Existence of point processes via entropy bounds

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Message of this talk:

- One should not always use the weak topology of measures!
- Instead, it is often convenient to use a notion of convergence of measures that exploits only the measurable structure of the underlying space.
- In this way one avoids continuity assumptions.
- Relative entropy is then a useful tool.

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- Instead, it is often convenient to use a notion of convergence of measures that exploits only the measurable structure of the underlying space.
- In this way one avoids continuity assumptions.
- Relative entropy is then a useful tool.

Arose from my work on Gibbs measures and LDPs '88, '93

Used for point processes jointly with
Zessin '93, Häggström '96, Dereudre '09, '12, Thäle '13, ...

Setwise convergence

$(\Omega, \mathcal{F})$ measurable space, $P_n, P \in \mathscr{P} := \mathscr{P}(\Omega, \mathcal{F})$

Definitions

$P_n \to P$ **setwise** $\Leftrightarrow \forall\, A \in \mathcal{F} : \; P_n(A) \to P(A)$

$\Leftrightarrow \forall\, f \in M^b(\Omega, \mathcal{F}) : \; \int f\, dP_n \to \int f\, dP$

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$(P_n)_{n \geq 1}$ equicontinuous

$$\Leftrightarrow \forall\, A_j \in \mathcal{F}, A_j \downarrow \varnothing : \quad \limsup_{n \to \infty} \; P_n(A_j) \downarrow 0 \quad \text{as } j \uparrow \infty$$

. . . , Gänßler '71

Every equicontinuous sequence in $\mathscr{P}$ admits a convergent subsequence.

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Proof.

As $P_n \in [0,1]^{\mathcal{F}}$, $\exists$ subnet (n_α) $\exists$ $P \in [0,1]^{\mathcal{F}}$: $P_{n_\alpha} \to P$ setwise.

P is additive and, by equicontinuity of (P_n), σ-additive.

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To find a subsequence, take $Q \in \mathscr{P}$ with $P_n = g_n Q$, $P = gQ$.

Let $\mathcal{G} = \sigma(g, g_n : n \geq 1)$, $\mathcal{A}$ a countable algebra generating $\mathcal{G}$.

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$P_{n_\alpha} \to P$ setwise $\Rightarrow$ $\exists$ n_k $\forall$ $A \in \mathcal{A}$: $P_{n_k}(A) \to P(A)$

Monotone class $\Rightarrow$ $P_{n_k}|_{\mathcal{G}} \to P|_{\mathcal{G}}$ setwise $\Rightarrow$

$$\forall\, A \in \mathcal{F} : \; P_{n_k}(A) = \int Q(A|\mathcal{G})\, dP_{n_k} \to \int Q(A|\mathcal{G})\, dP = P(A) \qquad \diamond$$

Local convergence

Assume from now on:

$(\Omega, \mathcal{F}) = \underset{\ell\to\infty}{\text{proj-lim}}\ (\Omega_\ell, \mathcal{F}_\ell)$ projective limit of 'local' Borel spaces

Definition

$P_n \to P$ locally $\Leftrightarrow$ $\forall\, \ell :$ $P_n^{(\ell)} \to P^{(\ell)}$ setwise

(P_n) locally equicontinuous $\Leftrightarrow$ $\forall\, \ell :$ $(P_n^{(\ell)})_{n\geq 1}$ equicontinuous

Corollary

Every locally equicontinuous sequence in $\mathscr{P}$ admits a locally convergent subsequence.

(Use Gänßler's theorem, diagonal method, Kolmogorov extension)

Entropy

Fix a reference p.m. $Q \in \mathscr{P}$

Local relative entropy of P rel. to Q

$$H(P^{(\ell)}; Q^{(\ell)}) := \sup_{g \in M^b(\Omega_\ell, \mathcal{F}_\ell)} \Big[\int g \, dP^{(\ell)} - \log \int e^g \, dQ^{(\ell)} \Big]$$

Definition

$(P_n)_{n \geq 1}$ locally entropy-bounded relative to $Q \in \mathscr{P}$

$$\Leftrightarrow \quad \forall \, \ell \geq 1 : \; \limsup_{n \to \infty} \; H(P_n^{(\ell)}; Q^{(\ell)}) =: c_\ell < \infty$$

Proposition

$(P_n)_{n\geq 1}$ locally entropy-bd. rel. to Q $\Rightarrow$ $(P_n)_{n\geq 1}$ admits a locally convergent subsequence

Proof.

$\forall\ \ell \geq 1,\ A_j \in \mathcal{F}_\ell,\ A_j \downarrow \varnothing$

$$\forall\ a > 0: \quad a\, P_n^{(\ell)}(A_j) \leq H(P_n^{(\ell)}; Q^{(\ell)}) + \log \int e^{a 1_{A_j}}\, dQ^{(\ell)}$$

$$\leq c_\ell + 1 \qquad \text{for large } n \text{ and } j$$

$\Rightarrow$ $(P_n^{(\ell)})_{n\geq 1}$ equicontinuous $\diamond$

Application to point processes

- (E, d) complete separable metric space, $\mathcal{E}$ Borel
- $\Omega := \mathscr{M}_p(E, \mathcal{E})$, $\mathcal{F} = \sigma(N_\Delta : \Delta \in \mathcal{E}_b)$
- Q_ϱ Poisson p.p. with intensity measure $\varrho \in \mathscr{M}(E, \mathcal{E})$
- $\Lambda_\ell \in \mathcal{E}_b$, $\Lambda_\ell \uparrow E$, $\Omega_\ell := \mathscr{M}_p(\Lambda_\ell, \mathcal{E} \cap \Lambda_\ell)$

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Theorem

$(P_n)_{n\geq 1}$ locally entropy-bd. rel. to Q_ϱ $\Rightarrow$

$$\exists\, n_k, P \;\; \forall\, f \in \mathscr{L} : \int f\, dP_{n_k} \to \int f\, dP$$

$f \in \mathscr{L} \;\; \Leftrightarrow \;\; \exists\, \ell, b_1, b_2$ s.t. $f(\omega) = f(\omega_{\Lambda_\ell})$, $|f| \leq b_1 + b_2\, N_{\Lambda_\ell}$

No continuity of f required!

Consequence: Campbell measures also converge locally!

Proof.

Proposition $\Rightarrow$ $\exists\, n_k$ s.t. convergence holds $\forall\, f \in \mathscr{L}^b$.

General f : Wlog $|f| \leq N_{\Lambda_\ell}$. Consider $f_m = f\, 1_{\{|f| \leq m\}} \in \mathscr{L}^b$.

$$\sup_n \Big| \int f\, dP_n - \int f_m\, dP_n \Big| \leq \sup_n \int N_{\Lambda_\ell}\ 1_{\{|N_{\Lambda_\ell}| \geq m\}}\ dP_n$$

$$\leq \frac{c_\ell}{a} + \frac{1}{a} \log \int e^{a N_{\Lambda_\ell}\, 1_{\{|N_{\Lambda_\ell}| \geq m\}}} dQ$$

$$\leq \varepsilon \quad \text{if} \ \ a, m \ \ \text{sufficiently large.}$$

$\diamond$

Case of stationary marked point processes:

$$E = \mathbb{R}^d \times M,\ \varrho = z\,\lambda \otimes \mu,\ \Lambda_\ell = [-2^{\ell-1}, 2^{\ell-1}]^d \times M$$

$$P \text{ stationary } \Rightarrow \exists\ h_\varrho(P) = \uparrow\lim_{\ell\to\infty}\ 2^{-\ell}\ H(P^{(\ell)}; Q_\varrho^{(\ell)})$$

Corollary

$\forall\ c \geq 0 :\ \big\{P \in \mathscr{P}_\Theta :\ h_\varrho(P) \leq c\big\}$ *sequentially compact*

Gibbsian point processes

Definitions

$\gamma : E \times \Omega \to \mathbb{R}_+$ Papangelou intensity $\Leftrightarrow$

$$\forall\, x, y \in E,\ \omega \in \Omega :\ \gamma(x,\omega)\,\gamma(y,\omega+\delta_x)\ \text{ symmetric in } x, y$$

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P Gibbs for γ and ϱ $\quad\Leftrightarrow\quad \mathrm{C}^!_P = \gamma(\varrho \otimes P) \quad\Leftrightarrow\quad \forall\, f \geq 0$

$$\int P(d\omega) \int \omega(dx)\, f(x, \omega{-}\delta_x) = \int \varrho(dx) \int P(d\omega)\, \gamma(x,\omega)\, f(x,\omega)$$

(GNZ)

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Basic Existence Theorem

γ *s.t.* (i) strong stability: $\gamma(x,\omega) \leq c$

(ii) finite range: $\forall\, x \in E\ \exists\, \ell\ \forall\, \omega \in \Omega : \gamma(x,\omega) = \gamma(x, \omega_{\Lambda_\ell})$

$\Rightarrow$ $\exists$ *Gibbs p.p. for* γ *and every* ϱ

Proof. Define $P_n := Z_n^{-1}\ \hat{\gamma}(\cdot, 0)\ Q_{\varrho|\Lambda_n}$ (free b.c.)

where for $\alpha = \{x_1, \ldots x_n\}$

$$\hat{\gamma}(\alpha, \omega) := \gamma(x_1, \omega)\, \gamma(x_2, \omega+\delta_{x_1}) \cdots \gamma(x_n, \omega+\delta_{x_1}+\cdots+\delta_{x_{n-1}})$$

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Then $\forall\ \ell < n,\ \omega$:

$$P_n(\,\cdot\,|\,\omega \text{ in } \Lambda_n\setminus\Lambda_\ell) = \underbrace{Z_{\Lambda_\ell}(\omega)}_{\geq\ e^{-\varrho(\Lambda_\ell)}}{}^{-1}\ \underbrace{\hat\gamma(\cdot\,,\omega)}_{\leq\ c^{N_{\Lambda_\ell}}}\ Q_{\varrho|\Lambda_\ell}$$

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$$\Rightarrow\ H(P_n^{(\ell)}; Q_\varrho^{(\ell)}) \leq c\,\varrho(\Lambda_\ell)\ \Rightarrow\ P_{n_k} \to \text{some } P \ \text{ locally.}$$

Since also $\mathrm{C}_{P_{n_k}} \to \mathrm{C}_P$ locally, P is Gibbsian for γ. $\diamond$

Standard examples

- Repulsive pair interaction of finite range:

  $E = \mathbb{R}^d$, $\gamma(x,\omega) = z\, e^{-\int \omega(dy)\, \varphi(x,y)}$ with

  (i) either $\varphi \geq 0$, or $\varphi \geq -c$ and $\varphi(x,y) = \infty$ for $|x-y| \leq r$

  (ii) $\varphi(x,y) = 0$ for $|x-y| > R$.

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- Two-type Widom-Rowlinson gas:

  $E = \mathbb{R}^d \times \{-1,1\}$, $\gamma(x,\sigma;\omega) = z\, 1_{\left\{\omega\left(B_r(x)\times\{-\sigma\}\right) = 0\right\}}$

Variations & extensions

A. Quermaß interactions

$E = \mathbb{R}^2 \times \mathbb{R}_+$, $\varrho = z\lambda \otimes \mu$, μ subexponential

$$\Gamma(\omega) := \bigcup_{(x,r)\in\omega} B_r(x) \qquad \text{(Boolean model)}$$

ω 'nice' $\Rightarrow$ $\log \gamma(x,r;\omega)$ '=' $W\big(\Gamma(\omega+\delta_{(x,r)})\big) - W\big(\Gamma(\omega)\big)$

where W is a linear combination of Minkowski functionals

Dereudre '09

$h_\varrho(P_n) \le c$ for suitable P_n, c. Hence $\exists$ stationary Gibbs p.p. for γ.

(Energy estimates for point collections)

Variations & extensions

B. Geometric interactions

$E = \mathbb{R}^2$, $\varrho = z\lambda$, $\Gamma(\omega) :=$ Delaunay or Voronoi graph of ω

ω 'nice' $\Rightarrow$ $\log \gamma(x, \omega)$ '$=$' $W\big(\Gamma(\omega{+}\delta_x)\big) - W\big(\Gamma(\omega)\big)$

where

$$W(\omega) = - \sum_{\alpha \text{ clique of } \Gamma(\omega)} \varphi(\alpha, \omega) \ ,$$

E.g.: $\varphi(\alpha, \omega) = \infty$ if α is a Delaunay triangle with smallest angle $< \frac{\pi}{3} - \delta$

Dereudre, Drouilhet, Georgii '12

φ reasonable, $z > z(\varphi) \geq 0$ $\Rightarrow$ $\exists$ stationary Gibbs p.p. for γ.

Variations & extensions

C. Branching tessellations

Cells of a tessellation of $\mathbb{R}^d$ are randomly divided by hyperplanes.

Reference model $Q = \text{STIT}$:

Cells split independently without memory

Gibbsian branching:

Cell $c \in T_s$ is divided by H at time s with rate $\Psi(s, c, T_s, dH)$

Georgii, Schreiber, Thäle '13

Ψ reasonable $\Rightarrow$ $\exists$ Gibbsian branching tessellation for Ψ.

Proof via some conditional relative entropy.