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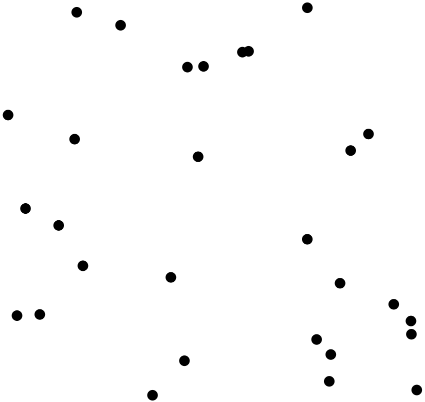
DFG FORSCHERGRUPPE
Geometry & Physics of Spatial Random Systems

Second order properties and central limit theorems for geometric functionals of Boolean models

joint work with Daniel Hug and Günter Last

Matthias Schulte

Boolean model



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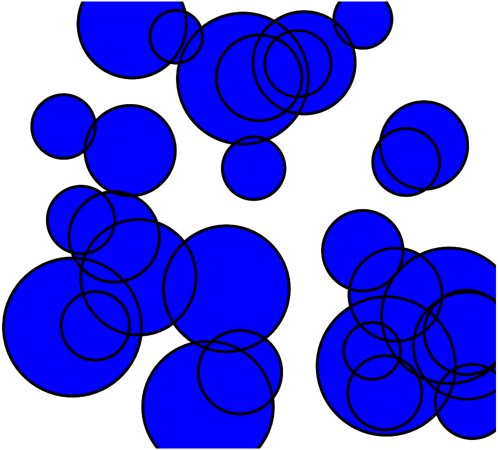

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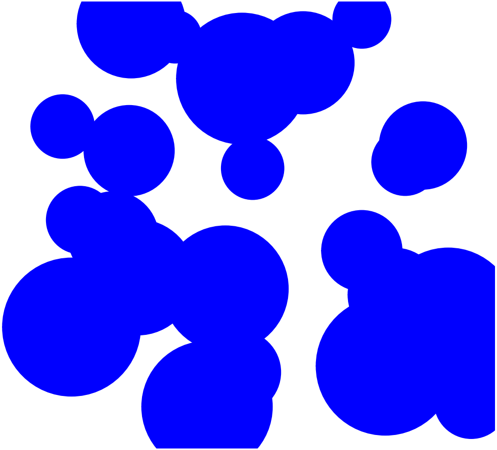

Boolean model

- $\mathcal{K}^d$ compact convex sets in $\mathbb{R}^d$
- $\mathbb{Q}$ probability measure on $\mathcal{K}^d$
- Measure Λ on $\mathcal{K}^d$ such that

$$\Lambda(\cdot) = \gamma \int_{\mathbb{R}^d} \int_{\mathcal{K}^d} \mathbf{1}\{x + K \in \cdot\} \, \mathbb{Q}(\mathrm{d}K) \, \mathrm{d}x$$

  with $\gamma > 0$
- η Poisson process on $\mathcal{K}^d$ with intensity measure Λ

η is stationary and any locally finite stationary Poisson process in $\mathcal{K}^d$ has an intensity measure of the form of Λ.

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Let the Boolean model Z be given by $Z = \bigcup_{K \in \eta} K$.

Geometric functionals of Boolean models

Let $\mathcal{R}^d$ denote the convex ring and let $r(K)$ be the inradius of K.

For $W \in \mathcal{K}^d$ and $\psi : \mathcal{R}^d \to \mathbb{R}$ we are interested in $\psi(Z \cap W)$ and, in particular, in its asymptotic behaviour as $r(W) \to \infty$.

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We assume that ψ is geometric, that is

- additive, i.e. $\psi(A \cup B) = \psi(A) + \psi(B) - \psi(A \cap B)$ for $A, B \in \mathcal{R}^d$;
- translation invariant, i.e. $\psi(x + A) = \psi(A)$ for all $x \in \mathbb{R}^d$ and $A \in \mathcal{R}^d$;
- locally bounded, i.e.

$$\sup\{|\psi(x + K)| : K \in \mathcal{K}^d, K \subset [0, 1]^d, x \in \mathbb{R}^d\} < \infty;$$

- measurable.

Geometric functionals of Boolean models

Examples of geometric functionals are

- volume and surface area,
- intrinsic volumes $V_i(\cdot)$, $i \in \{0, \dots, d\}$, which are given by

$$V_d(K + \varepsilon B^d) = \sum_{i=0}^{d} \kappa_{d-i} \varepsilon^{d-i} V_i(K), \quad K \in \mathcal{K}^d, \varepsilon > 0,$$

  and are even rigid motion invariant on $\mathcal{R}^d$,
- mixed volumes,
- integrals of surface area measures,
- ...

Expectation

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The typical grain Z_0 is a random set with distribution $\mathbb{Q}$. We assume that

$$v_i := \mathbb{E} V_i(Z_0) = \int_{\mathcal{K}^d} V_i(K) \, \mathbb{Q}(\mathrm{d}K) < \infty, \quad i \in \{0, \dots, d\}.$$

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Theorem: Miles 1976, Davy 1978

Assume that Z is isotropic and let $j \in \{0, \ldots, d\}$. Then

$$\mathbb{E} V_j(Z \cap W) - V_j(W) = -(1-p) \sum_{k=j}^{d} V_k(W) P_{j,k}(\gamma v_j, \ldots, \gamma v_{d-1})$$

for any $W \in \mathcal{K}^d$, where $P_{j,k}$ is a polynomial of degree $k-j$ on $\mathbb{R}^{d-j}$ and $p := \mathbb{E} V_d(Z \cap [0,1]^d) = \mathbb{P}(0 \in Z)$.

Expectation

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Basic examples:

For any $W \in \mathcal{K}^d$,

$$\mathbb{E} V_d(Z \cap W) = p V_d(W)$$

and

$$\mathbb{E} V_{d-1}(Z \cap W) = V_d(W)(1-p)\gamma v_{d-1} + V_{d-1}(W)p.$$

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Basic examples:

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For any geometric functional $\psi : \mathcal{R}^d \to \mathbb{R}$ the limit

$$\delta_\psi := \lim_{r(W)\to\infty} \frac{\mathbb{E}\psi(Z \cap W)}{V_d(W)}$$

exists.

Moment conditions

In the sequel, we will use the following moment conditions:

$$(\mathrm{M2}) \qquad \mathbb{E} V_i(Z_0)^2 < \infty, \quad i \in \{0, \dots, d\}$$

$$(\mathrm{M3}) \qquad \mathbb{E} V_i(Z_0)^3 < \infty, \quad i \in \{0, \dots, d\}$$

$$(\mathrm{M3} + \varepsilon) \qquad \mathbb{E} V_i(Z_0)^{3+\varepsilon} < \infty, \quad i \in \{0, \dots, d\}$$

Covariance structure

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Theorem: HLS 2013

Let ψ_1, ψ_2 be geometric functionals and assume (M2). Then the limit

$$\sigma_{\psi_1,\psi_2} := \lim_{r(W)\to\infty} \frac{\mathrm{Cov}(\psi_1(Z\cap W), \psi_2(Z\cap W))}{V_d(W)}$$

exists. Under (M3), there is a constant c_{ψ_1,ψ_2} such that

$$\left| \frac{\mathrm{Cov}(\psi_1(Z\cap W), \psi_2(Z\cap W))}{V_d(W)} - \sigma_{\psi_1,\psi_2} \right| \leq \frac{c_{\psi_2,\psi_2}}{r(W)}$$

for $W \in \mathcal{K}^d$ with $r(W) \geq 1$.

Covariance structure

More precisely,

$$\sigma_{\psi_1,\psi_2} = \sum_{n=1}^{\infty} \frac{\gamma}{n!} \int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^{n-1}} \psi_1^*(K_1 \cap \ldots \cap K_n) \\ \psi_2^*(K_1 \cap \ldots \cap K_n) \, \Lambda^{n-1}(\mathrm{d}(K_2, \ldots, K_n)) \, \mathbb{Q}(\mathrm{d}K_1)$$

where $\psi^* : \mathcal{K}^d \to \mathbb{R}$ is given by

$$\psi^*(K) = \mathbb{E}\psi(Z \cap K) - \psi(K), \quad K \in \mathcal{K}^d.$$

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where $\psi^* : \mathcal{K}^d \to \mathbb{R}$ is given by

$$\psi^*(K) = \mathbb{E}\psi(Z \cap K) - \psi(K), \quad K \in \mathcal{K}^d.$$

For $i, j \in \{0, \ldots, d\}$ we define

$$\sigma_{i,j} := \lim_{r(W)\to\infty} \frac{\mathrm{Cov}(V_i(Z \cap W), V_j(Z \cap W))}{V_d(W)}.$$

Idea of the proof

It follows from the Fock space representation (see Last/Penrose 2011) that

$$\begin{aligned}&\operatorname{Cov}(\psi_1(Z\cap W),\psi_2(Z\cap W))\\&=\sum_{n=1}^{\infty}\frac{1}{n!}\int_{(\mathcal{K}^d)^n}\mathbb{E}D^n_{K_1,\dots,K_n}\psi_1(Z\cap W)\\&\qquad\qquad\mathbb{E}D^n_{K_1,\dots,K_n}\psi_2(Z\cap W)\,\Lambda^n(\mathrm{d}(K_1,\dots,K_n))\end{aligned}$$

with

$$D^n_{K_1,\dots,K_n}\psi(Z\cap W)=\sum_{I\subset\{1,\dots,n\}}(-1)^{n-|I|}\psi(Z(\eta+\sum_{i\in I}\delta_{K_i})\cap W).$$

Idea of the proof

It turns out that

$$\mathbb{E} D^n_{K_1,\dots,K_n} \psi(Z \cap W) = (-1)^n \psi^*(K_1 \cap \ldots \cap K_n \cap W).$$

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It turns out that

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Combining the dominated convergence theorem with some new integral geometric inequalities, we show that

$$\begin{aligned}\lim_{r(W)\to\infty} \frac{1}{V_d(W)} \int_{(\mathcal{K}^d)^n} & \psi_1^*(K_1 \cap \dots \cap K_n \cap W) \\ & \psi_2^*(K_1 \cap \dots \cap K_n \cap W)\, \Lambda^n(\mathrm{d}(K_1, \dots, K_n)) \\ = \gamma \int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^{n-1}} & \psi_1^*(K_1 \cap \dots \cap K_n) \\ & \psi_2^*(K_1 \cap \dots \cap K_n)\, \Lambda^{n-1}(\mathrm{d}(K_2, \dots, K_n))\, \mathbb{Q}(\mathrm{d}K_1).\end{aligned}$$

Positive definiteness

Theorem: HLS 2013

Let ψ_k, $k \in \{0, \dots, d\}$, be homogeneous of degree k and satisfy

$$|\psi_k(K)| \geq \tilde{\beta}(\psi_k) r(K)^k, \quad K \in \mathcal{K}^d,$$

with constants $\tilde{\beta}(\psi_k) > 0$. Moreover, assume (M2) and that

$$(P) \qquad \mathbb{P}(V_d(Z_0) > 0) > 0.$$

Then the covariance matrix $\Sigma = (\sigma_{\psi_i,\psi_j})_{i,j \in \{0,\dots,d\}}$ is positive definite.

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Corollary: HLS 2013

If (P) and (M2) are satisfied, the covariance matrix $\Sigma = (\sigma_{i,j})_{i,j \in \{0,\dots,d\}}$ is positive definite.

Variance of the volume

Let $C_d(x) = \mathbb{E} V_d(Z_0 \cap (Z_0 + x))$, $x \in \mathbb{R}^d$, and $p := \mathbb{P}(0 \in Z)$. Then

$$\sigma_{d,d} = (1-p)^2 \int_{\mathbb{R}^d} e^{\gamma C_d(x)} - 1 \, dx.$$

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Proposition: HLS 2013

Under (M2) and (P) there is a constant $c_{d,d} > 0$ such that

$$\left| \sigma_{d,d} - \frac{\operatorname{Var} V_d(Z \cap W)}{V_d(W)} \right| \geq \frac{c_{d,d}}{r(W)}$$

for $W \in \mathcal{K}^d$ with $r(W) \geq 1$.

Covariance structure

For $i, j \in \{0, \ldots, d\}$ we define

$$\rho_{ij} = \sum_{n=1}^{\infty} \frac{\gamma}{n!} \int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^{n-1}} V_i(K_1 \cap K_2 \cap \ldots \cap K_n)$$

$$V_j(K_1 \cap K_2 \cap \ldots \cap K_n)\, \Lambda^{n-1}(\mathrm{d}(K_2, \ldots, K_n))\, \mathbb{Q}(\mathrm{d}K_1).$$

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Theorem: HLS 2013

Assume (M2) and let Z be isotropic. Then

$$\sigma_{i,j} = (1-p)^2 \sum_{k=i}^{d} \sum_{l=j}^{d} P_{i,k}(\gamma v_i, \dots, \gamma v_{d-1}) P_{j,l}(\gamma v_j, \dots, \gamma v_{d-1}) \rho_{k,l}$$

for $i, j \in \{0, \dots, d\}$. For $i, j \in \{d-1, d\}$ the formula remains true without isotropy.

Integral formulas

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Theorem: HLS 2013

The numbers $\rho_{i,j}$, $i, j \in \{0, \ldots, d\}$, can be expressed as

$$\rho_{i,j} = \int_{\mathbb{R}^d} e^{\gamma C_d(x)} H_{i,j}(\mathrm{d}x)$$

with certain multiple curvature measures $H_{i,j}$.

Covariance structure

Corollary: Heinrich/Molchanov 1999, HLS 2013

If (M2) is satisfied and the typical grain is almost surely full-dimensional, the asymptotic variance of the surface area is given by

$$\sigma_{d-1,d-1} = (1-p)^2\gamma^2 v_{d-1}^2 \int_{\mathbb{R}^d} \left(e^{\gamma C_d(x)} - 1\right) dx$$
$$+ (1-p)^2\gamma^2 \int_{(\mathbb{R}^d)^2} e^{\gamma C_d(x-y)} (C_{d-1}(x-y) - 2v_{d-1})\, M_{d-1,d}(d(y,x))$$
$$+ (1-p)^2\gamma \int_{\mathbb{R}^d} e^{\gamma C_d(x-y)}\, M_{d-1,d-1}(d(x,y))$$

with the second moment area measure

$$M_{d-1,d-1} := \mathbb{E} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}\{(x,y) \in \cdot\}\, \Phi_{d-1}(Z_0; dx)\, \Phi_{d-1}(Z_0; dy)$$

and the mean area-covariogram $C_{d-1}(x) := \mathbb{E}\Phi_{d-1}(Z_0; Z_0 + x)$.

Covariance structure

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Corollary: HLS 2013

Under (M2) the asymptotic covariance between volume and surface area is given by

$$\sigma_{d-1,d} = -(1-p)^2 \gamma v_{d-1} \int_{\mathbb{R}^d} \left(e^{\gamma C_d(x)} - 1 \right) dx + (1-p)^2 \gamma \int_{(\mathbb{R}^d)^2} e^{\gamma C_d(x-y)} M_{d-1,d}(d(x,y))$$

with the mixed moment measure

$$M_{d-1,d} := \mathbb{E} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}\{(x,y) \in \cdot\} \, \Phi_{d-1}(Z_0; dx) \, \Phi_d(Z_0; dy).$$

Spherical Boolean model

We consider volume and surface area of a spherical Boolean model with fixed radius in dependence on the intensity.

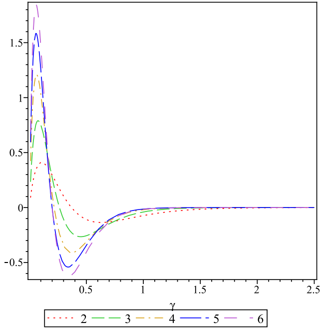

Figure : Covariance between volume and surface area for different dimensions

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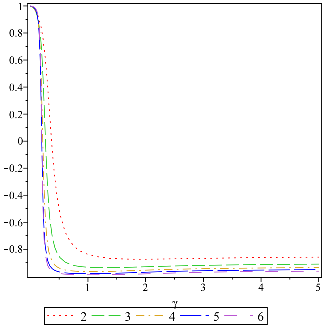

Figure : Correlation between volume and surface area for different dimensions

Covariances for $d = 2$

Corollary: HLS 2013

Assume (M2) and that Z is isotropic and $d = 2$. Then

$$\sigma_{0,2} = p(1-p)\gamma - (1-p)^2\left(\gamma - \frac{\gamma^2 v_1^2}{\pi}\right)\int_{\mathbb{R}^2}\left(e^{\gamma C_2(x)} - 1\right)dx$$
$$-(1-p)^2\frac{2\gamma^2 v_1}{\pi}\int_{(\mathbb{R}^2)^2} e^{\gamma C_2(x-y)}\, M_{1,2}(d(x,y)),$$

where we recall the mixed moment measure

$$M_{1,2} := \mathbb{E}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathbf{1}\{(x,y)\in\cdot\}\,\Phi_1(Z_0;dx)\,\Phi_2(Z_0;dy).$$

Covariances for $d = 2$

Corollary: HLS 2013

If additionally the typical grain is almost surely full-dimensional,

$$\sigma_{0,1} = (1-p)^2 \gamma v_1 + (1-p)^2 \left(\gamma^2 v_1 - \frac{\gamma^3 v_1^3}{\pi}\right) \int_{\mathbb{R}^2} \left(e^{\gamma C_2(x)} - 1\right) dx$$
$$+ (1-p)^2 \int_{(\mathbb{R}^2)^2} \tilde{\chi}(x-y)\, M_{1,2}(d(y,x))$$
$$- (1-p)^2 \frac{2\gamma^2 v_1}{\pi} \int_{(\mathbb{R}^2)^2} e^{\gamma C_2(x-y)}\, M_{1,1}(d(x,y)),$$

where

$$\tilde{\chi}(x) := e^{\gamma C_2(x)} \left(\frac{3\gamma^3 v_1^2}{\pi} - \frac{2\gamma^3 v_1}{\pi} C_1(x) - \gamma^2\right),$$
$$M_{1,1} := \mathbb{E} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbf{1}\{(x,y) \in \cdot\}\, \Phi_1(Z_0; dx)\, \Phi_1(Z_0; dy),$$

and $C_1(x) := \mathbb{E}\Phi_1(Z_0; Z_0 + x)$.

Covariances for $d = 2$

Corollary: HLS 2013

If additionally the typical grain is almost surely full-dimensional,

$$\begin{aligned}\sigma_{0,0} &= (1-2p)(1-p)\gamma + (1-p)(2p-3)\frac{\gamma^2 v_1^2}{\pi} \\ &+ (1-p)^2\Big(\gamma - \frac{\gamma^2 v_1^2}{\pi}\Big)^2 \int_{\mathbb{R}^2} \big(e^{\gamma C_2(x)} - 1\big)\,dx \\ &+ (1-p)^2 \int_{(\mathbb{R}^2)^2} \chi(x-y)\, M_{1,2}(d(y,x)) \\ &+ \frac{4}{\pi^2}(1-p)^2\gamma^3 v_1^2 \int_{(\mathbb{R}^2)^2} e^{\gamma C_2(x-y)}\, M_{1,1}(d(x,y)),\end{aligned}$$

where

$$\chi(x) := e^{\gamma C_2(x)}\Big(\frac{4\gamma^4 v_1^2}{\pi^2}(C_1(x) - v_1) + \frac{4\gamma^3 v_1}{\pi}\Big).$$

Spherical Boolean model

We consider a planar spherical Boolean model with fixed radius in dependence on the intensity.

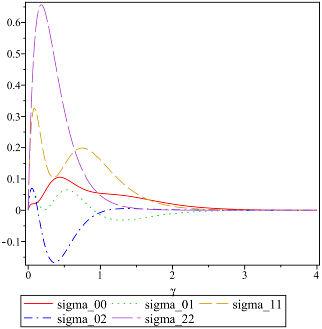

Figure : Covariances between intrinsic volumes

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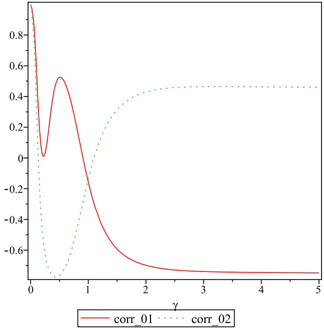

Figure : Correlations between intrinsic volumes

Probability distances

For two m-dimensional random vectors Y_1, Y_2 such that $\mathbb{E}\|Y_1\|^2 < \infty$ and $\mathbb{E}\|Y_2\|^2 < \infty$ we define

$$\mathbf{d_3}(Y_1, Y_2) = \sup_{h \in \mathcal{H}_m} |\mathbb{E}h(Y_1) - \mathbb{E}h(Y_2)|,$$

where $\mathcal{H}_m$ is the set of all thrice continuously differentiable functions $h : \mathbb{R}^m \to \mathbb{R}$ such that the second and third partial derivatives are bounded by one.

For two random variables Y_1, Y_2 such that $\mathbb{E}Y_1^2, \mathbb{E}Y_2^2 < \infty$ we define

$$\mathbf{d_W}(Y_1, Y_2) = \sup_{h \in \mathrm{Lip}(1)} |\mathbb{E}h(Y_1) - \mathbb{E}h(Y_2)|,$$

where $\mathrm{Lip}(1)$ is the set of all $h : \mathbb{R} \to \mathbb{R}$ whose Lipschitz constant is at most one.

Multivariate central limit theorem

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Theorem: HLS 2013

Let $\psi_1, \dots, \psi_m$ be geometric functionals and let $\Psi := (\psi_1, \dots, \psi_m)$. Assume (M2) and let N be a centred Gaussian random vector with covariance matrix $\Sigma = (\sigma_{\psi_i,\psi_j})_{i,j \in \{1,\dots,m\}}$. Then

$$\frac{1}{\sqrt{V_d(W)}}(\Psi(Z \cap W) - \mathbb{E}\Psi(Z \cap W)) \xrightarrow{d} N \quad \text{as} \quad r(W) \to \infty.$$

If (M3+ε) is satisfied, there is a constant $c_{\psi_1,\dots,\psi_m,\varepsilon} > 0$ such that

$$\mathbf{d_3}\left(\frac{1}{\sqrt{V_d(W)}}(\Psi(Z \cap W) - \mathbb{E}\Psi(Z \cap W)), N\right) \leq \frac{c_{\psi_1,\dots,\psi_m,\varepsilon}}{r(W)^{\min\{\varepsilon d/2, 1\}}}$$

for $W \in \mathcal{K}^d$ with $r(W) \geq 1$.

Univariate central limit theorem

Theorem: HLS 2013

Let ψ be additive, locally bounded and measurable and assume that

$$\liminf_{r(W)\to\infty} \frac{\operatorname{Var}\psi(Z\cap W)}{V_d(W)} > 0.$$

Assume (M2) and let N be a standard Gaussian random variable. Then

$$\frac{\psi(Z\cap W) - \mathbb{E}\psi(Z\cap W)}{\sqrt{\operatorname{Var}\psi(Z\cap W)}} \xrightarrow{d} N \quad \text{as} \quad r(W)\to\infty.$$

If (M3+ε) is satisfied, there are constants $c_{\psi,\varepsilon}$ and r_0 such that

$$\mathbf{d_W}\left(\frac{\psi(Z\cap W) - \mathbb{E}\psi(Z\cap W)}{\sqrt{\operatorname{Var}\psi(Z\cap W)}}, N\right) \leq \frac{c_{\psi,\varepsilon}}{V_d(W)^{\min\{\varepsilon/2,1/2\}}}$$

for $W\in\mathcal{K}^d$ with $r(W)\geq r_0$.

Central limit theorems

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Remark:

- CLTs for volume or surface area by Baddeley 1980, Mase 1982, Molchanov 1995, Heinrich/Molchanov 1999, Heinrich 2005, Baryshnikov/Yukich 2005, Penrose 2007, Heinrich/Spiess 2009.
- The rate of convergence in the multivariate case is optimal for $\varepsilon \geq 1$.
- The multivariate CLT holds for non-translation invariant functionals if the asymptotic covariance matrix exists.
- The rate of convergence in the univariate CLT is better than the rate in the multivariate CLT for $d \geq 3$ and $\varepsilon \geq 1$.
- The CLTs still hold for some non-stationary underlying Poisson processes.

Wiener-Itô chaos expansion

Let η be a Poisson process over a measurable space $(X, \mathcal{X})$ with a σ-finite intensity measure λ. Let I_n stand for the n-th Wiener-Itô integral.

Theorem: Last/Penrose 2011

Let $F \in L^2(\mathbb{P})$ be a Poisson functional. Then $f_n : X^n \to \mathbb{R}$, $n \in \mathbb{N}$, given by

$$f_n(x_1, \dots, x_n) = \frac{1}{n!} \mathbb{E} D^n_{x_1,\dots,x_n} F = \frac{1}{n!} \mathbb{E} \sum_{I \subset \{1,\dots,n\}} (-1)^{n-|I|} F(\eta + \sum_{i \in I} \delta_{x_i})$$

is in $L^2_s(X^n)$ and

$$F = \mathbb{E}F + \sum_{n=1}^{\infty} I_n(f_n),$$

where the right-hand side converges in $L^2(\mathbb{P})$. This implies that

$$\text{Var } F = \sum_{n=1}^{\infty} n! \|f_n\|_n^2.$$

Malliavin-Stein method

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Theorem: Peccati/Sole/Taqqu/Utzet 2010

Let $F \in L^2(\mathbb{P})$ be such that $F \in \operatorname{dom} D$ (i.e. $\sum_{n=1}^{\infty}(n+1)!\|f_n\|_n^2 < \infty$) and $\mathbb{E}F = 0$ and let N be a standard Gaussian random variable. Then

$$\mathbf{d_W}(F,N) \leq \mathbb{E}\Big|1 - \int_X D_xF(-D_xL^{-1}F)\,\lambda(\mathrm{d}x)\Big| + \mathbb{E}\int_X (D_xF)^2\,|DL^{-1}F|\,\lambda(\mathrm{d}x),$$

where

$$D_xF = \sum_{n=1}^{\infty} n\, I_{n-1}(f_n(x,\cdot)) \quad \text{and} \quad D_xL^{-1}F = -\sum_{n=1}^{\infty} I_{n-1}(f_n(x,\cdot)).$$

Malliavin-Stein method

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Theorem: HLS 2013

Let $F \in L^2(\mathbb{P}) \cap \operatorname{dom} D$ be such that

$$\int_{X^{|\sigma|}} |(f_i \otimes f_i \otimes f_j \otimes f_j)_\sigma| \, d\lambda^{|\sigma|} < \infty, \quad \sigma \in \Pi_{ij}, i, j \in \mathbb{N},$$

and assume that there are constants $a > 0$ and $b \geq 1$ such that

$$\int_{X^{|\sigma|}} |(f_i \otimes f_i \otimes f_j \otimes f_j)_\sigma| \, d\lambda^{|\sigma|} \leq \frac{a\, b^{i+j}}{(i!)^2 (j!)^2}, \quad \sigma \in \tilde{\Pi}_{ij}, i, j \in \mathbb{N}.$$

Let N be a standard Gaussian random variable. Then

$$\mathbf{d_W}\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq 2^{\frac{13}{2}} \sum_{i=1}^{\infty} i^{17/2} \frac{b^i}{\lfloor i/14 \rfloor !} \frac{\sqrt{a}}{\operatorname{Var} F}.$$

Idea of the proofs of the CLTs

- There is a multivariate version based on Peccati/Zheng 2010.
- We have to bound some asymptotic integrals, which can be treated similar as in the proof of the formula for the asymptotic covariances.
- So far, we must require the integrability condition (M3+ε). Using a truncation argument, this assumption can be weakened to (M2).

Thank you!

Thank you!

D. Hug, G. Last and M. Schulte: Second order properties and central limit theorems for geometric functionals of Boolean models, arXiv: 1308.6519.