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Percolation on stationary tessellations

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joint work with
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1. Stationary tessellations

Setting

X is a face-to-face tessellation of $\mathbb{R}^d$, that is a random collection of convex and bounded polytopes (called cells) covering the whole space and such that for any different $C, C' \in X$ the intersection $C \cap C'$ is either empty, or a face of both C and C'.

Definition

For $k \in \{0, \ldots, d\}$ let X_k denote the point process (on the space $\mathcal{P}^d$ of convex polytopes) of k-faces of X and let

$$\eta^{(k)} := \{s(F) : F \in X_k\}$$

denote the point process (on $\mathbb{R}^d$) of Steiner points of X_k.

Assumptions

The tessellation X is stationary, that is

$$X + x := \{C + x : C \in X\} \stackrel{d}{=} X, \quad x \in \mathbb{R}^d.$$

Moreover, for all compact sets $K \subset \mathbb{R}^d$,

$$\sum_{k=0}^{d} \mathbb{E} \sum_{F \in X_k} \mathbf{1}\{F \cap K \neq \emptyset\} < \infty.$$

2. Face percolation

Definition

Let $p \in [0, 1]$ and $n \in \{0, \ldots, d\}$. Given a tessellation X, we declare the polytopes in X_n independently black with probability p. All other polytopes in X_n are white. If $n \leq d - 1$ and $i \in \{n + 1, \ldots, d\}$, then we colour $F \in X_i$ black if all its $(i - 1)$-faces are black. Let

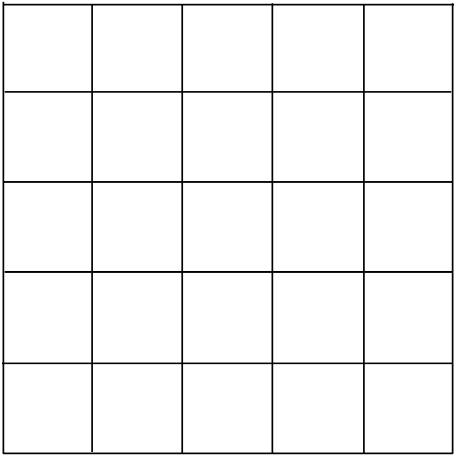
$$X_k^1 := \{F \in X_k : F \text{ is black}\}, \quad k \in \{0, \ldots, d\},$$

and

$$Z := \bigcup_{k=0}^{d} \bigcup_{F \in X_k^1} F.$$

Vertex percolation

Given X, the vertices are independently declared open with probability p. An edge is declared open if its endpoints are open. A cell is open if all its vertices are open.



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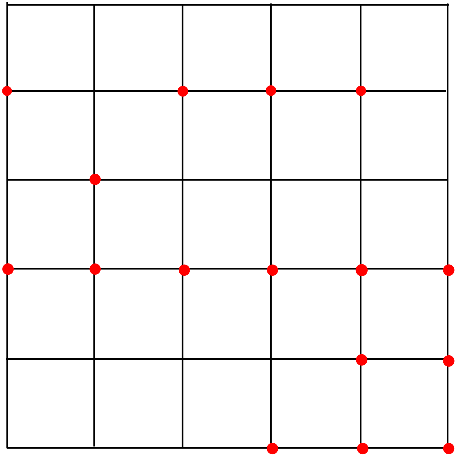

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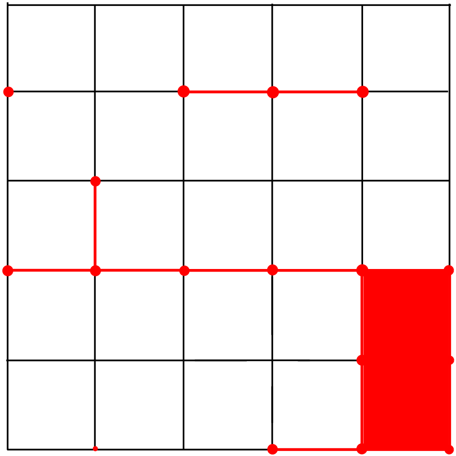
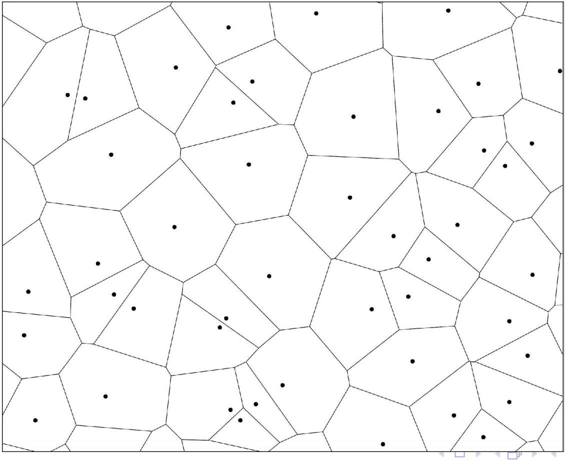


Voronoi percolation

Let X be a Voronoi tessellation. Declare the cells in X independently open with probability p and let Z be the union of all open cells.



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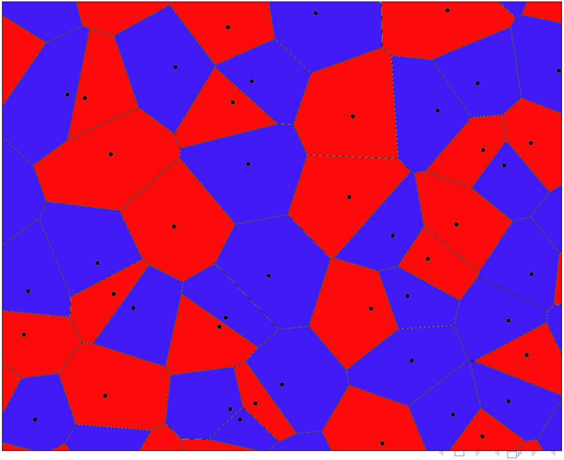

Theorem (Bollobás & Riordan '06)

Consider planar Poisson Voronoi percolation. Then $p_c = 1/2$. At this critical density there is no percolation, while above there is exactly one unbounded component.

3. Mean intrinsic volumes

Definition

Let $K \subset \mathbb{R}^d$ be compact and convex. The intrinsic volumes of K are the numbers $V_0(K), \ldots, V_d(K)$ uniquely determined by the Steiner formula

$$V_d(K + rB^d) = \sum_{j=0}^{d} r^j \kappa_j V_{d-j}(K), \quad r \geq 0,$$

where κ_j is the (j-dimensional) volume of the Euclidean unit ball B^j in $\mathbb{R}^j$.

Remark

$V_d(K)$ is the Lebesgue measure of K. If K has non-empty interior, then $V_{d-1}(K)$ is half the surface area of K. Moreover, $V_0(K) = \mathbf{1}\{K \neq \emptyset\}$.

Remark

The intrinsic volumes satisfy the additivity property

$$V_i(K \cup L) = V_i(K) + V_i(L) - V_i(K \cap L)$$

whenever $K, L, K \cup L$ are convex. Using the inclusion-exclusion formula the intrinsic volumes can be extended (uniquely!) to finite unions K of convex and compact sets. Then $V_{d-1}(K)$ is still half the surface area of K while $V_0(K)$ is the Euler characteristic of K.

Definition

Let $k \in \{0, \dots, d\}$ and denote by

$$\gamma_k := \mathbb{E}\eta^{(k)}[0,1]^d$$

the intensity of $\eta^{(k)}$. Let $\mathbb{P}_k^0$ denote the Palm probability measure of $\eta^{(k)}$. The expectation with respect to $\mathbb{P}_k^0$ is denoted by $\mathbb{E}_k^0$.

Definition

Let $x \in \mathbb{R}^d$. Then there are unique $k \in \{0, \dots, d\}$ and $F(x) := F \in X_k$ such that $x \in \mathrm{relint}(F)$. Under $\mathbb{P}_k^0$ the origin is almost surely in the relative interior of the k-face $F(0)$. The distribution

$$\mathbb{P}_k^0(F(0) \in \cdot)$$

is the distribution of the typical k-face.

Assumption

We assume that

$$\sum_{i,k=0}^{d} \mathbb{E}_k^0 V_i(F(0))^2 < \infty.$$

Definition (Face star)

Let $x \in \mathbb{R}^d$ and $l \in \{0, \dots, d\}$. Let k be the dimension of $F(x)$. If $l \geq k$ (resp. $l < k$) then we let $\mathcal{S}_l(x)$ be the set of all l-dimensional faces G such that $F(x) \subset G$ (resp. $G \subset F(x)$).

Theorem

Consider n-percolation on X. Let $W \subset \mathbb{R}^d$ be convex with $V_d(W) > 0$ and $i \in \{0, \dots, d\}$. Then the limit

$$\delta_i(p) := \lim_{t\to\infty} \frac{\mathbb{E} V_i(Z \cap tW)}{V_d(tW)}$$

exists and is given by

$$\delta_i(p) = \sum_{k=i}^{n} (-1)^{i+k} \gamma_k \mathbb{E}_k^0 \left[(1 - (1-p)^{|\mathcal{S}_n(0)|}) V_i(F(0)) \right]$$
$$+ \sum_{k=n+1}^{d} (-1)^{i+k} \gamma_k \mathbb{E}_k^0 \left[p^{|\mathcal{S}_n(0)|} V_i(F(0)) \right].$$

One idea of the proof: Using Groemer's (1972) extension of intrinsic volumes we have almost surely that

$$
\begin{aligned}
V_i(Z \cap tW) &= V_i(Z \cap \mathrm{int}(tW)) + V_i(Z \cap \partial tW) \\
&= \sum_{k=0}^{d} \sum_{F \in X_k^1} V_i(\mathrm{relint}(F \cap W_t)) + V_i(Z \cap \partial tW) \\
&= \sum_{k=0}^{d} (-1)^{i+k} \sum_{F \in X_k^1} V_i(F \cap tW) + V_i(Z \cap \partial tW),
\end{aligned}
$$

where we recall that X_k^1 is the set of black k-faces.

Example

For cell percolation on a planar and normal tessellation

$$\delta_0(p) = \gamma_2 p(1-p)(1-2p).$$

Example

For cell percolation on a planar line tessellation

$$\delta_0(p) = 3\gamma_2 p - 9\gamma_2 p^2 + 8\gamma_2 p^3 - 2\gamma_2 p^4.$$

4. Covariance structure

Assumption

We consider a normal stationary tessellation X and a convex body $W \subset \mathbb{R}^d$ of volume 1 (assumed to be a polytope if $d \geq 3$) such that the following limits exist for all $i, j \in \{0, \dots, d\}$:

$$\rho_{i,j}^{k,l} := \lim_{t \to \infty} \frac{1}{V_d(tW)} \mathbb{C}\mathrm{ov}\Big(\int V_i(F(x) \cap tW)\eta^{(k)}(dx), \int V_j(F(x) \cap tW)\eta^{(l)}(dx)\Big).$$

Remark

General tessellations require more efforts but can be treated as well.

Definition

For $i, j \in \{0, \dots, d\}$ we define asymptotic covariances

$$\sigma_{i,j}(p) := \lim_{t\to\infty} \frac{\mathbb{C}\mathrm{ov}(V_i(Z \cap tW), V_j(Z \cap tW))}{V_d(tW)}.$$

Definition

Let $x \in \mathbb{R}^d$, $l, n \in \{0, \dots, d\}$ and $m \in \mathbb{N}$. Define $\mathcal{S}_l^{m,n}(x)$ as the system of all l-dimensional faces sharing m n-faces with the face $F(x)$. Further let

$$S_{j,l}^{m,n} := \int V_j(F(x))\mathbf{1}\{F(x) \in \mathcal{S}_l^{m,n}(0)\}\eta^{(l)}(dx)$$

the total j-th intrinsic volumes of those faces.

Theorem

Consider n-percolation on X. Under suitable integrability assumptions the asymptotic covariances exist and are given by

$$\sigma_{i,j}(p) = \sum_{k=i}^{d}\sum_{l=j}^{d}(-1)^{i+j+k+l} f_{k,l}(p)\rho_{i,j}^{k,l}$$

$$+ \sum_{k=i}^{d}\sum_{l=j}^{d}(-1)^{i+j+k+l} \sum_{m=1}^{d+1-\max(k,l)} g_{k,l,m}(p)\gamma_k \mathbb{E}_k^0 V_i(F(0)) S_{j,l}^{m,n},$$

where $f_{k,l}$ and $g_{k,l,m}$ are explicitly given polynomials not depending on the distibution of X.

Remark

The above polynomials are given by

$$\begin{aligned} f_{k,l}(p) :=& ((1-(1-p)^{d-k+1})\mathbf{1}\{k<n\} + p^{d-k+1}\mathbf{1}\{k\geq n\}) \\ &\times ((1-(1-p)^{d-l+1})\mathbf{1}\{l<n\} + p^{d-l+1}\mathbf{1}\{l\geq n\}), \end{aligned}$$

and

$$\begin{aligned} g_{k,l,m}(p) :=& (1-p)^{2d-k-l-m+2}(1-(1-p)^m)\mathbf{1}\{k,l<n\} \\ &+ p^{d-k+1}(1-p)^{d-l+1}\mathbf{1}\{k\geq n, l<n\} \\ &+ (1-p)^{d-k+1}p^{d-l+1}\mathbf{1}\{k<n, l\geq n\} \\ &+ p^{2d-k-l-m+2}(1-p^m)\mathbf{1}\{l,k\geq n\}). \end{aligned}$$

The maximal degree (for $k=l=0$) is $2d+2$.

Corollary

The asymptotic covariance between volume and the j-th intrinsic volume is given by

$$\sigma_{d,j}(p) = \sum_{l=j}^{n-1} (-1)^{j+l} p(1-p)^{d-l+1} \gamma_d \mathbb{E}_d^0 V_d(F(0)) S_{j,l}^{1,n}$$
$$+ \sum_{l=\max(n,j)}^{d} (-1)^{j+l} p^{d-l+1}(1-p) \gamma_d \mathbb{E}_d^0 V_d(F(0)) S_{j,l}^{1,n}.$$

In particular we have for cell percolation (on arbitrary stationary tessellations)

$$\sigma_{d,d}(p) = p(1-p)\gamma_d \mathbb{E}_d^0[V_d(F(0))^2],$$
$$\sigma_{d,d-1}(p) = p(1-p)(1-2p)\gamma_d \mathbb{E}_d^0[V_d(F(0))V_{d-1}(F(0))].$$

5. Cell percolation on planar normal tessellations

Setting

In this section we consider cell percolation on a planar and normal tessellation.

Theorem

Under suitable integrability assumptions the asymptotic covariance between area and Euler characteristic is given by

$$\begin{aligned}\sigma_{0,2}(p) =& p(1-p)\gamma_2 \mathbb{E}_2^0 V_2(F(0)) \\ & - p^2(1-p)^2\gamma_2 \mathbb{E}_2^0[V_2(F(0)) f_0(F(0))],\end{aligned}$$

where $f_0(F(0))$ is the number of the vertices of the (typical cell) $F(0)$.

Theorem

The asymptotic covariance between surface length and Euler characteristic and the variance of the Euler characteristic are given by

$$\begin{aligned}
\sigma_{0,1}(p) =& p^2(1-p)^2(1-2p)(\rho_{1,0}^{2,2} - \gamma_2 \mathbb{E}_2^0[V_1(F(0)) f_0(F(0))]) \\
&+ p(1-p)(1-p-3p^2+2p^3)\gamma_2 \mathbb{E}_2^0[V_1(F(0))], \\
\sigma_{0,0}(p) =& \gamma_2 \mu_2 p^3(1-p)^3 \\
&+ \gamma_2 p(1-p)(1-9p-p^2+20p^3-10p^4) \\
&+ \rho_0 p^2(1-p)^2(1-2p)^2,
\end{aligned}$$

where $\mu_2 := \mathbb{E}_2^0 f_0(F(0))^2$ *and* $\rho_0 := \rho_{0,0}^{2,2}$ *is the asymptotic variance of* $\eta^{(2)}$.

Proof: By Euler's formula and normality

$$\gamma_0 = 2\gamma_2, \quad \gamma_1 = 3\gamma_2$$

and

$$\rho_{0,0}^{0,0} = 4\rho_0, \quad \rho_{0,0}^{0,1} = 6\rho_0,$$
$$\rho_{0,0}^{0,2} = 2\rho_0, \quad \rho_{0,0}^{1,1} = 9\rho_0, \quad \rho_{0,0}^{1,2} = 3\rho_0.$$

The result follows from the general theorem.

Remark

For a planar Poisson Voronoi tessellation $\rho_0 = \gamma_2$ and $\mu_2 \approx 37.78$ (Heinrich and Muche, 2008).

Corollary

The covariance $\sigma_{2,0}$ has a global minimum at $1/2$ while the variance $\sigma_{0,0}$ has a global maximum at $1/2$ if

$$\mu_2 > \frac{86}{3} + \frac{4\rho_0}{3\gamma_2}.$$

Remark

Jensen's inequality and $\mathbb{E}_2^0 f_0(F(0)) = 6$ imply that

$$\mu_2 \geq 36.$$

6. Poisson Voronoi percolation

Setting

In this section we consider cell percolation on the Voronoi tessellation generated by a stationary Poisson process η of intensity 1.

Definition

Let $\eta^x := \eta \cup \{x\}$ and $\eta^{0,x} := \eta \cup \{0, x\}$, $x \in \mathbb{R}^d$, and define a stochastic kernel κ by

$$\kappa(x, \cdot) := \mathbb{P}((C(\eta^{0,x}, 0), C(\eta^{0,x}, x)) \in \cdot), \quad x \in \mathbb{R}^d,$$

and the random variables

$$V_i^{(k)}(x) := V_i(\mathcal{F}_k(C(\eta^x, x))).$$

Theorem

The limits $\rho_{i,j}^{k,l}$ exist and are given by

$$(d-k+1)(d-l+1)\rho_{i,j}^{k,l} = \mathbb{E} V_i^{(k)}(0) V_j^{(l)}(0)$$
$$+ \int \Big[\int V_i(\mathcal{F}_k(C)) V_j(\mathcal{F}_l(C')) \, \kappa(x, d(C, C')) - \mathbb{E} V_i^{(k)}(0) \mathbb{E} V_j^{(l)}(0) \Big] \, \mathrm{d}x.$$

7. A central limit theorem

Theorem

Consider cell percolation on a Poisson Voronoi tessellation. Then the vector

$$\xi_t := (V_0(Z \cap tW), \dots, V_d(Z \cap tW))$$

of intrinsic volumes satisfies the central limit theorem

$$t^{-1/2}(\xi_t - \mathbb{E}\xi_t) \xrightarrow{d} N(p) \quad \text{as } t \to \infty,$$

where $N(p)$ is a centred normal distribution with covariance matrix $(\sigma_{ij}(p))$. For $p \in (0,1)$ this matrix is positive definite.

Idea of the proof: Stabilization theory (Penrose and Yukich, 2005).

8. References

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