What is Typical?

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Let G be a compact second countable topological group equipped with the Borel σ-algebra $\mathcal{G}$.

For a measure μ on $(G, \mathcal{G})$ and a set $C \in \mathcal{G}$ such that $\mu(C) > 0$, define $\mu(\cdot \mid C)$ by

$$\mu(A \mid C) = \mu(A \cap C)/\mu(C), \quad A \in \mathcal{G}.$$

For $t \in G$, define $t\mu$ by $\quad t\mu(A) := \mu(t^{-1}A), \quad A \in \mathcal{G}$.

Let $\lambda \neq 0$ be a left-invariant Haar measure. Since G is compact, λ is finite and also right-invariant.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space supporting all random elements on this talk.

Let X be a random element in a space on which G acts, for instance $X = (X_s)_{s \in G}$ and $tX = (X_{t^{-1}s})_{s \in G}$.

Call X ***stationary*** if $tX \overset{D}{=} X$ for each $t \in G$.

Call S a ***typical*** location in G if $\mathbf{P}(S \in \cdot) = \lambda(\cdot \mid G)$.
And typical location ***for*** X if $\mathbf{P}(S \in \cdot \mid X) = \lambda(\cdot \mid G)$.

Theorem 1: If S is a typical location for X, then $S^{-1}X$ is stationary.

Proof: If S is a typical location for X then so is St^{-1} for each $t \in G$. Thus $(St^{-1})^{-1}X \overset{D}{=} S^{-1}X$.
But $(St^{-1})^{-1}X = t(S^{-1}X)$. Thus $t(S^{-1}X) \overset{D}{=} S^{-1}X$.

Call S a *typical* location in G if $\mathbf{P}(S \in \cdot) = \lambda(\cdot \mid G)$.
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Say the origin is at a typical location for X
if $S^{-1}X \stackrel{D}{=} X$ where S is a typical location for X.

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Theorem 2: The origin is at a typical location for X
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Theorem 2: The origin is at a typical location for X
if and only if X is stationary.

Proof: Let S be typical location for X. If $S^{-1}X \overset{D}{=} X$ then X is stationary since $S^{-1}X$ is stationary.

Conversely, if X is stationary then $S^{-1}X \overset{D}{=} X$ follows from stationarity and the independence of S and X.

Now let ξ be a nontrivial random measure on G.
Call ξ ***stationary*** if $t\xi \stackrel{D}{=} \xi$ for all $t \in G$.

For $t \in G$ put $t(X, \xi) = (tX, t\xi)$.
Call (X, ξ) ***stationary*** if $t(X, \xi) \stackrel{D}{=} (X, \xi)$ for all $t \in G$.

Call S a *typical* location in the *mass* of ξ if

$$\mathbf{P}(S \in \cdot \mid \xi) = \xi(\cdot \mid G)$$

and say that the origin is at a typical location in the mass of ξ if also

$$S^{-1}\xi \stackrel{D}{=} \xi.$$

Call S a typical location *for* X in the mass of ξ if

$$\mathbf{P}(S \in \cdot \mid X, \xi) = \xi(\cdot \mid G)$$

and say that the origin is at a typical location for X in the mass of ξ if also

$$S^{-1}(X, \xi) \stackrel{D}{=} (X, \xi).$$

The origin is at a typical location for X in the mass of ξ if and only if the same holds on certain sets C placed uniformly at random around the origin:

Theorem 3: The origin is at a typical location for X in the mass of ξ if and only if for all λ-**continuity** sets $C \in \mathcal{G}$, $\lambda(C) > 0$,

$$\big(V_C^{-1}(X, \xi), U_C V_C\big) \overset{D}{=} ((X, \xi), U_C)$$

where U_C and V_C are such that

(i) $\mathbf{P}(U_C \in \cdot \mid X, \xi) = \lambda(\cdot \mid C)$

(ii) $\mathbf{P}(V_C \in \cdot \mid X, \xi, U_C) = \xi(\cdot \mid U_C^{-1} C)$.

Proof of the only-if claim: Assume that the origin is at a typical location for X in the mass of ξ. Fix the set C and a measurable $f \geqslant 0$. We must prove that

$$\mathbf{E}\big[f\big(V_C^{-1}(X,\xi), U_C V_C\big)\big] = \mathbf{E}\big[f\big((X,\xi), U_C\big)\big]. \qquad (\star)$$

Use (i) $\mathbf{P}(U_C \in \cdot \mid X, \xi) = \lambda(\,\cdot \mid C)$

(ii) $\mathbf{P}(V_C \in \cdot \mid X, \xi, U_C) = \xi(\,\cdot \mid U_C^{-1} C)$

to take the first step towards establishing $(\star)$:

$$\mathbf{E}\big[f\big(V_C^{-1}(X,\xi), U_C V_C\big)\big]$$

$$= \mathbf{E}\Big[\iint 1_{\{u \in C\}} 1_{\{v \in u^{-1} C\}} f\left(v^{-1}(X,\xi), uv\right) \frac{\xi(dv)}{\xi(u^{-1}C)} \frac{\lambda(du)}{\lambda(C)}\Big]$$

Take S **such that** $\mathbf{P}(S \in \cdot \mid X, \xi) = \xi(\cdot \mid G)$.
Then $S^{-1}(X, \xi) \overset{D}{=} (X, \xi)$ **which yields the second step**

$$\mathbf{E}\Big[\iint 1_{\{u \in C\}} 1_{\{v \in u^{-1}C\}} f\big(v^{-1}(X, \xi), uv\big) \,\frac{\xi(dv)}{\xi(u^{-1}C)} \frac{\lambda(du)}{\lambda(C)}\Big]$$

$$= \mathbf{E}\Big[\iint 1_{\{u \in C\}} 1_{\{v \in u^{-1}C\}} f\big(v^{-1} S^{-1}(X, \xi), uv\big) \,\frac{S^{-1}\xi(dv)}{S^{-1}\xi(u^{-1}C)} \frac{\lambda(du)}{\lambda(C)}\Big]$$

Rewriting yields the third step:

$$\mathbf{E}\Big[\iint 1_{\{u\in C\}}1_{\{v\in u^{-1}C\}}f\big(v^{-1}S^{-1}(X,\xi),uv\big)$$

$$\frac{S^{-1}\xi(dv)}{S^{-1}\xi(u^{-1}C)}\frac{\lambda(du)}{\lambda(C)}\Big]$$

$$=\mathbf{E}\Big[\iint 1_{\{u\in C\}}1_{\{S^{-1}v\in u^{-1}C\}}f\big((S^{-1}v)^{-1}S^{-1}(X,\xi),uS^{-1}v\big)$$

$$\frac{\xi(dv)}{S^{-1}\xi(u^{-1}C)}\frac{\lambda(du)}{\lambda(C)}\Big]$$

Now use $\mathbf{P}(S \in \cdot \mid X, \xi) = \xi(\,\cdot \mid G)$ for the fourth step:

$$\mathbf{E}\Big[\iint 1_{\{u\in C\}} 1_{\{S^{-1}v \in u^{-1}C\}} f\big((S^{-1}v)^{-1} S^{-1}(X,\xi), uS^{-1}v\big) \frac{\xi(dv)}{S^{-1}\xi(u^{-1}C)} \frac{\lambda(du)}{\lambda(C)} \Big]$$

$$= \mathbf{E}\Big[\iiint 1_{\{u\in C\}} 1_{\{s^{-1}v \in u^{-1}C\}} f\big(v^{-1}(X,\xi), us^{-1}v\big) \frac{\xi(dv)}{s^{-1}\xi(u^{-1}C)} \frac{\lambda(du)}{\lambda(C)} \frac{\xi(ds)}{\xi(G)} \Big]$$

Make variable substitution $r = us^{-1}v \quad (u = rv^{-1}s)$ and use right-invariance of λ to take for the fifth step:

$$\mathbf{E}\Big[\iiint 1_{\{u\in C\}} 1_{\{s^{-1}v\in u^{-1}C\}} f\big(v^{-1}(X,\xi), us^{-1}v\big) \frac{\xi(dv)}{s^{-1}\xi(u^{-1}C)} \frac{\lambda(du)}{\lambda(C)} \frac{\xi(ds)}{\xi(G)}\Big]$$

$$= \mathbf{E}\Big[\iiint 1_{\{v^{-1}s\in r^{-1}C\}} 1_{\{r\in C\}} f\big(v^{-1}(X,\xi), r\big) \frac{\xi(dv)}{v^{-1}\xi(r^{-1}C)} \frac{\lambda(dr)}{\lambda(C)} \frac{\xi(ds)}{\xi(G)}\Big]$$

Rewrite to take the sixth step:

$$\mathbf{E}\Big[\iiint 1_{\{v^{-1}s\in r^{-1}C\}}1_{\{r\in C\}}f\big(v^{-1}(X,\xi),r\big)$$

$$\frac{\xi(dv)}{v^{-1}\xi(r^{-1}C)}\frac{\lambda(dr)}{\lambda(C)}\frac{\xi(ds)}{\xi(G)}\Big]$$

$$=\mathbf{E}\Big[\iiint 1_{\{s\in r^{-1}C\}}1_{\{r\in C\}}f\big(v^{-1}(X,\xi),r\big)$$

$$\frac{\xi(dv)}{v^{-1}\xi(r^{-1}C)}\frac{\lambda(dr)}{\lambda(C)}\frac{v^{-1}\xi(ds)}{\xi(G)}\Big]$$

Use $\mathbf{P}(S \in \cdot \mid X, \xi) = \xi(\cdot \mid G)$ **for the seventh step:**

$$\mathbf{E}\Big[\iiint 1_{\{s \in r^{-1}C\}} 1_{\{r \in C\}} f\big(v^{-1}(X,\xi), r\big) \frac{\xi(dv)}{v^{-1}\xi(r^{-1}C)} \frac{\lambda(dr)}{\lambda(C)} \frac{v^{-1}\xi(ds)}{\xi(G)}\Big]$$

$$= \mathbf{E}\Big[\iint 1_{\{s \in r^{-1}C\}} 1_{\{r \in C\}} f\big(S^{-1}(X,\xi), r\big) \frac{S^{-1}\xi(ds)}{S^{-1}\xi(r^{-1}C)} \frac{\lambda(dr)}{\lambda(C)}\Big]$$

Now, use $S^{-1}(X,\xi) \stackrel{D}{=} (X,\xi)$ for the eighth step:

$$\mathbf{E}\Big[\iint 1_{\{s\in r^{-1}C\}} 1_{\{r\in C\}} f\big(S^{-1}(X,\xi),r\big) \frac{S^{-1}\xi(ds)}{S^{-1}\xi(r^{-1}C)}\frac{\lambda(dr)}{\lambda(C)}\Big]$$

$$= \mathbf{E}\Big[\iint 1_{\{s\in r^{-1}C\}} 1_{\{r\in C\}} f\big((X,\xi),r\big) \frac{\xi(ds)}{\xi(r^{-1}C)}\frac{\lambda(dr)}{\lambda(C)}\Big]$$

Rewrite to take the ninth and tenth step:

$$\mathbf{E}\Big[\iint 1_{\{s\in r^{-1}C\}}1_{\{r\in C\}}f\big((X,\xi),r\big)\frac{\xi(ds)}{\xi(r^{-1}C)}\frac{\lambda(dr)}{\lambda(C)}\Big]$$

$$=\mathbf{E}\Big[\int\Big(\int 1_{\{s\in r^{-1}C\}}\frac{\xi(ds)}{\xi(r^{-1}C)}\Big)1_{\{r\in C\}}f\big((X,\xi),r\big)\frac{\lambda(dr)}{\lambda(C)}\Big]$$

$$=\mathbf{E}\Big[\int 1_{\{r\in C\}}f\big((X,\xi),r\big)\frac{\lambda(dr)}{\lambda(C)}\Big]$$

Finally, use (i) $\mathbf{P}(U_C \in \cdot \mid X, \xi) = \lambda(\,\cdot \mid C)$ **for the eleventh step:**

$$\mathbf{E}\Big[\int 1_{\{r\in C\}} f\big((X,\xi),r\big)\frac{\lambda(dr)}{\lambda(C)}\Big] = \mathbf{E}\big[f\big((X,\xi\big),U_C)\big],$$

that is, $(\star)$ **holds:**

$$\mathbf{E}\big[f\big(V_C^{-1}(X,\xi),U_CV_C\big)\big] = \mathbf{E}\big[f\big((X,\xi\big),U_C\big)\big]. \qquad (\star)$$

We have just established the following theorem.

Theorem 3: **The origin is at a typical location for X in the mass of ξ if and only if for all λ-continuity sets $C \in \mathcal{G}$, $\lambda(C) > 0$,**

$$\big(V_C^{-1}(X,\xi), U_C V_C\big) \stackrel{D}{=} ((X,\xi), U_C)$$

where U_C and V_C are such that

(i) $\mathbf{P}(U_C \in \cdot \mid X, \xi) = \lambda(\cdot \mid C)$

(ii) $\mathbf{P}(V_C \in \cdot \mid X, \xi, U_C) = \xi(\cdot \mid U_C^{-1} C)$.

Now let G be only locally compact.
Then λ and ξ are only σ-finite
so the previous typicality definitions do not work.

Definition 1: Say that the origin is at a *typical* location *for* X in the *mass* of ξ if for all relatively compact λ-continuity sets $C \in \mathcal{G}$ with $\lambda(C) > 0$,

$$\big(V_C^{-1}(X,\xi), U_C V_C\big) \stackrel{D}{=} ((X,\xi), U_C)$$

where U_C and V_C are such that

(i) $\mathbf{P}(U_C \in \cdot \mid X, \xi) = \lambda(\cdot \mid C)$

(ii) $\mathbf{P}(V_C \in \cdot \mid X, \xi, U_C) = \xi(\cdot \mid U_C^{-1}C)$.

Definition 1: Say that the origin is at a *typical* location *for* X in the *mass* of ξ if for all **relatively compact** λ-continuity sets $C \in \mathcal{G}$ with $\lambda(C) > 0$,

$$\big(V_C^{-1}(X,\xi), U_C V_C\big) \stackrel{D}{=} ((X,\xi), U_C) \qquad (\star)$$

where (i) $\mathbf{P}(U_C \in \cdot \mid X, \xi) = \lambda(\cdot \mid C)$

(ii) $\mathbf{P}(V_C \in \cdot \mid X, \xi, U_C) = \xi(\cdot \mid U_C^{-1} C)$.

The condition $(\star)$ is exactly the property used in Last and Thorisson (Ann. Probab. 2009) to define *mass-stationarity*: (X, ξ) is called mass-stationary if the origin is a typical location for X in the mass of ξ in the sense of Definition 1.

Recall that we are now only
assuming that G is locally compact.

Theorem: (X, λ) is mass-stationary
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Theorem: (X, ξ) is mass-stationary
if and only if
(X, ξ) is the Palm version of a stationary (Y, η)

Recall that a pair (X, ξ) is called a *Palm version* of a stationary pair (Y, η) if for all measurable $f \geqslant 0$ and all compact $A \in \mathcal{G}$ with $\lambda(A) > 0$,

$$\mathbf{E}[f(X, \xi)] = \mathbf{E}\Big[\int_A f\big(t^{-1}(Y, \eta)\big)\eta(dt)\Big]\Big/\lambda(A).$$

In this definition (X, ξ) and (Y, η) are allowed to have distributions that are only σ-finite and not necessarily probability measures.

The distribution of (X, ξ) is finite if and only if η has finite intensity, that is, if and only if $\mathbf{E}[\eta(A)] < \infty$ for compact A. In this case the distribution of (X, ξ) can be normalized to a probability measure.