle\cdot,\cdot\rangle_J := \omega(J\cdot,\cdot)$ is a riemannian metric.

For any $H\in \mathcal{C}^\infty(S^1\times M,\mathbb{R})$ we denote by $\mathcal{P}(H)$ the set of contractible 1-periodic orbits of the Hamiltonian flow generated by H, i.e. the flow of the time dependent vector field X_H where $\omega(X^t_H,\cdot)=-dH^t$.

The aim of these notes is to show that the set of H such that

$$\mathcal{M}_H(x_-,x_+)=\left\{ u: \mathbb{R}\times S^1 \xrightarrow{\mathcal{C}^\infty} M \;\middle|\; \begin{array}{l}\frac{\partial u}{\partial s}+J(u)\left(\frac{\partial u}{\partial t}+X^t_H(u)\right)=0 \\ \lim_{s\to\pm\infty} u(s,\cdot)=x_\pm\end{array}\right\}$$

is a finite dimensional smooth manifold for all $x_-,x_+\in\mathcal{P}(H)$.

is "$\mathcal{C}^\infty$ generic". We will be more precise in I.

The idea to prove this is the following:

Remark: However the proof we will give won't follow exactly the order we present now. Our proof will be 1, 2, 4-5, 3, 6.

1. For any non degenerate Hamiltonian function and for any $x_-,x_+\in\mathcal{P}(H)$ we consider a Banach manifold

$\mathcal{B}^{1,p}(x_-,x_+)\subset W^{1,p}(\mathbb{R}\times S^1,M)$ and a Banach bundle

$\mathcal{E}^p \longrightarrow \mathcal{B}^{1,p}(x_-,x_+)$ such that

$\partial_H : \mathcal{B}^{1,p}(x_-,x_+)\longrightarrow \mathcal{E}^p$ smooth "Fredholm" section

and $\partial_H^{-1}\{0\}=\mathcal{M}_H(x_-,x_+)$.

2. If $(\text{then } \partial_H) \pitchfork \{0\}$

$\overset{\text{IFT}}{\Longrightarrow}$ $\partial_H^{-1}\{0\}$ smooth manifold and of finite dimension.

3. To show that it happens for generic H we have to make H varies. More precisely we will do as follow:

Fix H_0 non degenerate, we define $\mathcal{HP}(H_0) \subset C^\infty(S^1\times M,\mathbb{R})$ a space of deformation of H_0 s.t. $\forall H \in \mathcal{HP}(H_0)$, $\mathcal{P}(H) = \mathcal{P}(H_0)$ and $\mathcal{HP}(H_0)$ is a separable Banach manifold "C^∞" dense in $C^\infty(S^1\times M,\mathbb{R})$ and consider

$$\delta : \mathcal{B}^{1,p}(x_-,x_+)\times \mathcal{HP}(H_0) \longrightarrow \mathcal{E}$$
$$(u, H) \longmapsto \partial_{J,H}(u)$$

We show that then $\delta \pitchfork \{0\}$

$\overset{\text{IFT}}{\Longrightarrow}$ $\delta^{-1}\{0\}$ Banach manifold.

4. $H \in \mathcal{HP}(H_0)$ is a regular value of $\Pi : \delta^{-1}\{0\} \to \mathcal{HP}(H_0)$ $\Longleftrightarrow$ then $\partial_{J,H} \pitchfork \{0\}$

5. We use Sard-Smale theorem to conclude that $\mathcal{HP}_{reg}(H_0)$ is dense in $\mathcal{HP}(H_0)$ for C^∞ topology.

---

1). follows almost exclusively from what Luca has done.

2), 4), 5) are general functional analysis theorems.

Our main task will be to prove 3) especially the first square, we formalize it in Theorem 3.3 below.

The plan will be the following:

I. The setting (The Banach manifolds)

a. The Sobolev space $W^{1,p}$ and $\mathcal{B}^{1,p}(x_-, x_+)$

b. The space of deformation and the precise statement of the theorem

II. Reformulation of the Theorem

a. IFT for sections of Banach bundle

b. Reformulation of the Theorem via IFT

III. Proof of the Theorem

a. Sard-Smale theorem + Transversality

b. Proof of the theorem.

I. The setting.

a. $\mathcal{B}^{1,p}(x_-,x_+)$

- $\overline{\partial}_{J,H}^{\infty} : \mathcal{C}^{\infty}(S^1\times\mathbb{R}, M) \longrightarrow \mathcal{E}^{\infty}$

$$u \longmapsto \partial_s u + J(u)\left(\partial_t u + X_H^t(u)\right)$$

where $\mathcal{E}^{\infty}$ is Fréchet bundle $\mathcal{E}^{\infty}_u := \Gamma^{\infty}(u^* TM)$ is the operator that pops out when we transform the ODE of gradient lines of the functional $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$ with respect to the metric on $\mathcal{L}M$ defined by

(1) $\langle \xi, \eta \rangle = \int_0^1 \omega_{x(t)}\big(J(x(t))\xi(t), \eta(t)\big)\,dt \;,\; \forall \xi,\eta \in \Gamma^{\infty}(x^*\mathcal{L}M)$

to a PDE.

- $\mathcal{M}_H(x_-,x_+) = \overline{\partial}_{J,H}^{\infty\,-1}\{0\} \cap \mathcal{C}^{\infty}(x_-,x_+)$

where $\mathcal{C}^{\infty}(x_-,x_+) = \{u : \mathbb{R}\times S^1 \to M \mid \lim_{s\to\pm\infty} u(s,\cdot) = x_\pm\}$.

- To be able to do analysis we have to work with Banach manifolds. It is natural to complete $\mathcal{L}M$ with respect to the metric defined by (1) and so to work in $W^{1,2}(S^1, M)$.

- However to complete $\mathcal{C}^{\infty}(x_-,x_+) \subset \mathcal{C}^{\infty}(S^1\times\mathbb{R}, M)$ since $\dim(S^1\times\mathbb{R}) = 2$, it is natural to work with $W^{1,p}$ maps for $p > 2$ so that $W^{1,p}(x_-,x_+) \subset W^{1,p}(S^1\times\mathbb{R}, M) \subset \mathcal{C}^0(S^1\times\mathbb{R}, M)$

(also if we define $W^{1,p}$ thanks to an embedding $M \hookrightarrow \mathbb{R}^N$ - instead of in terms of distribution - when $p > 2$ the definition does not depend on the embedding).

More precisely we will work with:

$\mathcal{B}^{1,p}(x_-,x_+) \subset W^{1,p}(x_-,x_+)$ where

$$\mathcal{B}^{1,p}(x_-,x_+) = \Big\{ u \in W^{1,p}_{loc}(x_-,x_+) \;\Big|\; \exists\, \xi^- \in W^{1,p}(]-\infty,-s_0] \times S^1, x^{-*}TM)$$
$$\xi^+ \in W^{1,p}([s_0,+\infty[\times S^1, x^{+*}TM)$$
$$u(s,t) = \exp_{x_\pm(t)}\big(\xi^\pm(s,t)\big),\ |s| \geq s_0 \Big\}$$

Proposition: $\forall H \in C^\infty(S^1 \times M, \mathbb{R})$ and $\forall x_\pm \in \mathcal{P}(H)$

- $\mathcal{B}^{1,p}(x_-,x_+)$ is a smooth Banach manifold with
$$T_u \mathcal{B}^{1,p}(x_-,x_+) = W^{1,p}(\mathbb{R} \times S^1, u^*TM)$$
- $\mathcal{M}_H(x_-,x_+) \subset \mathcal{B}^{1,p}(x_-,x_+)$ (exponential decay)
- $$\bar\partial_{J,H} : \mathcal{B}^{1,p}(x_-,x_+) \longrightarrow \mathcal{E}^p$$
$$u \longmapsto \partial_s u + J(u)\big(\partial_t u + X_H^t(u)\big)$$
where $\mathcal{E}^p$ Banach bundle $\mathcal{E}^p_u = L^p(\mathbb{R} \times S^1, u^*TM)$ is smooth.

We admit this proposition.

Remark: - Is the extension of $\bar\partial^\infty_{J,H}$ to $\mathcal{B}^{1,p}$ in the last proposition can be done by density of $C^\infty_c(x_-,x_+) \subset \mathcal{B}^{1,p}(x_-,x_+)$? In the way we have done it $\partial_s u$ and $\partial_t u$ designates the weak derivative of $u \in \mathcal{B}^{1,p}(x_-,x_+)$.

Moreover one can use the Calderon - Zygmund inequality to prove elliptic regularity, i.e if $u \in \mathcal{B}^{1,p}(x_-,x_+)$ s.t $\bar\partial_{J,H}(u) = 0 \Rightarrow u \in C^\infty(S^1 \times \mathbb{R}, M)$

In other words:

Theorem 1.1: $\bar\partial_{J,H}^{-1}\{0\} = \mathcal{M}_H(x_-,x_+)$.

b. Precise statement

For this part we follow mainly Audin-Damian 8.3

Since $\mathcal{C}^\infty(S^1\times M, \mathbb{R})$ does not admit a natural Banach structure, we will consider for any $H \in \mathcal{C}^\infty(S^1\times M, \mathbb{R})$ a subspace $\mathcal{H}(H_0) \subset \mathcal{C}^\infty(S^1\times M, \mathbb{R})$ that admits such structure.

Consider $\mathcal{E} := (\mathcal{E}_i)_{i\in\mathbb{N}}$ a sequence $\subset \mathbb{R}_{>0}$. For any $H \in \mathcal{C}^\infty(S^1\times M, \mathbb{R})$ we define the norm $\|H\|_\mathcal{E} := \sum_{k\geq 0} \mathcal{E}_k \|d^k H\|_\infty$ where $\|d^k H\|_\infty$ is defined as follow

Fix a covering charts $(\mathcal{U}_i, \varphi_i)_{i\in[1,N]}$ of $S^1\times M$, $\varphi_i : \mathcal{U}_i \xrightarrow{\sim} \mathcal{B}_0(1)$ (closed ball) and $\| d^k H \|_\infty := \max_{i,z,|\alpha|=k} \left| \frac{\partial^k H\circ\varphi_i^{-1}}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_k}}(z)\right|$. We then denote by $\mathcal{C}^\infty_\mathcal{E}(S^1\times M, \mathbb{R})$ the vector space consisting of $H \in \mathcal{C}^\infty(S^1\times M, \mathbb{R})$, s.t $\|H\|_\mathcal{E} < +\infty$. It is "clearly" a Banach space, i.e $\left(\mathcal{C}^\infty_\mathcal{E}(S^1\times M, \mathbb{R}), \|\cdot\|_\mathcal{E}\right)$ is complete. Moreover we have the next proposition that we admit

Proposition: There exists a sequence $\mathcal{E}$ such that $\mathcal{C}^\infty_\mathcal{E}(S^1\times M, \mathbb{R})$ is dense in $\mathcal{C}^\infty(S^1\times M, \mathbb{R})$ for the $\mathcal{C}^\infty$ topology.

(* see next page for an explicit sequence $\mathcal{E}$)

Fix such an $\mathcal{E}$ and $H_0 \in \mathcal{C}^\infty(S^1\times M)$. Consider $\mathcal{H}_\delta(H_0) \subset \mathcal{C}^\infty_\mathcal{E}(S^1\times M, \mathbb{R})$, defined by $\forall H \in \mathcal{H}_\delta(H_0)$ $H(t,x) = H_0(t,x)$, $\forall x$ in a neighborhood of $\mathcal{P}(H_0)$, $\forall t \in S^1$ and $\|H - H_0\|_\mathcal{E} < \delta$. $\mathcal{H}_\delta(H_0)$ endowed with $\|\cdot\|_\mathcal{E}$ is a Banach manifold with $T_H\mathcal{H}(H_0) = \{\hat{H} \in \mathcal{C}^\infty(S^1\times M, \mathbb{R}) \mid \hat{H}|_{S^1\times\mathcal{V}(\mathcal{P}_{H_0})} \equiv 0\}$.

Moreover for δ small enough $\forall H \in \mathcal{H}_\delta(H_0)$, $\mathcal{P}(H) = \mathcal{P}(H_0)$. We then denote $\mathcal{H}(H_0)$.

We can finally state properly the theorem we want to show:

Theorem:

Let H_0 be a non degenerate Hamiltonian function. The set $\mathcal{H}_{reg}(H_0)$ of $H \in \mathcal{H}(H_0)$ s.t. $\bigcup \mathcal{M}_H(x_-, x_+)$ is a smooth finite dimensional manifold is residual in $\mathcal{H}(H_0)$ endowed with the $\mathcal{C}^\infty$ topology.

To explicit the sequence ε of the previous proposition we use and admit the next lemma proven in Audin-Damian.

Lemma: Endowed with the $\mathcal{C}^\infty$ topology the space $\mathcal{C}^\infty(M \times S^1, \mathbb{R})$ is separable, i.e. it admits a dense sequence.

—

Remark: For the $\mathcal{C}^0$ topology this result can be deduced from Stone-Weierstraß after embedding $M \times S^1 \hookrightarrow [-1,1]^N$.

—

So consider $(f_k) \subset \mathcal{C}^\infty(S^1 \times M, \mathbb{R})$ a dense sequence for the $\mathcal{C}^\infty$ topo and put

$$\varepsilon_n := \frac{1}{2^n \max_{k \leq n} \| f_k \|_{\mathcal{C}^n}}$$

where $\|f\|_{\mathcal{C}^n} = \sum_{i=0}^{n} \|d^i f\|_\infty$ as we defined in the beginning of the section.

II. Reformulation of the Theorem via IFT

a. IFT for Banach Bundle.

Let $\mathcal{E} \xrightarrow{\pi} \mathcal{B}$ be a Banach bundle.

Denote by $\sigma_0 : \mathcal{B} \to \mathcal{E}$ the 0-section, and consider the canonical splitting of $T_{\sigma_0(x)}\mathcal{E} \xrightarrow{\sim} T_x\mathcal{B} \oplus \mathcal{E}_x$ and denote by $\pi_x : T_{\sigma_0(x)}\mathcal{E} \to \mathcal{E}_x$.
Then for any section $\sigma : \mathcal{B} \to \mathcal{E}$ and any $x \in \sigma^{-1}\{\sigma_0(\mathcal{B})\}$ we define the vertical derivative of σ at x to be
$\sigma_0(\mathcal{B}) = 0_B$

$$d^v\sigma(x) : T_x\mathcal{B} \longrightarrow \mathcal{E}_x \qquad u \longmapsto \pi_x\left(d_x\sigma(u)\right).$$

<u>Theorem 2.1</u>: Let $\sigma : \mathcal{B} \to \mathcal{E}$ be a section. Suppose that for all $x \in \sigma^{-1}\{0_B\}$ $d^v\sigma_x$ is surjective and admits a right inverse then $\sigma^{-1}\{0_B\}$ is a Banach manifold. If moreover $d^v\sigma_x$ is Fredholm then it is a manifold of (finite) dimension equals to the Fredholm index.

Remark/Idea of the proof: • $d^v\sigma_x$ surj for all $x \in \sigma^{-1}\{0_B\}$ and admits right inverse
$\Leftrightarrow \operatorname{Im}\sigma \pitchfork 0_B$, i.e. $d_x\sigma(T_x\mathcal{B}) \oplus T_{\sigma_0(x)}0 = T_{\sigma_0(x)}\mathcal{E}$.
The condition about admitting right inverse is superfluous in finite dimensional settings.

- A continuous operator $A : E \to F$ between two Banach spaces is Fredholm if $\dim \operatorname{Ker} A < +\infty$ and $\dim F/\operatorname{Im} A < +\infty$.

$$\operatorname{Index} A = \dim \operatorname{Ker} A - \dim\left(F/\operatorname{Im} A\right).$$

b. Reformulation of the theorem

Thanks to the Calderon-Zygmund inequality one can prove that

<u>Theorem 2.2</u>: Let H be a non degenerate Hamiltonian function. Then $\forall x_-, x_+ \in \mathcal{P}(H)$, and $\forall u \in \mathcal{M}_H(x_-, x_+) = \bar\partial_{J,H}^{-1}\{0_B\}$
$d^v\bar\partial_{J,H}(u)$ is Fredholm.

Thanks to the 2 previous theorems one can reformulate the theorem

Theorem

Let H_0 be a non degenerate Hamiltonian function. The set $\mathcal{H}_{reg}(H_0)$ of $H \in \mathcal{H}(H_0)$ s.t. $d^v \mathcal{F}_{J,H}(u)$ is surjective $\forall x_\pm \in \mathcal{P}(H)$, $\forall u \in \mathcal{M}_H(x_-, x_+)$ manifold is residual in $\mathcal{H}(H_0)$ endowed with the C^∞ topology.

III. Proof of the Theorem

a. Sard-Smale Theorem.

We say that $y \in Y$ is a regular value of $f : X \to Y$, if $f^{-1}\{y\} = \emptyset$ or $d_x f$ is surj $\forall x \in f^{-1}\{y\}$

Theorem 3.1: Let X and Y be separable Banach manifolds, i.e. they admit dense sequences, and $f : X \to Y$ a smooth Fredholm map, i.e $d_x f : T_x X \to T_{f(x)} Y$ is Fredholm for all $x \in X$.

Then $Y_{reg}(f) = \{ y \in Y$ is a regular value of $f \}$ is residual in Y.

We admit this previous theorem. This allows to deduce the following theorem

Theorem 3.2: Let $\mathcal{E} \xrightarrow{\pi} \mathcal{B} \times \mathcal{C}$ be a Banach bundle (everything separable) and $\Sigma : \mathcal{B} \times \mathcal{C} \to \mathcal{E}$ $\quad (b,x) \longmapsto \Sigma(b,x) = \sigma_x(b)$ a smooth section.

a) $\forall (b,x) \in \Sigma^{-1}\{0\}$, the vertical derivative of Σ at (b,x) is surjective.

b) $\forall x \in \mathcal{C}$, $\forall b \in \sigma_x^{-1}\{0\}$ the vertical derivative of σ_x at b is Fredholm.

Then the set $\mathcal{C}_{reg} = \{ x \in \mathcal{C} \mid d^v \sigma(x)$ is surjective (and admits a right inverse) $\}$ is a residual subset of $\mathcal{C}_{reg}$.

Rough idea of the proof of Theorem 3.2:

1) The hypothesis a) + b) allows to make sure that the vertical derivative of Σ admits a right inverse at any $(b,x) \in \Sigma^{-1}\{0\}$.

2) This allows to apply IFT and deduce that $\Sigma^{-1}\{0\} =: \mathcal{M}$ is a manifold.

3) $\pi : \mathcal{M} \longrightarrow \mathcal{E}$ is Fredholm of same index as the vertical derivative of σ_x and moreover « x regular value of $\pi \Leftrightarrow d^v\sigma(x)$ surjective ».

4) Use the Sard-Smale theorem to conclude.

b) Proof of $\boxed{\text{Theorem}}$

We would like to apply the previous Theorem 3.2 for the map

$\mathcal{S} : \mathcal{B}^{1,p}(x_-, x_+) \times \mathcal{H}^{\ell}(H_0) \longrightarrow \mathcal{E}^p \qquad (u,H) \mapsto \mathcal{S}(u,H) := \bar{\partial}_H(u).$

Since we already know that this map satisfies assumption b) of Theorem 3.2 thanks to Theorem 2.2, it remains to show that it satisfies also assumption a). More precisely

Theorem 3.3 The vertical derivative of $\mathcal{S}$ at any point $(u,H) \in \mathcal{S}^{-1}\{0\}$ is surjective.

Indeed, showing Theorem 3.3 then with Theorem 3.2 together with Theorem 2.1. allows to prove $\boxed{\text{Theorem}}$.

Recall that if F is a subvector space of a vector space E we denote by $F^{ann} = \{ f \in E^* = \mathcal{L}(E,\mathbb{R}) \mid f_{|F} = 0 \}$.

Lemma: Let F be closed. Then $F = E \Leftrightarrow F^{ann} = \{0\}$.

Proof: Use Hahn-Banach.

Since $\forall H \in \mathcal{H}^{\ell}(H_0)$, H is non degenerate, for all $(u,H) \in \mathcal{S}^{-1}\{0\}$ $d^v \bar{\partial}_H(u)$ is Fredholm and so $\operatorname{Im}(d^v \mathcal{S}(u,H))$ is closed.

So thanks to the previous lemma to show the surjectivity of $d^v \mathcal{S}(u,H)$ it is enough to show that

$$\left(\mathrm{Im}\left(d^v \mathcal{S}(u,H)\right)\right)^{ann} = \left\{ w \in L^p(S^1 \times M, u^* TM)^* \,\middle|\, w_{|\mathrm{Im}\, d^v \mathcal{S}(u,H)} = 0 \right\} = \{0\}$$

Since $(L^p)^*$ is naturally isomorphic to L^q (once we fix a measure on $S^1 \times \mathbb{R}$ and a metric on M) where $\frac{1}{p} + \frac{1}{q} = 1$ via

$$L^q \longrightarrow (L^p)^* \qquad w \longmapsto \left(f \mapsto \iint_{\mathbb{R} \times S^1} \langle w, f \rangle \, ds\, dt \right).$$

Using this identification

$$\left(\mathrm{Im}\; d^v \mathcal{S}(u,H)\right)^{ann} = \left\{ w \in L^q \,\middle|\, \iint_{\mathbb{R} \times S^1} \langle d^v \mathcal{S}(u,H)[\hat{u}, \hat{H}], w \rangle \, ds\, dt = 0 \quad (1) \right.$$
$$\left. \forall (\hat{u}, \hat{H}) \in T_{(u,H)} \left(\mathcal{B}^{1,p}(x_-, x_+) \times \mathcal{H}(H_0) \right) \right\}.$$

We will show that if $w \neq 0$ then $\iint_{\mathbb{R} \times S^1} \langle \partial_s u, w \rangle \, ds\, dt = +\infty$ contradicting Hölder inequality.

1) First one can see that $d^v \mathcal{S}(u,H)(\hat{u}, \hat{H}) = d^v \partial_H(u)(\hat{u}) - \nabla \hat{H}_t(u)$, so putting $\hat{H} \equiv 0$ we get

$$\iint_{\mathbb{R} \times S^1} \langle d^v \partial_H(u)(\hat{u}), w \rangle \, ds\, dt = 0, \quad \forall \hat{u}$$
$$\parallel$$
$$\iint_{\mathbb{R} \times S^1} \langle \hat{u}, (d^v \partial_H(u))^* w \rangle_J \, ds\, dt = 0, \ \forall \hat{u} \Rightarrow (d^v \partial_H(u))^* w = 0$$

$\Rightarrow$ by elliptic regularity that w is smooth. Actually more is true: one can find a trivialization $\Phi : \mathbb{R} \times S^1 \times \mathbb{R}^{2n} \xrightarrow{\sim} u^* TM$ such that in these coordinates $(d^v \partial_J(u))^* w = 0 \Leftrightarrow -\frac{\partial \hat{w}}{\partial s} + J_0 \frac{\partial \hat{w}}{\partial t} + S(s,t)\hat{w} = 0$ where $\hat{w}(s,t) = \Phi_{s,t}^{-1}(w(s,t))$

J_0 the standard complex structure of $\mathbb{R}^{2n}$ and $S : S^1 \times \mathbb{R}^{2n} \longrightarrow \mathrm{End}(\mathbb{R}^{2n})$.

So $\hat{w}$ satisfies a « perturbed Cauchy-Riemann equation » so it shares some properties of holomorphic map. In particular $\{(s,t)\in \mathbb{R}\times S^1 \mid d^k w(s,t)=0 \text{ for all } k\}$ is open and closed. It can be proven using Carleman-Similarity principle, we will admit it.

2) Now putting $\hat{u}=0$ equation (1) gives

$$\iint_{\mathbb{R}\times S^1} \langle -\nabla \hat{H}_t(u), w\rangle_J \, ds\, dt = 0 \quad \text{for all } \hat{H}\in T_H \mathcal{H}(H_0).$$

$$= \iint_{\mathbb{R}\times S^1} d\hat{H}_t(u)\, w \, ds\, dt = 0 \quad \text{for all } \hat{H}\in T_H \mathcal{H}(H_0) \qquad (3).$$

$$\boxed{\begin{array}{l}\text{Claim: This implies that } \exists\, \alpha : S^1 \longrightarrow \mathbb{R}, \\ \qquad w(s,t) = \alpha(t)\, \partial_s u(s,t) \quad \text{for all } (s,t)\in \mathbb{R}\times S^1\end{array}}$$

We admit this claim for the moment and give a sketch of the proof below.

3) Let show that $\dfrac{d}{ds}\displaystyle\int_{S^1} \langle \partial_s u(s_0,t), w(s_0,t)\rangle_J \, dt = 0$ for any s_0.

Indeed first remark that

$$d^v \partial_H(u)[\partial_s u] = 0 \quad \text{and} \quad (d^v \partial_H(u))^* w = 0 \qquad (3.1).$$

We already proved the second equality, the first one comes from: Since $\partial_H(u)=0$ this implies that $\partial_H(u_z)=0$ where $u_z(s,t) := u(s+z,t)$ so

$$\frac{d}{dz}\Big|_{z=0} \partial_H(u_z) = d\partial_H(u)(\partial_s u) = 0 \text{ which implies } d^v \partial_H(u)(\partial_s u) = 0.$$

So using the trivialization Φ and putting $\hat{u}(s_0,t) = \Phi^{-1}_{s_0,t}(\partial_s u(s_0,t))$, $\hat{w}(s_0,t) = \Phi^{-1}_{s_0,t}(w(s_0,t))$.

we have that $\partial_s \hat{u} + J_0 \partial_t \hat{u} + {}^t S \hat{u} = 0$ and $-\partial_s \hat{w} + J_0 \partial_t \hat{w} + S\hat{w} = 0$

$$\text{So } \frac{d}{ds}\int_{S^1} \langle \hat{u}, \hat{w}\rangle \, dt = \int_{S^1} \langle \partial_s \hat{u}, \hat{w}\rangle + \langle \hat{u}, \partial_s \hat{w}\rangle \, dt$$

$$= \int_{S^1} \langle -J_0 \partial_t \hat{u} - {}^t S \hat{u}, \hat{w}\rangle + \langle \hat{u}, J_0 \partial_t \hat{w} + S\hat{w}\rangle \, dt$$

$$= \int_{S^1} \langle -J_0 \partial_t \hat{u}, \hat{w} \rangle - \langle \hat{u}, \cancel{S\hat{w}} \rangle + \langle \hat{u}, \cancel{S\hat{w}} \rangle + \langle \hat{u}, J_0 \partial_t \hat{w} \rangle \, dt$$

$$= \int_{S^1} \frac{d}{dt} \langle \hat{u}, J_0 \hat{w} \rangle \, dt = 0 .$$

4) Suppose that $w \neq 0$. Then $\alpha : S^1 \to \mathbb{R} \setminus \{0\}$. Indeed suppose $\exists t_0$, $\alpha(t_0) = 0$, then $\forall s \in \mathbb{R}$, $w(s, t_0) = 0$
$\Rightarrow \frac{\partial^k w}{\partial s^k}(s, t_0)$ for all $k \in \mathbb{N}$. Now using again $(d^v \partial_H(u))^* w = 0$ and the trivialization $-\partial_s \hat{w} + J_0 \partial_t \hat{w} + S\hat{w} = 0$ we see by induction that $\partial_t^k w(s,t_0) = 0 \Rightarrow d^k w(s, t_0) = 0$, $\forall k$ and s.
So by 1) this would imply that $w = 0$. ↯

5) $\displaystyle \int_{S^1} \langle \partial_s u, w \rangle_J \, dt = \int_{S^1} \langle \partial_s u, \alpha(t) \partial_s u \rangle \, dt = \int_{S^1} \alpha(t) \|\partial_s u\|^2 \, dt > 0$

by 2). And so by 3) $\displaystyle \iint_{\mathbb{R} \times S^1} \langle \partial_s u, w \rangle_J \, dt = +\infty$

which contradicts Hölder inequality. ↯. $\square$

It remains to prove the $\boxed{\text{Claim}}$ of point 2). To do so we have to focus on two sets $R(u)$ and $C(u)$ associated to $u \in \mathcal{M}_H(x_-, x_+)$.

$$C(u) = \{ (s,t) \in \mathbb{R} \times S^1 \mid \partial_s u(s,t) = 0 \}$$

$$R(u) = \left\{ (s,t) \in \mathbb{R} \times S^1 \;\middle|\; \begin{array}{l} 1.\ \partial_s u(s,t) \neq 0 \\ 2.\ \nexists s',\ u(s',t) \neq u(s,t) \end{array} \quad 3.\ u(s,t) \neq x^{\pm}(t) \right\}$$

Since $d^v \partial_H(u)(\partial_s u)$ see (3.1) one can use again Carleman-Similarity principle and the principle of analytic continuation to get the next theorem that we admit:

Theorem 3.4: If $x_+ \neq x_-$ then $C(u)$ is discrete and $R(u)$ is open and dense.

Proof of the $\boxed{\text{Claim}}$. We will proceed as follow

1) We prove the existence of $\alpha : C(u)^c \xrightarrow{C^\infty} \mathbb{R}$ such that
$$w(s,t) = \alpha(s,t)\, \partial_s u(s,t) \text{ for all } (s,t) \in \mathbb{R} \times S^1.$$

2) We prove that $\partial_s \alpha = 0$ so it can be seen as a function $\alpha : S^1 \longrightarrow \mathbb{R}$.

Both proofs will be by contradiction:

Proof of 1):

Suppose by contradiction that w and $\partial_s u$ are linearly independant at $(s_0, t_0) \in C(u)^c$

$\Rightarrow$ Then $f :]-\varepsilon, \varepsilon[\times \mathcal{V} \hookrightarrow S^1 \times M$
$$(r, s, t) \longmapsto \left(t, \exp_{u(s,t)}(r w(s,t))\right) \text{ is an embedding}$$
for $\varepsilon > 0$ and $\mathcal{V}$ neighbourhood of (s_0, t_0) small enough, indeed
$$\left.\frac{d}{dr}\right|_{r=0} f_t(r,s) = w(s,t) \quad \text{and} \quad \frac{d}{ds} f_t(0,s) = \partial_s u(s,t).$$

Construct H with support inside $\mathcal{U}$ a small neighbourhood of $\operatorname{Im} f$ s.t.
$$\hat{H}(f(r,s,t)) = r\,\beta(s,t) \text{ for } \beta \text{ a a positive function on } \mathcal{V}.$$

then
$$\left.\frac{d}{dr}\right|_{r=0} \hat{H}(f(r,s,t)) = \beta(s,t) = d\hat{H}(f(0,s,t))(w(s,t))$$
$$= d\hat{H}(u(s,t))(w(s,t)).$$

And so $\displaystyle\iint_{\mathbb{R}\times S^1} d\hat{H}_t(u(s,t))(w(s,t))\, ds\, dt$

Cheat (see the Remark below)

$$= \iint_{\mathcal{V}} d\hat{H}_t(u(s,t))(w(s,t))\, ds\, dt > 0.$$ So this contradicts (3) (modulo the fact that we haven't checked $\hat{H} \in T_H \mathcal{H}(H_0)$, see the remark below) and we deduce that $\exists\, \alpha : C(u)^c \longrightarrow \mathbb{R},\ w(s,t) = \alpha(s,t)\partial_s u$

Remark: We have cheated for the previous equality, indeed it can happen that for $(s,t) \notin \mathcal{V}$, $(t, u(s,t)) \in \mathcal{U}$! To solve this problem we should first do the same thing but constructing $\alpha : R(u) \to \mathbb{R}$, not that $R(u) \subset C(u)^c$. Indeed in this case one can make sure that for $\mathcal{V}$ small enough and $\mathcal{U}$ small enough $(t, u(s,t)) \in \mathcal{U} \Leftrightarrow (t,s) \in \mathcal{V}$. Moreover in this case we can make sure that $\mathcal{U} \cap S^1 \times \mathcal{P}(H) = \phi$ since $(s_0,t_0) \in R(u) \Rightarrow u(s_0,t_0) \neq x_\pm(t_0)$. And so $\hat{H} \in T_H \mathcal{H}(H_0)$.

Then we have $\alpha : R(u) \longrightarrow \mathbb{R}$ $\quad (s,t) \longmapsto \dfrac{\langle w, \partial_s u\rangle}{\|\partial_s u\|^2}$.

One can then extend α to $C(u)^c$ and since $R(u)$ is dense we still have that $w(s,t) = \alpha(s,t)\,\partial_s u(s,t)$ for all $(s,t) \in C(u)^c$.

$\square$

Proof of $\mathcal{L}$): Suppose that $\partial_s \alpha \neq 0$.

$$\Rightarrow \exists (s_0,t_0) \in R(u),\ \partial_s \alpha(s_0,t_0) \neq 0.$$

Then there exists $k : \mathbb{R} \times S^1 \longrightarrow \mathbb{R}_{\geq 0}$ supported in $\mathcal{V}$ s.t.

$$0 \neq \iint_{S^1 \times \mathbb{R}} (\partial_s \alpha) k\, ds\, dt = -\int_{S^1}\int_{\mathbb{R}} (\partial_s k)\alpha\, ds\, dt \quad (\text{IBP}).$$

As before we consider $\mathcal{V}$ small enough s.t.:

$$\mathcal{V} \hookrightarrow S^1 \times M \qquad (s,t) \mapsto (t, u(s,t)) \text{ embedding}$$

and $\mathcal{U}$ a neighbourhood of this map s.t. $(t, u(s,t)) \in \mathcal{U} \Leftrightarrow (s,t) \in \mathcal{V}$.

Then we can construct $\hat{H} : S^1 \times M \longrightarrow \mathbb{R}$ supported in $\mathcal{U}$ s.t.
$\hat{H}(t, u(s,t)) = k(s,t)$ for all $(s,t) \in \mathcal{V}$ and extend it arbitrarily.

So $\frac{d}{ds} \hat{H}(t, u(s,t)) = d\hat{H}^t(u(s,t))(\partial_s u(s,t)) = \partial_s k(s,t)$

$$\iint_{\mathbb{R} \times S^1} d\hat{H}^t(u(s,t))(w(s,t))\, ds\, dt = \iint_{\mathcal{V}} d\hat{H}^t(u(s,t))(w(s,t))\, ds\, dt$$

$$= \iint_{\mathcal{V}} d\hat{H}^t(u(s,t))(\alpha(s,t)\, \partial_s u(s,t))\, ds\, dt$$

$$= \iint_{\mathcal{V}} \alpha(s,t)\, d\hat{H}^t(u(s,t))(\partial_s u(s,t))\, ds\, dt$$

$$= \iint_{\mathcal{V}} \alpha(s,t)\, \partial_s k(s,t)\, ds\, dt \neq 0,$$ and then

contradicting (3). Since $R(u)$ is dense in $C(u)^c$
$\partial_s \alpha = 0$ also on $C(u)^c$. Since $C(u)$ is discrete, $C(u)^c$ is
connected $\Rightarrow$ α can be seen as a function $S^1 \longrightarrow \mathbb{R}$. $\square$