1.1 Introduction

(M, ω) closed symplectic manifold. We assume

$$\omega|_{\pi_2(M)} = 0. \qquad \text{(symplectically aspherical)}$$

Non-deg. Arnold conjecture $\varphi : M \circlearrowleft$ Hamiltonian diffeomorphi[sm] ($\varphi = \phi_1$, where $\frac{d}{dt}\phi_t = X_{H_t}(\phi_t)$, $\phi_0 = id$, for some $H : \mathbb{T} \times M \to \mathbb{R}$) such that all contractible fix points are non-degenerate. Then

$$|\mathrm{Fix}_c(\varphi)| \geq \sum_{j=0}^{\dim M} \beta_j(M; \mathbb{F}).$$

Rmk. $x \in \mathrm{Fix}_c(\varphi)$ if $\phi_1(x) = x$ and $\{t \mapsto \phi_t(x)\}$ contractib[le]
Non-degenerate means that "linearized Ham. equations along $\phi_t(x)$" does not have non-trivial 1-periodic solutions or, equivalently, that $id - d\phi_1(x) : T_xM \to T_xM$ invertible.

Contractible fix points of ϕ correspond to "critical points" of $\mathbb{A}_H : \Lambda M := C^\infty_{con}(\mathbb{T}, M) \to \mathbb{R}$,

$$\mathbb{A}_H(\gamma) := -\int_{\mathbb{D}} \Gamma^* \omega + \int_{\mathbb{T}} H(t, \gamma(t))\, dt$$

where $\Gamma : \mathbb{D} \to M$ is such that $\Gamma|_{\partial \mathbb{D}} = \gamma$.

Goal Do Morse theory with $\mathbb{A}_H$.

- ΛM not a (Hilbert) manifold
- Natural candidate $H^{1/2}(\mathbb{T}, M)$ not a manifold
- "Morse index of critical points" is always ∞.

1.2 The Floer equation

Def. An almost complex structure (acs) on M is a section J of $\mathrm{Hom}(TM) \to M$ s.t. $J^2 = id$. J is called compatible with ω if $g(\cdot, \cdot) := \omega(\cdot, J\cdot)$ is a Riem. metric.

Prop. The space $\mathcal{J}_\omega := \{ J \mid J \text{ acs on } M \text{ compatible with } \omega \}$ is non-empty and contractible.

Fix $J \in \mathcal{J}_\omega$ and compute

$$d\mathbb{A}_H(\gamma)[\xi] = \int_{\mathbb{T}} \langle J(\gamma(t))\,\dot\gamma(t) + \nabla H_t(\gamma(t)), \xi(t) \rangle \, dt$$
$$= \langle \nabla_{L^2} \mathbb{A}_H(\gamma), \xi \rangle_{L^2}$$

where

$$\nabla_{L^2} \mathbb{A}_H(\gamma) = J(\gamma)\cdot\dot\gamma + \nabla H_t(\gamma).$$

The corresponding evolution equation

$$\frac{d}{ds} u = -\nabla_{L^2} \mathbb{A}_H(u), \qquad u : \mathbb{R} \to \left\{ \begin{matrix} \text{loops in} \\ M \end{matrix} \right\},$$

is not well-posed, but can regard as a PDE

$$\begin{cases} \partial_s u + J(u)\cdot \partial_t u + \nabla H_t(u) = 0, \\ u : \mathbb{R}\times\mathbb{T} \to M \end{cases} \qquad \text{(FE)}$$

which is called the <u>Floer equation</u>.

<u>Rmk.</u> (FE) is a zero-order perturbation of

$$\partial_s u + J(u)\,\partial_t u = 0,$$

whose solutions are called <u>pseudo-holomorphic curves</u> (Gromov).

<u>Floer's idea</u> In order to build a chain complex as in Morse homology we don't need a globally defined gradient flow, but just a "nice structure" for the space of solutions of (FE) s.t. $u(s,\cdot) \xrightarrow[s\to\pm\infty]{} \gamma_\pm$, for given p.o. $\gamma_\pm$.

<u>Key steps</u>

(i) Define $\mathcal{M}(\gamma_-,\gamma_+) := \left\{ u \;\middle|\; \begin{matrix} u \text{ solves (FE)} \\ u(s,\cdot) \xrightarrow[s\to\pm\infty]{} \gamma_\pm \end{matrix} \right\}$ and show that for generic choice of J is a finite dim. mfd.

(ii) For u solution of (FE) set

$$E(u) := \int_{\mathbb{R}} \int_{\mathbb{T}} \|\partial_s u(s,t)\|^2 \, dt\, ds = \|\partial_s u\|^2_{L^2} \quad \underline{\text{energy}}$$

and define

$$\mathcal{M} := \left\{ u \;\middle|\; \begin{array}{l} u \text{ solves (FE)} \\ E(u) < +\infty \end{array} \right\}.$$

Show that

$$\mathcal{M} = \bigcup_{\gamma_-, \gamma_+} \mathcal{M}(\gamma_-, \gamma_+)$$

and that $\mathcal{M}$ is compact in $C^\infty_{loc}(\mathbb{R} \times \mathbb{T}, M)$ (Gromov compactness).

(iii) Study $\overline{\mathcal{M}(\gamma_-, \gamma_+)}$ (broken Floer trajectories and gluing)

(iv) Build a boundary operator on the free module generated by the periodic orbits, indexed by the Conley-Zehnder index.

(v) Compute the resulting homology (continuation).

1.3 The linearized Floer equation

We assume $(M, \omega, J) = (\mathbb{R}^{2m}, \omega_0, J_0)$ and formally linearize (FE) along a solution u:

$$\partial_s v + J_0 \cdot \partial_t v + \nabla^2 H_t(u) \cdot v = 0. \qquad \text{(LFE)}$$

Set $S(s,t) := \nabla^2 H_t(u(s,t))$ and linear operator

$$D_S : W^{1,p}(\mathbb{R} \times \mathbb{T}, \mathbb{R}^{2m}) \longrightarrow L^p(\mathbb{R} \times \mathbb{T}, \mathbb{R}^{2m}), \qquad p \geq 2.$$
$$v \longmapsto \partial_s v + J_0 \partial_t v + S \cdot v$$

Theorem Let $S \in C^0(\overline{\mathbb{R}} \times \mathbb{T}, \mathrm{Hom}(\mathbb{R}^{2m}))$ be such that $S(\pm\infty, t)$ are loops of symmetric matrices with

$$\dot{\xi}(t) = J_0 S(\pm\infty, t) \cdot \xi(t)$$

non-degenerate linear Hamiltonian systems. Then, D_S is Fredholm, $\forall\, p \geq 2$, and $\mathrm{ind}(D_S) = \mu_{CZ}(\Phi_+) - \mu_{CZ}(\Phi_-)$.

Q. What are $\mu_{CZ}(\Phi_+)$, $\mu_{CZ}(\Phi_-)$?

Let $\phi_\pm : [0,1] \to Sp(2n)$ be the solution to

$$\begin{cases} \frac{d}{dt} \phi_\pm(t) = J_0 \cdot S(\pm\infty, t) \cdot \phi_\pm(t), \\ \phi_\pm(0) = id . \end{cases}$$

By non-degeneracy, $id - \phi_\pm(1)$ is invertible. We have that $\phi_\pm(1)$ can be connected to either W_0 or W_1 in $Sp(2n)$ by a path $\Psi_\pm : [0,1] \to Sp(2n)$ s.t. $id - \Psi(t)$ invertible $\forall\, t \in [0,1]$, where $W_0 = -id$ and $W_1 = \operatorname{diag}(2, -1, \dots, -1, -1/2, -1, \dots, -1)$. Denote $\tilde{\phi}_\pm := \phi_\pm \# \Psi_\pm$ and set the Conley-Zehnder index extension of $\det : U(n) \to S^1$

$$\mu_{CZ}(\Phi_\pm) := \deg\left(\det{}^2 \circ \tilde{\phi}_\pm : \mathbb{R}/_{2\mathbb{Z}} \longrightarrow S^1\right).$$

μ_{CZ} measures the number of half-windings in $Sp(2n)$.

Now, set

$$\mathcal{B} := \{ u \in W^{1,p}(\mathbb{R} \times \mathbb{T}, M) \mid u(s,\cdot) \xrightarrow[s \to \pm\infty]{} \gamma_\pm \}$$

$$\mathcal{E} \xrightarrow{\tau} \mathcal{B} \quad \text{Banach bundle with fibre}$$

$$\mathcal{E}_u := \overline{C^\infty_c(\mathbb{R} \times \mathbb{T}, u^* TM)}^{L^p}$$

Then,

$$\bar{\partial}_{J,H} : \mathcal{B} \longrightarrow \mathcal{E}, \quad \bar{\partial}_{J,H} u = \partial_s u + J(u)\partial_t u + \nabla H_t(u)$$

is a smooth section and we are interested in $\bar{\partial}_{J,H}^{-1}(\{0\})$.
The theorem implies that $\bar{\partial}_{J,H}$ is a non-linear Fredholm operator with index $\operatorname{ind}(\bar{\partial}_{J,H}) = \mu_{CZ}(\gamma_+) - \mu_{CZ}(\gamma_-)$.
Therefore, if 0 is a regular value, the IFT implies that

$$\mathcal{M}(\gamma_-, \gamma_+) = \bar{\partial}_{J,H}^{-1}(\{0\})$$

is a smooth manifold of dimension $\operatorname{ind}(\bar{\partial}_{J,H})$.
(see <u>Transversality</u>).